

Bistable Deployable Composite Booms With Parabolic Cross-Sections

Andrew J. Lee*

North Carolina State University, Raleigh, NC 27695, USA

Juan M. Fernandez[†]

NASA Langley Research Center, Hampton, VA 23681, USA

Jacob G. Daye[‡]

North Carolina State University, Raleigh, NC 27695, USA

This paper investigates how stable equilibrium states in the extended and coiled configurations can be predicted in thin-shelled composite booms with parabolic cross-sections. These conic shapes potentially offer greater stiffness properties when compared to circular cross-sections, which is critical for improving the load bearing performance of deployed booms. Inducing bistability through the choice of composite shell layups in parabolic booms would allow for controllable self-deployment due to a less energetic coiled state when compared to monostable booms. An inextensional analytical model is used to predict the stable coiled diameters of tape spring and Collapsible Tubular Mast (CTM) booms with parabolic cross-sections. The parabolic section is discretized into circular segments using biarc spline interpolation, which allows them to be integrated into the strain energy minimization procedure used to obtain the equilibrium states. When the parabolic booms are parametrically compared against circular booms with identical layups, flattened height, and mass, the former are found to generally have better stiffness performance while being less efficient in stowed volume as evidenced by larger coiled diameters. Analytical coiled diameters and their strain energy are verified with finite element simulations for optimal parabolic tape spring and CTM booms.

I. Introduction

Coilable thin-shell composite booms with high packaging and mass efficiencies are widely utilized to deploy large planar spacecraft structures such as flexible blanket solar arrays [1, 2] and solar sails [3, 4]. The simplest example is a tape spring with uniform curvature in the transverse direction and a subtended angle that is typically less than 225°. Rollable booms with open cross-sections such as the Storable Tubular Extensible Member (STEM) [5] and the Triangular Rollable and Collapsible (TRAC) [6, 7] booms can achieve better mechanical performance and mitigate manufacturing costs for greater scalability. However, their low torsional stiffness, which can cause instability and low shape accuracy, has driven the development of booms with closed cross-sections such as the Collapsible Tubular Mast (CTM) [8–10] and more recently the Corrugated Rollable Tubular (COROTUB) booms [11].

Bistability, or the existence of two stable equilibrium states in the extended and coiled configurations, can be induced in cylindrical composite shells through the choice of laminate layup. This property is advantageous for deployable thin-shell booms when compared to monostable variants where only their as-manufactured extended state is stable. This is due to having lower constraint requirements under stowage and exhibiting a more controllable self-deployment from having less strain energy in the coiled state. An inextensional model with strain energy minimization [12] has been widely adopted to predict the stability of the deployed and stowed states of composite tape springs [13–16] and multi-shelled booms [17, 18]. A common aspect of these bistable booms is that the cross-sectional geometry is always cylindrical or consists of cylindrical segments. Depending on the complexity, the cross-section can be modelled as some combination of circular arcs with uniform curvature and flat segments with zero curvature. However, deployable bistable booms with conic cross-sections have not been studied and they may yield stiffness advantages in the extended configuration [19] as demonstrated in previous monostable CTM designs that included ellipsoidal [20] and parabolic

*Assistant Professor, Department of Mechanical and Aerospace Engineering, 1840 Entrepreneur Dr., Campus Box 7910. AIAA Member.

[†]Research Aerospace Engineer, Structural Dynamics Branch, NASA LaRC, 4 West Taylor St., MS 230. Senior AIAA Member.

[‡]Graduate Student, Department of Mechanical and Aerospace Engineering, 1840 Entrepreneur Dr., Campus Box 7910. AIAA Student Member.

segments [21]. This would translate to increased load carrying capability, especially with regards to local shell and global Euler buckling, and lower stowage induced deformations aided by more favorable strain energy profiles.

The objective of this paper is to model the bistable configurations of composite booms with parabolic cross-sections, characterize their stiffness performance in the deployed state, and their volume efficiency and degree of stability in the coiled state relative to booms with cylindrical cross-sections. Although the proposed modelling approach can account for other conic sections including ellipses and hyperbolas, they are not the focus of this paper. An analytical model is derived to parametrically find optimal geometries for a parabolic tape spring and the existence of a stable coiled state is verified with finite element analysis. Its stiffness properties and coiled shapes are compared against a circular tape spring with identical mass through matching materials, layups, and flattened heights. This comparison and finite element analysis is extended to CTM booms consisting of parabolic and circular segments in both the inner and outer composite shells. The inner and outer shells are defined as the boom half closest and furthest away from the coiling axis or center of the spool about which the boom wraps around.

II. Analytical Model

This section derives the analytical model for obtaining the stable coiled configuration and stiffness properties of deployable composite booms with parabolic cross-sections. An inextensional model [12] is extended to predict the coiled diameter and strain energy states of a parabolic tape spring. The parabolic segments are then integrated into the inner and outer shells of CTM booms to model their extended and coiled states. For a cylindrical shell that is under plane stress and initially stress free, its mid-surface is assumed to only bend without stretching which implies that all deformations are uniform, inextensional, and the Gaussian curvature remains unchanged. This means that every shell configuration could be fitted to the surface of an underlying cylinder, as shown in Fig. 1(a), and these configurations are defined by the cylinder's curvature C and the shell's orientation angle θ relative to the cylinder's longitudinal axis.

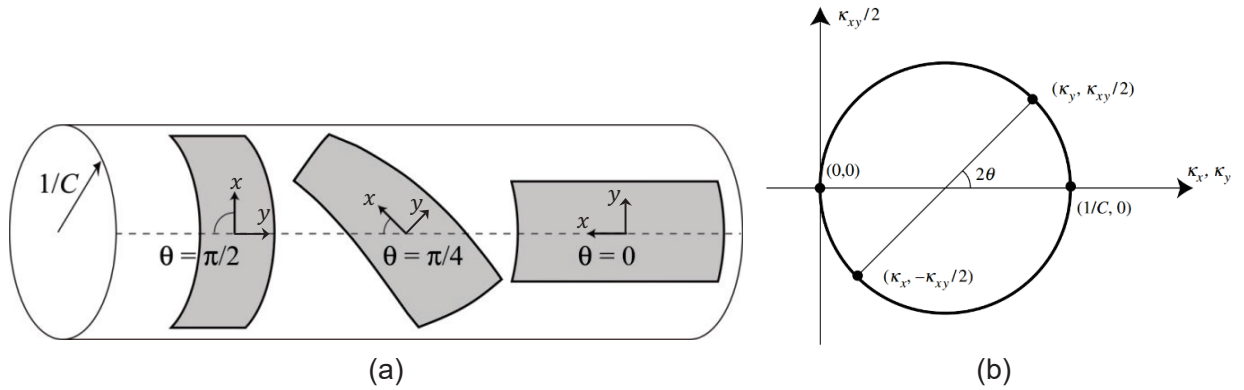


Fig. 1 (a) Coordinate system defining shell configurations of each circular arc on an underlying cylinder of curvature C . (b) Mohr's circle of curvature [12].

The two parameter formulation requires the shell curvature to be constant at any given state which restricts the modelling approach to booms with circular cross-sections. Since the curvature of a parabola does not satisfy this condition, its exact definition cannot be directly implemented. Instead, the parabolic cross-section is discretized into a series of circular arcs using biarc spline interpolation [22] so that each arc can be modelled with C and θ in a piecewise approach. This is illustrated in Fig. 2(a) for a parabolic tape spring where its circular segments are denoted with the subscript $i = 1, \dots, n$ and each arc is defined by its own subtended angle α_i , radius R_i , and length L_i . A biarc is a pair of circular arcs having the same tangent vector at their connecting joint, and it is often used in curve fitting applications [23–25]. This paper adopts an efficient algorithm [26] for approximating a continuous bounded function of monotonic curvature with one sign through a series of interpolating biarcs. Only the key parameters and equations required for the interpolation procedure are presented here and their derivations are excluded.

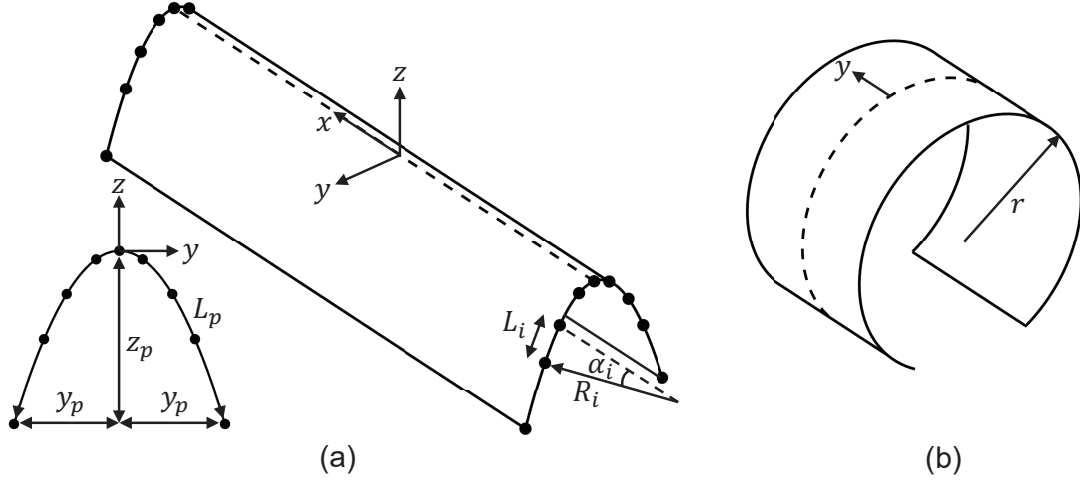


Fig. 2 Bistable tape spring with parabolic cross-section that is discretized with circular arcs denoted with the subscript $i = 1, \dots, n$. Each segment has its own subtended angle α_i and radius R_i . (a) Initially extended state with parabolic height z_p , half width y_p , and length L_p and (b) coiled state with radius of curvature r .

A. Biarc Spline Interpolation

Since a parabola has non-monotonic curvature, only half of the curve is interpolated and the other half is found with symmetry about the origin. The parabolic arc length L_p and half width y_p are chosen as independent parameters which define the parabolic shape of the boom's cross-section. Then the parabolic function is defined in the $y - z$ coordinate system and is given as:

$$p(y) = fy^2 \quad (1)$$

where it is bounded by $-y_p \leq y \leq y_p$. The coefficient f is numerically found from the parabolic arc length definition as given by:

$$L_p = 2 \int_0^{y_p} \sqrt{1 + \left(\frac{dp}{dy}\right)^2} dy \quad (2)$$

Fig. 3 shows the composition of a biarc which connects the start point $P_1(y_1, z_1)$ to the end point $P_3(y_3, z_3)$ and it is constructed from two circular arcs of radii R_1 and R_2 tangentially sharing the mid point $P_2(y_2, z_2)$. Geometric matching conditions are imposed at the two ends such that the biarc $s(y)$ and the parabolic curve $p(y)$ share P_1 and P_3 and their tangent vectors:

$$s(y_1) = p(y_1) = z_1 \quad (3a)$$

$$s(y_3) = p(y_3) = z_3 \quad (3b)$$

$$\frac{ds}{dy}(y_1) = \frac{dp}{dy}(y_1) = 2fy_1 \quad (3c)$$

$$\frac{ds}{dy}(y_3) = \frac{dp}{dy}(y_3) = 2fy_3 \quad (3d)$$

Initially, $y_1 = z_1 = 0$, $y_3 = y_p$, and $z_3 = fy_p^2$ so that the two ends of the biarc coincide with those of the half parabola. Let $Q(y_Q, z_Q)$ be the intersection point between the tangent vectors at P_1 and P_3 . Assuming that the radius of curvature is monotonically increasing with y and $\|P_1Q\| \leq \|P_3Q\|$, the coordinates for Q are given by:

