

Dynamic Aeroelastic Flight Dynamic Modeling of Mach 0.745 Transonic Truss-Braced Wing

Nhan Nguyen*

NASA Ames Research Center, Moffett Field, CA 94035

Juntao Xiong[†]

KBR Wyle, Inc., Moffett Field, CA 94035

This paper presents a modeling approach for dynamic aeroelastic flight dynamic analysis of the Mach 0.745 Transonic Truss-Braced Wing. The modeling approach is based on a transonic correction method to correct the Theodorsen's theory for transonic flow. CFD unsteady Reynolds-averaged Navier-Stokes equations (RANS) simulations are conducted using FUN3D solver for a series of wing sections from the Mach 0.745 Transonic Truss-Braced Wing in pitch and plunge oscillations. Unsteady lift and pitching moment coefficients are obtained and used to develop the correction terms in the Theodorsen's theory to account for transonic aerodynamics. The unsteady lift and pitching moment derivatives with respect to the unsteady angle of attack are obtained as functions of the reduced frequency. These derivatives are used to compute the unsteady lift and pitching moment contributions by the angle of attack, pitch rate, and roll rate. They are then approximated using a frequency-domain regression to obtain the dynamic stability derivatives for the Mach 0.745 Transonic Truss-Braced Wing. The structural dynamic mode shapes of the Mach 0.745 TTBW are extracted from a NAS-TRAN finite-element model. These mode shapes are used to compute the generalized unsteady aerodynamic forces. The aerodynamic mass, damping, and stiffness and the aerodynamic lag states are constructed to couple the dynamic aeroelastic contribution to the flight dynamic model of the Mach 0.745 Transonic Truss-Braced Wing. The coupled dynamic aeroelastic flight dynamic equations of motion are formulated. The eigenvalues of the coupled system are computed. All the flight dynamic modes and structural dynamic modes are stable at Mach 0.745. The effect of transonic aerodynamics generally causes all the dynamic modes to have lower damping values.

I. Introduction

The Subsonic Ultra Green Aircraft Research (SUGAR) Transonic Truss-Braced Wing (TTBW) aircraft concept is a Boeing-developed N+3 aircraft configuration funded by the NASA ARMD Advanced Air Transport Technologies (AATT) project.^{1,2} The TTBW aircraft concept is designed to be aerodynamically efficient by employing a wing aspect ratio of about 19.55 which is significantly greater than those of cantilever wing transport configurations. For example, the latest Boeing 777-X is reported to have a wing aspect ratio of 11. Without structural bracing, the increase in the wing root bending moment would require a significant structural reinforcement which would lead to an increase in the structural weight that would offset the aerodynamic benefit of the high aspect ratio wing. Thus, the design of a truss-braced structure is a multidisciplinary design optimization process that strives to achieve a delicate balance between aerodynamic efficiency and structural efficiency. In the SUGAR configuration, the trade between the aerodynamic performance and structural design results in a truss-braced configuration with the wings braced at approximately mid-span by two main struts. In addition, two jury struts, one on each wing, provide the additional structural reinforcement. Two versions of the TTBW configurations are currently being developed by Boeing; a Mach 0.745 version and a Mach 0.8 version. Figure 1 is an illustration of the Mach 0.745 TTBW aircraft.

*Senior Research Scientist and Technical Group Lead of Advanced Control and Engineering Systems Group, Intelligent Systems Division, nhan.t.nguyen@nasa.gov

[†]Aerospace Engineer, Intelligent System Division, juntao.xiong@nasa.gov



Figure 1. Boeing SUGAR Transonic Truss-Braced Wing Aircraft Concept

This study examines the dynamic stability of the Mach 0.745 TTBW. To analyze dynamic stability of transonic aircraft, capabilities for addressing transonic aerodynamics in the evaluation of both steady-state and dynamic stability derivatives are needed. Toward this end, a transonic correction method is being proposed for addressing this need. A new method for extending the classical Theodorsen's theory³ of unsteady aerodynamics for incompressible flow to transonic flow is initially proposed by the author.⁴⁻⁶ Further development has been conducted and leads to the current method.⁷ The method modifies the Theodorsen's theory to correct for changes in the amplitude and phase shift of the lift and pitching moment of an oscillating airfoil in transonic flow. The amplitude and phase shift corrections are generally a function of the flow parameters such as the Mach number, the reduced frequency, and the instantaneous angle of attack as well the airfoil characteristics such as the thickness and the type of airfoil. To derive the unsteady corrections, unsteady RANS CFD simulations are performed using FUN3D for selected reduced frequencies and Mach numbers for several transonic airfoil wing sections of the Boeing TTBW aircraft configuration.^{1,2}

