

A Nonlinear Viscoelastic–Viscoplastic Constitutive Model for Epoxy Polymers

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Motivation and Objectives

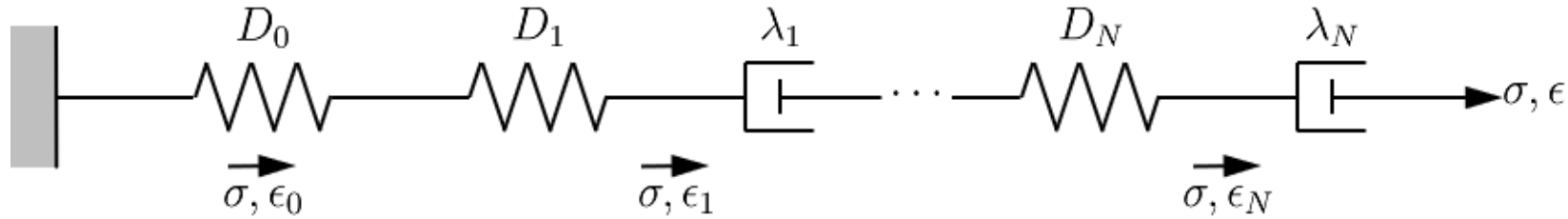
Motivation

- Deployable composite booms undergo permanent shape changes after stowage
- Polymer matrices may exhibit stress-dependent viscoelasticity when undergoing strains beyond 1%
- Necessary to investigate nonlinear viscoelasticity and viscoplasticity

Objectives

1. Develop a nonlinear viscoelasticity model for polymer matrices
2. Validate the nonlinear viscoelasticity model
3. Evaluate the viscoelastic characterization of PMT-F7 epoxy
4. Develop a viscoelastic–viscoplastic constitutive model for polymer matrices
5. Demonstrate the present constitutive model's capabilities

3D Schapery Integral



$$\epsilon' = \frac{1}{2}g_0J_0\sigma' + \frac{1}{2}g_1 \int_0^t \Delta J [\psi(t) - \psi(s)] \frac{d(g_2\sigma')}{ds} ds$$

$$\epsilon_m = \frac{1}{3}g_0B_0\sigma_m + \frac{1}{3}g_1 \int_0^t \Delta B [\psi(t) - \psi(s)] \frac{d(g_2\sigma_m)}{ds} ds$$

$$\Delta J(\psi) = \sum_{i=1}^N J_i [1 - \exp(-\lambda_i\psi)] \quad \text{and} \quad \Delta B(\psi) = \sum_{i=1}^N B_i [1 - \exp(-\lambda_i\psi)]$$

$(\bullet)'$ and $(\bullet)_m$ — deviatoric and mean parts of a tensor

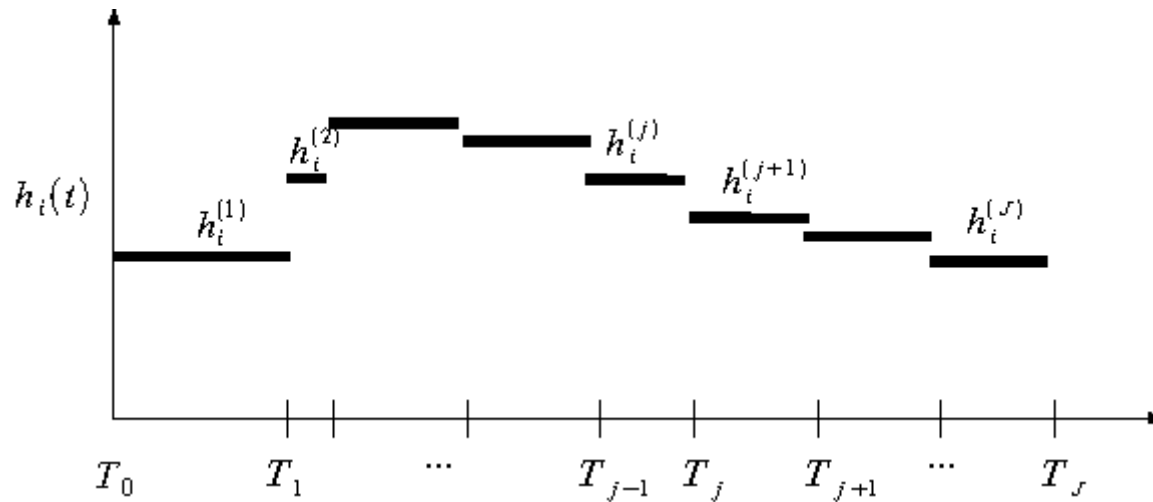
g_0, g_1, g_2 , and ψ — stress-dependent functions

$J_0, B_0, \Delta J$, and ΔB — instantaneous and transient shear and bulk compliances

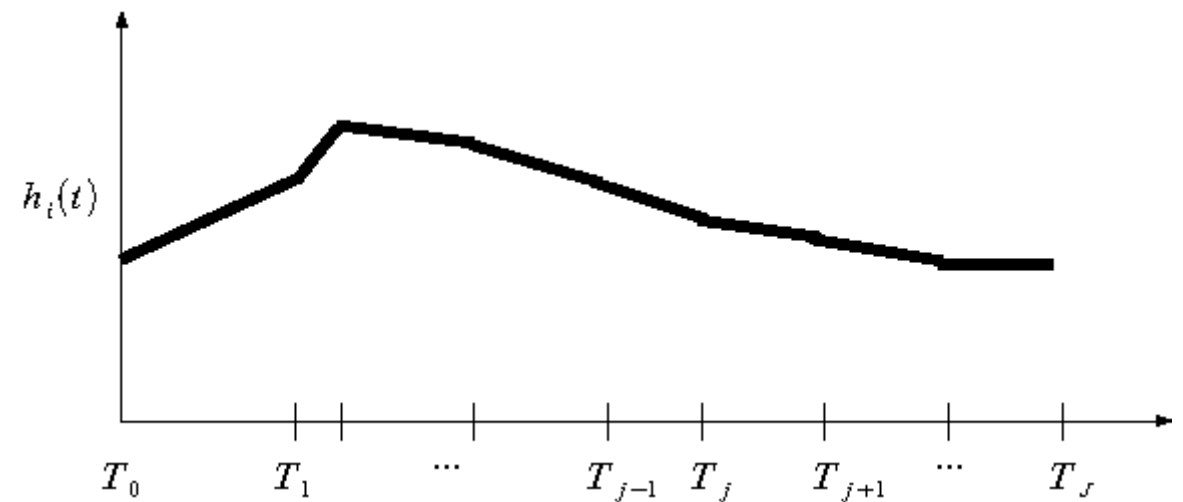
Improved Nonlinear Viscoelasticity Model

- Improved Lai and Bakker (1996)'s model by abandoning the assumption that all stress-dependent functions remain constant over each small time interval
- The integration scheme used to be switched between the forward and the backward Euler during creep–recovery, and now it is the forward Euler

Lai, J. and Bakker, A., "3-D Schapery representation for non-linear viscoelasticity and finite element implementation," *Computational Mechanics*, Vol. 18, No. 3, 1996, pp. 182–191.



Piecewise constant



Piecewise linear

Deviatoric and Mean Strains

Decompose deviatoric and mean strains

$$\boldsymbol{\epsilon}' = \boldsymbol{\epsilon}'_0 + \sum_{i=1}^N \boldsymbol{\epsilon}'_i \quad \text{and} \quad \epsilon_m = (\epsilon_m)_0 + \sum_{l=1}^N (\epsilon_m)_l$$

0th and i^{th} deviatoric and mean strains

$$\boldsymbol{\epsilon}'_0 = \frac{1}{2}g_0J_0\boldsymbol{\sigma}' + \frac{1}{2}g_1 \int_0^t \sum_{i=1}^N J_i \frac{d(g_2\boldsymbol{\sigma}')}{ds} ds$$

$$\boldsymbol{\epsilon}'_i = -\frac{1}{2}g_1 \int_0^t J_i \exp \{ -\lambda_i [\psi(t) - \psi(s)] \} \frac{d(g_2\boldsymbol{\sigma}')}{ds} ds$$

$$(\epsilon_m)_0 = \frac{1}{3}g_0B_0\sigma_m + \frac{1}{3}g_1 \int_0^t \sum_{i=1}^N B_i \frac{d(g_2\sigma_m)}{ds} ds$$

$$(\epsilon_m)_i = -\frac{1}{3}g_1 \int_0^t B_i \exp \{ -\lambda_i [\psi(t) - \psi(s)] \} \frac{d(g_2\sigma_m)}{ds} ds$$

Relate Strains to Stress Increments over $[t_n, t_{n+1}]$

Von Mises stress

$$\|\boldsymbol{\sigma}'\| = \sqrt{\frac{3}{2} \boldsymbol{\sigma}' : \boldsymbol{\sigma}'}$$