$$y_Q = y_1 + \frac{2fy_3(y_3 - y_1) - z_3 + z_1}{2f(y_3 - y_1)} = y_3 + \frac{2fy_1(y_3 - y_1) + z_3 - z_1}{2f(y_3 - y_1)} \quad (4a)$$

$$z_Q = z_1 + 2fy_1(y_Q - y_1) = z_3 - 2fy_3(y_3 - y_Q) \quad (4b)$$

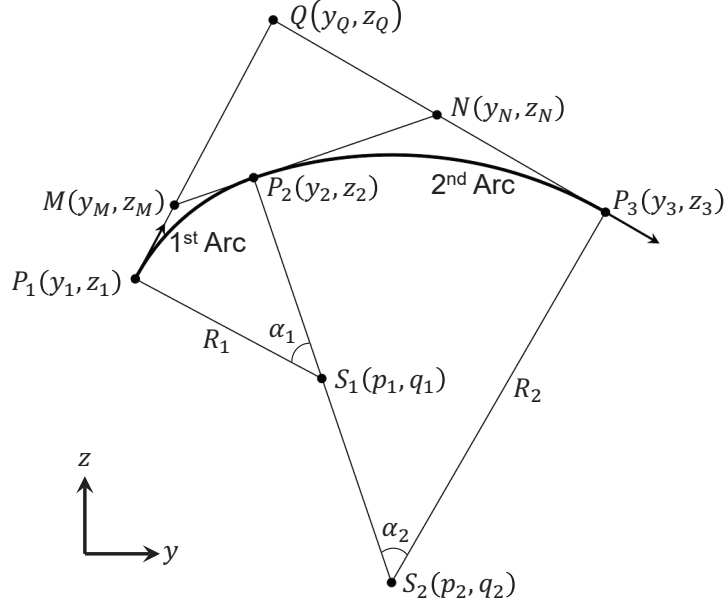


Fig. 3 Diagram of biarc spline consisting of two circular arcs joined tangentially at $P_2 (x_2, y_2)$.

Point $M (y_M, z_M)$ lies on the segment P_1Q and its coordinate y_M is a free parameter which determines the joint location of the two circular arcs at P_2 . Initially, it is chosen to be the midpoint between y_1 and y_Q such that:

$$y_M = y_1 + \frac{1}{2} (y_Q - y_1) \quad (5)$$

and then z_M is found with:

$$z_M = z_1 + 2fy_1 (y_M - y_1) \quad (6)$$

To find the location of P_2 , the slope m of the line segment MN is derived from the relationship $\|P_1M\| + \|P_3N\| = \|MN\|$ and is found to be:

$$m = 2fy_3 + \frac{2A \left(2fy_3A + B\sqrt{1 + 4f^2y_3^2} \right)}{B^2 - A^2} \quad (7)$$

where segment MN is tangent to the biarc at P_2 . The definitions of A and B in Eq. 7 are the following:

$$A = z_3 - z_1 - 2fy_1 (y_M - y_1) - 2fy_3 (y_3 - y_M) \quad (8a)$$

$$B = (y_M - y_1) \sqrt{1 + 4f^2y_1^2} + (y_3 - y_M) \sqrt{1 + 4f^2y_3^2} \quad (8b)$$

The coordinates of P_2 can now be derived from $\|P_1M\| = \|P_2M\|$ and they are found to be:

$$y_2 = y_M + (y_M - y_1) \sqrt{\frac{1 + 4f^2y_1^2}{1 + m^2}} \quad (9a)$$

$$z_2 = z_M + m (y_2 - y_M) = z_1 + (y_M - y_1) \left(2fy_1 + m \sqrt{\frac{1 + 4f^2y_1^2}{1 + m^2}} \right) \quad (9b)$$

To evaluate the accuracy of the constructed biarc, its interpolation error is evaluated as the maximum magnitude of the error function $e (y)$ defined as difference between the parabolic curve and the biarc for $y \in [y_1, y_3]$:

$$e (y) = p (y) - s (y) \quad (10)$$

Instead of numerically estimating the error from Eq. 10, which can be computationally costly, the error function is interpolated with two cubic polynomial segments $w_1(y)$ and $w_2(y)$ that are tangentially continuous at y_2 and remain zero at y_1 and y_3 . These are given by:

$$w_1(y) = a_1(y - y_1)^2(y - b_1) \quad (11a)$$

$$w_2(y) = a_2(y - y_3)^2(y - b_2) \quad (11b)$$

The tangent continuity conditions between the polynomials are:

$$w_1(y_2) = w_2(y_2) = e(y_2) = fy_2^2 - z_2 \quad (12a)$$

$$\frac{dw_1}{dy}(y_2) = \frac{dw_2}{dy}(y_2) = \frac{de}{dy}(y_2) = 2fy_2 - m \quad (12b)$$

which are used to find the definitions for a_1 , a_2 , b_1 , and b_2 :

$$a_1 = \frac{(2fy_2 - m)(y_2 - y_1) - 2(fy_2^2 - z_2)}{(y_2 - y_1)^3} \quad (13a)$$

$$a_2 = \frac{(2fy_2 - m)(y_3 - y_2) + 2(fy_2^2 - z_2)}{(y_3 - y_2)^3} \quad (13b)$$

$$b_1 = y_2 - \frac{(fy_2^2 - z_2)(y_2 - y_1)}{(2fy_2 - m)(y_2 - y_1) - 2(fy_2^2 - z_2)} \quad (13c)$$

$$b_2 = y_2 - \frac{(fy_2^2 - z_2)(y_3 - y_2)}{(2fy_2 - m)(y_3 - y_2) + 2(fy_2^2 - z_2)} \quad (13d)$$

Besides y_1 and y_3 , the extreme points ξ_1 and ξ_2 of Eq. 11(a) and (b), respectively, are found by solving $\frac{dw_1}{dy}(\xi_1) = 0$ and $\frac{dw_2}{dy}(\xi_2) = 0$:

$$\xi_1 = \frac{1}{3}(2b_1 + y_1) \quad (14a)$$

$$\xi_2 = \frac{1}{3}(2b_2 + y_3) \quad (14b)$$

The interpolation error can now be defined as the maximum of d_1 and d_2 , which are the maximum magnitudes for the two cubic polynomials representing the error function. These are given by:

$$d_1 = \begin{cases} |w_1(\xi_1)| & \xi_1 \in (y_1, y_2) \\ 0 & \xi_1 \notin (y_1, y_2) \end{cases} \quad (15a)$$

$$d_2 = \begin{cases} |w_2(\xi_2)| & \xi_2 \in (y_2, y_3) \\ 0 & \xi_2 \notin (y_2, y_3) \end{cases} \quad (15b)$$

If both d_1 and d_2 are less than or equal to the specified error tolerance e_s , then the required accuracy is achieved by the constructed biarc in approximating the parabolic function. The coordinates of the center points $S_1(p_1, q_1)$ and $S_2(p_2, q_2)$ for the two adjacent circular arcs are then found with:

$$q_1 = z_1 + \frac{y_2 - y_1 + m(z_2 - z_1)}{m - 2fy_1} \quad (16a)$$

$$q_2 = z_3 + \frac{y_3 - y_2 + m(z_3 - z_2)}{m - 2fy_3} \quad (16b)$$

$$p_1 = y_1 - 2fy_1(q_1 - z_1) \quad (16c)$$

$$p_2 = y_3 - 2fy_3(q_2 - z_3) \quad (16d)$$

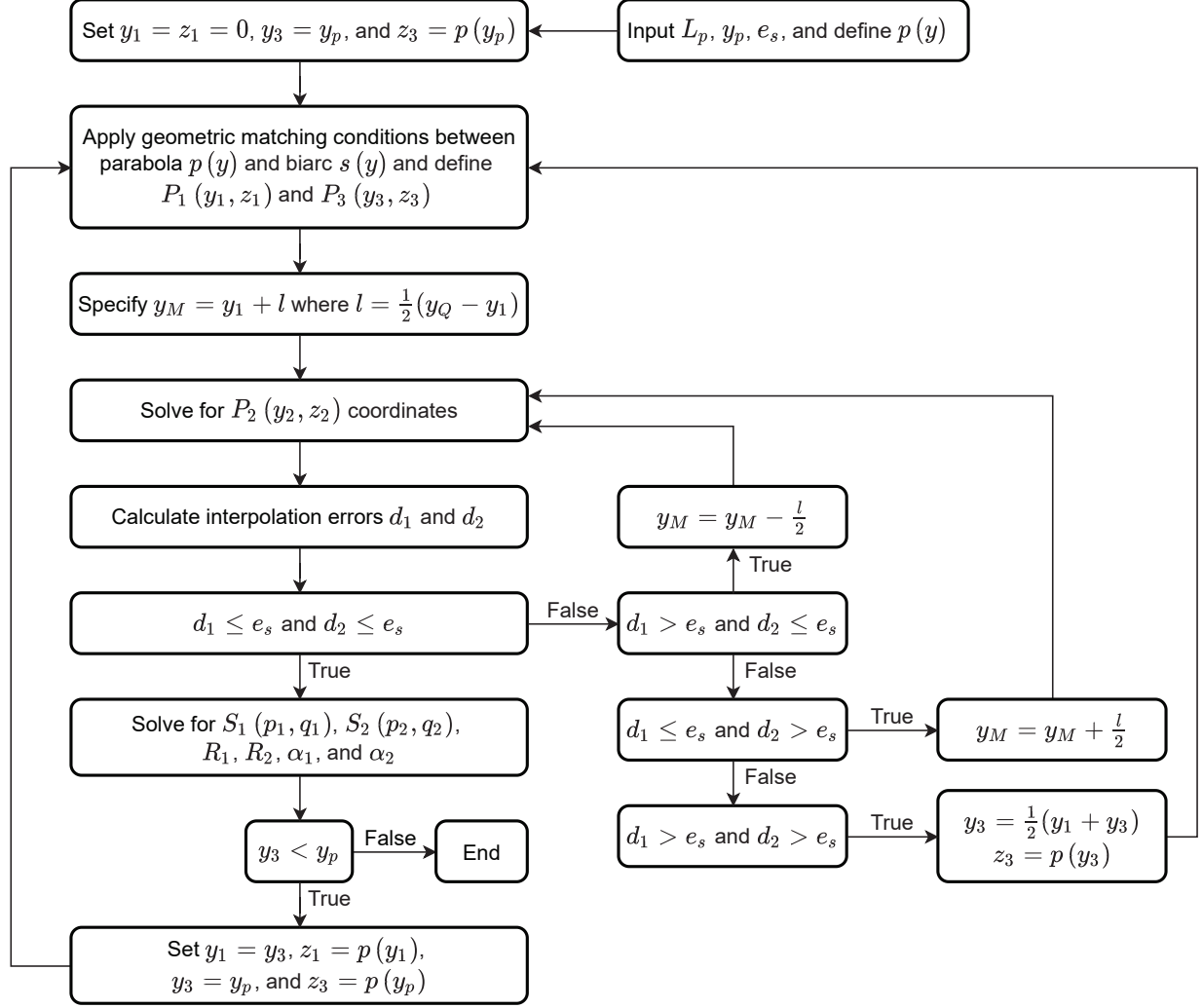


Fig. 4 Flowchart describing algorithm for biarc spline interpolation.

and the corresponding radii R_1 and R_2 for each circular arc are given by:

$$R_1 = (q_1 - z_1) \sqrt{1 + 4f^2 y_1^2} \quad (17a)$$

$$R_2 = (q_2 - z_3) \sqrt{1 + 4f^2 y_3^2} \quad (17b)$$

Finally, the subtended angles α_1 and α_2 can be found and are the following:

$$\alpha_1 = 2 \arcsin \left(\frac{\sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}}{2R_1} \right) \quad (18a)$$

$$\alpha_2 = 2 \arcsin \left(\frac{\sqrt{(y_3 - y_2)^2 + (z_3 - z_2)^2}}{2R_2} \right) \quad (18b)$$

Fig. 4 shows a flowchart describing the algorithm for the biarc spline interpolation procedure. If either d_1 or d_2 is greater than the error tolerance e_s , then the initially specified y_M value, which determines the joint location and ultimately the biarc shape, is altered through the bisection method and the interpolation procedure is repeated from Eq. 6. Specifically, if $d_1 > e_s$ and $d_2 \leq e_s$, then y_M in the next iteration is at the midpoint between y_1 and the current y_M . If $d_1 \leq e_s$ and $d_2 > e_s$, then y_M in the next iteration is at the midpoint between the current y_M and y_Q . This procedure for modifying y_M is repeated until the error tolerance is met through $d_1 \leq e_s$ and $d_2 \leq e_s$.

