

A Nonlinear Viscoelastic–Viscoplastic Constitutive Model for Epoxy Polymers

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The objective of this paper is to develop a nonlinear viscoelastic–viscoplastic constitutive model for epoxy polymers. The classic nonlinear viscoelasticity model is reformulated to yield a closed-form incremental constitutive relation, which relates the stress increments to the viscoelastic strain increments. A viscoplasticity model, which consists of the Drucker–Prager yield function, nonlinear isotropic and kinematic hardening laws, and the Perzyna viscosity function, is subsequently developed. The nonlinear viscoelasticity and the viscoplasticity models are then implemented in a radial return algorithm. A closed-form algorithmic tangent operator is derived to facilitate model implementation. The nonlinear viscoelasticity model is validated by reproducing the test data for polymethyl methacrylate (PMMA). The present constitutive model’s capabilities are demonstrated through modeling viscoelastic–viscoplastic PMMA loaded at different strain rates. The mechanics of structure genome-based micromechanics approach, embedded with the present constitutive model, is used to homogenize a unidirectional fiber-reinforced composite with a PMMA matrix, subjected to uniaxial and shear loading at different strain rates. The viscoelastic characterization of PMT-F7 epoxy is evaluated by comparing the predicted creep–recovery responses by the present constitutive model with the experimental ones.

I. Introduction

Deployable composite booms are designed to be self-deployed in space after (1) flattened and coiled for stowage and (2) stowed for a long time before launch. Thin-ply high strain composites (TP-HSCs) are widely used in booms thanks to their high strength-to-weight ratios, low manufacturing costs, and ability to sustain large curvatures without failure. Despite these advantages, TP-HSCs in composite booms may be degraded by different mechanisms during boom coiling and stowage, and the composite booms only partially recover their cross-sections when deployed. Specifically:

1. When a TP-HSC is bent during boom coiling, its resin matrix may undergo finite, viscoplastic deformation.

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2. When a TP-HSC has been stowed for a long time, its resin matrix may undergo considerable stress relaxation.
3. Once a TP-HSC is considerably degraded, the composite boom may exhibit reduced bending/torsional stiffness and not be successfully deployed.

It is critical but challenging to predict the extent of viscoplastic deformation/stress relaxation in a TP-HSC. For one thing, there is still a need for a constitutive model for the resin matrix exhibiting material and geometric nonlinearities in practice. For another, the multiscale modeling of a composite boom often spans multiple length scales (e.g. fiber, yarn, ply, laminate, and boom structure in Figure 1). In this paper, focus is placed on the development of (1) a nonlinear viscoelasticity model, (2) a viscoplasticity model for epoxy polymers, and (3) a redial return algorithm unifying these two models.

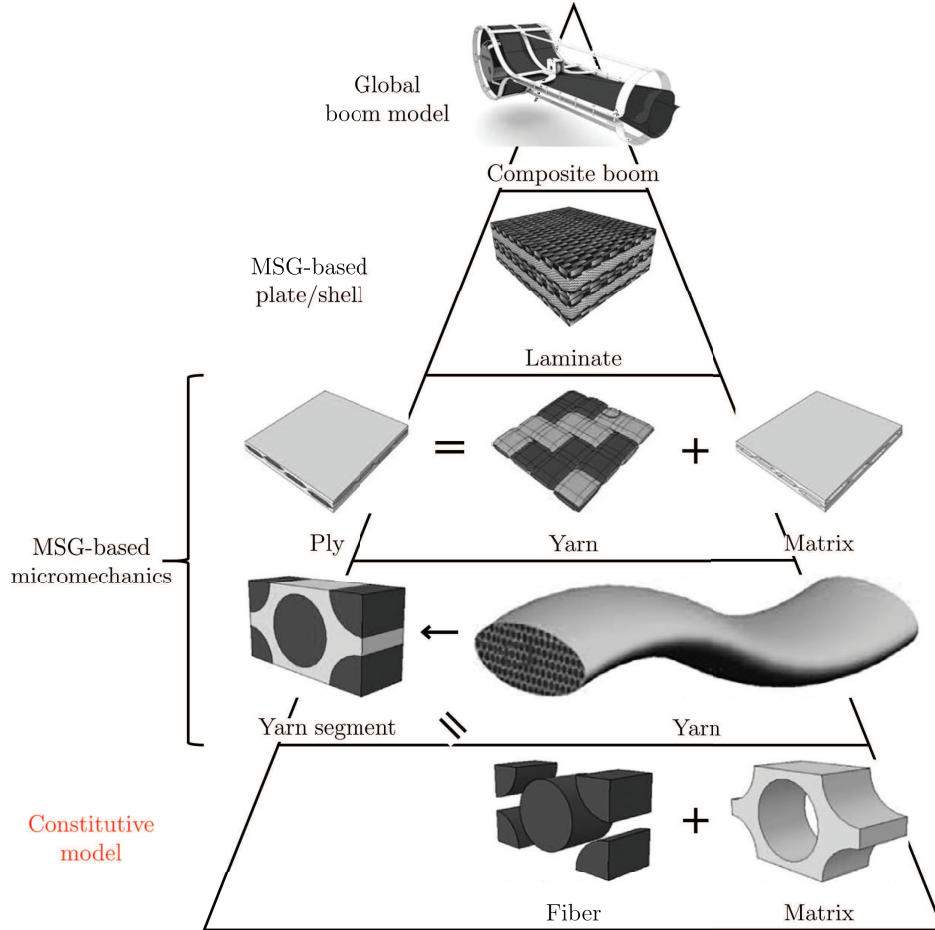


Figure 1. Bottom-up mechanics of structure genome (MSG)-based multiscale modeling.

A linear viscoelasticity model (e.g., the Boltzmann integral) is applicable to materials undergoing strains up to about 1%. Experimental observations indicate that materials often exhibit stress-dependent viscoelasticity when undergoing strains beyond 1%. Shapery¹ constructed the most widely used nonlinear viscoelasticity model (i.e., the Shapery integral) based on the thermodynamics of irreversible processes. The Shapery integral characterizes stress dependence (1) by introducing four stress-dependent functions and (2) by relating the strains to not only the stresses but also these functions. Several authors developed constitutive models based on the Shapery integral for its finite element (FE) implementation. Starting from the Shapery integral, Lai and Bakker² (1) divided the time domain into numerous small intervals, (2) assumed that all stress-dependent functions remained constant over each small time interval, and (3) subsequently derived a closed-form incremental constitutive relation. On the one hand, the closed-form constitutive relation can be either unified with a viscoplasticity model or conveniently implemented in an FE framework. On the other

hand, the above assumption severely limits the accuracy of this model. Noting this shortcoming, Haj-Ali and Muliana³ abandoned the above assumption and developed an implicit time integration algorithm for the Shapery integral. Despite improved accuracy, this integration algorithm cannot be conveniently unified with a viscoplasticity model. Therefore, it is necessary to derive a closed-form constitutive relation without the above assumption.

A viscoplasticity model consists of the yield criterion, the flow rule, the viscosity function, and the hardening law(s). Different chosen options on these components result in different models. Some authors^{4–6} chose the von Mises criterion, an associated flow rule, the Johnson–Cook strain rate dependence, and an isotropic hardening law. This model is widely used for polymers and is readily available in a commercial FE code, Abaqus. However, the von Mises criterion is for metals, and the viscosity function and the hardening law have no thermodynamic foundation and therefore limit the accuracy of this model. Some others^{7–9} chose the Drucker–Prager criterion, an associated/non-associated flow rule, differently assumed viscosity functions, and a nonlinear isotropic hardening law. The Drucker–Prager criterion is suitable for polymers because it can characterize polymer yielding mechanisms (i.e., shear yielding and crazing due to shear and mean stresses). On the other hand, the hardening law has no thermodynamic foundation. Krairi and Doghri¹⁰ chose then von Mises criterion, a non-associated flow rule, the Perzyna viscosity function, and nonlinear isotropic and kinematic hardening laws. This model has a rigorous thermodynamic foundation and is very versatile, but the von Mises criterion is for metals. In conclusion, there is still a need for a viscoplasticity model that (1) has a rigorous thermodynamic foundation and (2) is especially developed for epoxy polymers.

An integration scheme involves computing, updating, and storing all state variables, in each load step. Radial return is among the most popular integration schemes for viscoplasticity models. This is because radial return balances accuracy and efficiency well. In the linear (stress-independent) regime, the radial return algorithm is well established,^{11,12} but in the nonlinear regime, there is still no radial return algorithm. In this paper, the linear radial return algorithm is modified to accommodate the current objective, and the nonlinear viscoelasticity and the viscoplasticity models are implemented in the modified algorithm subsequently. An incremental constitutive relation relates the stress increments to the strain increments. It is needed for the assembly of the global stiffness matrix, during the FE implementation of a constitutive model. Due to the complexity of the problem, none of competing constitutive models has a closed-form incremental constitutive relation. The lack of a closed-form incremental constitutive relation is problematic for implementation since it requires treating the plastic hardening variables as additional degrees of freedom (DOFs), customizing the FE solver to consider the additional DOFs, and solving the displacements and hardening variables in a staggered manner. In this paper, a closed-form incremental constitutive relation is derived to facilitate model calibration and implementation.