Stress-dependent functions

$$g_0 = 1 + \sum_{i=1}^N \alpha_i \left\langle \frac{\|\boldsymbol{\sigma}'\|}{\sigma_0} - 1 \right\rangle^i, \quad g_1 = 1 + \sum_{i=1}^N \beta_i \left\langle \frac{\|\boldsymbol{\sigma}'\|}{\sigma_0} - 1 \right\rangle^i$$
$$g_2 = 1 + \sum_{i=1}^N \alpha_i \left\langle \frac{\|\boldsymbol{\sigma}'\|}{\sigma_0} - 1 \right\rangle^i, \quad a_\sigma = 1 + \sum_{i=1}^N \delta_i \left\langle \frac{\|\boldsymbol{\sigma}'\|}{\sigma_0} - 1 \right\rangle^i, \quad \psi = \int_0^t \frac{ds}{a_\sigma(s)}$$

Use the chain rule and a change of variables

$$dg_1 = \frac{dg_1}{dt} \Delta t = \frac{dg_1}{d\boldsymbol{\sigma}} : \Delta \boldsymbol{\sigma}, \quad \text{etc.}$$

Incremental Constitutive Relation over $[t_n, t_{n+1}]$

Deviatoric and mean strain increments

$$\Delta \boldsymbol{\epsilon}' = \tilde{\boldsymbol{\mathcal{J}}} : \Delta \boldsymbol{\sigma} - \sum_{i=1}^N [1 - \exp(-\lambda_i \Delta \psi)] (\boldsymbol{\epsilon}'_i)_n \quad \text{and} \quad \Delta \epsilon_m = \tilde{\boldsymbol{B}} : \Delta \boldsymbol{\sigma} - \sum_{i=1}^N [1 - \exp(-\lambda_i \Delta \psi)] [(\epsilon_m)_i]_n$$

$$\tilde{\boldsymbol{\mathcal{J}}} = \frac{1}{2} \left[J_0 \left(\boldsymbol{\sigma}'_n \otimes \frac{dg_0}{d\boldsymbol{\sigma}} + g_0 \boldsymbol{\mathcal{I}}' \right) + g_1 \sum_{i=1}^N J_i [1 - \exp(-\lambda_i \Delta \psi)] \left(\boldsymbol{\sigma}'_n \otimes \frac{dg_2}{d\boldsymbol{\sigma}} + g_2 \boldsymbol{\mathcal{I}}' \right) \right]$$

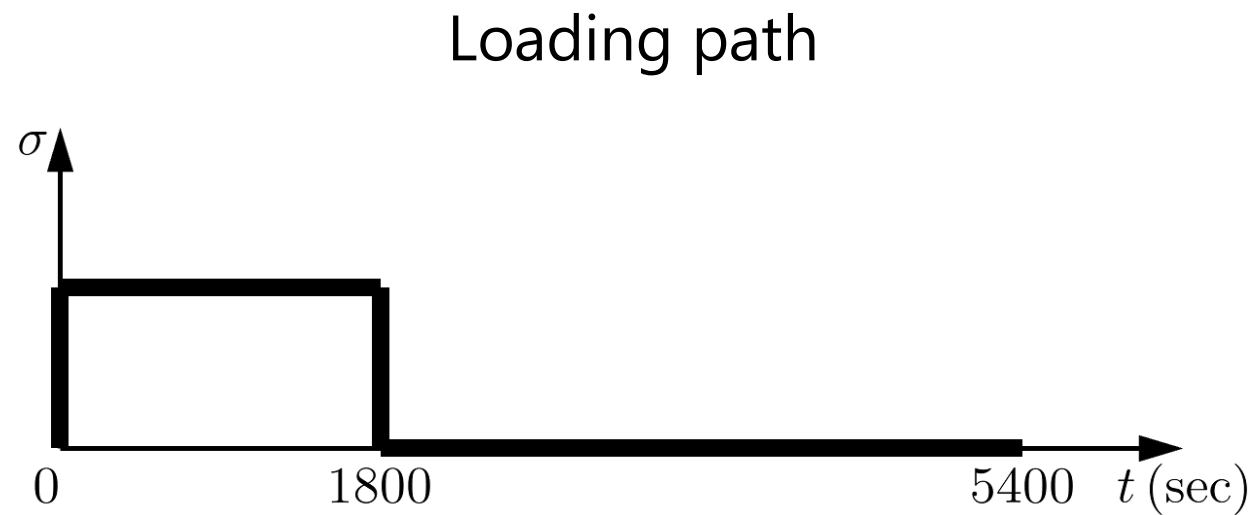
where

$$\tilde{\boldsymbol{B}} = \frac{1}{3} \left\{ B_0 \left[(\sigma_m)_n \frac{dg_0}{d\boldsymbol{\sigma}} + \frac{1}{3} g_0 \boldsymbol{\mathcal{I}} \right] + g_1 \sum_{i=1}^N B_i [1 - \exp(-\lambda_i \Delta \psi)] \left[(\sigma_m)_n \frac{dg_2}{d\boldsymbol{\sigma}} + \frac{1}{3} g_2 \boldsymbol{\mathcal{I}} \right] \right\}$$

Incremental constitutive relation

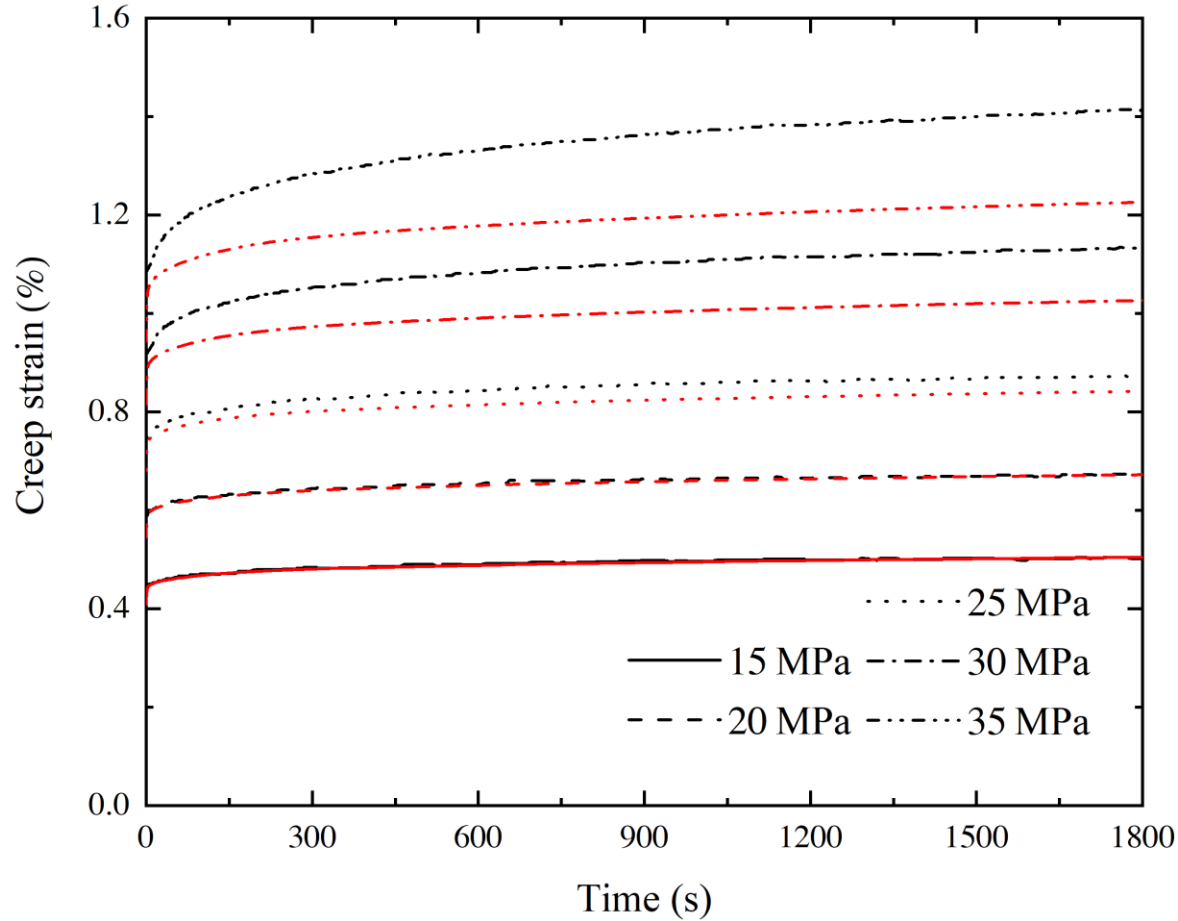
$$\Delta \boldsymbol{\sigma} = \tilde{\boldsymbol{\mathcal{C}}} : \Delta \boldsymbol{\epsilon} + \sum_{i=1}^N [1 - \exp(-\lambda_i \Delta \psi)] \tilde{\boldsymbol{\mathcal{C}}} : (\boldsymbol{\epsilon}_i)_n \quad \text{where} \quad \tilde{\boldsymbol{\mathcal{C}}}^{-1} = \tilde{\boldsymbol{\mathcal{J}}} + \boldsymbol{\mathcal{I}} \otimes \tilde{\boldsymbol{B}}$$