If both d_1 and d_2 are greater than the error tolerance e_s , then the parabolic curve is subdivided into two segments with identical y ranges, and the interpolation procedure is restarted on the first segment. This means y_1 remains the same at zero, but y_3 becomes the midpoint between y_1 and y_p in the next iteration. If the error requirement is satisfied with a suitable biarc, then the interpolation is implemented with the second segment. If the error requirement is still not satisfied ($d_1 > e_s$ and $d_2 > e_s$), then the subdivision continues with the first segment, where y_3 is continuously adjusted, until either d_1 or d_2 becomes less than or equal to e_s , or both $d_1 \leq e_s$ and $d_2 \leq e_s$. Then the interpolation scheme resumes with the remaining portion of the curve, which itself goes through the subdivision process until the specified accuracy is achieved. This procedure is sequentially implemented through the remaining segments, where both y_1 and y_3 are continuously adjusted, until the error tolerance is met for the entire parabolic range. Due to the geometric matching conditions imposed for each interpolating biarc, the entire circular spline approximating the curve retains tangent continuity. If the error tolerance e_s is lowered, then the discretization density of the curve will increase.

B. Strain Energy Minimization

The biarc interpolation procedure yields n number of circular arcs approximating the parabolic cross-section of the deployable composite boom within a specified error tolerance. The arcs are denoted by the subscript $i = 1, \dots, n$ and each segment consists of a subtended angle α_i , radius R_i , and length $L_i = R_i \alpha_i$. It is assumed that each circular arc retains its shape through the longitudinal x direction. The non-dimensional change in shell curvature for the i th circular arc in the $x - y$ coordinate system is denoted as $\hat{\kappa}_i$. This is found with Mohr's circle of curvature in Fig. 1(b) [12] and the delta is between the initially extended state at $\theta = 0$ and the final coiled state at $0 < \theta < \pi$. It is given by:

$$\hat{\kappa}_i = R_e \Delta \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \frac{\hat{C}}{2} \begin{bmatrix} 1 - \cos 2\theta \\ \cos 2\theta + 1 - \frac{2}{\hat{C}} \left(\frac{R_e}{R_i} \right) \\ 2 \sin 2\theta \end{bmatrix} \quad (19)$$

where each circular arc's change in shell curvature is normalized by the radius R_e through $\hat{C} = CR_e$. R_e is chosen to be the value that places the initially extended state at $\hat{C} = 1$. Note that all non-dimensional variables are denoted by the hat symbol. The mean radius of the parabolic function $p(y)$ serves as the representative geometric parameter to be compared against the circular boom's constant radius. It is computed with the average value of the curvature function $\kappa(y)$ as given by:

$$\kappa(y) = \left| \frac{d^2 p}{dy^2} \left[1 + \left(\frac{dp}{dy} \right)^2 \right]^{-\frac{3}{2}} \right| \quad (20)$$

$$R_m = \frac{y_p}{\int_0^{y_p} \kappa(y) dy} \quad (21)$$

In the parabolic cross-section, it is assumed that the curvature of its constituent arcs will all be in the same direction. This means that the curvature change in Eq. 19 assumes that every shell segment bends and coils in the equal-sense direction and so $\Delta \kappa_x$ will always be positive. The curvature of the final coiled configuration will also be identical between all circular shell segments, which means that they will share the same coiled radius r as shown in Fig. 2(b).

For a parabolic composite tape spring, the bending strain energy per unit length $\hat{U}$ is the sum of the strain energy terms of every circular shell segment that results from the biarc interpolation. $\hat{U}$ is given by:

$$\hat{U} = \frac{1}{2L_p} \sum_{i=1}^n L_i \hat{\kappa}_i^T \hat{\mathbf{D}} \hat{\kappa}_i \quad (22)$$

where it is non-dimensionalized in terms of the parabolic length L_p , the reduced bending stiffness term D_{11}^* in the x

direction, and the mean radius of curvature for the cross-section R_m as shown below.

$$\hat{U} = \frac{UR_e^2}{L_p D_{11}^*} \quad (23a)$$

$$\hat{D} = \frac{D^*}{D_{11}^*} \quad (23b)$$

The D^* matrix in Eq. 23(b) is the reduced bending stiffness matrix from classical lamination theory as defined below:

$$D^* = D - B^T A^{-1} B \quad (24)$$

which allows coupling between bending and stretching for any laminates having a non-zero B matrix. D^* reduces to D if $B = 0$ and there are no in-plane strains.

Before the strain energy minimization is conducted, the normalizing radius R_e in Eq. 19 must first be found. If R_e is to yield $\hat{C} = 1$ for the stable extended state at $\theta = 0$, then it must satisfy the following condition:

$$\frac{\partial \hat{U}}{\partial \hat{C}} \left(\theta = 0, \hat{C} = 1 \right) = 0 \quad (25)$$

which dictates that the corresponding equilibrium state be aligned at $\hat{C} = 1$. R_e is then solved from Eq. 25. It should be noted that the curvature $C = 1/R_e$ is the energy minimizing value for the extended state which is identical for all discretized segments, just as C is identical for the coiled state and every configuration in between. However, these segments have different initial curvatures in the as-manufactured stress-free ($\hat{U} = 0$) state due to the parabolic cross-section. Therefore, the model will yield some artificial strain energy in the extended state due to the difference in curvatures when compared against the as-manufactured state. It can be quantified from the y-component of the curvature change $\hat{\kappa}_i$ in Eq. 19 having non-zero values when $\theta = 0$ as a result of the curvature variation between the segments in the extended state. This effect is mitigated by the fact that the as-manufactured stable geometry and strain energy are always initially known.

The strain energy minimizing equilibria are found by taking the variation of $\hat{U}$ with respect to the two independent parameters $\hat{C}$ and θ as shown below:

$$\delta \hat{U} = \frac{\partial \hat{U}}{\partial \theta} \delta \theta + \frac{\partial \hat{U}}{\partial \hat{C}} \delta \hat{C} = 0 \quad (26)$$

where the following conditions are imposed to satisfy Eq. 26, whose solutions for $\hat{C}$ and θ define the equilibrium states:

$$\frac{\partial \hat{U}}{\partial \theta} = 0 \quad (27a)$$

$$\frac{\partial \hat{U}}{\partial \hat{C}} = 0 \quad (27b)$$

The equilibrium solution is stable if the boom's non-dimensional stiffness matrix $\hat{K}$ given below is positive definite.

$$\hat{K} = \begin{bmatrix} \frac{\partial^2 \hat{U}}{\partial \theta^2} & \frac{\partial^2 \hat{U}}{\partial \theta \partial \hat{C}} \\ \frac{\partial^2 \hat{U}}{\partial \theta \partial \hat{C}} & \frac{\partial^2 \hat{U}}{\partial \hat{C}^2} \end{bmatrix} \quad (28)$$

C. Parabolic CTM Boom

The nominal CTM boom structure is composed of two omega-shaped thin shells forming a closed cross-section. The two web sections that bonds the inner and outer shells are flat and between the webs are three tangent circular arc segments per shell. As shown in Fig. 5, the parabolic CTM boom's middle segment is parabolic and the outer two segments that connect to the webs remain circular for both the inner and outer shells. Its cross-section is assumed to be symmetric about both planar directions. The parabolic segments are discretized with circular arcs using the biarc spline interpolation procedure outlined previously. This defines the last discretized arc's coordinates (y_n, z_n) and (p_n, q_n) , the radius R_n , angle α_n , and arc length L_n in Fig. 5.

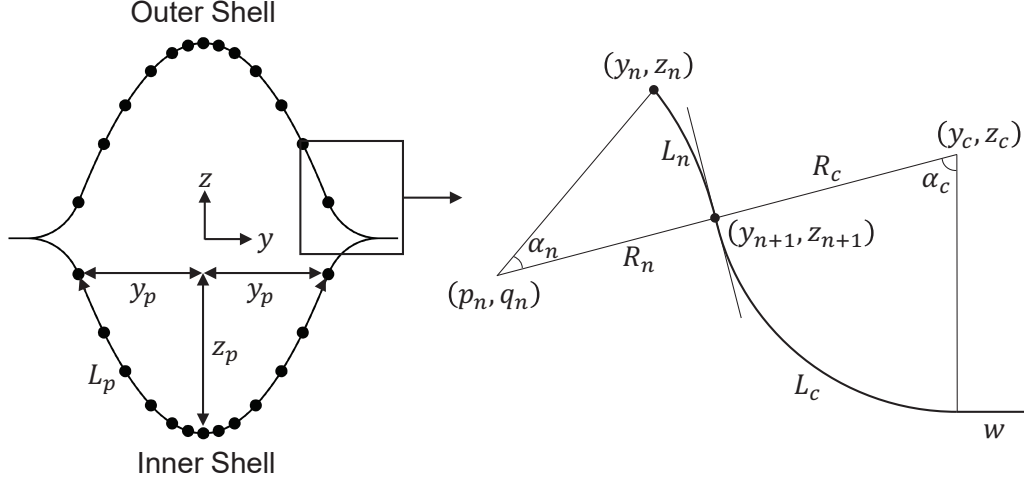


Fig. 5 Cross-section of a parabolic CTM boom with parabolic segments in the inner and outer shells. Also shown is the right hand side of the outer shell listing the geometric parameters for the last arc in the interpolated parabola connected to the circular segment and the web section.

The boom's flattened height h and the web width w are chosen to be design parameters that are initially specified. If tangent continuity is to be imposed and kinks are to be prevented between the n th arc of the parabolic segment and the circular segment, then the latter's arc length L_c , radius R_c , and subtended angle α_c must be computed based on specified values for h , w , and computed parameters for the n th arc. L_c is simply found from the definition of the flattened height as shown below:

$$h = 2(w + L_c) + L_p \quad (29)$$

The subtended angle α_c is found from a geometric relationship between the n th arc and circular segment:

$$\alpha_c = \arcsin\left(\frac{y_{n+1} - p_n}{R_n}\right) \quad (30)$$

and finally the radius R_c is found from $L_c = \alpha_c R_c$. With the circular segments defined, the non-dimensional changes in curvature can now be prescribed for each segment in the inner and outer shells. They are given below for the parabolic segments, circular segments, and the web section as $\hat{\kappa}_{iO,I}$, $\hat{\kappa}_{cO,I}$, and $\hat{\kappa}_w$, respectively, where the subscripts O and I denote the outer and inner shells, respectively.

$$\hat{\kappa}_{iO,I} = \frac{\hat{C}}{2} \begin{bmatrix} \pm (1 - \cos 2\theta) \\ \cos 2\theta + 1 - \frac{2}{\hat{C}} \left(\frac{R_e}{R_{iO,I}} \right) \\ 2 \sin 2\theta \end{bmatrix} \quad (31)$$

$$\hat{\kappa}_{cO,I} = \frac{\hat{C}}{2} \begin{bmatrix} \mp (1 - \cos 2\theta) \\ \cos 2\theta + 1 - \frac{2}{\hat{C}} \left(\frac{R_e}{R_{cO,I}} \right) \\ 2 \sin 2\theta \end{bmatrix} \quad (32)$$

$$\hat{\kappa}_w = \frac{\hat{C}}{2} \begin{bmatrix} 1 - \cos 2\theta \\ 0 \\ 2 \sin 2\theta \end{bmatrix} \quad (33)$$

The changes in curvature in every shell segment and the web sections are normalized by the radius R_e through $\hat{C} = CR_e$ as previously done in Eq. 19. The parabolic segment in the outer shell and the circular segments in the inner

shell bend in the equal-sense direction, and so their change in κ_x is positive. Conversely, the parabolic segment in the inner shell and the circular segments in the outer shell bend in the opposite-sense direction, and so their change in κ_x is negative. The web sections remain flat in the y direction and so they have no change in κ_y .