The unsteady lift and pitching moment derivatives with respect to the unsteady angle of attack are obtained as functions of the reduced frequency. The unsteady lift and pitching moment derivatives frequency response functions are integrated over the wing to account for the angle of attack, pitch rate, roll rate, and yaw rate contributions. The integrated frequency response functions are then approximated using Roger rational fraction approximation to obtain the dynamic stability derivatives of lift and pitching moment due to the angle of attack and pitch rate and the dynamic stability derivative for the rolling moment and yawing moment with respect to the roll rate and yaw rate. The steady-state stability derivatives are computed from VSPAERO solver for inviscid subsonic flow. A transonic correction is applied to steady-state stability derivatives using transonic small disturbance code TSFOIL coupled to an integral boundary method.⁸⁻¹⁰ The flight dynamic equations of motion are then modified to include the transonic corrected steady-state and dynamic stability derivatives. Linearization of the equations of motion is used to extract the eigenvalues of the flight dynamic modes for stability analysis.

II. Transonic Correction Method

The transonic correction method is developed for both unsteady lift and pitching moment. The correction functions are obtained from unsteady RANS CFD simulations of an oscillating airfoil in a pitching and plunging motion.

A. Pitching Motion

Consider an oscillating airfoil in a pitching motion with a sinusoidal pitch angle

$$\theta = \bar{\theta} + \tilde{\theta} \quad (1)$$

$$\tilde{\theta} = \theta_0 \sin \omega t = \theta_0 \sin k \tau \quad (2)$$

where $\tau = \frac{2V_\infty t}{c}$ is a non-dimensional time and $k = \frac{\omega c}{2V_\infty}$ is the reduced frequency.

1. Unsteady Lift Correction

The unsteady lift coefficient is expressed as

$$c_l = \bar{c}_l + \tilde{c}_l \quad (3)$$

where $\bar{c}_l = c_{l_0} + c_{l_\alpha} \bar{\theta}$ is the steady-state lift coefficient and $\tilde{c}_l$ denotes the unsteady lift coefficient.

We propose an unsteady transonic correction to the unsteady lift coefficient of the form

$$\tilde{c}_l = (U_\theta + iW_\theta) \frac{c_{l_\alpha}}{c_{l_\alpha}^*} \tilde{c}_l^* \quad (4)$$

where $U_\theta(k, M_\infty, \bar{\theta}, \theta_0)$ and $W_\theta(k, M_\infty, \bar{\theta}, \theta_0)$ are transonic correction functions to be applied to the Theodorsen's unsteady lift coefficient, denoted by $\tilde{c}_l^*$, of an oscillating airfoil in incompressible flow, $c_{l_\alpha}^* = 2\pi$, and c_{l_α} is the steady-state lift curve slope. The complex-valued correction provides both the amplitude and phase corrections to the unsteady lift by the functions $U_\theta(k, M_\infty, \bar{\theta}, \theta_0)$ and $W_\theta(k, M_\infty, \bar{\theta}, \theta_0)$, respectively. These functions depend on the reduced frequency k , Mach number M_∞ , the mean angle of attack $\bar{\alpha}$, the amplitude of the alternating component of the angle of attack θ_0 .

The unsteady lift coefficient $\tilde{c}_l^*$ is established by the Theodorsen's theory as

$$\tilde{c}_l^* = c_{l_c}^* + c_{l_{nc}}^* \quad (5)$$

where $c_{l_c}^*$ is the circulatory lift coefficient and $c_{l_{nc}}^*$ is the non-circulatory lift coefficient established by the Theodorsen's theory.

The circulatory lift coefficient $c_{l_c}^*$ is given by

$$c_{l_c}^* = C(k) c_{l_\alpha}^* \left(\bar{\theta} + 2 \frac{e_c}{c} \frac{c \dot{\theta}}{2V_\infty} \right) = [F(k) + iG(k)] c_{l_\alpha}^* \left(\bar{\theta} + 2 \frac{e_c}{c} \frac{d\theta}{d\tau} \right) \quad (6)$$

The non-circulatory lift coefficient $c_{l_{nc}}^*$ is due to the inertial force acting on the oscillating airfoil and is given by

$$c_{l_{nc}}^* = \pi \frac{c \dot{\theta}}{2V_\infty} + 2\pi \frac{e_m}{c} \frac{c^2 \ddot{\theta}}{4V_\infty^2} = \pi \frac{d\theta}{d\tau} + 2\pi \frac{e_m}{c} \frac{d^2\theta}{d\tau^2} \quad (7)$$