The objective of this paper is to develop a nonlinear viscoelastic–viscoplastic constitutive model for epoxy polymers. The classic nonlinear viscoelasticity model is reformulated to yield a closed-form incremental constitutive relation, which relates the stress increments to the viscoelastic strain increments. A viscoplasticity model, which consists of the Drucker–Prager yield function, nonlinear isotropic and kinematic hardening laws, and the Perzyna viscosity function, is subsequently developed. The nonlinear viscoelasticity and the viscoplasticity models are then implemented in a radial return algorithm. A closed-form algorithmic tangent operator is derived to facilitate model implementation. The nonlinear viscoelasticity model is validated by reproducing the test data for polymethyl methacrylate (PMMA). The present constitutive model’s capabilities are demonstrated through modeling viscoelastic–viscoplastic PMMA loaded at different strain rates. The MSG-based micromechanics approach, embedded with the present constitutive model, is used to homogenize a unidirectional fiber-reinforced composite (UDFRC) with a PMMA matrix, subjected to uniaxial and shear loading at different strain rates. The viscoelastic characterization of PMT-F7 epoxy is evaluated by comparing the predicted creep–recovery responses by the present constitutive model with the experimental ones.

II. Nonlinear Viscoelasticity Model

In this section, the classic nonlinear viscoelasticity model is reformulated to yield a closed-form incremental constitutive relation, which relates the stress increments to the viscoelastic strain increments. For brevity, assume that the deformation is entirely viscoelastic throughout this section. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ denote the

stress and the strain tensor, respectively. The 1D Schapery integral reads¹

$$\epsilon = g_0 D_0 \sigma + g_1 \int_0^t \Delta D [\psi(t) - \psi(s)] \frac{d(g_2 \sigma)}{ds} ds, \quad (1)$$

where D_0 and ΔD denote the instantaneous and the transient compliance, respectively, g_0 , g_1 , and g_2 are stress-dependent functions, and ψ is a stress-dependent effective time. The constitutive relation for an isotropic, linearly elastic material reads

$$\epsilon = \frac{1}{2} J \sigma' + \frac{1}{3} B \sigma_m \mathbf{I} \otimes \mathbf{I}, \quad (2)$$

where J and B denote the shear and the bulk compliance, respectively, $(\cdot)'$ and $(\cdot)_m$ the deviatoric part and mean of a second-order tensor, respectively, and $\mathbf{I}$ the second-order identity tensor. Figure 2 depicts a 1D generalized Maxwell model, where $(\cdot)_i$ denotes a quantity in the i^{th} element, D_i the transient compliance, λ_i the inverse of the characteristic time, and ϵ_i the strain. Following Ref. [2], decompose ΔD as

$$\Delta D(\psi) = \sum_{i=1}^N D_i [1 - \exp(-\lambda_i \psi)]. \quad (3)$$

J_i 's and B_i 's can be related to D_i 's by Poisson's ratio, ν , i.e.,

$$J_i = 2(1 + \nu) D_i \quad \text{and} \quad B_i = 3(1 - 2\nu) D_i. \quad (4)$$

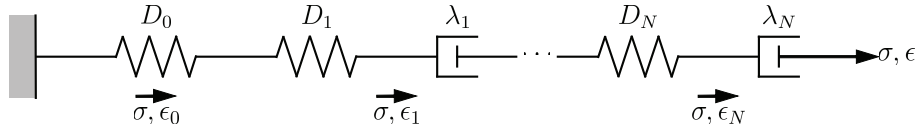


Figure 2. 1D generalized Maxwell model.

The 3D Schapery integral can be formulated similarly to the 1D one as

$$\epsilon' = \frac{1}{2} g_0 J_0 \sigma' + \frac{1}{2} g_1 \int_0^t \Delta J [\psi(t) - \psi(s)] \frac{d(g_2 \sigma')}{ds} ds, \quad (5)$$

$$\epsilon_m = \frac{1}{3} g_0 B_0 \sigma_m + \frac{1}{3} g_1 \int_0^t \Delta B [\psi(t) - \psi(s)] \frac{d(g_2 \sigma_m)}{ds} ds, \quad (5')$$

where

$$\Delta J(\psi) = \sum_{i=1}^N J_i [1 - \exp(-\lambda_i \psi)] \quad \text{and} \quad \Delta B(\psi) = \sum_{i=1}^N B_i [1 - \exp(-\lambda_i \psi)]. \quad (6)$$

Following Refs. [2,3], let the stress-dependent functions in Eq. (5) take the forms of

$$g_0 = 1 + \sum_{i=1}^N \alpha_i \left\langle \frac{\|\sigma'\|}{\sigma_0} - 1 \right\rangle^i, \quad g_1 = 1 + \sum_{i=1}^N \beta_i \left\langle \frac{\|\sigma'\|}{\sigma_0} - 1 \right\rangle^i, \quad (7)$$

$$g_2 = 1 + \sum_{i=1}^N \alpha_i \left\langle \frac{\|\sigma'\|}{\sigma_0} - 1 \right\rangle^i, \quad a_\sigma = 1 + \sum_{i=1}^N \delta_i \left\langle \frac{\|\sigma'\|}{\sigma_0} - 1 \right\rangle^i, \quad \psi = \int_0^t \frac{ds}{a_\sigma(s)}, \quad (7')$$

where

$$\langle x \rangle = \begin{cases} x & x \geq 0, \\ 0 & x < 0, \end{cases} \quad (8)$$

is the Macaulay bracket, $\|\sigma'\| = \sqrt{\frac{3}{2} \sigma' : \sigma'}$ the von Mises stress, and σ_0 the stress threshold of nonlinear viscoelasticity. Differentiating both sides of the last equation in Eq. (7) gives

$$d\psi = \frac{dt}{a_\sigma(t)}. \quad (9)$$

ϵ' and ϵ_m can be decomposed as

$$\epsilon' = \epsilon'_0 + \sum_{i=1}^N \epsilon'_i \quad \text{and} \quad \epsilon_m = (\epsilon_m)_0 + \sum_{l=1}^N (\epsilon_m)_l, \quad (10)$$

respectively. Combining Eqs. (5), (6), and (10) gives

$$\epsilon'_0 = \frac{1}{2}g_0 J_0 \sigma' + \frac{1}{2}g_1 \int_0^t \sum_{i=1}^N J_i \frac{d(g_2 \sigma')}{ds} ds, \quad (11)$$

$$\epsilon'_i = -\frac{1}{2}g_1 \int_0^t J_i \exp\{-\lambda_i [\psi(t) - \psi(s)]\} \frac{d(g_2 \sigma')}{ds} ds, \quad (12)$$

$$(\epsilon_m)_0 = \frac{1}{3}g_0 B_0 \sigma_m + \frac{1}{3}g_1 \int_0^t \sum_{i=1}^N B_i \frac{d(g_2 \sigma_m)}{ds} ds, \quad (13)$$

$$(\epsilon_m)_i = -\frac{1}{3}g_1 \int_0^t B_i \exp\{-\lambda_i [\psi(t) - \psi(s)]\} \frac{d(g_2 \sigma_m)}{ds} ds. \quad (14)$$

Suppose that all variables at a given instant of time, t_n , are known. Let $(\cdot)_n$ denote a quantity at t_n , and let $\Delta(\cdot) = (\cdot)_{n+1} - (\cdot)_n$, e.g., $t_{n+1} = t_n + \Delta t$. Let $\Delta\epsilon$ be prescribed strain increments over $[t_n, t_{n+1}]$. For notational convenience, omit the subscript $n+1$ on each quantity at t_{n+1} . The task is to relate $\Delta\epsilon$ to $\Delta\sigma$ by a closed-form expression. Similarly to ϵ' and ϵ_m , $\Delta\epsilon'$ and $\Delta\epsilon_m$ can be decomposed as

$$\Delta\epsilon' = \Delta\epsilon'_0 + \sum_{i=1}^N \Delta\epsilon'_i \quad \text{and} \quad \Delta\epsilon_m = \Delta(\epsilon_m)_0 + \sum_{l=1}^N \Delta(\epsilon_m)_l, \quad (15)$$

respectively. First relate $\Delta\epsilon'_i$ to $\Delta\sigma$. Eq. (12) can be rewritten as

$$\epsilon'_i = -\frac{1}{2}g_1 \int_0^{t_n} J_i \exp\{-\lambda_i [\psi(t) - \psi(s)]\} \frac{d(g_2 \sigma')}{ds} ds - \frac{1}{2}g_1 \int_{t_n}^t J_i \exp\{-\lambda_i [\psi(t) - \psi(s)]\} \frac{d(g_2 \sigma')}{ds} ds. \quad (16)$$

The first term on the right side of Eq. (16) can be approximated using the forward Euler method as

$$-\frac{1}{2}g_1 \int_0^{t_n} J_i \exp\{-\lambda_i [\psi(t) - \psi(s)]\} \frac{d(g_2 \sigma')}{ds} ds = \exp(-\lambda_i \Delta\psi) (\epsilon'_i)_n, \quad (17)$$

and the second integral can be similarly approximated as

$$\int_{t_n}^t J_i \exp\{-\lambda_i [\psi(t) - \psi(s)]\} \frac{d(g_2 \sigma')}{ds} ds = J_i \exp(-\lambda_i \Delta\psi) \left(\frac{dg_2}{dt} \sigma'_n + g_2 \mathcal{I}' \right) \Delta t, \quad (18)$$

where $\Delta\psi = \psi_n \Delta t$. Eq. (7) indicates that g_0 , g_1 , and g_2 are functions of $\|\sigma'\|$ (and therefore σ). Also note that σ varies with time t . dg_0 , dg_1 , and dg_2 can then be expressed using the chain rule and a change of variables as

$$dg_0 = \frac{dg_0}{dt} \Delta t = \frac{dg_0}{d\sigma} : \Delta\sigma, \quad dg_1 = \frac{dg_1}{dt} \Delta t = \frac{dg_1}{d\sigma} : \Delta\sigma, \quad dg_2 = \frac{dg_2}{dt} \Delta t = \frac{dg_2}{d\sigma} : \Delta\sigma. \quad (19)$$