Following the same approach as the parabolic tape spring, the parabolic CTM boom's bending strain energy per unit length is the sum of the strain energy terms for every shell segment and web section. This is given by:

$$\hat{U} = \frac{1}{h} \left[L_{cO} \hat{\mathbf{k}}_{cO}^T \hat{\mathbf{D}}_{cO} \hat{\mathbf{k}}_{cO} + L_{cI} \hat{\mathbf{k}}_{cI}^T \hat{\mathbf{D}}_{cI} \hat{\mathbf{k}}_{cI} + w \hat{\mathbf{k}}_w^T \hat{\mathbf{D}}_w \hat{\mathbf{k}}_w + \frac{1}{2} \sum_{i=1}^n \left(L_{iO} \hat{\mathbf{k}}_{iO}^T \hat{\mathbf{D}}_{pO} \hat{\mathbf{k}}_{iO} + L_{iI} \hat{\mathbf{k}}_{iI}^T \hat{\mathbf{D}}_{pI} \hat{\mathbf{k}}_{iI} \right) \right] \quad (34)$$

where it is non-dimensionalized in terms of the flattened height h , the outer shell parabolic segment's reduced bending stiffness term D_{pO11}^* in the x direction, and the mean radius of curvature for the parabolic segment R_m as shown below.

$$\hat{U} = \frac{UR_e^2}{hD_{pO11}^*} \quad (35a)$$

$$\hat{\mathbf{D}}_{w,p,cO,I} = \frac{\mathbf{D}_{w,p,cO,I}^*}{D_{pO11}^*} \quad (35b)$$

The reduced bending stiffness matrix is distinguished between the web sections, parabolic segments, and circular segments in the inner and outer shells to allow each segment to have its own laminate layup. The stable equilibrium configurations are then found with the strain energy minimization procedure using Eqs. 26 to 28.

D. Second Moment of Area and Torsional Constant

The bending and torsional stiffnesses are evaluated by computing the boom's second moments of area about the principal axes, I_{yy} and I_{zz} , and the torsional constant, J , based on the cross-sectional geometry in the extended configuration [11, 17, 18]. The effective elastic and shear moduli are dependent on the boom's material properties and layup, which are held constant while the cross-sectional parameters are varied to optimize the stiffness properties through parametric analysis. The geometric centers of the cross-sections are placed at the origin of the $y - z$ coordinate system as shown in Figs. 2 and 5 for the parabolic tape spring and CTM booms, respectively. The cross-section is discretized into (y, z) coordinates, where each segment consisting of a pair of (y, z) coordinates is idealized as a rectangle rotated about the origin by the angle β . For the parabolic curves, the interpolated circular arcs are further discretized into rectangular segments. Then the second moments of area for each segment about its own centroid relative to the y and z axes, I_{y_j} and I_{z_j} , are given by:

$$I_{y_j} = \frac{1}{12} l_j t_j \left(l_j^2 \sin^2 \beta + t_j^2 \cos^2 \beta \right) \quad (36)$$

$$I_{z_j} = \frac{1}{12} l_j t_j \left(l_j^2 \cos^2 \beta + t_j^2 \sin^2 \beta \right) \quad (37)$$

where the m number of rectangular segments are denoted with the subscript $j = 1, \dots, m$ and l_j and t_j are the length and thickness of each segment, respectively.

The boom's second moments of area I_{yy} and I_{zz} are found by summing I_{y_j} or I_{z_j} through the parallel axis theorem, as shown by:

$$I_{yy,zz} = \sum_{j=1}^m I_{y,z_j} + A_j d_j^2 \quad (38)$$

where A_j is the area of each rectangular segment and d_j is the distance from each segment's centroid to the y or z axis.

The torsional constant J is the sum of the torsional constants for the web as open sections J_O and the arced segments as the closed sections J_C as given by:

$$J = J_O + J_C = \frac{1}{3} \sum_{j=1}^m l_j t_j^3 + \frac{4A_E^2}{\sum_{j=1}^m \frac{l_j}{t_j}} \quad (39)$$

where A_E is the total enclosed area of the closed section.

III. Parametric Analysis

The geometry of parabolic tape spring and CTM booms is evaluated to determine how it influences stiffness properties, bistability, and the stable coiled diameter $\emptyset = 2r$. The material description and properties for the composite plies and adhesive used for the parametric analysis are presented in Table 1.

Table 1 Material properties of thin-ply composites and adhesive.

Label	Material Form	Fiber/Resin	E_1 (GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)	Thickness t (μm)
C	Unidirectional Carbon Fiber	MR60H/PMT-F7	144.1	5.2	0.335	2.8	40.0
PW _C	Plain Weave Carbon Fiber	M30S/PMT-F7	89.0	89.0	0.035	4.2	58.2
A	Hysol EA9696 Film Epoxy	N/A	2.14	2.14	0.030	0.62	85.0

Maximizing the second moments of area and the torsional constant is how the optimal cross-sections are obtained. Only designs that yield a stable coiled configuration are considered through the bistability criterion, which are conditions in which the coiled equilibrium solution at $\theta = \frac{\pi}{2}$ must satisfy in order to be in a stable state: $\hat{C} > 0$ and $S = \frac{\partial^2 \hat{U}}{\partial \theta^2} > 0$. These conditions ensure that Eq. 28 remains positive definite. Specifically, the S value represents the concavity of the second strain energy well and a higher value means that the boom can more easily reach and remain in the coiled configuration. It is iteratively computed since the analytical form depends on the number of discretized arcs in the parabolic cross-section. All monostable designs are excluded in the analysis. Additionally, any designs containing a circular segment with a radius less than 12 mm are also removed since this is a manufacturing and laminate strain to yield or failure constraint.

A. Parabolic Tape Spring Boom

The layup [45PW_C/0_C/45PW_C] is chosen for the design study of the parabolic tape spring boom. Fig. 6 shows I_{yy} , I_{zz} , J , S , $\emptyset$, and the mean radius of the cross-section R_m as functions of two independent parameters that define the parabolic shape. These are the flattened height h , which is identical to the parabolic arc length L_p for tape spring booms, and the parabolic height z_p . The latter is interchangeable with the parabolic half width y_p , which is the actual parameter that is iterated as an input for the biarc spline interpolation procedure. The interpolation error is always set to $e_s = 0.01$ mm and the step sizes for h and y_p are 2 mm and 0.5 mm, respectively. For each value of h , increasing z_p will lower the required minimum radius among the circular segments until the 12 mm radius limit is exceeded. This constraint is what determines the upper limit of allowable z_p values per flattened height. The lower limit for z_p is formed by removing any designs with $z_p/h < 0.2$, which ensures that the parabolic tape spring will have sufficient cross-sectional depth.

It can be seen from Fig. 6 that there is no single design which simultaneously optimizes all metrics of interest. Therefore for each flattened height, maximizing the lower of I_{yy} and I_{zz} is considered to be optimal since the extended boom must maintain stiffness and withstand loading in both the y and z directions during operation. In this case, I_{yy} is the critical metric that is always optimized at the maximum parabolic height, or the minimum mean parabolic radius R_m . This does not conflict with the torsional constant J since it does not vary with respect to z_p . Maximizing I_{yy} also minimizes the coiled diameter $\emptyset$ at each flattened height, which optimizes the packaged volume efficiency. The bistability criterion S is found to be 1.547 for every design point, which indicates that the shape of the parabola does not affect the tendency for the boom to stay in the stable coiled state. It is actually found that only the layup and material properties of the parabolic tape spring influences the S metric.

To determine how the parabolic tape spring performs relative to a circular tape spring, the boom metrics in Fig. 6 are compared between these two cross-sectional shapes in Fig. 7. Specifically, only the optimal designs are compared where I_{yy} is maximized at each flattened height h . Both booms share the same [45PW_C/0_C/45PW_C] layup and h , which means that the mass is always kept identical during the comparisons. Note that for the circular tape spring, the two independent parameters that define its cross-section are chosen to be the flattened height h and the radius R , where the latter parameter is varied to maximize I_{yy} . For each h , the lower limit of radius R is either the 12 mm manufacturing constraint or the value dictated by the upper subtended angle limit of $\alpha = 180^\circ$ (from $R = h/\alpha$), whichever is larger. Indeed, it is at these lower radii limits, or the maximum possible cross-sectional curvatures, where I_{yy} is always maximized for the circular tape spring.

When considering the critical metric I_{yy} , the parabolic tape spring outperforms its circular counterpart for $h \geq 59$ mm and this margin increases as the flattened height approaches 100 mm. Conversely, the circular tape spring has

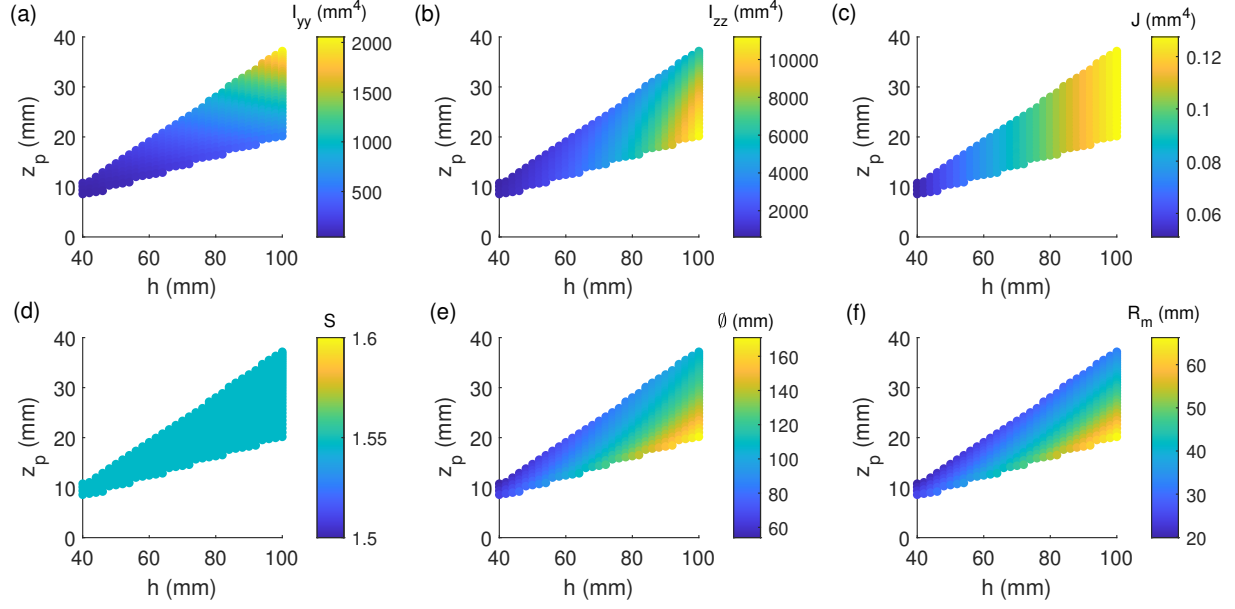


Fig. 6 Parametric analysis of $[45PW_C/0_C/45PW_C]$ parabolic tape spring for the metrics (a) I_{yy} , (b) I_{zz} , (c) J , (d) bistability criterion S , (e) coiled diameter ϕ , and (f) mean radius R_m as functions of flattened height h and parabolic height z_p .