Use Incremental Analysis during Loading and Unloading

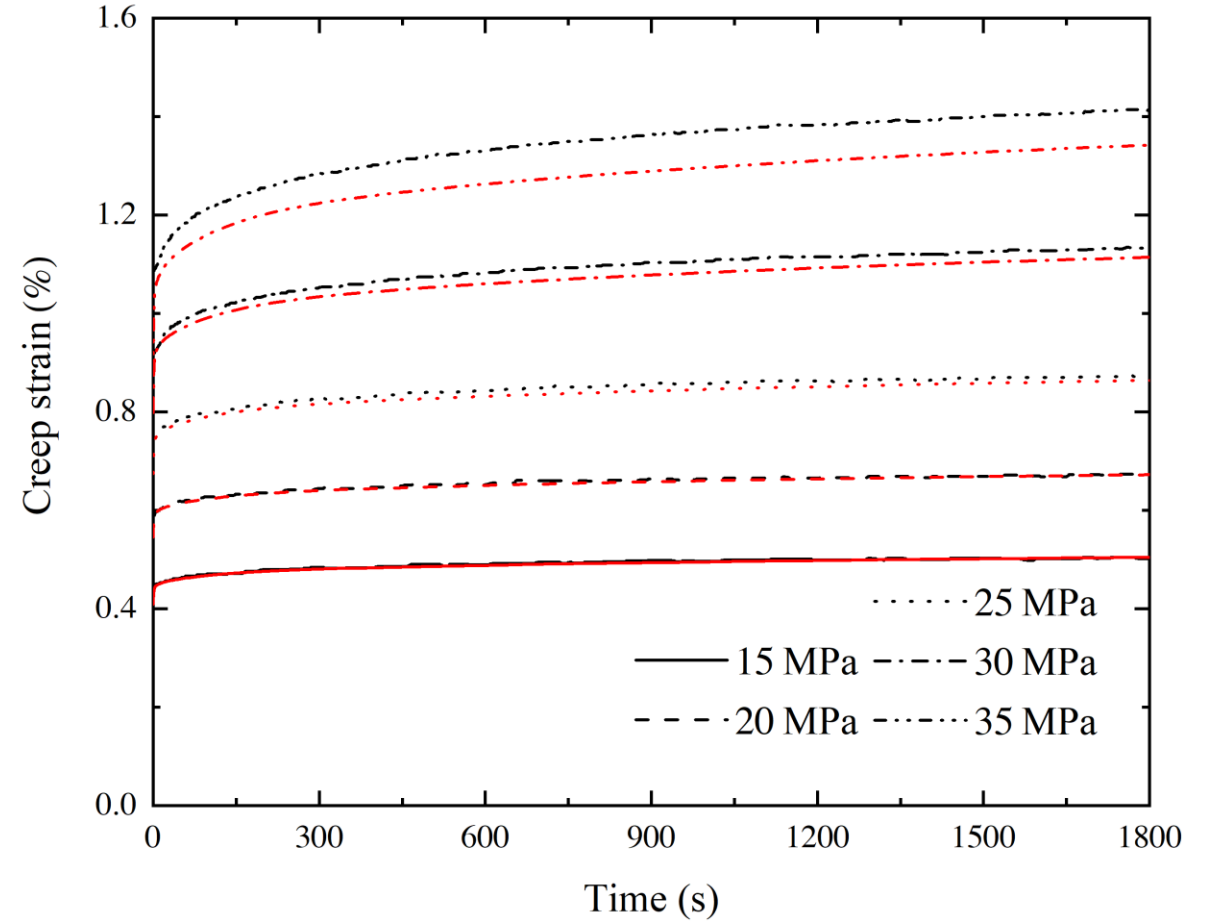


	Loading	Creep	Unloading	Recovery	Integration scheme
Two-stage	1 step in 0 s	100 steps in 1800 s	1 step in 0 s	200 steps in 3600 s	Forward Euler during unloading and backward elsewhere
Four-stage	100 steps in 1 ms	100 steps in 1800 s	100 steps in 1 ms	200 steps in 3600 s	Forward Euler

Creep Responses of Polymethyl Methacrylate (PMMA)