Substituting Eqs. (17)–(19) into Eq. (16) gives

$$\epsilon'_i = \exp(-\lambda_i \Delta\psi) (\epsilon'_i)_n - \frac{1}{2}g_1 J_i \exp(-\lambda_i \Delta\psi) \left(\sigma'_n \otimes \frac{dg_2}{d\sigma} + g_2 \mathcal{I}' \right) : \Delta\sigma, \quad (20)$$

where $\mathcal{I}' = \mathcal{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}$ denotes the fourth-order deviatoric projection operator with $\mathcal{I}$ denoting the fourth-order identity tensor. Subtracting both sides of Eq. (20) by $(\epsilon'_i)_n$ gives

$$\Delta\epsilon'_i = -\frac{1}{2}g_1 J_i \exp(-\lambda_i \Delta\psi) \left(\sigma'_n \otimes \frac{dg_2}{d\sigma} + g_2 \mathcal{I}' \right) : \Delta\sigma - [1 - \exp(-\lambda_i \Delta\psi)] (\epsilon'_i)_n. \quad (21)$$

Similarly to $\Delta\epsilon'_i$, $\Delta(\epsilon_m)_i$, $\Delta\epsilon'_0$, and $\Delta(\epsilon_m)_0$ can be related to $\Delta\sigma$ by

$$\Delta(\epsilon_m)_i = -\frac{1}{3}g_1B_i \exp(-\lambda_i\Delta\psi) \left[(\sigma_m)_n \frac{dg_2}{d\sigma} + \frac{1}{3}g_2\mathbf{I} \right] : \Delta\sigma - [1 - \exp(-\lambda_i\Delta\psi)] [(\epsilon_m)_i]_n, \quad (22)$$

$$\Delta\epsilon'_0 = \left[\frac{1}{2}J_0 \left(\sigma'_n \otimes \frac{dg_0}{d\sigma} + g_0\mathbf{I}' \right) + \frac{1}{2}g_1 \sum_{i=1}^N J_i \left(\sigma'_n \otimes \frac{dg_2}{d\sigma} + g_2\mathbf{I}' \right) \right] : \Delta\sigma, \quad (23)$$

$$\Delta(\epsilon_m)_0 = \left\{ \frac{1}{3}B_0 \left[(\sigma_m)_n \frac{dg_0}{d\sigma} + \frac{1}{3}g_0\mathbf{I} \right] + \frac{1}{3}g_1 \sum_{i=1}^N B_i \left[(\sigma_m)_n \frac{dg_2}{d\sigma} + \frac{1}{3}g_2\mathbf{I} \right] \right\} : \Delta\sigma, \quad (24)$$

respectively. Substituting Eqs. (21)–(24) into Eq. (15) gives

$$\Delta\epsilon' = \tilde{\mathcal{J}} : \Delta\sigma - \sum_{i=1}^N [1 - \exp(-\lambda_i\Delta\psi)] (\epsilon'_i)_n, \quad (25)$$

$$\Delta\epsilon_m = \tilde{\mathcal{B}} : \Delta\sigma - \sum_{i=1}^N [1 - \exp(-\lambda_i\Delta\psi)] [(\epsilon_m)_i]_n, \quad (25')$$

where

$$\tilde{\mathcal{J}} = \frac{1}{2} \left[J_0 \left(\sigma'_n \otimes \frac{dg_0}{d\sigma} + g_0\mathbf{I}' \right) + g_1 \sum_{i=1}^N J_i [1 - \exp(-\lambda_i\Delta\psi)] \left(\sigma'_n \otimes \frac{dg_2}{d\sigma} + g_2\mathbf{I}' \right) \right] \quad (26)$$

and

$$\tilde{\mathcal{B}} = \frac{1}{3} \left\{ B_0 \left[(\sigma_m)_n \frac{dg_0}{d\sigma} + \frac{1}{3}g_0\mathbf{I} \right] + g_1 \sum_{i=1}^N B_i [1 - \exp(-\lambda_i\Delta\psi)] \left[(\sigma_m)_n \frac{dg_2}{d\sigma} + \frac{1}{3}g_2\mathbf{I} \right] \right\}. \quad (27)$$

Introduce tangent stiffness tensor $\tilde{\mathcal{C}}$ such that

$$\tilde{\mathcal{C}}^{-1} = \tilde{\mathcal{J}} + \mathbf{I} \otimes \tilde{\mathcal{B}}. \quad (28)$$

Eq. (25) can then be rewritten as

$$\Delta\epsilon = \tilde{\mathcal{C}}^{-1} : \Delta\sigma - \sum_{i=1}^N [1 - \exp(-\lambda_i\Delta\psi)] (\epsilon_i)_n. \quad (29)$$

Premultiplying both sides of Eq. (29) by $\tilde{\mathcal{C}}$ gives

$$\Delta\sigma = \tilde{\mathcal{C}} : \Delta\epsilon + \sum_{i=1}^N [1 - \exp(-\lambda_i\Delta\psi)] \tilde{\mathcal{C}} : (\epsilon_i)_n, \quad (30)$$

which is the incremental constitutive relation when the deformation is entirely viscoelastic.

III. Viscoplasticity Model

In this section, a viscoplasticity model, which consists of the Drucker–Prager yield function, nonlinear isotropic and kinematic hardening laws, and the Perzyna viscosity function, is developed. The yield function of a material exhibiting combined nonlinear isotropic–kinematic hardening takes the general form of

$$f = f(\sigma, \mathbf{X}, R), \quad (31)$$

where $\mathbf{X}$ denotes the back stress tensor due to kinematic hardening, and R is related to the current yield stress. Assume that there exist a pseudo-viscoplastic potential, $F(\sigma, \mathbf{X}, R)$, and a viscoplastic multiplier, $\dot{v}$, such that the viscoplastic evolution laws are given by¹³

$$\dot{\epsilon}^{vp} = \dot{v} \frac{\partial F}{\partial \sigma}, \quad \dot{\alpha} = -\dot{v} \frac{\partial F}{\partial \mathbf{X}}, \quad \dot{r} = -\dot{v} \frac{\partial F}{\partial R}, \quad (32)$$

where ϵ^{vp} denotes the viscoplastic strain, α is a second-order tensor describing kinematic hardening, and r is a scalar describing isotropic hardening. Once f , F , and $\dot{v}$ are specified, the viscoplastic evolution laws can be fully determined.

Let the material obey the Drucker–Prager yield function, i.e.,

$$f(\sigma, \mathbf{X}, R) = \frac{1}{1-\alpha} (\|\beta'\| + 3\alpha\beta_m) - \sigma_c - R, \quad (33)$$

where

$$\beta = \sigma - \mathbf{X}, \quad \|\beta'\| = \sqrt{\frac{3}{2}\beta' : \beta'}, \quad \alpha = \frac{\sigma_c - \sigma_t}{\sigma_c + \sigma_t}, \quad (34)$$

and σ_c and σ_t denote the initial yield stress in uniaxial compression and that in uniaxial tension, respectively. Meanwhile, set¹³

$$F = f + \frac{3a}{4C}\mathbf{X} : \mathbf{X} + \frac{R^2}{2Q}, \quad (35)$$

where C and a are two kinematic hardening parameters, and Q is an isotropic hardening parameter. Substituting Eq. (35) into Eq. (32) gives

$$\dot{\epsilon}^{vp} = \dot{v}\mathbf{n}, \quad \dot{\alpha} = \dot{\epsilon}^{vp} - \frac{3a}{2C}\mathbf{X}\dot{v}, \quad \dot{r} = \left(1 - \frac{R}{Q}\right)\dot{v}, \quad (36)$$

where $\mathbf{n} = \partial f / \partial \sigma = \partial F / \partial \sigma$ denotes the normal to the yield surface. Assume that¹³

$$\mathbf{X} = \frac{2}{3}C\alpha \quad \text{and} \quad R = bQr, \quad (37)$$

where b is another isotropic hardening parameter. Substituting the last two equations of Eq. (36) into the rate form of Eq. (37) gives

$$\dot{\mathbf{X}} = \frac{2}{3}C\dot{\epsilon}^{vp} - a\mathbf{X}\dot{v} \quad \text{and} \quad \dot{R} = b(Q - R)\dot{v}, \quad (38)$$

which are the Chaboche hardening law¹⁴ and the rate form of the Voce hardening law,¹⁵ respectively. Integrating the second equation of Eq. (38) with respect to time gives the Voce hardening law,

$$R = Q[1 - \exp(-bv)]. \quad (39)$$

Differentiating both sides of Eq. (33) with respect to σ gives

$$\mathbf{n} = \frac{1}{1-\alpha} \left(\frac{\partial \|\beta'\|}{\partial \sigma} + 3\alpha \frac{\partial \beta_m}{\partial \sigma} \right). \quad (40)$$

The partial derivatives in Eq. (40) can be expressed as

$$\frac{\partial \|\beta'\|}{\partial \sigma} = \frac{1}{2\|\beta'\|} \frac{\partial \|\beta'\|^2}{\partial \sigma} = \frac{1}{2\|\beta'\|} \left(\frac{3}{2} 2 \frac{\partial \sigma'}{\partial \sigma} : \beta' \right) = \frac{3\mathcal{I}' : \beta'}{2\|\beta'\|} = \frac{3\beta'}{2\|\beta'\|}, \quad (41)$$

$$\frac{\partial \beta_m}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\beta : \mathbf{I}}{3} \right) = \frac{\mathbf{I}}{3}. \quad (41')$$

Substituting Eq. (41) into Eq. (40) gives

$$\mathbf{n} = \frac{1}{1-\alpha} \left(\frac{3\beta'}{2\|\beta'\|} + \alpha \mathbf{I} \right). \quad (42)$$

Suppose that the material is subject to uniaxial compression in the x_1 -direction, for which β takes the form of

$$\beta = \beta_{11}\mathbf{e}_1 \otimes \mathbf{e}_1. \quad (43)$$

It can be verified that $|\dot{\epsilon}_{11}^{vp}| = \dot{v}$, indicating that the hardening parameters can be calibrated via cyclic uniaxial compression tests.