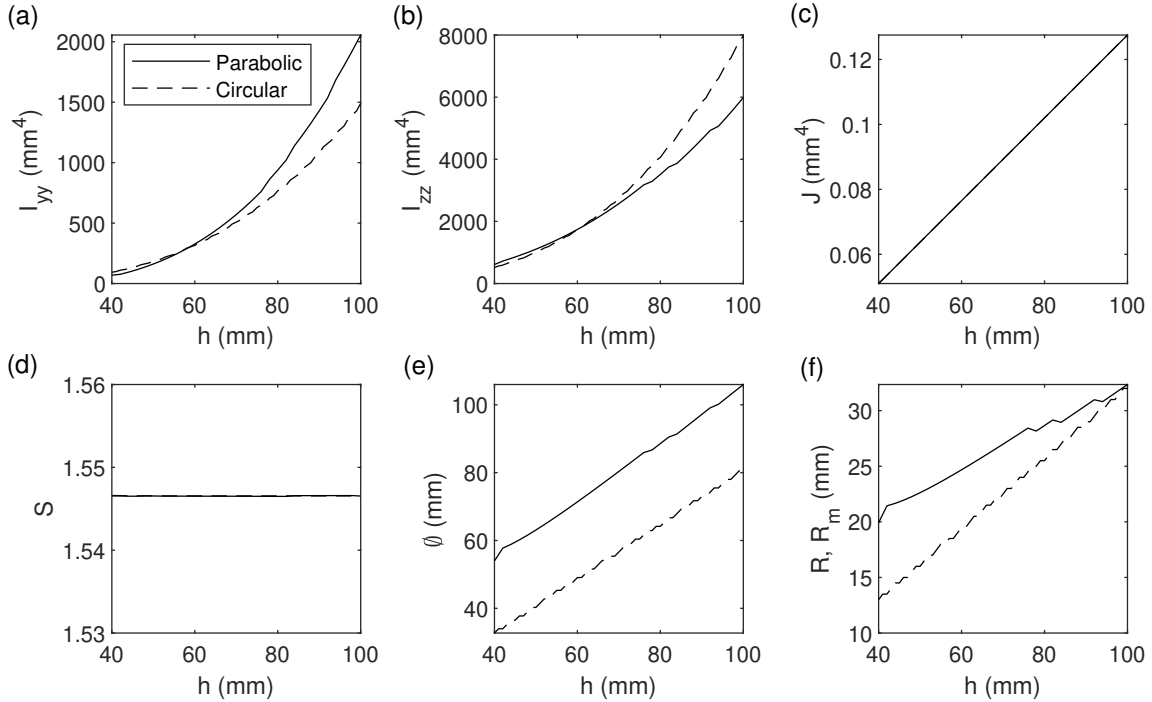


Fig. 7 Optimal design's (a) I_{yy} , (b) I_{zz} , (c) J , (d) coiled diameter ϕ , (e) bistability criterion S , and (f) initial radius R and R_m for the $[45PW_C/0_C/45PW_C]$ parabolic and circular tape springs as a function of the flattened height h .

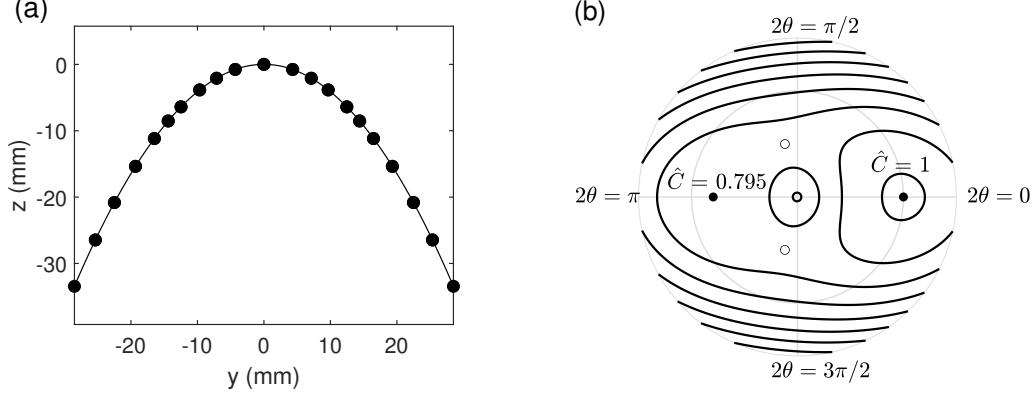


Fig. 8 (a) Optimal cross-section of the $[45PW_C/0_C/45PW_C]$ parabolic tape spring with a flattened height of $h = 92$ mm and (b) corresponding polar contour plot of $\hat{U}$ as a function of $\hat{C}$ and θ . The black dots and circles denote stable and unstable equilibrium points, respectively. The contour lines are separated by the constant $\Delta\hat{U} = 0.1$.

superior stiffness properties for $h < 59$ mm, but the difference is significantly less at smaller flattened heights. These results are driven by the difference in effective radius, or R and R_m for respectively the circular and parabolic booms, across various flattened heights. The $R_i \geq 12$ mm constraint for the discretized segments limits how small the mean radius R_m of the parabolic tape spring can be relative to the circular tape spring's radius R . This is clearly observed at lower values of h , where $R < R_m$, and the larger corresponding curvature of the circular tape spring results in higher values of I_{yy} . However as h approaches 100 mm, the manufacturing limit becomes less significant as R converges with R_m , and here the parabolic tape spring retains the superior stiffness performance. Regardless of what the flattened height is, the circular tape spring always retains a smaller coiled diameter and so it has more favorable packaging efficiency. However, the bistability criterion are always identical between the two booms, as well as their torsional constant.

To examine a specific parabolic tape spring design, Fig. 8(a) presents the optimal cross-section with a flattened height of $h = 92$ mm, which corresponds to a parabola height of $z_p = 33.44$ mm, or a parabola half width of $y_p = 28.5$ mm. The parabolic coefficient f in Eq. 1 can then be approximated as -0.0412 and this cross-section yields the second moments of area values of $I_{yy} = 1534$ mm⁴, $I_{zz} = 4926$ mm⁴, torsional constant of $J = 0.117$ mm⁴, and mean radius of $R_m = 30.98$ mm. With the interpolation error of $e_s = 0.01$ mm, the parabolic shape is successfully interpolated with 20 circular segments. Fig. 8(b) shows a polar contour plot of the non-dimensional bending strain energy per unit length $\hat{U}$ as a function of the non-dimensional curvature $\hat{C}$ and the shell angle relative to an underlying cylinder θ . Note that the normalizing radius was found to be $R_e = 39.38$ mm. The stable extended state is located at $\theta = 0$ and $\hat{C} = 1$. The second strain energy well is the stable coiled state located at $\theta = \frac{\pi}{2}$ and $\hat{C} = 0.795$ which results in a coiled diameter of $\emptyset = 99.11$ mm and bistability criterion value of $S = 1.547$. There are also two unstable saddle equilibrium points that are marked as circles and they are located between the coiled and extended states that are denoted as black dots.

B. Parabolic CTM Boom

The parabolic CTM boom that is evaluated in the parametric analysis is summarized in Table 2. The plainweave (PW) lamina is always the outermost surface ply for the boom, and thus the labeling convention for each shell segment's laminate is to list the plies from the outermost to the innermost. Bistable layups are designated for the inner shell circular segments to aid the outer shell bistable parabolic segment in generating a second stable state in the coiled configuration. This is due to both segments bending in the equal-sense direction relative to their extended configurations. Ply drops are placed between the parabolic and circular segments of the outer shell to reduce the stiffness of the circular segment and web, which are monostable. This mitigates the negation of the second strain energy well that is generated by the parabolic segment. The web section is modelled as a single laminate and it consists of the outer and inner web laminates with the adhesive layer in between.

With the flattened height h and web width w fixed, the two independent parameters that define the CTM geometry are chosen to be the parabolic arc length L_p and the parabolic height z_p , the latter of which is interchangeable with the

Table 2 Parabolic CTM boom evaluated in parametric analysis.

Outer Parabolic Segment	Outer Circular Seg. and Web	Inner Parabolic Segment	Inner Circular Seg. and Web	Height h (mm)	Web w (mm)
[45PW _C /0C/45PW _C]	[45PW _C /0C]	[45PW _C /90C]	[45PW _C /45PW _C]	130	4.5

parabolic half width y_p . Specifying these two parameters will yield a single set of the circular segment's radius R_c , subtended angle α_c , and arc length L_c that satisfies tangent continuity through the cross-section. The interpolation error and step sizes are set to be identical to the tape spring's parametric analysis. The variation of L_p and z_p is identical between the inner and outer shells to maintain a symmetric cross-sectional shape. An additional volume requirement is imposed where four co-wrapped booms per spool, each with a length of 16.55 m, do not exceed an outer coiled diameter of 305 mm [18]. A packaging efficiency of 60% is assumed for all booms.