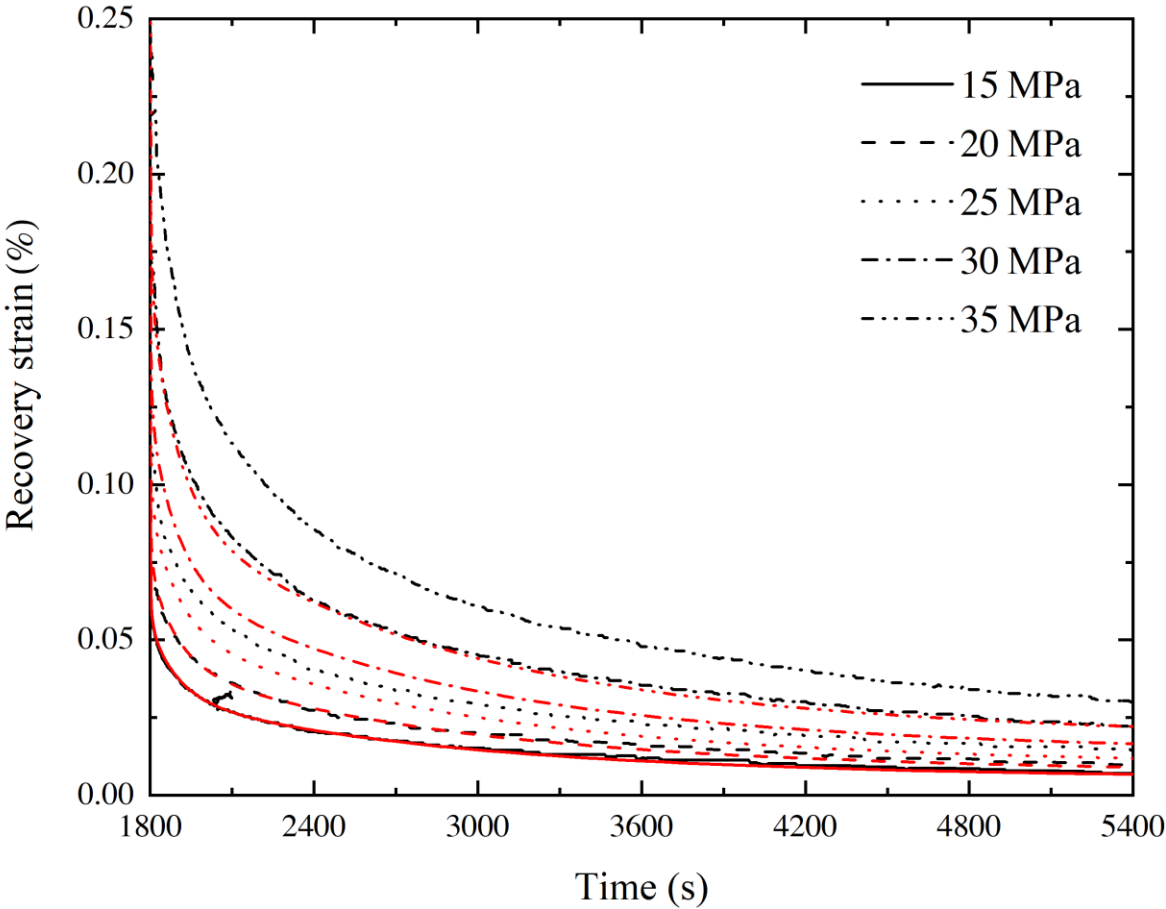
Two-stage



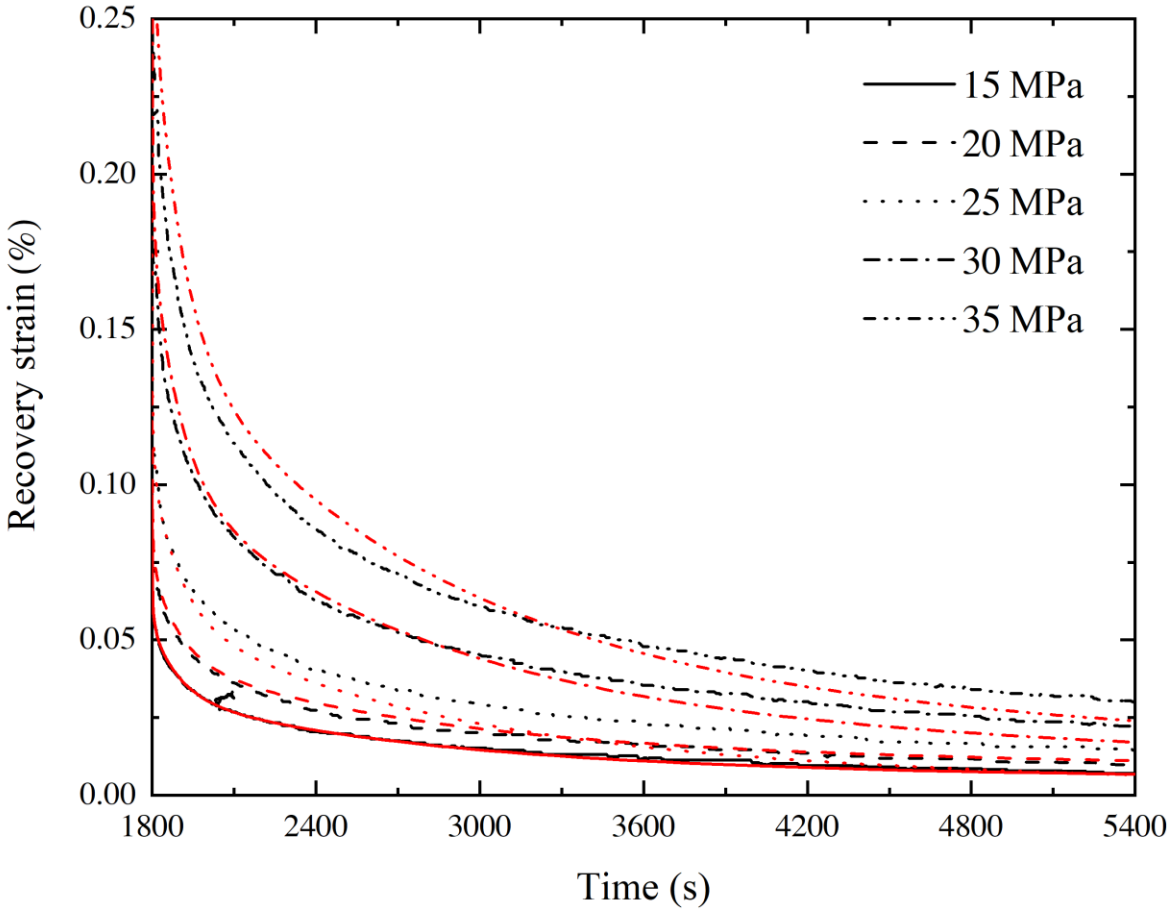
Four-stage

Black — Experimental by Lai and Bakker (1996) and Red — Predicted

Recovery Responses of Polymethyl Methacrylate (PMMA)



Two-stage



Four-stage

Calibrated Parameters

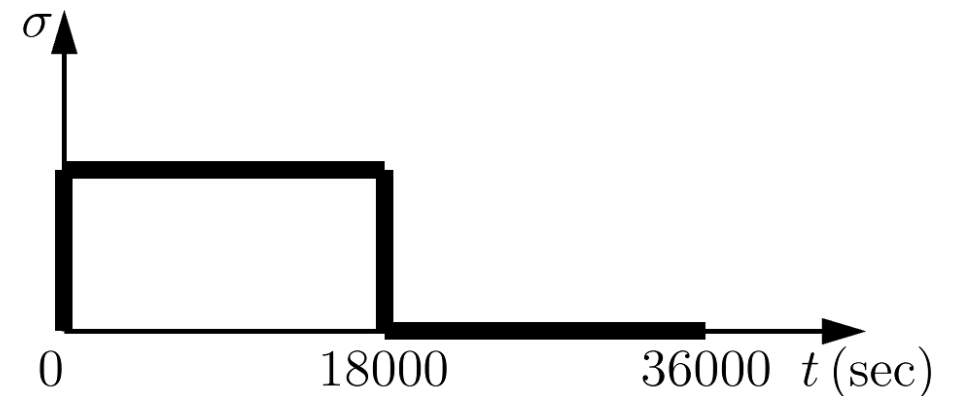
Compliances and characteristic times

i	$\lambda_i \text{ (s}^{-1}\text{)}$	$D_i \text{ (MPa}^{-1}\text{)}$
0	N/A	2.72E-04
1	1.333E-02	1.480E-05
2	1.667E-03	2.920E-07
3	1.000E-03	6.052E-06
4	1.000E-04	6.146E-06
5	1.000E-05	1.965E-05

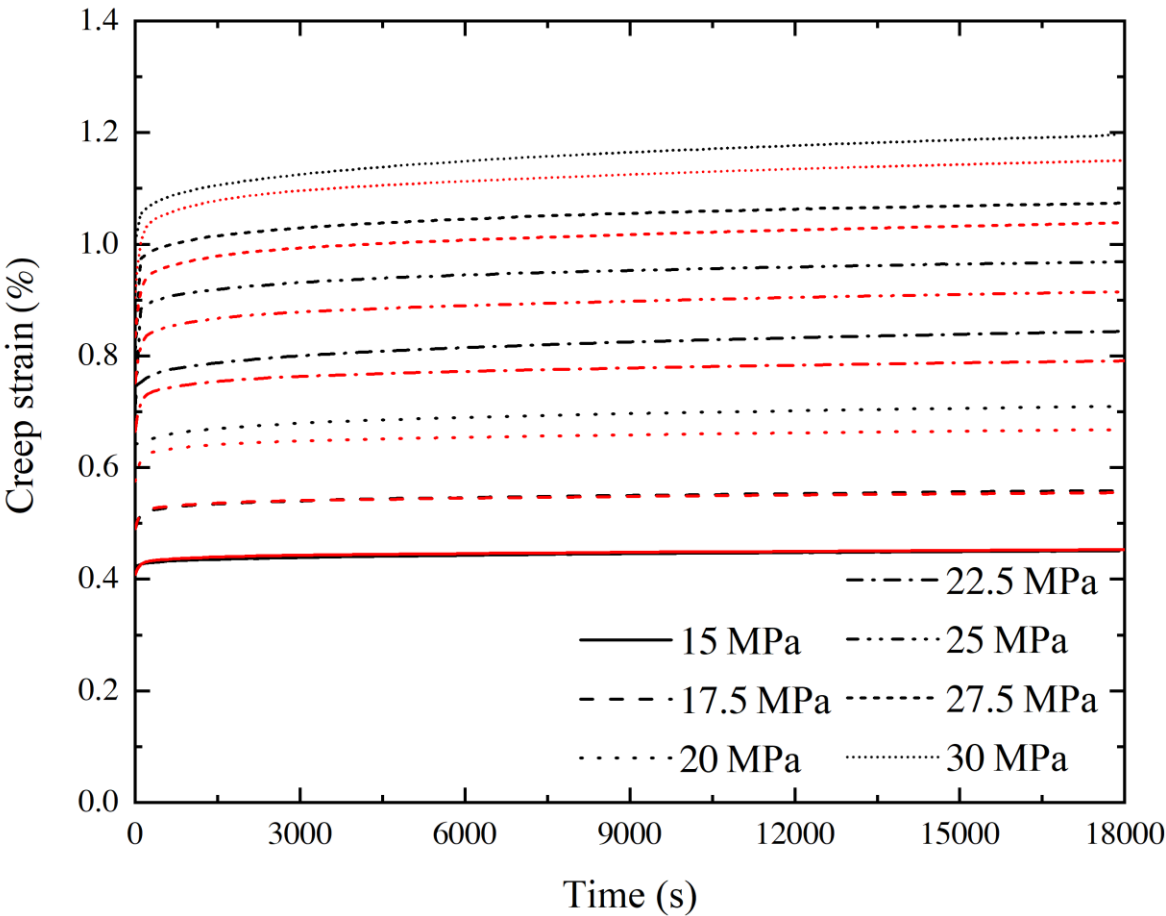
Coefficients of stress-dependent functions

	1	2	3
g_0	0.1672	0.0831	-0.1429
g_1	1.8649	-0.8597	0.6718
g_2	1.3637	-0.3236	0
a_σ	0	0	0