Last set¹⁶

$$\dot{v} = \begin{cases} \gamma \left(\frac{f}{\sigma_y + R} \right)^n & f \geq 0, \\ 0 & f < 0, \end{cases} \quad (44)$$

where γ denotes a viscosity parameter, and n denotes a rate-sensitivity parameter.

IV. Radial Return Algorithm

In this section, the nonlinear viscoelasticity and the viscoplasticity models are first implemented in a radial return algorithm, and a closed-form algorithmic tangent operator is then derived to facilitate model implementation. Still consider time interval $[t_n, t_{n+1}]$. Here the radial return algorithm consists of the following steps:

1. trial viscoelastic step—assume that the deformation is entirely viscoelastic, apply so-called trial viscoelastic strains to the material, and compute the corresponding trial stresses
2. radial return step—reduce the trial strains along the viscoplastic flow direction such that the reduced strains and the resulting hardening variable(s) satisfy all constraints imposed by the viscoplasticity model

These steps are iterated to convergence (see Figure 3).

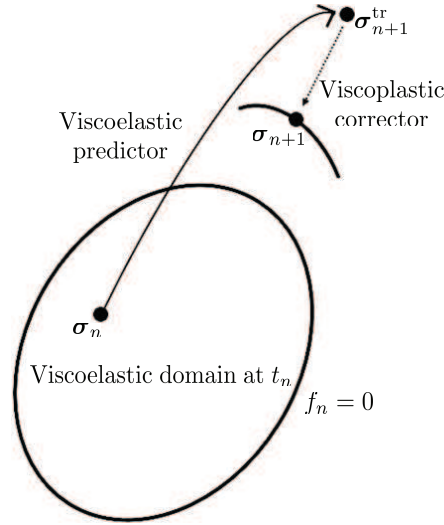


Figure 3. Radial return over $[t_n, t_{n+1}]$.

When updating the variables, ϵ is held fixed. The trial stress tensor, σ^{tr} , can be obtained by rewriting Eq. (30) as

$$\sigma^{\text{tr}} = \tilde{\mathcal{C}} : (\epsilon - \epsilon_n) + \sum_{i=1}^N [1 - \exp(-\lambda_i \Delta\psi)] \tilde{\mathcal{C}} : (\epsilon_i^{ve})_n + \sigma_n, \quad (45)$$

where some ϵ 's in Section II become viscoelastic strain tensor ϵ^{ve} in this section because the deformation becomes viscoelastic–viscoplastic. The integration scheme can be formulated as solving the following equation set for σ and Δv :

$$\sigma = \sigma^{\text{tr}} - \tilde{\mathcal{C}} : \Delta\epsilon^{vp}, \quad (46)$$

$$\Delta v = \dot{v} \Delta t. \quad (47)$$

Eqs. (36) and (37) suggest the following:

$$\Delta\epsilon^{vp} = \Delta v \mathbf{n}, \quad (48)$$

$$\Delta \mathbf{X} = \frac{2}{3} C \Delta\epsilon^{vp} - a \mathbf{X} \Delta v = \frac{2}{3} C \Delta v \mathbf{n} - a \mathbf{X} \Delta v. \quad (49)$$

Eq. (49) can be rewritten as

$$\mathbf{X}(\beta, \Delta v) = \frac{1}{1 + a \Delta v} \left(\mathbf{X}_n + \frac{2}{3} C \Delta v \mathbf{n} \right). \quad (50)$$

Combining Eqs. (46) and (48) and noting that $\beta = \sigma - \mathbf{X}$ give

$$\beta + \mathbf{X} = \sigma^{\text{tr}} - \Delta v \tilde{\mathcal{C}} : \mathbf{n}. \quad (51)$$

The task can then be reformulated as solving the following equation set for β and Δv :

$$\Psi(\beta, \Delta v) = \beta - \sigma^{\text{tr}} + \Delta v \tilde{\mathcal{C}} : \mathbf{n} + \mathbf{X}(\beta, \Delta v) = 0, \quad (52)$$

$$P(\beta, \Delta v) = \Delta v - \dot{v} \Delta t = 0. \quad (53)$$

Here Newton's method is used to find the solution. Require

$$\Psi(\beta_{\text{old}} + d\beta, \Delta v_{\text{old}} + d\Delta v) = \Psi(\beta_{\text{old}}, \Delta v_{\text{old}}) + \frac{\partial \Psi}{\partial \beta} : d\beta + \frac{\partial \Psi}{\partial \Delta v} d\Delta v = \mathbf{0}, \quad (54)$$

$$P(\beta_{\text{old}} + d\beta, \Delta v_{\text{old}} + d\Delta v) = P(\beta_{\text{old}}, \Delta v_{\text{old}}) + \frac{\partial P}{\partial \beta} : d\beta + \frac{\partial P}{\partial \Delta v} d\Delta v = 0, \quad (54')$$

where

$$\frac{\partial \Psi}{\partial \beta} = \mathbf{I} + \Delta v \tilde{\mathcal{C}} : \frac{\partial \mathbf{n}}{\partial \beta} + \frac{\partial \mathbf{X}}{\partial \beta}, \quad \frac{\partial \Psi}{\partial \Delta v} = \tilde{\mathcal{C}} : \mathbf{n} + \frac{\partial \mathbf{X}}{\partial \Delta v}, \quad (55)$$

$$\frac{\partial P}{\partial \beta} = -\frac{\partial \dot{v}}{\partial \beta} \Delta t, \quad \frac{\partial P}{\partial \Delta v} = 1 - \frac{\partial \dot{v}}{\partial \Delta v} \Delta t \quad (55')$$

with

$$\frac{\partial \mathbf{X}}{\partial \beta} = \frac{2C\Delta v}{3(1+a\Delta v)} \frac{\partial \mathbf{n}}{\partial \beta}, \quad (56)$$

$$\frac{\partial \mathbf{X}}{\partial \Delta v} = -\frac{1}{(1+a\Delta v)^2} \left(a\mathbf{X}_n - \frac{2}{3}C\mathbf{n} \right), \quad (56')$$

$$\frac{\partial \dot{v}}{\partial \beta} = n\gamma \left(\frac{f}{\sigma_y + R} \right)^{n-1} \frac{\partial}{\partial \beta} \left(\frac{f}{\sigma_y + R} \right) \quad (56'')$$

$$\begin{aligned} &= \frac{n\gamma}{\sigma_y + R} \left(\frac{f}{\sigma_y + R} \right)^{n-1} \mathbf{n}, \\ \frac{\partial \dot{v}}{\partial \Delta v} &= n\gamma \left(\frac{f}{\sigma_y + R} \right)^{n-1} \frac{\partial}{\partial \Delta v} \left(\frac{f}{\sigma_y + R} \right) \\ &= -\frac{n\gamma(f + \sigma_y + R)}{(\sigma_y + R)^2} \left(\frac{f}{\sigma_y + R} \right)^{n-1} \frac{dR}{d\Delta v}. \end{aligned} \quad (56''')$$

Differentiating $\beta' / \|\beta'\|$ with respect to β gives

$$\begin{aligned} \frac{\partial}{\partial \beta} \left(\frac{\beta'}{\|\beta'\|} \right) &= \beta' \otimes \frac{\partial}{\partial \beta} \left(\frac{1}{\|\beta'\|} \right) + \frac{1}{\|\beta'\|} \frac{\partial \beta'}{\partial \beta} \\ &= \beta' \otimes \left(-\frac{1}{\|\beta'\|^2} \frac{\partial \|\beta'\|}{\partial \beta} \right) + \mathcal{I}' = \frac{1}{\|\beta'\|} \left(\mathcal{I}' - \frac{3\beta' \otimes \beta'}{2\|\beta'\|^2} \right). \end{aligned} \quad (57)$$

$\partial \mathbf{n} / \partial \beta$ can then be expressed as

$$\frac{\partial \mathbf{n}}{\partial \beta} = \frac{3}{2(1-\alpha)\|\beta'\|} \left(\mathcal{I}' - \frac{3\beta' \otimes \beta'}{2\|\beta'\|^2} \right). \quad (58)$$

Introduce matrix notation

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{22} & \beta_{13} & \beta_{23} & \beta_{33} \end{bmatrix}^T. \quad (59)$$

Eq. (54) can be written in matrix form as

$$\begin{Bmatrix} \Psi(\beta_{\text{old}}, \Delta v_{\text{old}}) \\ SP(\beta_{\text{old}}, \Delta v_{\text{old}}) \end{Bmatrix} + J \begin{Bmatrix} d\beta \\ Sd\Delta v \end{Bmatrix} = 0, \quad (60)$$

where

$$J = \begin{bmatrix} \frac{\partial \Psi}{\partial \beta} & \frac{1}{S} \frac{\partial \Psi}{\partial \Delta v} \\ S \frac{\partial P}{\partial \beta} & \frac{\partial P}{\partial \Delta v} \end{bmatrix} \quad (61)$$

is a 7×7 Jacobian matrix, and $S = dR/d\Delta v$ is a scale factor making J well-conditioned. Rearranging Eq. (60) gives

$$\begin{Bmatrix} d\beta \\ S d\Delta v \end{Bmatrix} = -J^{-1} \begin{Bmatrix} \Psi(\beta_{\text{old}}, \Delta v_{\text{old}}) \\ SP(\beta_{\text{old}}, \Delta v_{\text{old}}) \end{Bmatrix}. \quad (62)$$