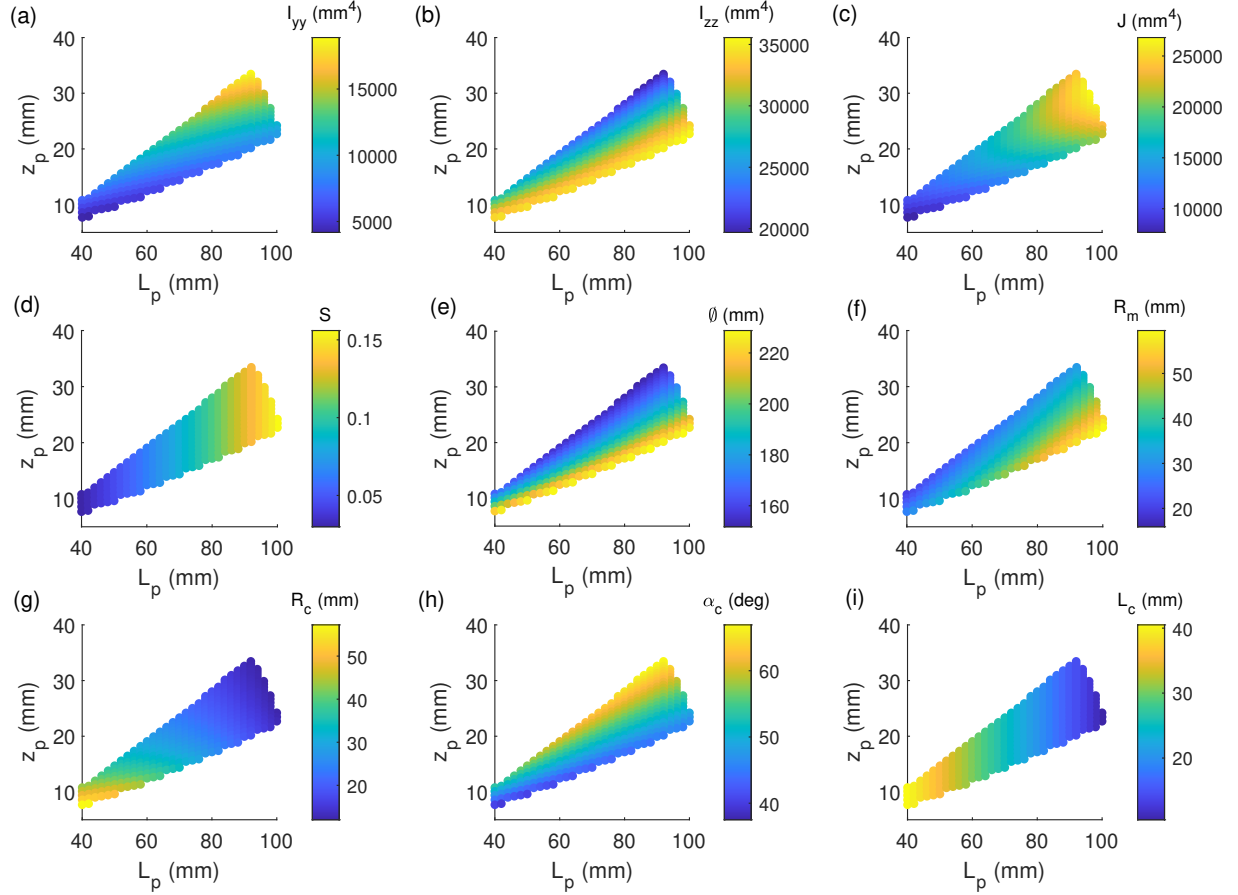


Fig. 9 Parametric analysis of the parabolic CTM boom for the metrics (a) I_{yy} , (b) I_{zz} , (c) J , (d) bistability criterion S , (e) coiled diameter $\emptyset$, (f) mean radius R_m of the parabolic segment, circular segment (g) radius R_c , (h) subtended angle α_c , and (i) arc length L_c as functions of the parabolic arc length L_p and parabolic height z_p .

Fig. 9 shows the variation of the second moments of area I_{yy} and I_{zz} , torsional constant J , bistability criterion S , stable coiled diameter $\emptyset$, parabolic mean radius R_m , circular segment radius R_c , subtended angle α_c , and arc length L_c for the parabolic CTM boom. Since $h = 130$ mm and $w = 4.5$ mm for every design, the parameters R_c , α_c , and L_c describe the size proportion of the circular segments relative to the parabolic segment in the inner and outer shells. For each value of L_p , the lower limit of z_p is set by the stowed volume requirement for the outer coiled diameter of the spool since the stable coiled diameter per boom increases as its parabolic segment loses cross-sectional depth. The

upper z_p limit is set by the 12 mm radius constraint where the discretized segment with the lowest radius (among R_i for $i = 1, \dots, n$) is the restricting factor for $L_p \leq 92$ mm and the circular segment radius R_c is the restricting factor for $L_p > 92$ mm. As implemented in the previous section, maximizing the lower of I_{yy} and I_{zz} results in optimal designs for each value of L_p since there is no single global optimum across all metrics of interest. This means I_{yy} is the critical metric that is optimized at the maximum parabolic height (or minimum R_m), which also minimizes the coiled diameter, and these findings follow that of the parabolic tape spring. Maximizing I_{yy} also does not conflict with the bistability criterion, which is constant for each L_p value. Larger parabolic heights also tend to favor J , where the maximum z_p value for each L_p yields a torsional constant found to either be the maximum possible value or sufficiently close to the maximum value.

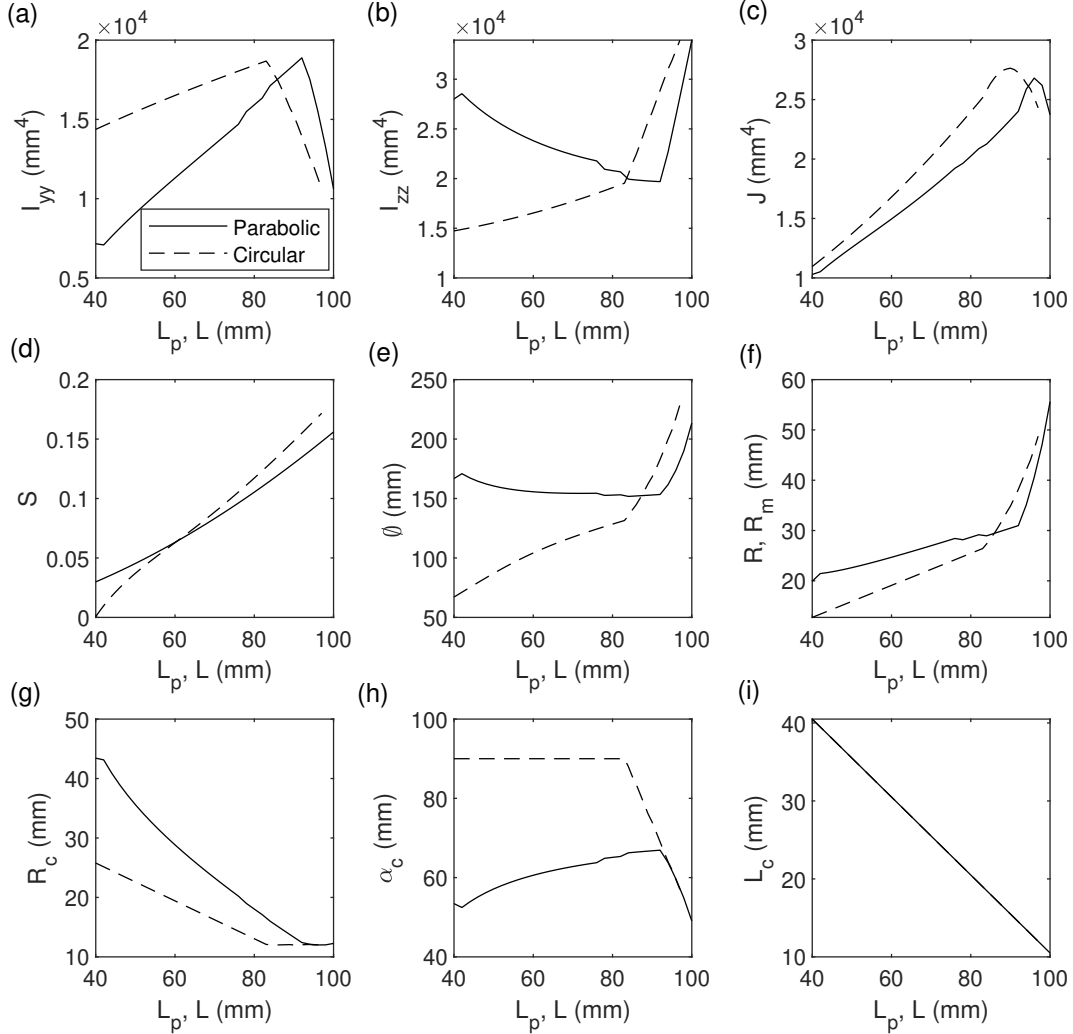


Fig. 10 Optimal design's (a) I_{yy} , (b) I_{zz} , (c) J , (d) bistability criterion S , (e) coiled diameter ϕ , (f) primary radius R_m and R , secondary (g) radius R_c , (h) subtended angle α_c , and (i) arc length L_c for the parabolic and circular CTM booms as a function of L_p and L .

The circular CTM booms were previously analyzed in [18] and they share the same cross-sectional shape as the parabolic CTM with the exception of a single cylindrical arc being in place of the parabolic segment. To generate the metrics for the circular CTM boom and facilitate a performance comparison with its parabolic counterpart, a similar parametric analysis as Fig. 9 is conducted with identical shell layups, flattened height, and web width as listed in Table 2. The two independent parameters are the arc length L and the subtended angle α of the primary circular segment at the center of the inner and outer shells. The secondary circular segment is considered to be the section

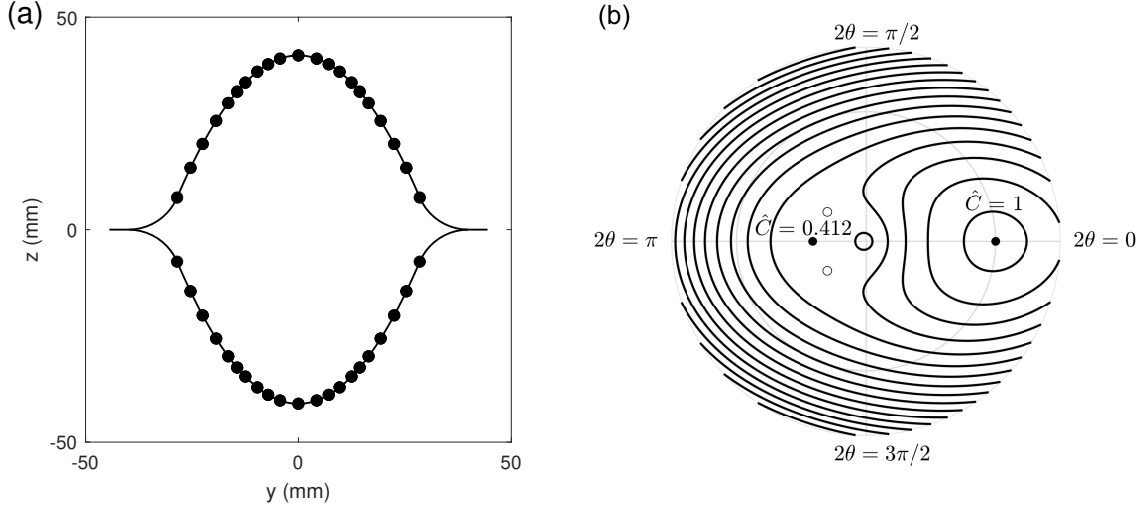


Fig. 11 (a) Optimal cross-section of the parabolic CTM boom with $L_p = 92$ mm and (b) corresponding polar contour plot of $\hat{U}$ as a function of $\hat{C}$ and θ . The black dots and circles denote stable and unstable equilibrium points, respectively. The contour lines are separated by the constant $\Delta \hat{U} = 0.1$.

between the primary segment and the web and denoted by the subscript c as done for the parabolic CTM. Specifying these two variables will yield a single set of geometrically admissible primary segment radius $R = L/\alpha$, secondary segment radius R_c , subtended angle α_c , and arc length $L_c = R_c \alpha_c$. In the parametric analysis, I_{yy} is found to be maximized at the maximum possible subtended angle α for each circular arc length L , which is constrained by either the upper manufacturing limit of $\alpha = 180^\circ$ or the 12 mm radius limit on R_c . This follows the established trend of larger cross-sectional curvatures favoring the I_{yy} metric.

The boom metrics in Fig. 9 are compared between the optimized parabolic and circular CTM booms in Fig. 10 for the purpose of comparing their stiffness properties and coiled volume efficiency. These optimal designs correspond to points with maximum I_{yy} values for each parabolic and circular arc length, or L_p and L respectively. In the Fig. 10 comparisons, the L is identical to L_p so that equal segment length to flattened height ratio is maintained between the parabolic and circular CTM booms, such that $L_p/h = L/h$. It should be noted that there are no viable circular CTM designs for $L > 97$ mm due to the volume restriction involving the outer coiled diameter of the spool. Also, the mass is always kept identical during the comparisons since the booms share the same flattened height and layups. The findings are similar to those of the parabolic vs. circular tape spring comparisons from Fig. 7, where the parabolic CTM outperforms the circular CTM on the critical I_{yy} metric in the upper range of arc lengths, or $L_p, L \geq 86$ mm and vice versa for $L_p, L < 86$ mm. This is correlated by $R_m > R$ for $L_p, L < 86$ mm and $R_m < R$ for $L_p, L \geq 86$ mm, where the lower radii and larger corresponding curvatures contribute to greater I_{yy} values. Again, the 12 mm radius constraint is more restricting for the discretized segments of the parabolic CTM when compared to the circular CTM segments. The two boom shapes also yield similar torsional constants and bistability criterion values, where the circular CTM maintains a small advantage for these metrics across the majority of the L_p and L parameter space. One difference to the tape spring comparisons is that the parabolic CTM manages to have a smaller stable coiled diameter for $L_p, L \geq 88$ mm, revealing a set of designs that have greater packaged volume efficiency.