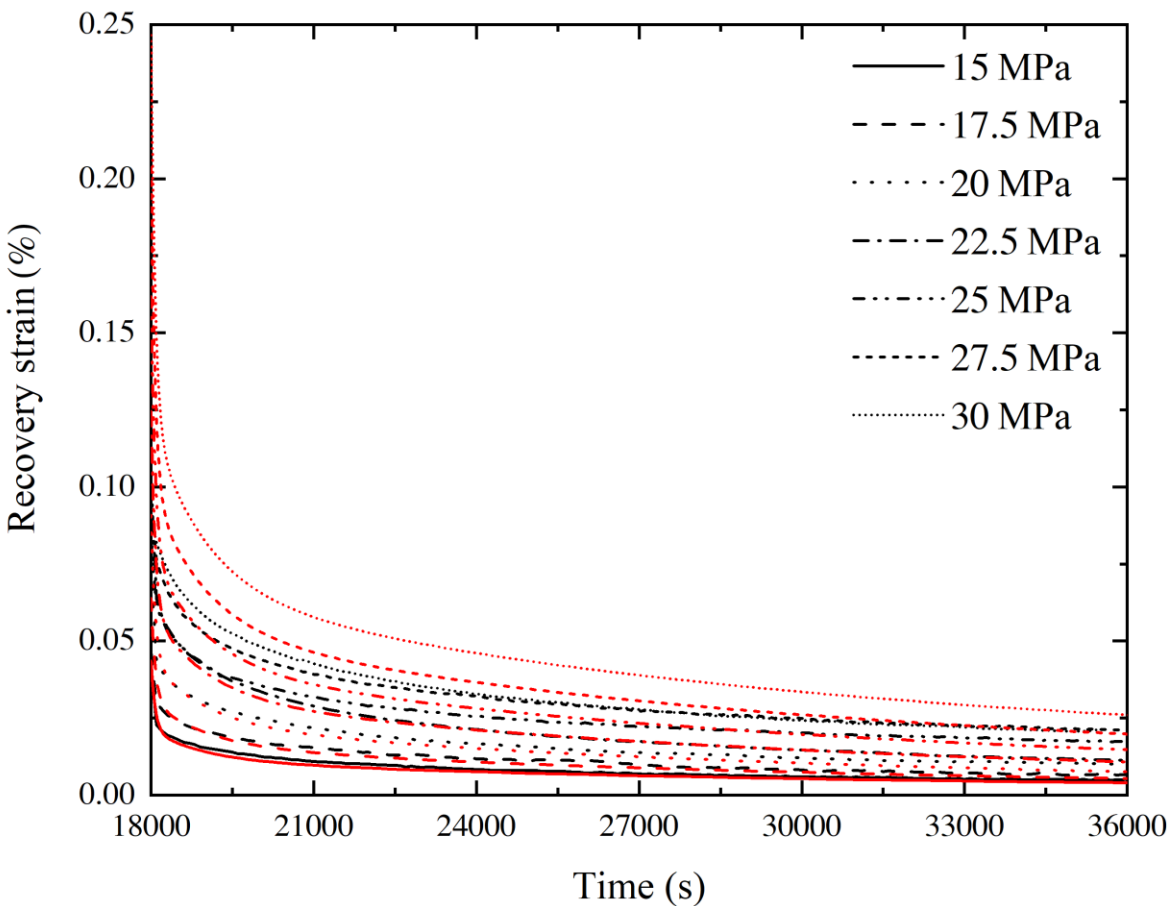
Loading path



Responses of PMT-F7 Epoxy



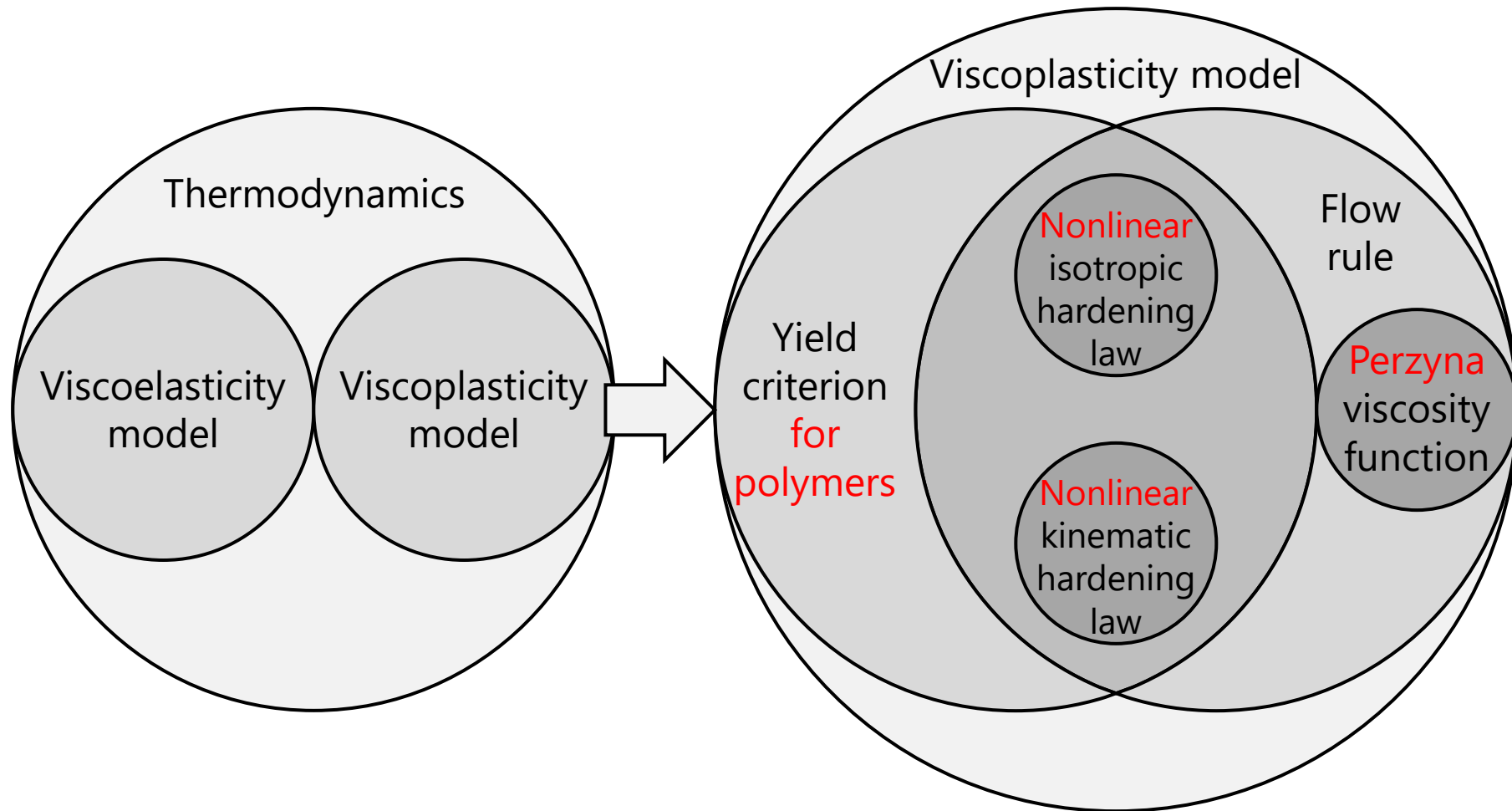
Creep



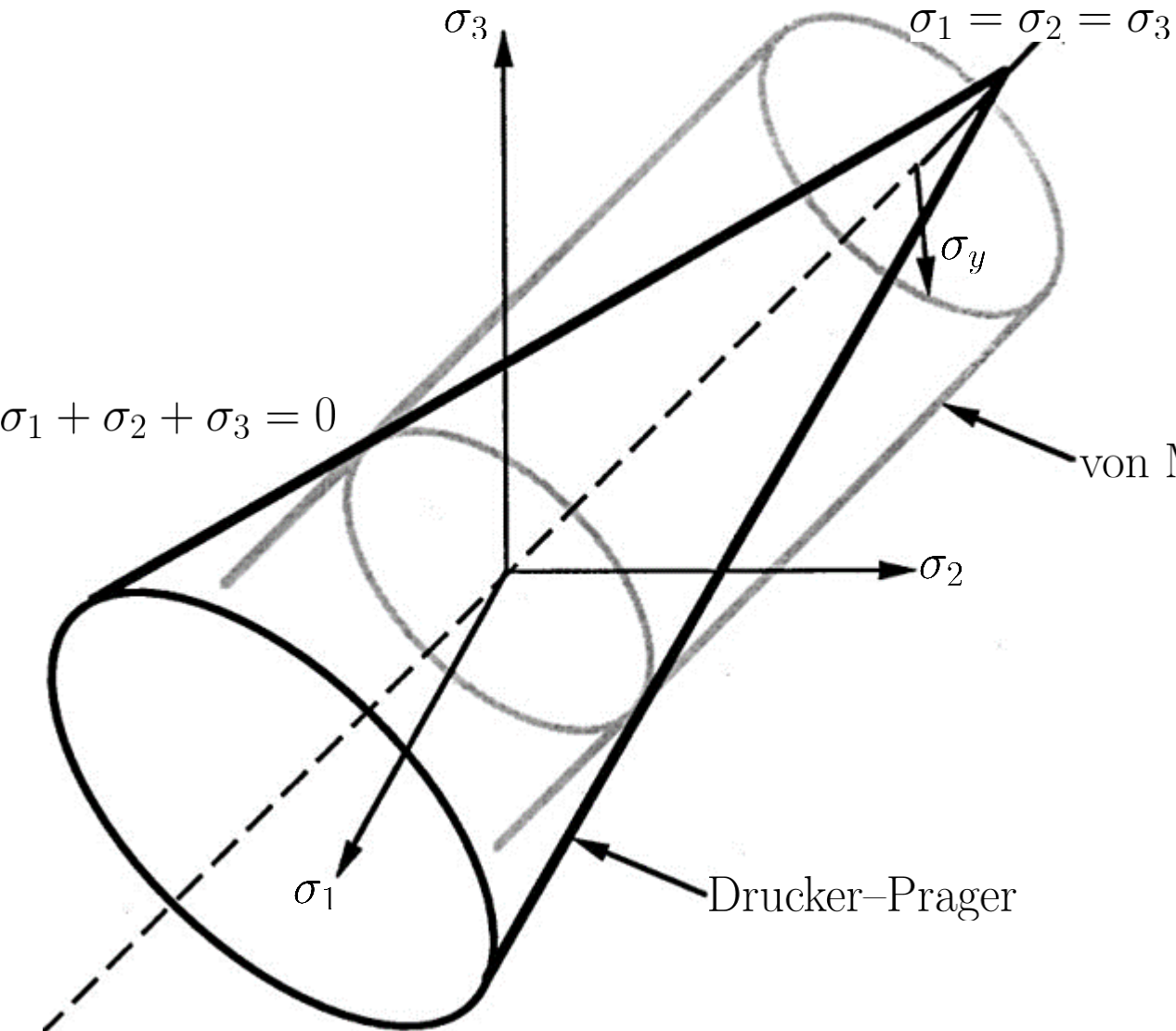
Recovery

Black — Experimental and Red — Predicted

Advances in Viscoplasticity Model



Viscoplasticity Model



Drucker–Prager yield function

$$f(\boldsymbol{\sigma}, \mathbf{X}, R) = \frac{1}{1-\alpha} (\|\boldsymbol{\beta}'\| + 3\alpha\beta_m) - \sigma_c - R$$

where

$$\boldsymbol{\beta} = \boldsymbol{\sigma} - \mathbf{X} \quad \text{and} \quad \|\boldsymbol{\beta}'\| = \sqrt{\frac{3}{2} \boldsymbol{\beta}' : \boldsymbol{\beta}'}$$

$(\bullet)'$ — deviatoric part of a tensor

$$\alpha = \frac{\sigma_c - \sigma_t}{\sigma_c + \sigma_t}$$

σ_c and σ_t — initial yield stress in uniaxial compression/tension

Viscoplasticity Model