For a harmonic motion with $\theta = e^{ik\tau}$, we have $\frac{d\theta}{d\tau} = ik\theta$ and $\frac{d^2\theta}{d\tau^2} = ik \frac{d\theta}{d\tau} = -k^2\theta$. The unsteady lift coefficient $\tilde{c}_l^*$ is obtained as

$$\tilde{c}_l^* = c_{l_\alpha}^* \alpha_0 (F_k \sin k\tau + G_k \cos k\tau) \quad (8)$$

where

$$F_k = F(k) - 2k \frac{e_c}{c} G(k) - k^2 \frac{e_m}{c} \quad (9)$$

$$G_k = G(k) + 2k \frac{e_c}{c} F(k) + \frac{k}{2} \quad (10)$$

The unsteady lift coefficient with the transonic correction is expressed as

$$\tilde{c}_l = c_{l_\alpha} \theta_0 (F_k U_\theta - G_k W_\theta) \sin k\tau + c_{l_\alpha} \theta_0 (G_k U_\theta + F_k W_\theta) \cos k\tau \quad (11)$$

The Fourier sine and cosine transforms of the unsteady lift coefficient are given by

$$\int_0^{\frac{2n\pi}{k}} \tilde{c}_l \sin k\tau d\tau = \frac{n\pi}{k} c_{l_\alpha} \theta_0 (F_k U_\theta - G_k W_\theta) \quad (12)$$

$$\int_0^{\frac{2n\pi}{k}} \tilde{c}_l \cos k\tau d\tau = \frac{n\pi}{k} c_{l_\alpha} \theta_0 (G_k U_\theta + F_k W_\theta) \quad (13)$$

where n is the number of periods of oscillation.

Thus, the transonic amplitude and phase lift correction functions U_θ and W_θ are obtained as

$$U_\theta = \frac{k \left(F_k \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \sin k\tau d\tau + G_k \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \cos k\tau d\tau \right)}{n\pi c_{l\alpha} \theta_0 (F_k^2 + G_k^2)} \quad (14)$$

$$W_\theta = \frac{k \left(F_k \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \cos k\tau d\tau - G_k \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \sin k\tau d\tau \right)}{n\pi c_{l\alpha} \theta_0 (F_k^2 + G_k^2)} \quad (15)$$

2. Unsteady Pitching Moment Correction

The Theodorsen's theory establishes the pitching moment coefficient about the pitch center as

$$c_m = \bar{c}_m + c_{m_c} + c_{m_{nc}} \quad (16)$$

where $\bar{c}_m$ is the steady-state pitching moment coefficient, c_{m_c} is the circulatory pitching moment coefficient, and $c_{m_{nc}}$ is the non-circulatory pitching moment coefficient.

For incompressible flow, the aerodynamic center is at the quarter-chord point. For supersonic flow, the aerodynamic center moves to the mid-chord point. Thus, for transonic flow, the aerodynamic center moves between the quarter-chord and mid-chord points. The motion of the aerodynamic center may be modeled as

$$x_{ac} = \bar{x}_{ac} + x_0 \sin(\omega t - \phi) = \bar{x}_{ac} + x_0 \sin(k\tau - \phi) \quad (17)$$

where ϕ is the phase delay angle between the angle of attack motion and the aerodynamic center motion. We make an assumption that the unsteady motion of the aerodynamic center is proportional to the unsteady angle of attack such that

$$\frac{x_{ac}}{c} = \frac{\bar{x}_{ac}}{c} + (A_\theta + iB_\theta) \frac{\bar{\theta}}{\theta_0} \quad (18)$$

where the functions $A_\theta(k, M_\infty, \bar{\theta}, \theta_0)$ and $B_\theta(k, M_\infty, \bar{\theta}, \theta_0)$ represent the amplitude and phase angle of the motion of the aerodynamic center.

The steady-state pitching moment coefficient about the pitch center can be expressed as

$$\bar{c}_m = c_{m_{ac}} + \frac{\bar{e}}{c} \bar{c}_l \quad (19)$$

where $\bar{e} = x_e - \bar{x}_{ac}$ is the distance between the pitch center and mean aerodynamic center.

From the steady-state aerodynamics, we obtain

$$\frac{\partial \bar{c}_m}{\partial \bar{\alpha}} = \frac{\partial c_{m_{ac}}}{\partial \bar{\alpha}} + \frac{\bar{e}}{c} c_{l\alpha} \quad (20)$$

By the definition of an aerodynamic center, $\frac{\partial c_{m_{ac}}}{\partial \bar{\alpha}} = 0$. Therefore, we write

$$\bar{c}_m = \bar{c}_{m_0} + c_{m_\alpha} \bar{\alpha} \quad (21)$$

where $\bar{c}_{m_0} = c_{m_{ac}} + \frac{\bar{e}}{c} \bar{c}_{l_0}$ is the pitching moment coefficient about the pitch center at the zero angle of attack and c_{m_α} is the pitching moment curve slope about the pitch center.