Table 3 Optimal parabolic and circular CTM boom metrics and geometry.

CTM Boom	I_{yy} (mm ⁴)	I_{zz} (mm ⁴)	J (mm ⁴)	S	$\emptyset$ (mm)	R, R_m (mm)	R_c (mm)	α_c (deg)	L_c (mm)	L_p, L (mm)
Parabolic	18884	19692	24015	0.135	153.38	30.98	12.41	66.92	14.5	92
Circular	18682	19535	25166	0.126	131.47	26.42	12.10	90	19	83

If the stiffness properties are optimized by maximizing I_{yy} across L_p and L for each boom, then their metrics and

geometry from Fig. 10 are presented in Table 3. Overall, the parabolic CTM boom offers a slight advantage in stiffness while being outperformed in terms of the coiled volume efficiency. Specifically, it has slightly larger I_{yy} and I_{zz} and smaller J values while the circular CTM retains a lower coiled diameter. The optimal parabolic CTM boom has a parabolic arc length of $L_p = 92$ mm and a parabolic height of $z_p = 33.44$ mm, which results in a parabolic segment that is identical to the case shown for the tape spring in Fig. 8. The cross-sectional geometry for this optimal case is shown in Fig. 11(a) which includes the 20 interpolated circular segments that compose the parabolic segments in both the inner and outer shells. Its polar contour plot of the non-dimensional strain energy per unit length $\hat{U}$ is presented in Fig. 11(b) and the normalizing radius R_e is 31.60 mm. Besides the first stable extended state at $\hat{C} = 1$ and $\theta = 0$, the second energy minimum is found to be at $\theta = \pi/2$ and $\hat{C} = 0.412$, which yields a stable coiled diameter of $\emptyset = 153.38$ mm. This value is much larger than that of the parabolic tape spring ($\emptyset = 99.11$ mm) due to the inner shell being monostable and causing the bistable outer shell's coiled diameter to increase when coupled as a single structure.

IV. Finite Element Analysis

An implicit quasi-static finite element analysis (FEA) is conducted in Abaqus/Standard 2020 to verify the coiled shape and associated strain energy of the optimal parabolic tape spring and CTM booms presented in Fig. 8 and Fig. 11, respectively. Fig. 12 shows the finite element model and general analysis approach, which is to coil each boom to a diameter that is less than its stable coiled diameter $\emptyset$, and then release the constraints to let the boom settle into the stable coiled configuration. The parabolic tape spring and CTM boom lengths are specified as 500 mm and 800 mm, respectively. Both the boom and the rigid hub consist of 4 node reduced integration shell elements (S4R) in each simulation. The mesh is uniform with an element size of 2.3 mm for the tape spring and 3.07 mm for the CTM boom. The hub length is specified as 150 mm while its diameter is 60 mm and 140 mm for the tape spring and CTM boom, respectively. The parabolic cross-section is generated with spline interpolation of the discretized nodal coordinates shown in Fig. 8(a). Note that the **ABD** matrices resulting from the shell layups are directly inputted into the general shell stiffness section for faster and more reliable convergence. For the CTM boom, the inner and outer shell web section nodes are coupled together through tie constraints once they are independently defined and meshed. Frictionless contact is always imposed between the hub and the side of the boom facing the hub. For the tape spring, this is between its inner surface and the hub. For the CTM, there is contact between the hub and the bottom surfaces of the inner and outer shells. Due to this contact formulation, both booms have no self-contact during coiling since the simulations only serve to verify the analytical model.

There are five analysis steps in the parabolic tape spring's coiling simulation shown in Fig. 12(a) and the associated boundary conditions and loading are now described. In the first step, the tape spring root node B and reference point C are kinematically coupled, tip node A and reference point C are fixed, and reference point D is tied to the hub's geometric center. Reference point C is positioned at a distance of the parabolic height z_p above reference point D. The hub undergoes upward rigid body translation of magnitude z_p in the vertical z direction while a downward pressure load of 100 kPa is applied to the boom edge elements until it is flattened and reference points C and D are coincident. The second step frees node A in the axial x direction and applies a tip force of 20 N to tension the boom. The third step applies angular rotation to both reference points C and D until the boom is nearly coiled. The fourth step removes the tip force, pressure load, and the boundary conditions at node A while rotating reference points C and D by a minimal amount to initiate the release of the coiled boom. Then the final step fixes reference points C and D and lets the boom unfurl to reach static equilibrium in its stable coiled shape. Note that node B remains kinematically coupled to reference point C during the entire simulation.

The parabolic CTM boom simulation shown in Fig. 12(b) also contains five analysis steps and it generally follows the procedure for the parabolic tape spring. It has additionally defined root side nodes R_1 and R_2 and tip side nodes T_1 and T_2 . The boundary conditions and coupling are identical in the first step and reference point C is placed above reference point D at a distance equal to the cross-sectional height of the entire CTM boom. The hub undergoes upward rigid body translation by this same distance while 20 N tension loads are placed on nodes R_1 and R_2 to flatten the boom's root edge against the rigid hub. In the second step, the root tension is replaced with a downward 100 kPa pressure load on the inner and outer shell's edge elements to keep them flattened. Simultaneously, node A is freed in the x direction and a tip force of 10 N longitudinally tensions the boom. Additional 30 N side loads are placed on the tip nodes T_1 and T_2 to flatten the rest of the boom. Maintaining a flattened profile through the end of coiling is found to be necessary for getting the boom to enter the strain energy well associated with the coiled stable state. The third coiling and fourth release steps are identical to those of the parabolic tape spring simulation. During the fifth step, the T_1 and T_2 side loads are removed to ensure that the cross-section is no longer constrained in any manner, and the boom unfurls to

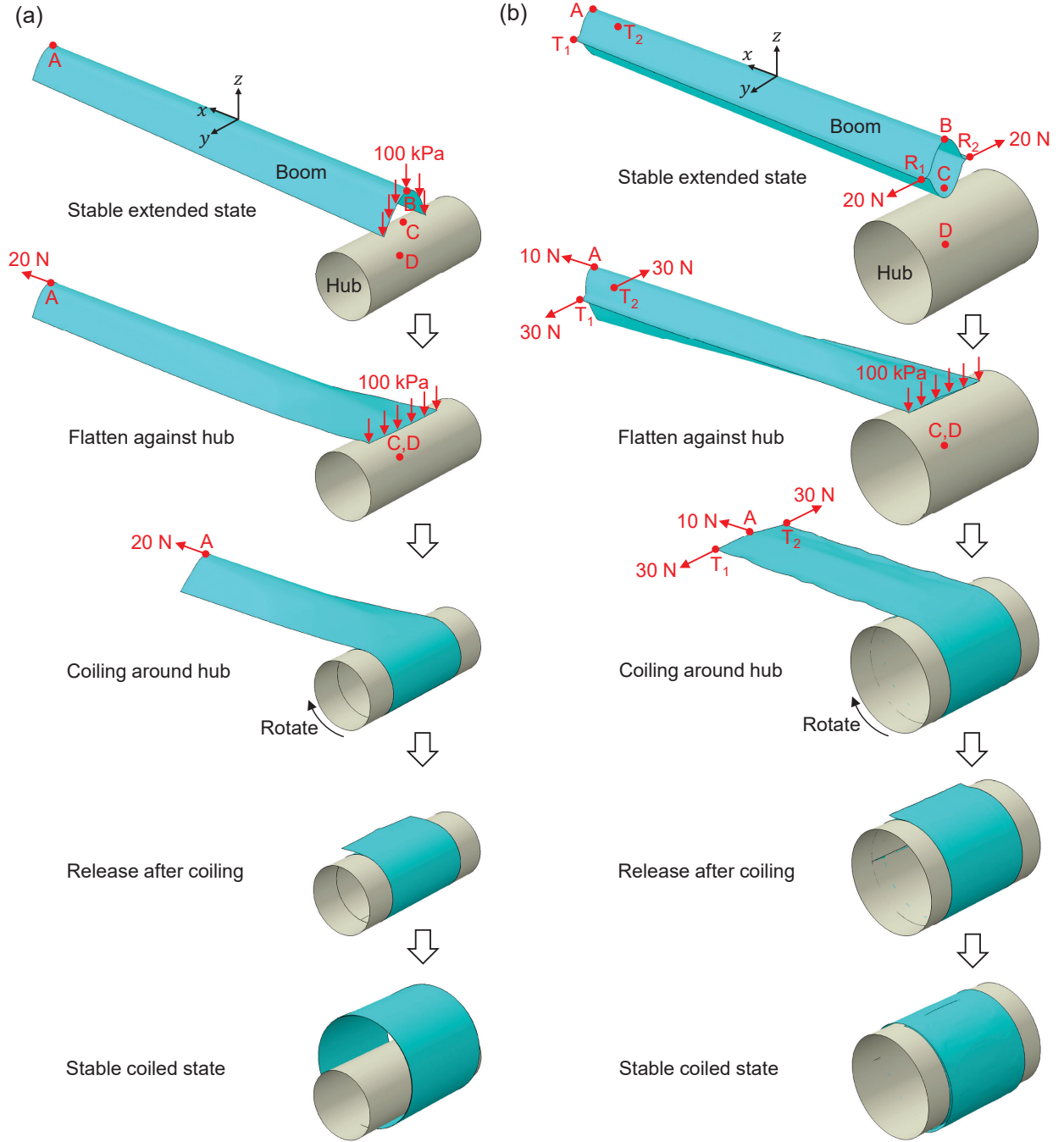


Fig. 12 Quasi-static finite element simulations of coiling the optimal (a) 500 mm parabolic tape spring and (b) 800 mm parabolic CTM boom to reach their stable coiled configurations. The tape spring cross-section is shown in Fig. 8(a) and its rigid hub diameter is 60 mm. The CTM cross-section is shown in Fig. 11(a) and its rigid hub diameter is 140 mm.

reach its final stable equilibrium state corresponding to the coiled configuration. To verify that these simulations can distinguish between bistable and monostable configurations, the same FEA was conducted for isotropic monostable booms and they were confirmed to fully uncoil back to their extended shapes during the release step.