The corrections can then be computed and added to the solutions, i.e.,

$$\beta_{\text{new}} = \beta_{\text{old}} + d\beta \quad \text{and} \quad \Delta v_{\text{new}} = \Delta v_{\text{old}} + d\Delta v. \quad (63)$$

Once the process is iterated to convergence, the variables are updated as

$$\Delta \epsilon^{vp} = \Delta v \mathbf{n}, \quad (64)$$

$$\mathbf{X} = \frac{1}{1 + a\Delta v} \left(\mathbf{X}_n + \frac{2}{3} C \Delta \epsilon^{vp} \right), \quad (64')$$

$$\boldsymbol{\sigma} = \boldsymbol{\beta} + \mathbf{X}. \quad (64'')$$

When deriving the algorithmic tangent operator, ϵ is no longer held fixed. Totally differentiating both sides of Eq. (52) with respect to β , Δv , and ϵ gives

$$d\Psi = \frac{\partial \Psi}{\partial \beta} : d\beta + \frac{\partial \Psi}{\partial \Delta v} d\Delta v + \frac{\partial \Psi}{\partial \epsilon} : d\epsilon = 0, \quad (65)$$

$$dP = \frac{\partial P}{\partial \beta} : d\beta + \frac{\partial P}{\partial \Delta v} d\Delta v + \frac{\partial P}{\partial \epsilon} : d\epsilon = 0, \quad (66)$$

where

$$\frac{\partial \Psi}{\partial \epsilon} = -\frac{\partial \boldsymbol{\sigma}^{\text{tr}}}{\partial \epsilon} = -\tilde{\mathcal{C}} \quad \text{and} \quad \frac{\partial P}{\partial \epsilon} = 0. \quad (67)$$

Solving Eq. (66) for $d\Delta v$ gives

$$d\Delta v = -\frac{\frac{\partial P}{\partial \beta}}{\frac{\partial P}{\partial \Delta v}} : d\beta. \quad (68)$$

Substituting Eq. (68) into Eq. (65) and solving the equation for $d\beta$ gives

$$d\beta = \left(\frac{\partial \Psi}{\partial \beta} - \frac{\frac{\partial \Psi}{\partial \Delta v} \otimes \frac{\partial P}{\partial \beta}}{\frac{\partial P}{\partial \Delta v}} \right)^{-1} : \tilde{\mathcal{C}} : d\epsilon, \quad (69)$$

Meanwhile, one has

$$d\boldsymbol{\sigma} = d\boldsymbol{\beta} + d\mathbf{X}, \quad (70)$$

$$d\mathbf{X} = \frac{\partial \mathbf{X}}{\partial \beta} : d\beta + \frac{\partial \mathbf{X}}{\partial \Delta v} d\Delta v. \quad (70')$$

Combining Eqs. (69) and (70) gives

$$\begin{aligned} d\boldsymbol{\sigma} &= d\boldsymbol{\beta} + \frac{\partial \mathbf{X}}{\partial \beta} : d\beta + \frac{\partial \mathbf{X}}{\partial \Delta v} d\Delta v = \left(\boldsymbol{\mathcal{I}} + \frac{\partial \mathbf{X}}{\partial \beta} - \frac{\frac{\partial \mathbf{X}}{\partial \Delta v} \otimes \frac{\partial P}{\partial \beta}}{\frac{\partial P}{\partial \Delta v}} \right) : d\beta \\ &= \left(\boldsymbol{\mathcal{I}} + \frac{\partial \mathbf{X}}{\partial \beta} - \frac{\frac{\partial \mathbf{X}}{\partial \Delta v} \otimes \frac{\partial P}{\partial \beta}}{\frac{\partial P}{\partial \Delta v}} \right) : \left(\frac{\partial \Psi}{\partial \beta} - \frac{\frac{\partial \Psi}{\partial \Delta v} \otimes \frac{\partial P}{\partial \beta}}{\frac{\partial P}{\partial \Delta v}} \right)^{-1} : \tilde{\mathcal{C}} : d\epsilon \\ &\equiv \mathcal{C}^{\text{alg}} : d\epsilon \end{aligned} \quad (71)$$

where $\mathcal{C}^{\text{alg}}$ denotes the fourth-order consistent tangent operator.

V. Numerical Examples

Lai and Bakker² conducted a series of creep–recovery tests on PMMA, at different stresses (σ 's), and calibrated the viscoelastic parameters of PMMA. First, the nonlinear viscoelasticity model is validated by reproducing these test data with the calibrated parameters. Then, the present constitutive model's capabilities are demonstrated through modeling viscoelastic–viscoplastic PMMA loaded at different strain rates. Next, the MSG-based micromechanics approach, embedded with the present constitutive model, is used to homogenize a UDFRC with a PMMA matrix, subjected to uniaxial and shear loading at different strain rates. Last, the viscoelastic characterization of PMT-F7 epoxy is evaluated by comparing the predicted creep–recovery responses by the constitutive model with the experimental ones.

A. Nonlinearly Viscoelastic PMMA

Table 1 lists the viscoelastic parameters of PMMA. According to Ref. [2], isotropic PMMA has an instantaneous compliance (D_0) of 270.9×10^{-6} MPa and a stress threshold of nonlinear viscoelasticity (σ_0) of 20 MPa. Assume that it has a Poisson's ratio (ν_0) of 0.39. Figure 4 depicts the loading history during a creep–recovery test. The loading history can be decomposed into loading, creep, unloading, and recovery. Table 2 lists these stages of loading, along with their respective duration and numbers of load steps chosen for time integration.

Table 1. Viscoelastic parameters of PMMA.

(a) Coefficients in the Prony series. ²				
i	$\lambda_i \text{ (s}^{-1}\text{)}$	$D_i \text{ (} 10^{-6} \text{ MPa}^{-1}\text{)}$		
1	1	23.6358		
2	10^{-1}	5.6602		
3	10^{-2}	14.8405		
4	10^{-3}	18.8848		
5	10^{-4}	28.5848		
6	10^{-5}	40.0569		
7	10^{-6}	60.4235		
8	10^{-7}	79.6477		
9	10^{-8}	162.1790		

(b) Stress-dependent parameters. ³				
	1	2	3	4
g_0	−0.1824	0.9852	−2.0171	1.0717
g_1	0.067	0.133	2.133	−2.133
g_2	−0.773	9.707	−15.787	8.533
a_σ	−0.373	2.58	−5.227	3.52

Table 2. Time integration for PMMA.

	Duration (sec)	Number of load steps
Loading	10^{-3}	100
Creep	1800	100
Unloading	10^{-3}	100
Recovery	3600	200



Figure 4. loading history of PMMA.

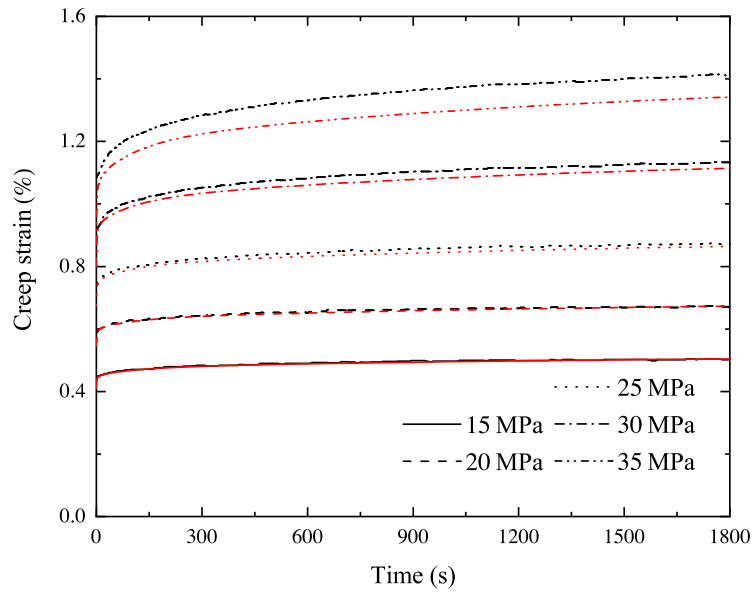


Figure 5. Experimental and predicted creep responses of PMMA, where the black and the red curves are experimental and predicted, respectively.

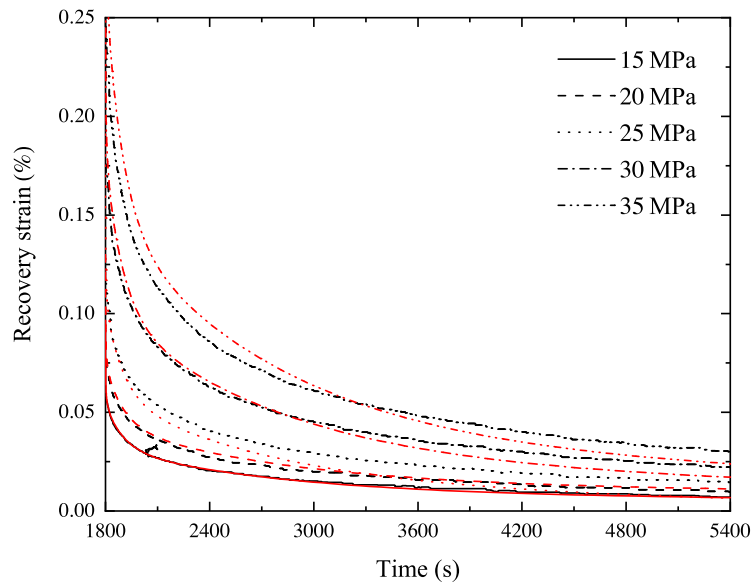


Figure 6. Experimental and predicted recovery responses of PMMA, where the black and the red curves are experimental and predicted, respectively.