Viscoplastic potential

$$F = f + \frac{3a}{4C} \mathbf{X} : \mathbf{X} + \frac{R^2}{2Q}$$

C and a — kinematic hardening parameters

Flow rule

$$\dot{\boldsymbol{\epsilon}}^{vp} = \dot{v} \frac{\partial F}{\partial \boldsymbol{\sigma}} \equiv \dot{v} \mathbf{n} \quad \text{where} \quad \mathbf{n} = \frac{1}{1-\alpha} \left(\frac{3\boldsymbol{\beta}'}{2\|\boldsymbol{\beta}'\|} + \alpha \mathbf{I} \right)$$

Kinematic hardening law

$$\dot{\mathbf{X}} = \frac{2}{3} C \dot{\boldsymbol{\epsilon}}^{vp} - a \mathbf{X} \dot{v}$$

Isotropic hardening law

$$R = Q [1 - \exp(-bv)]$$

Q and b — isotropic hardening parameters

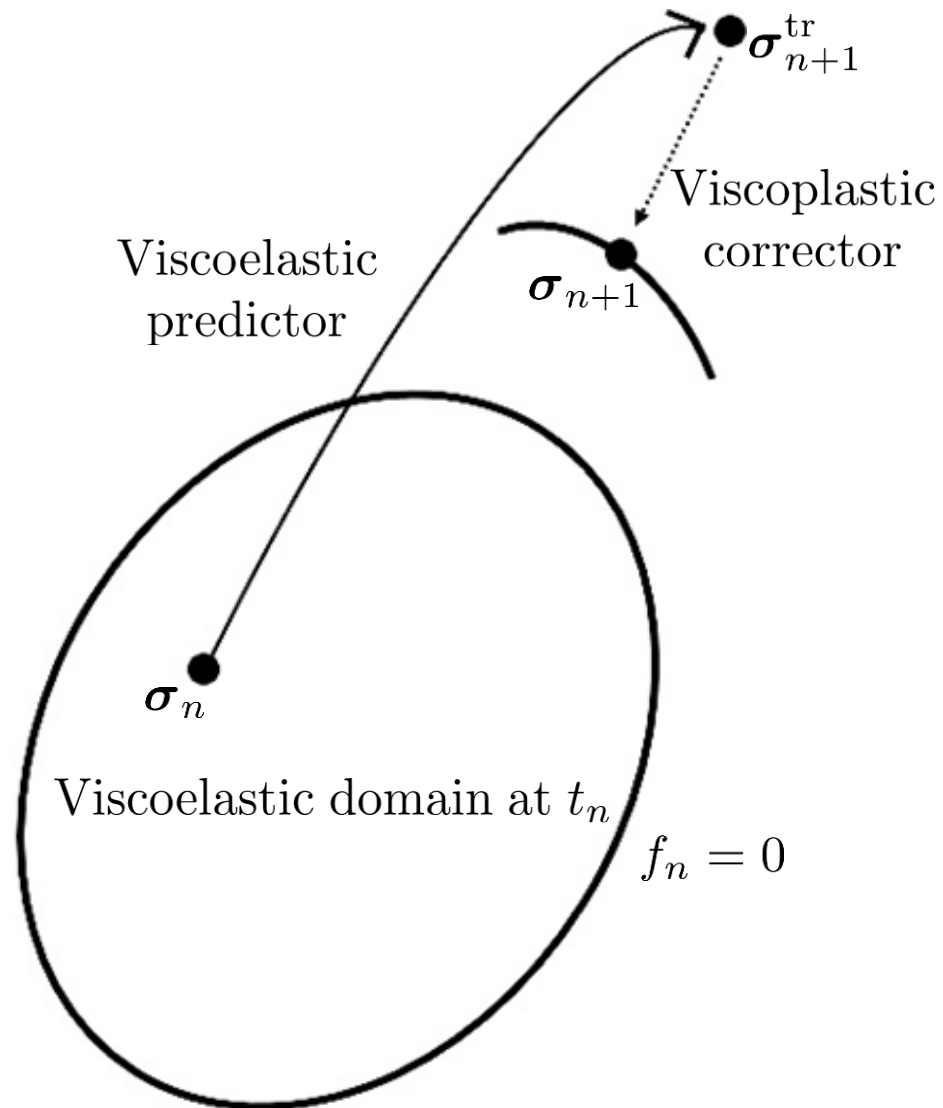
Perzyna viscosity function

$$\dot{v} = \begin{cases} \gamma \left(\frac{f}{\sigma_y + R} \right)^n & f \geq 0, \\ 0 & f < 0 \end{cases}$$