Fig. 13 shows the total strain energy U of the parabolic tape spring and CTM boom as a function of their average

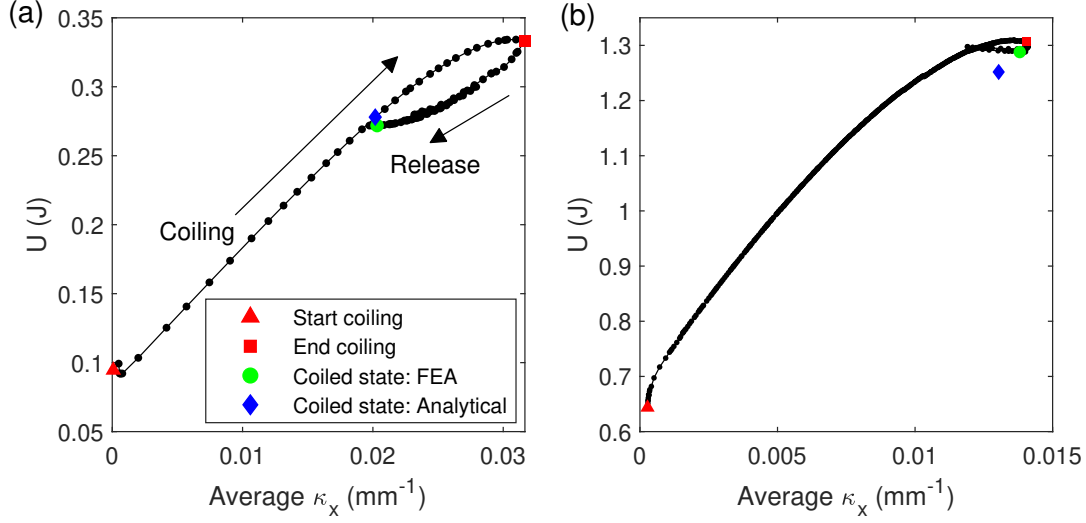


Fig. 13 The total strain energy U vs. average axial curvature κ_x for the optimal (a) parabolic tape spring and (b) parabolic CTM boom from their respective finite element simulations of quasi-static coiling and release, as shown in Fig. 12. The analytical predictions are shown for comparison.

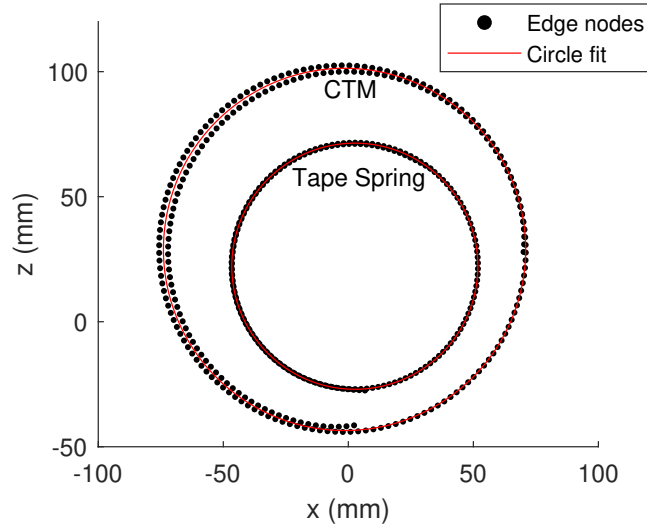


Fig. 14 Final stable coiled profiles of the optimal parabolic tape spring and parabolic CTM boom composed of edge nodes with numerical circle fits.

axial curvature κ_x from the FEA of quasi-static coiling and release into the stable configuration. The black markers indicate each simulation increment where the strain energy is recorded in the output, which is once every 10 time steps. The start of coiling corresponds to the beginning of the third analysis step, which consists of near-zero axial curvature and some amount of strain energy caused by the flattening and tensioning of the booms. The end of coiling matches the beginning of the fourth step for the parabolic tape spring in Fig. 13(a). This point is associated with near maximum amount of stored strain energy and the largest axial curvature observed in the simulation. Then the constraint release causes both strain energy and curvature loss as the tape spring unfurls to a larger diameter while it reaches static equilibrium in the stable coiled state. For the parabolic CTM boom in Fig. 13(b), the end of coiling is associated with

one of the largest amounts of stored strain energy and axial curvature, and the constraint release initially causes axial curvature and strain energy loss as the boom unfurls to a larger diameter. However, this loss is less than the parabolic tape spring because the hub diameter is already close to the stable coiled diameter that the boom is settling to. For both booms, the end of the simulation is labelled as the FEA coiled state and the analytical prediction is also shown for comparison.

The resulting stable coiled profiles are shown in Fig. 14 through the side edge nodes. A circle fit [27] of the edge nodes approximates the coiled diameter to be $\emptyset = 98.42$ mm for the parabolic tape spring, which closely matches the analytical prediction of $\emptyset = 99.11$ mm with a percent error of 0.70%. The analytical model's total strain energy of $U = 0.278$ J is also verified with the FEA's $U = 0.272$ J resulting in a percent error of 2.16%. The FEA coiled diameter for the parabolic CTM boom is found to be $\emptyset = 144.87$ mm, which is larger than the hub diameter of 140 mm, and again verifies the analytical prediction of $\emptyset = 153.38$ mm with a percent error of 5.55%. The corresponding strain energy level predicted by the finite element simulation is $U = 1.288$ J, which correlates with the analytical prediction of $U = 1.252$ J resulting in a percent error of 2.88%. So although the accuracy for the strain energy remains consistent between the two booms, the coiled diameter prediction becomes less accurate with the more complex parabolic CTM boom. Overall however, there is good agreement between the FEA simulations and the analytical model predicting the stable coiled configuration of booms with parabolic cross-sections.

V. Conclusions

An analytical model is derived to predict the stable coiled configurations of deployable composite booms with parabolic cross-sections. Biarc spline interpolation discretizes conic sections into a series of circular segments with tangent continuity and the strain energy of every segment is summed and minimized to obtain the boom's stable equilibrium states. This procedure is formulated for both parabolic tape springs and CTM booms consisting of parabolic shells. Parametric studies of the cross-sectional geometry reveal that the parabolic tape spring retains superior stiffness properties over a circular tape spring of identical layout, flattened height, and mass at larger values of the flattened height. However, this advantage comes at the cost of higher coiled diameters. These findings generally stay consistent, though the difference in relative stiffness is smaller when parabolic and circular CTM booms of identical layouts, flattened height, and mass are compared through their own parametric analysis. Finite element simulations are then developed for coiling the optimal design cases of the parabolic tape spring and CTM booms. The FEA verifies both the stored strain energy and the coiled diameter predictions made by the analytical model for the parabolic tape spring with low percent errors of 2.16% and 0.70%, respectively. The analytical model maintains its predictive accuracy for the parabolic CTM boom, where the percent errors for the stored strain energy and coiled diameter are found to be 2.88% and 5.55%, respectively.

Acknowledgments

Financial support from the Department of Mechanical and Aerospace Engineering and the College of Engineering at North Carolina State University is gratefully acknowledged.

References

- [1] Hoang, B., White, S., Spence, B., and Kiefer, S., "Commercialization of Deployable Space Systems' roll-out solar array (ROSA) technology for Space Systems Loral (SSL) solar arrays," *2016 IEEE Aerospace Conference*, IEEE, 2016, pp. 1–12.
- [2] Chamberlain, M. K., Kiefer, S. H., and Banik, J., "On-orbit structural dynamics performance of the roll-out solar array," *2018 AIAA Spacecraft Structures Conference*, 2018.
- [3] Straubel, M., Seefeldt, P., Spietz, P., and Huehne, C., "The design and test of the Gossamer-1 boom deployment mechanisms engineering model," *2nd AIAA Spacecraft Structures Conference*, 2015, p. 1837.
- [4] Fernandez, J. M., Rose, G., Stohman, O. R., Younger, C. J., Dean, G. D., Warren, J. E., Kang, J. H., Bryant, R. G., and Wilkie, K. W., "An advanced composites-based solar sail system for interplanetary small satellite missions," *2018 AIAA Spacecraft Structures Conference*, 2018.
- [5] Rimrott, F. P. J., "Storable tubular extendible member: a unique machine element," *Machine Design*, Vol. 37, No. 28, 1965, pp. 156–165.
- [6] Murphey, T. W., and Banik, J., "Triangular rollable and collapsible boom," , 2011. US Patent 7,895,795.

- [7] Leclerc, C., and Pellegrino, S., "Nonlinear elastic buckling of ultra-thin coilable booms," *International Journal of Solids and Structures*, Vol. 203, 2020, pp. 46–56.
- [8] Rennie, B. B., "New Closed Tubular Extendible Boom," *2nd Aerospace Mechanisms Symposium*, 1967, pp. 163–170.
- [9] Aguirre-Martinez, M., Bowen, D., Davidson, R., Lee, R., and Thorpe, T., "The development of a continuous manufacturing method for a deployable satellite mast in CFRP," *British Plastics Congress*, Citeseer, 1986, pp. 107–110.
- [10] Geppert, U., Biering, B., Lura, F., Block, J., Straubel, M., and Reinhard, R., "The 3-step DLR–ESA Gossamer road to solar sailing," *Advances in Space Research*, Vol. 48, No. 11, 2011, pp. 1695–1701.
- [11] Fernandez, J. M., and Volle, C. E., "Corrugated Rollable Tubular Booms," *AIAA Scitech 2021 Forum*, 2021.
- [12] Guest, S., and Pellegrino, S., "Analytical models for bistable cylindrical shells," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, Vol. 462, No. 2067, 2006, pp. 839–854.
- [13] Schultz, M. R., Hulse, M. J., Keller, P. N., and Turse, D., "Neutrally stable behavior in fiber-reinforced composite tape springs," *Composites Part A: Applied Science and Manufacturing*, Vol. 39, No. 6, 2008, pp. 1012–1017.
- [14] Fernandez, J. M., Viquerat, A., Lappas, V. J., and Daton-Lovett, A. J., "Bistable over the whole length (BOWL) CFRP booms for solar sails," *Advances in solar sailing*, Springer, 2014, pp. 609–628.
- [15] Knott, G., and Viquerat, A., "Curved bistable composite slit tubes with positive Gaussian curvature," *AIAA Journal*, Vol. 56, No. 4, 2018, pp. 1679–1688.
- [16] Shore, J., Viquerat, A., Richardson, G., and Aglietti, G., "An energy optimisation approach to modelling tape spring behaviour," *AIAA Scitech 2020 Forum*, 2020.
- [17] Lee, A., and Fernandez, J. M., "Mechanics of bistable two-shelled composite booms," *2018 AIAA Spacecraft Structures Conference*, 2018.
- [18] Lee, A. J., and Fernandez, J. M., "Inducing bistability in Collapsible Tubular Mast booms with thin-ply composite shells," *Composite Structures*, Vol. 225, 2019, p. 111166.
- [19] Fernandez, J. M., "Bistable Collapsible Tubular Mast Boom," , 2020. US Patent 11,199,005.
- [20] Herbeck, L., Leipold, M., Sickinger, C., Eiden, M., and Unkenbold, W., "Development and test of deployable ultra-lightweight cfrp-booms for a solar sail," *Spacecraft Structures, Materials and Mechanical Testing*, 2001.
- [21] Hillebrandt, M., Meyer, S., Zander, M., Straubel, M., and Hühne, C., "The boom design of the de-orbit sail satellite," *European Conference on Spacecraft Structures, Materials and Mechanical Testing*, 2014.
- [22] Bolton, K., "Biarcs curves," *Computer-Aided Design*, Vol. 7, No. 2, 1975, pp. 89–92.
- [23] Marciniak, K., and Putz, B., "Approximation of spirals by piecewise curves of fewest circular arc segments," *Computer-Aided Design*, Vol. 16, No. 2, 1984, pp. 87–90.
- [24] Meek, D. S., and Walton, D. J., "Approximating smooth planar curves by arc splines," *Journal of Computational and Applied Mathematics*, Vol. 59, No. 2, 1995, pp. 221–231.
- [25] Yang, X., "Efficient circular arc interpolation based on active tolerance control," *Computer-Aided Design*, Vol. 34, No. 13, 2002, pp. 1037–1046.
- [26] Jakubczyk, K., "Approximation of Smooth Planar Curves by Circular Arcs Splines," <http://kaj.uniwersytetradom.pl/docs/Biarcs.pdf>, 2012.
- [27] Pratt, V., "Direct least-squares fitting of algebraic surfaces," *ACM SIGGRAPH computer graphics*, Vol. 21, No. 4, 1987, pp. 145–152.