Figures 5 and 6 compare the predicted curves with the experimental ones. As can be seen:

- The predicted curves for 15 MPa and 20 MPa match their corresponding experimental curves.
- The other predicted curves slightly deviate from their corresponding experimental curves.

The first point indicates that, without stress dependence, the viscoelasticity model is relatively simple and can produce very accurate results. The second point indicates that, with stress dependence, the viscoelasticity model becomes more complex and produces small numerical errors. Notably, in Figure 6, the predicted curve for 25 MPa intersects those for 15 MPa and 20 MPa. This indicates that, when viscoelasticity shifts from linear to nonlinear, the accumulation of errors may cause inconsistent predictions. Considering the forward Euler method being used for time integration is first-order accurate, the prediction accuracy can be improved by replacing this first-order method by a second-order one in future work. On the whole, the predicted curves agree well with the experimental ones. This indicates that the nonlinear viscoelasticity model can characterize experimentally measured stress dependence.

B. Viscoelastic–Viscoplastic PMMA

Assume that isotropic PMMA obeys the von Mises yield function ($\alpha = 0$ in Eq. (33)). This assumption is made due to a lack of sufficient experimental data. Table 3 lists the viscoplastic parameters of PMMA, where C takes an excessively high value to demonstrate the effects of kinematic hardening, and the other parameters take realistic values. In this section, the material only exhibits linear viscoelasticity because all stress-dependent functions in Ref. [2] are unspecified over $\sigma \geq 40$ MPa.

Table 3. Viscoplastic parameters of PMMA.

σ_y (MPa)	Q (MPa)	b	C (MPa)	a	γ (s^{-1})	n
35	5.50879	53.2535	1035.167	118.6	1.00845×10^{-4}	8.27066

Figure 7 shows the stress–strain curves of PMMA subject to uniaxial loading. As can be seen, the material becomes stiffer as the major strain rate increases. This agrees with the theories of viscoelasticity and viscoplasticity. Notably, the material exhibits prominent nonlinear rate-dependent stress–strain responses in the viscoplastic regime.

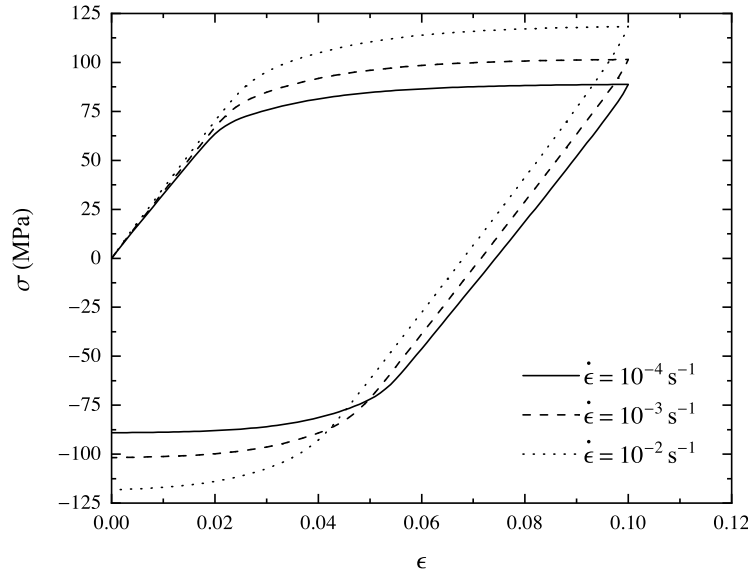


Figure 7. Stress–strain curves of PMMA subjected to uniaxial loading.

It is beneficial to investigate the effects of kinematic hardening. Figures 8 compares the stress–strain curves of PMMA exhibits isotropic hardening with those exhibiting combined isotropic–kinematic hardening.

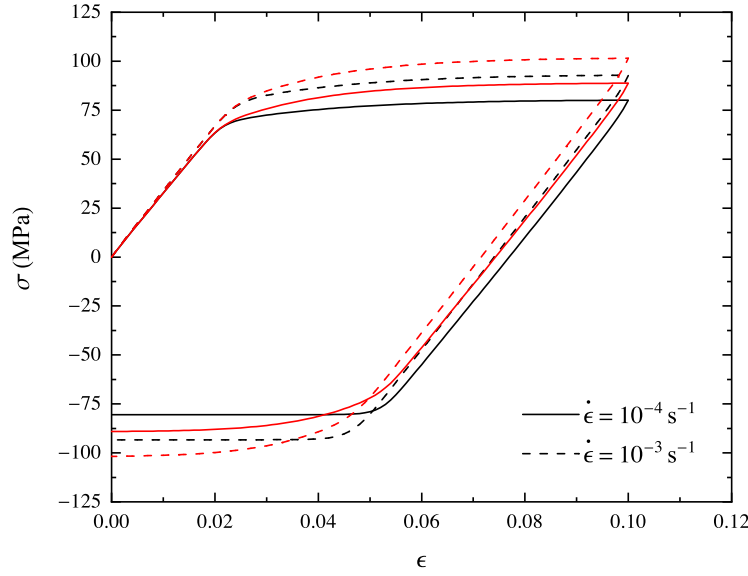


Figure 8. Stress–strain curves of PMMA exhibiting different plastic hardening, where the black curves are for isotropic hardening, and the red ones combined isotropic–kinematic hardening.

According to the theory of plasticity, isotropic hardening causes the yield surface to expand in stress space, while kinematic hardening causes it to translate. During cyclic loading:

- Isotropic hardening causes the yield stress to increase during loading and reverse loading.
- Kinematic hardening causes the yield stress to “increase” during loading but to “decrease” during reverse loading.

Accordingly, for the strain rate of 10^{-4} s^{-1} :

- Kinematic hardening makes the material stiffer in the viscoplastic regime.
- During loading, both curves become highly nonlinear at 65 MPa.
- During reverse loading, the solid curve becomes nonlinear at -80 MPa , while the dashed curve at -60 MPa .

Similar findings can be obtained for for the strain rate of 10^{-3} s^{-1} . In conclusion, Figures 7 and 8 indicate that the present constitutive model can handle viscoplastic rate-dependence and combined isotropic–kinematic hardening.

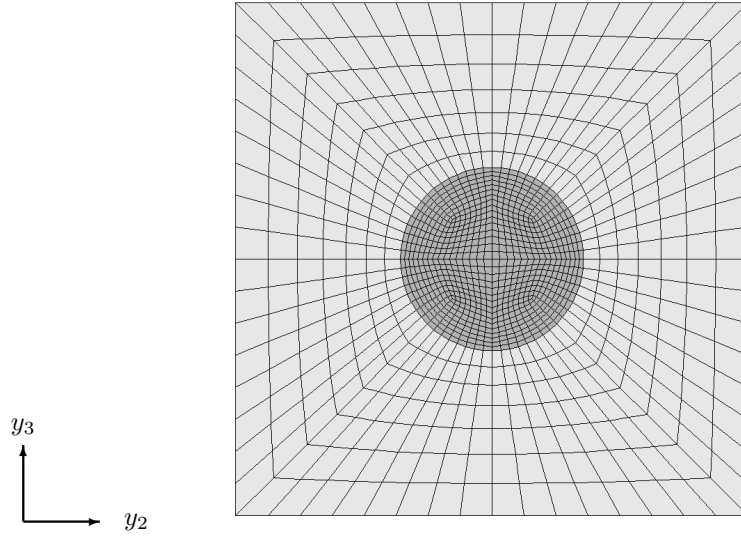
C. UDFRC

The present constitutive model can then be implemented in the MSG-based micromechanics approach for the constitutive modeling of 3D composite structures. Consider a UDFRC consisting of a PMMA matrix (with its material parameters listed in Tables 1 and 3) and numerous cylindrical, elastic fibers arranged in a square array. For demonstration purposes, the volume fraction of fibers is set to be 10%. This is because:

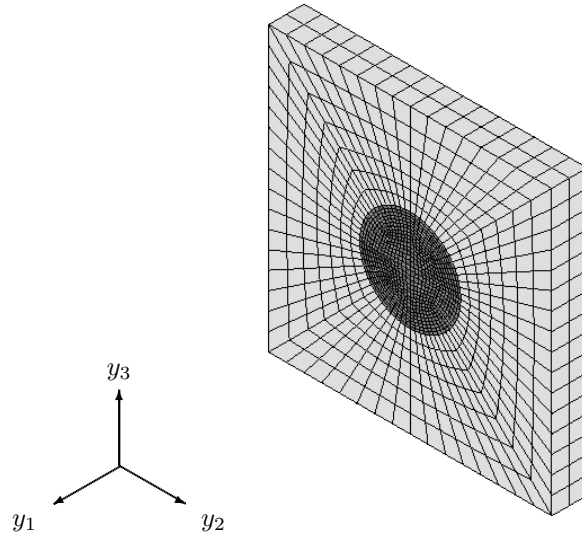
- There is a lack of experimental data for composite-level calibration or validation.
- Only PMMA contributes to the composite’s material nonlinearity and rate dependence.

Let the fibers have a Young’s modulus of 76 GPa and a Poisson’s ratio of 0.22. Let the structure gene (SG) of this composite consist of a square matrix and a circular fiber located at its center. Choose the center of the SG as the origin of the local coordinates, y_i , and the fiber direction and the length and width directions of the SG as the y_1 -, the y_2 -, and the y_3 -directions, respectively.