γ — viscosity parameter

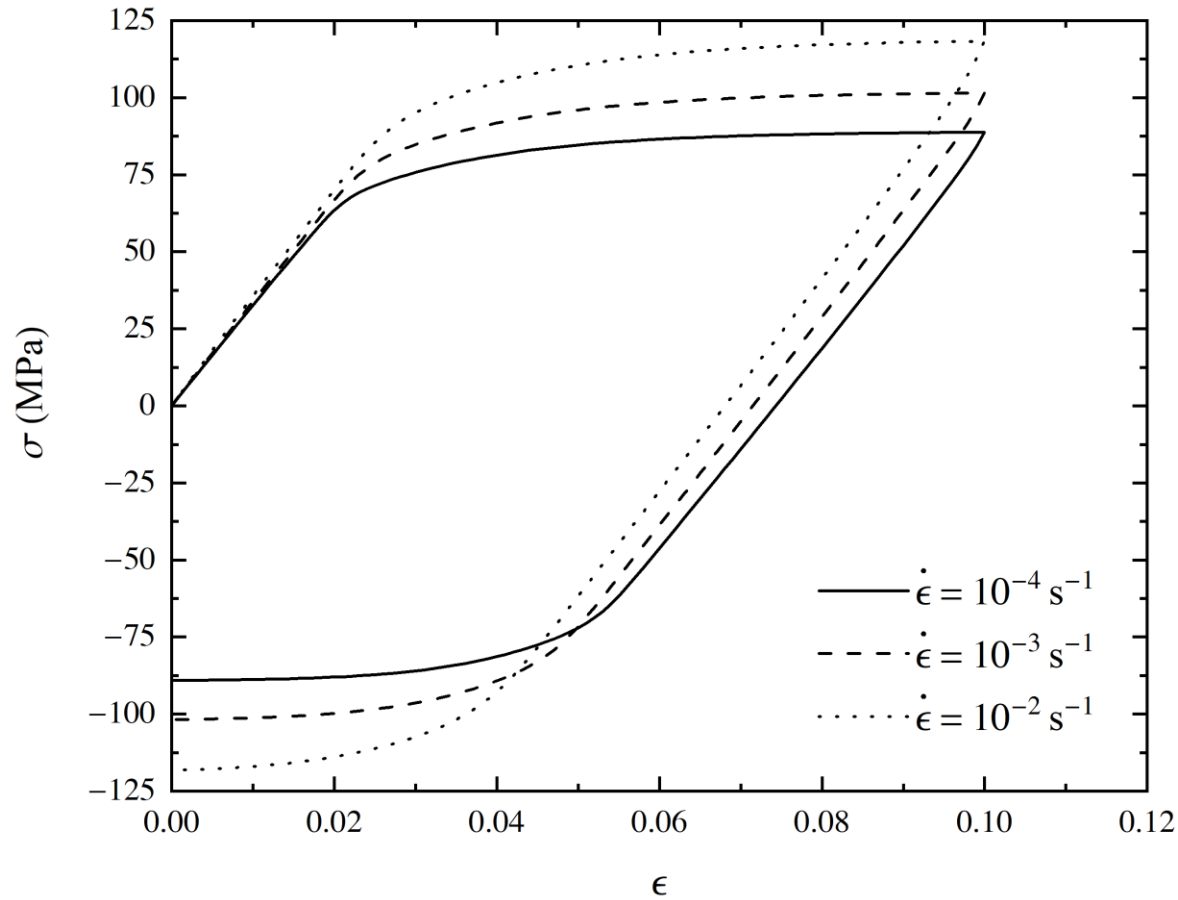
n — rate-sensitivity parameter

Radial Return over $[t_n, t_{n+1}]$

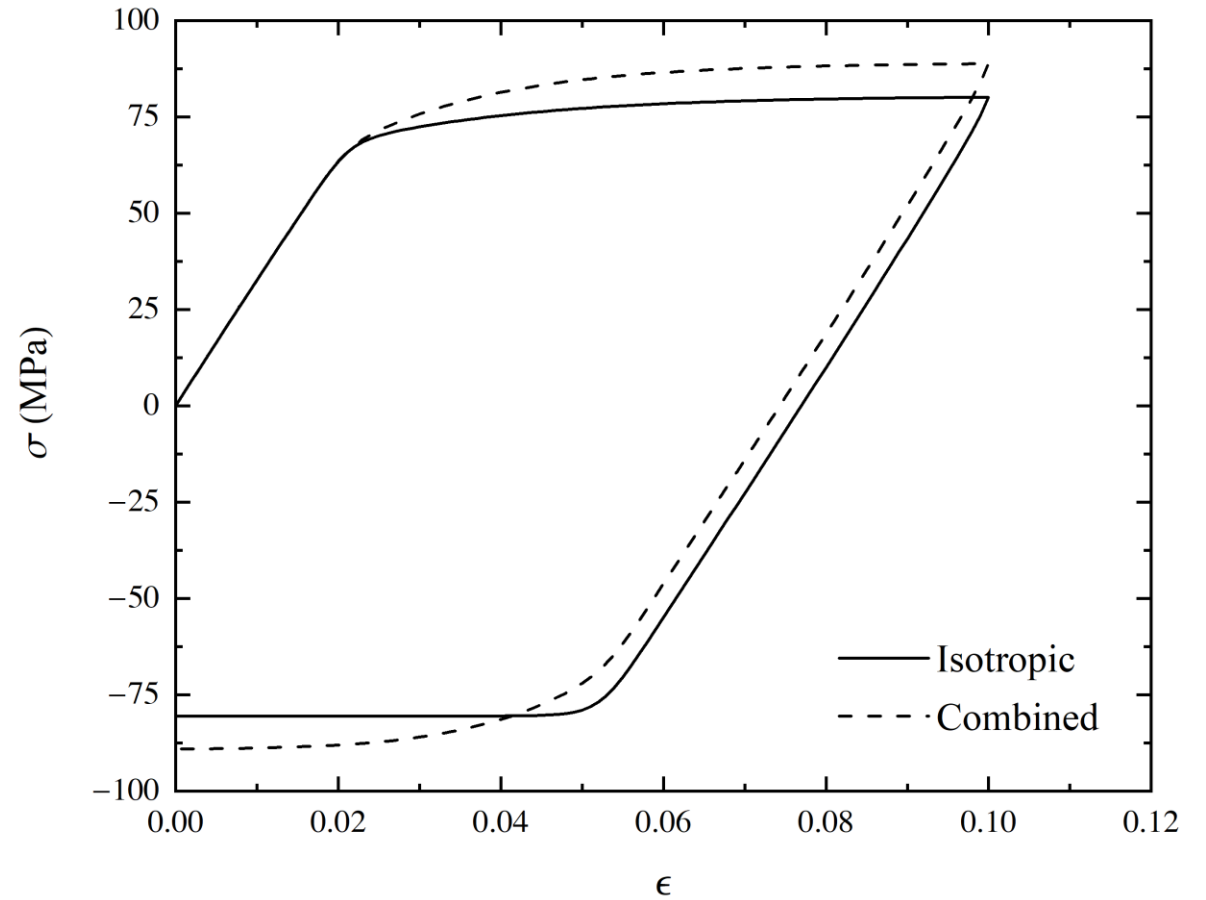


1. Apply trial viscoelastic strains, and compute trial stresses
2. Reduce trial strains along viscoplastic flow direction such that reduced strains and hardening variables satisfy all viscoplastic constraints
3. Iterate Steps 1 and 2 to convergence
4. Derive the closed-form algorithmic tangent operator to facilitate the finite element implementation

Stress–Strain Curves of Viscoelastic–Viscoplastic PMMA



Combined isotropic–kinematic



Isotropic vs. Combined at 10^{-4} s^{-1}

Conclusions

The improved viscoelasticity model, along with four-stage time integration:

- Is accurate and robust
- Better reproduced the experimental creep–recovery responses of PMMA than two-stage time integration

The prediction for PMT-F7 epoxy starts to deviate from the test data at a creep stress of 20 MPa

The constitutive model can also characterize viscoplastic rate-dependence and combined isotropic–kinematic hardening

We will consider temperature-dependent viscoelasticity and viscoplasticity in future work

Thank You!

Strain Increments over $[t_n, t_{n+1}]$

Strain increments obtained using the forward Euler method and chain rule

$$\Delta \epsilon'_0 = \left[\frac{1}{2} J_0 \left(\boldsymbol{\sigma}'_n \otimes \frac{dg_0}{d\boldsymbol{\sigma}} + g_0 \mathbf{I}' \right) + \frac{1}{2} g_1 \sum_{i=1}^N J_i \left(\boldsymbol{\sigma}'_n \otimes \frac{dg_2}{d\boldsymbol{\sigma}} + g_2 \mathbf{I}' \right) \right] : \Delta \boldsymbol{\sigma}$$

$$\Delta \epsilon'_i = -\frac{1}{2} g_1 J_i \exp(-\lambda_i \Delta \psi) \left(\boldsymbol{\sigma}'_n \otimes \frac{dg_2}{d\boldsymbol{\sigma}} + g_2 \mathbf{I}' \right) : \Delta \boldsymbol{\sigma} - [1 - \exp(-\lambda_i \Delta \psi)] (\epsilon'_i)_n$$

$$\Delta (\epsilon_m)_0 = \left\{ \frac{1}{3} B_0 \left[(\sigma_m)_n \frac{dg_0}{d\boldsymbol{\sigma}} + \frac{1}{3} g_0 \mathbf{I} \right] + \frac{1}{3} g_1 \sum_{i=1}^N B_i \left[(\sigma_m)_n \frac{dg_2}{d\boldsymbol{\sigma}} + \frac{1}{3} g_2 \mathbf{I} \right] \right\} : \Delta \boldsymbol{\sigma}$$

$$\Delta (\epsilon_m)_i = -\frac{1}{3} g_1 B_i \exp(-\lambda_i \Delta \psi) \left[(\sigma_m)_n \frac{dg_2}{d\boldsymbol{\sigma}} + \frac{1}{3} g_2 \mathbf{I} \right] : \Delta \boldsymbol{\sigma} - [1 - \exp(-\lambda_i \Delta \psi)] [(\epsilon_m)_i]_n$$