Figure 9(a) depicts the SG used in the micromechanics approach. The 2D SG is meshed with 4-node quadrilateral elements with 3 degrees of freedom (DOFs) at each node, and the meshed SG consists of 1216 elements. This FE model is found to be capable of producing converged results. In contrast, to simulate longitudinal shear deformation using FE analysis (FEA), one has to use a 3D SG as shown in Figure 9(b). The 3D SG is meshed with 8-node hexahedral elements, and the meshed SG consists of 2432 elements. Clearly, a 2D SG can have much less DOFs than a 3D SG and therefore reduce the computational effort.



(a) Micromechanics approach.



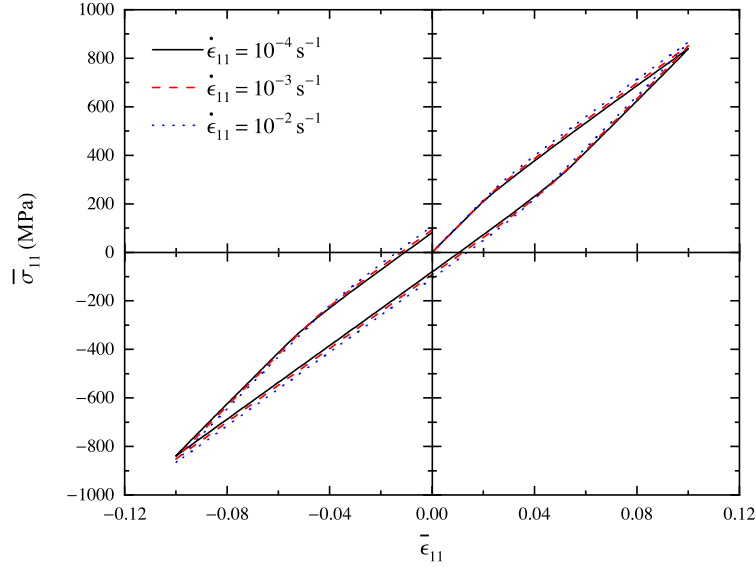
(b) FEA.

Figure 9. Meshed SGs of a UDRFC.

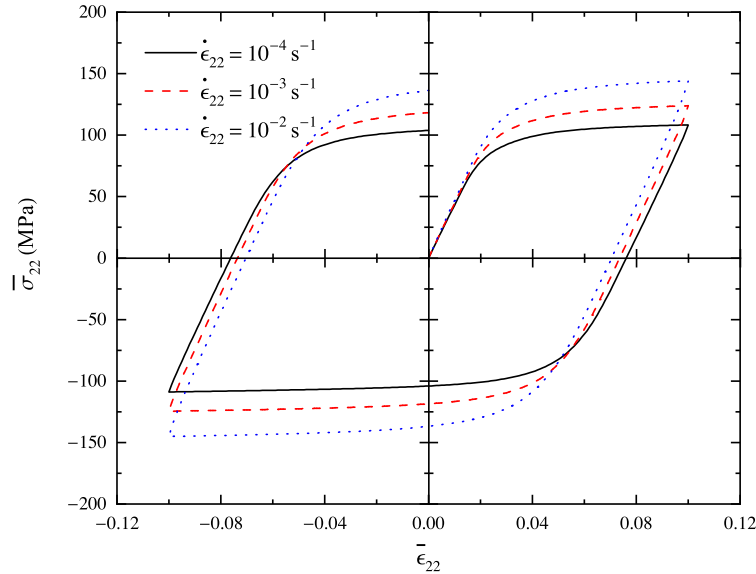
Figure 10 shows the stress-strain hysteresis loops of the composite subjected to uniaxial loading in the y_1 - and the y_2 -directions, respectively. Hereafter let each loading path consist of the following steps:

1. initial loading—the major strain(s) is increased from 0 to 0.1
2. initial unloading—the major strain(s) is decreased from 0.1 to 0
3. reverse loading—the major strain(s) is decreased from 0.0 to -0.1
4. reverse unloading—the major strain(s) is increased from -0.1 to 0

It can be seen that the composite exhibits an approximately linear stress–strain relationship when loaded in the y_1 -direction and a highly nonlinear one when loaded in the y_2 -direction. This can be understood by investigating the stress distribution in the SG. In Figure 10(a), the fiber sustains most of the external load, and accordingly, the composite behaves like a linearly elastic fiber. In Figure 10(b), the matrix sustains most of the external load, and the stress distribution in the matrix remains highly nonuniform. As a result, the matrix yields successively at different locations. In addition, in Figure 10(b), the composite becomes stiffer as the major strain rate increases, while in Figure 10(a), it seldom exhibits any rate dependence. This is again because here the rate-independent fiber sustains most of the external load.



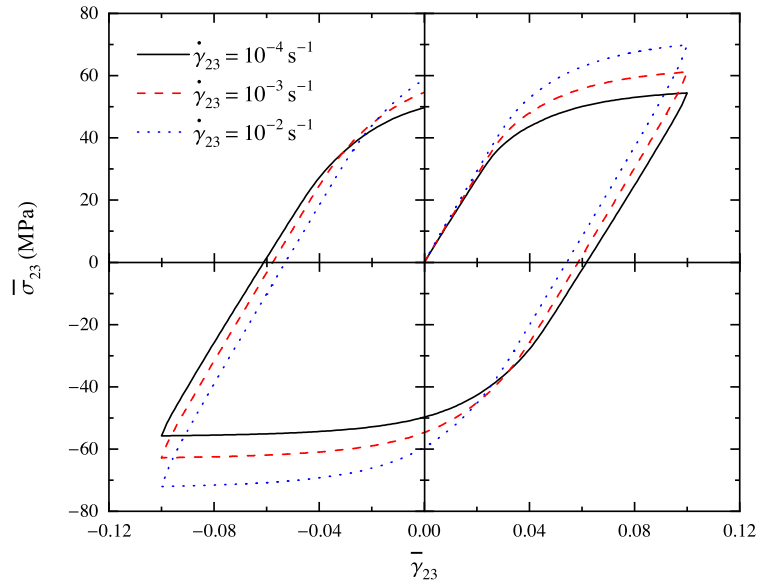
(a) y_1 -direction.



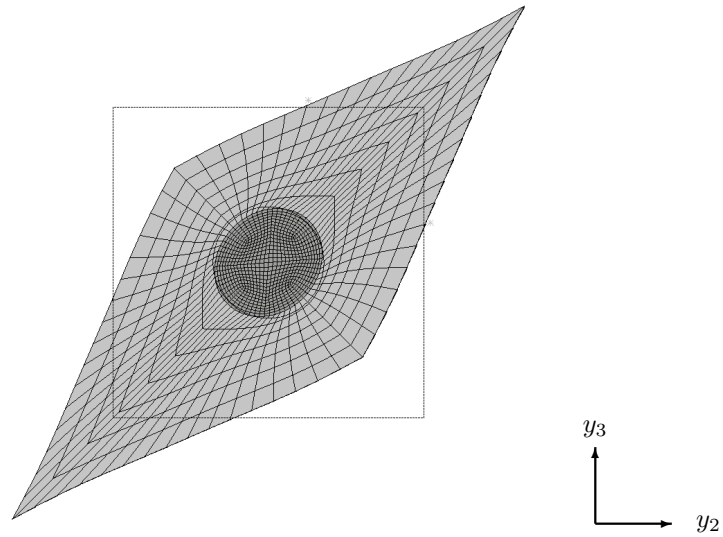
(b) y_2 -direction.

Figure 10. Stress–strain hysteresis loops of a UDRFC subjected to uniaxial loading.

Figure 11(a) shows the stress–strain hysteresis loops of the composite subjected to transverse shear loading, and Figure 11(b) depicts an SG undergoing exaggerated deformation. It can be seen that the stress–strain relationships here are similar to those in Figure 10(b). Therefore, the discussion for Figure 10(b) holds here. However, the stress levels here are much lower than those in Figure 10(b). This is because the matrix more easily yields under pure shear than under other loading conditions, such as uniaxial loading.



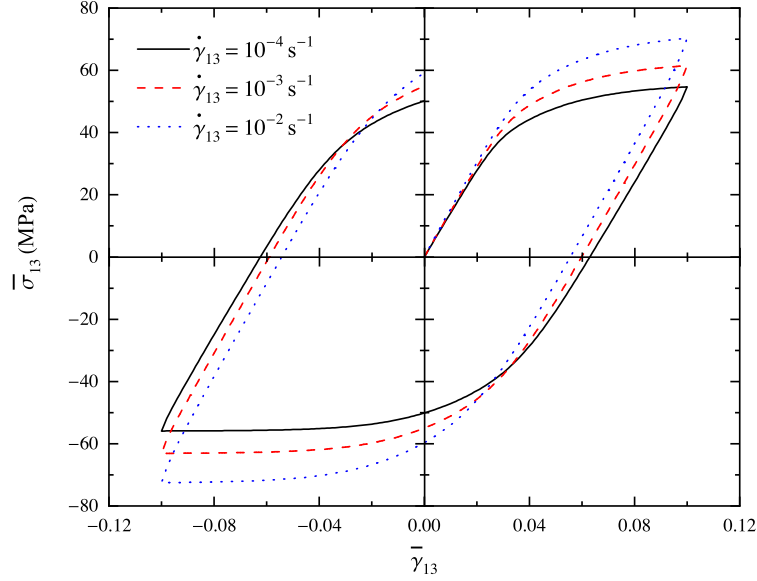
(a) Stress-strain hysteresis loops.



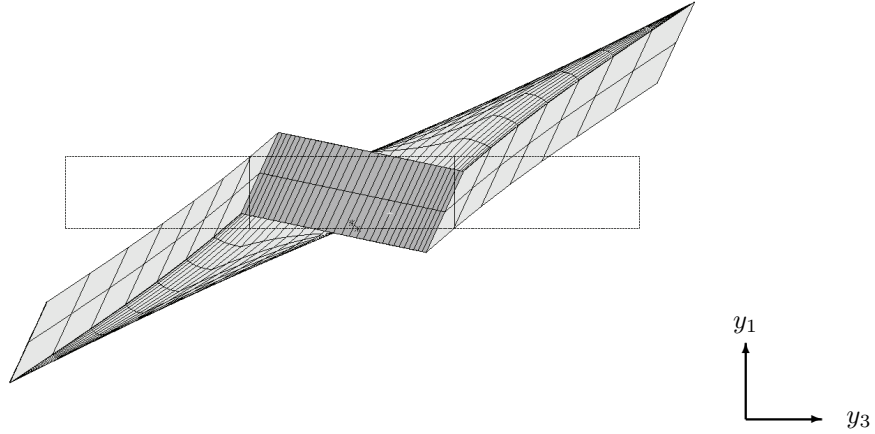
(b) Deformed shape and undeformed edge.

Figure 11. UDRFC subjected to transverse shear loading.

Figure 12(a) shows the stress–strain hysteresis loops of the composite subjected to longitudinal shear loading, and Figure 12(b) depicts an SG undergoing exaggerated deformation. It can be seen that the stress–strain relationships here are similar to those in Figures 10(b) and 11(a). Therefore, the discussion for Figure 10(b) and 11(a) holds here. As mentioned above, to simulate longitudinal shear loading using FEA, one has to at least use a 3D SG. This can be better understood by noting that, in Figure 12(b), the local displacements fluctuate in the y_1 -direction due to the existence of the fiber. This indicates that, compared with FEA, the micromechanics approach can often homogenize a composite in a simpler and more efficient way.



(a) Stress–strain hysteresis loops.



(b) Deformed shape and undeformed edge.

Figure 12. UDRFC subjected to longitudinal shear loading.

D. PMT-F7 Epoxy

Following the test and calibration method in Ref. [17], the authors conducted a series of creep–recovery tests on PMT-F7 epoxy, at stresses varying from 15 MPa to 30 MPa, and calibrated the viscoelastic parameters

of PMT-F7 epoxy. Table 4 lists the viscoelastic parameters of PMT-F7 epoxy. Isotropic PMT-F7 epoxy is measured to have an instantaneous compliance of 2.720×10^{-4} MPa, a stress threshold of nonlinear viscoelasticity of 15 MPa, a Poisson's ratio of 0.5, and a yield stress of 40 MPa. Since the creep stress never exceeded the yield stress, the specimens never yielded during the tests. Figure 13 depicts the loading history during a creep-recovery test. Table 5 lists all stages of loading, along with their respective duration and numbers of load steps chosen for time integration.

Table 4. Viscoelastic parameters of PMT-F7 epoxy.

(a) Coefficients in the Prony series.		
i	$\lambda_i \text{ (s}^{-1}\text{)}$	$D_i \text{ (MPa}^{-1}\text{)}$
1	1.333×10^{-2}	1.480×10^{-5}
2	1.667×10^{-3}	2.920×10^{-7}
3	1.0×10^{-3}	6.052×10^{-6}
4	1.0×10^{-4}	6.146×10^{-6}
5	1.0×10^{-5}	1.965×10^{-5}

(b) Stress-dependent parameters.			
	1	2	3
g_0	0.1672	0.0831	-0.1429
g_1	1.8649	-0.8597	0.6718
g_2	1.3637	-0.3236	0.0
a_σ	0.0	0.0	0.0

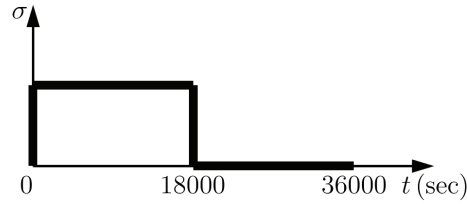


Figure 13. loading history of PMT-F7 epoxy.

Table 5. Time integration for PMT-F7 epoxy.

	Duration (sec)	Number of load steps
Loading	10^{-3}	100
Creep	18000	100
Unloading	10^{-3}	100
Recovery	18000	100

Figures 14 and 15 compare the predicted curves with the experimental ones. As can be seen:

- The predicted curves for 15 MPa and 17.5 MPa match their corresponding experimental curves.
- The other predicted curves slightly deviate from their corresponding experimental curves.

These results indicate the following:

- The coefficients in the Prony series (see Table 4(a)) are accurately calibrated.
- The predictions are accurate at low stresses but not so accurate at high stresses.

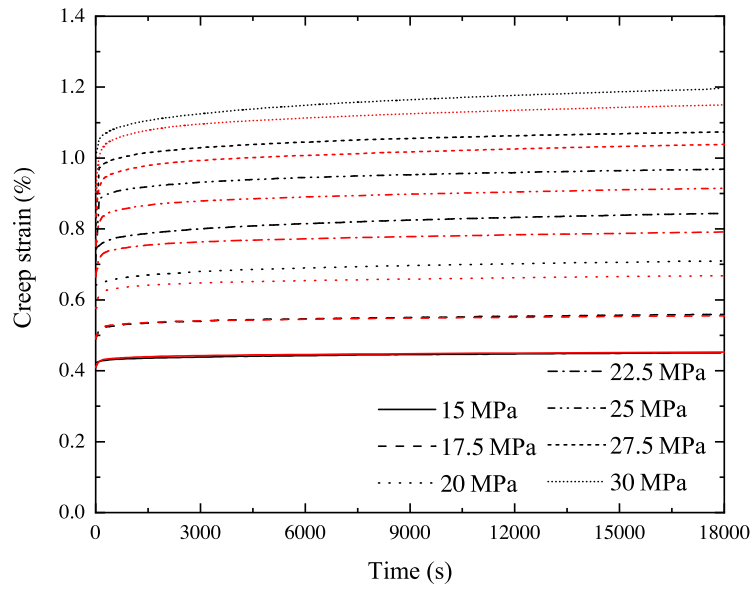


Figure 14. Experimental and predicted creep responses of PMT-F7 epoxy, where the black and the red curves are experimental and predicted, respectively.

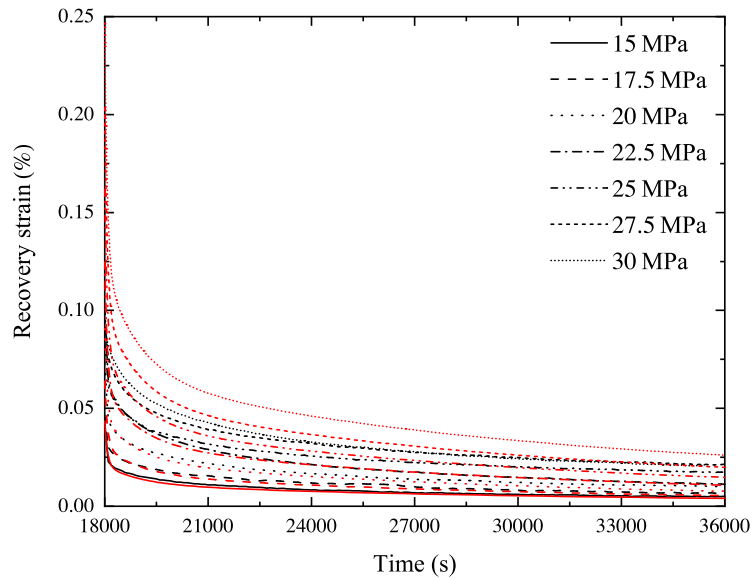


Figure 15. Experimental and predicted recovery responses of PMT-F7 epoxy, where the black and the red curves are experimental and predicted, respectively.

Unlike Figure 6, the predicted curves in Figure 15 never intersect. This can be understood by noting the following:

- In Figure 6, one predicted curve for a creep stress slightly greater than the stress threshold intersects those without stress dependence.
- In Figure 15, all creep stresses are greater than the stress threshold here, 15 MPa.

As mentioned earlier, when viscoelasticity shifts from linear to nonlinear, the accumulation of errors may cause inconsistent predictions. But in Figure 15, viscoelasticity remains nonlinear. On the whole, the predicted curves agree well with the experimental ones. As mentioned earlier, the prediction accuracy can be improved by replacing the forward Euler method being used by a second-order integration method in future work.

VI. Conclusions

In this paper, a nonlinear viscoelastic–viscoplastic constitutive model for epoxy polymers was developed. The classic nonlinear viscoelasticity model was reformulated to yield a closed-form incremental constitutive relation, which relates the stress increments to the viscoelastic strain increments. A viscoplasticity model, which consisted of the Drucker–Prager yield function, nonlinear isotropic and kinematic hardening laws, and the Perzyna viscosity function, was subsequently developed. The nonlinear viscoelasticity and the viscoplasticity models were then implemented in a radial return algorithm. A closed-form algorithmic tangent operator was derived to facilitate model implementation.

The following findings were obtained from the results:

- The nonlinear viscoelasticity model is found to be capable of characterizing experimentally measured stress dependence.
- The present constitutive model is found to be capable of handling viscoplastic rate-dependence and combined isotropic–kinematic hardening.
- The MSG-based micromechanics approach is demonstrated to be capable of handling rate dependence.

The present constitutive model could be conveniently implemented in an FE code to perform structural analysis. The micromechanics approach could be embedded in a multiscale modeling framework to perform multiscale modeling. The present framework could be further improved by implementing more sophisticated viscoelasticity and viscoplasticity models in future work.

Acknowledgments

This work was supported by NASA SBIR Phase I: T12.01-5412 (contract number: 80NSSC20C0583). The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsement, either expressed or implied, of the sponsor.

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