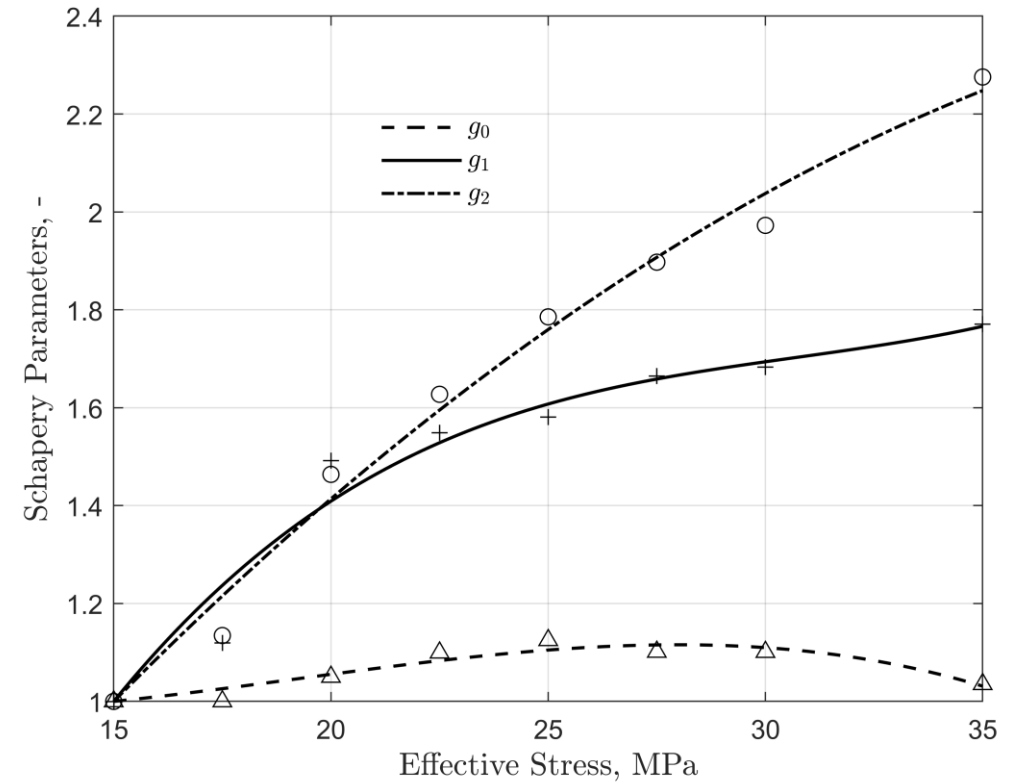
$(\bullet)_n$ — quantity at t_n

$\Delta(\bullet)$ — increment in a quantity over $[t_n, t_{n+1}]$

* For notational convenience, omit subscript $n+1$ on each quantity at t_{n+1}

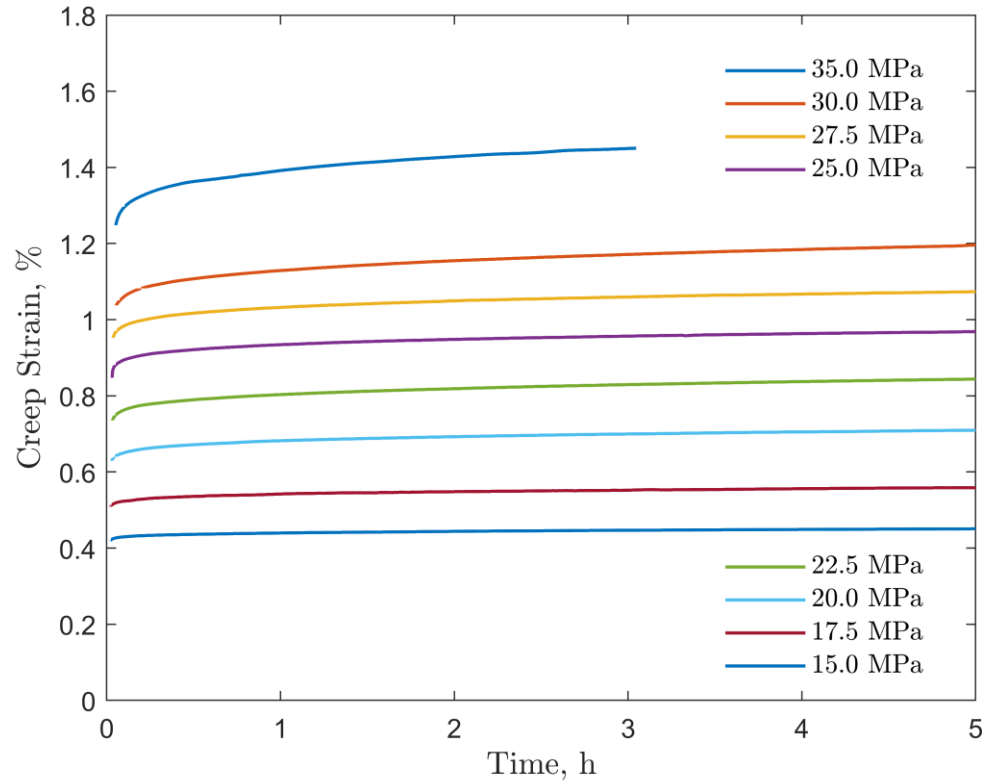
Viscoelastic Characterization

1. Conduct a 16-hour-long creep test with a creep stress of 15 MPa at room temperature
2. Measure the creep strain with a strain gauge
3. Calibrate D_i 's and λ_i 's by fitting a 6-term Prony series to the test data
4. For verification, conduct another 10-hour-long creep–recovery test with a creep stress of 15 MPa at room temperature
5. Calibrate the stress-dependent functions via creep–recovery tests with creep stresses above 15 MPa



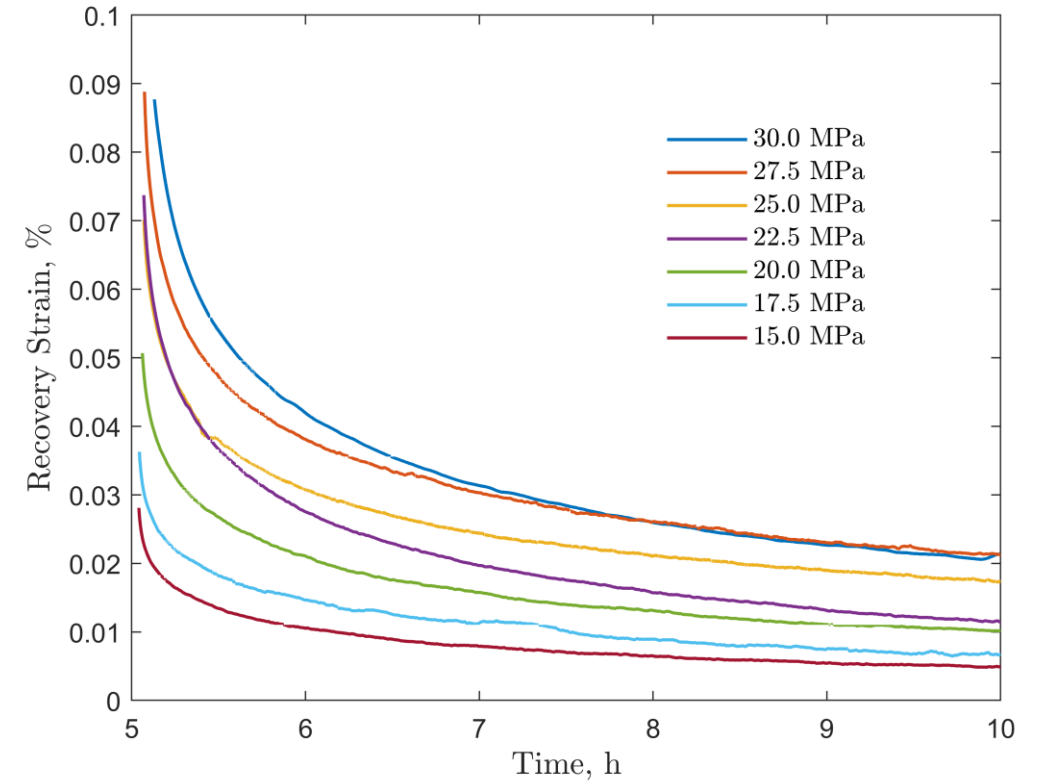
Stress-dependent functions

Viscoelastic Characterization



$$\varepsilon_c(t) = \left[g_0 D_0 + g_1 g_2 \tilde{D} \left(\frac{t}{a_\sigma} \right) \right] \sigma_0 H(t)$$

Experimental creep strains



$$\varepsilon_{cr}(t) = \left[\tilde{D} \left(\frac{t_1}{a_\sigma} + t - t_1 \right) - \tilde{D}(t - t_1) \right] g_2 \sigma_0$$

Experimental creep–recovery strains