Using the pitching moment curve slope c_{m_α} , the distance $\bar{e}$ between the pitch center and mean aerodynamic center can be determined as

$$\frac{\bar{e}}{c} = \frac{c_{m_\alpha}}{c_{l\alpha}} \quad (22)$$

Then, the pitching moment coefficient about the aerodynamic center is obtained as

$$c_{m_{ac}} = \bar{c}_m - \frac{c_{m_\alpha}}{c_{l\alpha}} \bar{c}_l \quad (23)$$

The circulatory pitching moment coefficient about the pitch center is given by

$$c_{m_c} = \frac{x_e - x_{ac}}{c} c_{l_c} - \frac{\bar{e}}{c} \bar{c}_l = \left[\frac{\bar{e}}{c} - (A_\theta + iB_\theta) \frac{\tilde{\alpha}}{\alpha_0} \right] c_{l_c} - (A_\theta + iB_\theta) \frac{\tilde{\alpha}}{\theta_0} \bar{c}_l \quad (24)$$

The circulatory lift coefficient is computed as

$$\begin{aligned} c_{l_c} &= (U_\theta + iW_\theta) \frac{c_{l_\alpha}}{c_{l_\alpha}^*} c_{l_c}^* \\ &= \left[F(k) - 2k \frac{e_c}{c} G(k) \right] c_{l_\alpha} \theta_0 (U_\theta \sin k\tau + W_\theta \cos k\tau) \\ &\quad + \left[G(k) + 2k \frac{e_c}{c} F(k) \right] c_{l_\alpha} \alpha \theta_0 (U_\theta \cos k\tau - W_\theta \sin k\tau) \end{aligned} \quad (25)$$

Then, the circulatory pitching moment coefficient about the pitch center is obtained as

$$\begin{aligned} c_{m_c} &= \left[\frac{\bar{e}}{c} - (A_\theta + iB_\theta) \frac{\tilde{\alpha}}{\alpha_0} \right] c_{l_\alpha} \theta_0 (F_c \sin k\tau + G_c \cos k\tau) \\ &= \frac{\bar{e}}{c} c_{l_\alpha} \theta_0 (F_c \sin k\tau + G_c \cos k\tau) \\ &\quad - \frac{1}{2} c_{l_\alpha} \theta_0 (F_c A_\theta - G_c B_\theta) + \frac{1}{2} c_{l_\alpha} \theta_0 (F_c A_\theta - G_c B_\theta) \cos 2k\tau \\ &\quad - \frac{1}{2} c_{l_\alpha} \theta_0 (G_c A_\theta + F_c B_\theta) \sin 2k\tau \end{aligned} \quad (26)$$

where

$$F_c = \left[F(k) - 2k \frac{e_c}{c} G(k) \right] U_\theta - \left[G(k) + 2k \frac{e_c}{c} F(k) \right] W_\theta \quad (27)$$

$$G_c = \left[G(k) + 2k \frac{e_c}{c} F(k) \right] U_\theta + \left[F(k) - 2k \frac{e_c}{c} G(k) \right] W_\theta \quad (28)$$

It is of interest to note that the circulatory pitching moment coefficient produces a steady-state component equal to $-\frac{1}{2} (F_c A - G_c B)$ due to the unsteady lift coefficient and the motion of the aerodynamic center. The second harmonic of the frequency response of the circulatory pitching moment coefficient is a nonlinear term due to the product of the unsteady angle of attack and unsteady lift.

We also propose a transonic correction for the non-circulatory pitching moment coefficient as

$$c_{m_{nc}} = (T_\theta + iV_\theta) c_{m_{nc}}^* \quad (29)$$

where $c_{m_{nc}}^*$ is the incompressible non-circulatory pitching moment given by the Theodorsen's theory as

$$\begin{aligned} c_{m_{nc}}^* &= -\pi \frac{e_c}{c} \frac{c \dot{\theta}}{2V_\infty} - \frac{\pi}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) \frac{c^2 \ddot{\theta}}{4V_\infty^2} = -\pi \frac{e_c}{c} \frac{d\theta}{d\tau} - \frac{\pi}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) \frac{d^2\theta}{d\tau^2} \\ &= \pi \theta_0 k \left[\frac{k}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) \sin k\tau - \frac{e_c}{c} \cos k\tau \right] \end{aligned} \quad (30)$$

Thus,

$$c_{m_{nc}} = \pi \theta_0 k \left[\frac{k}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) (T_\theta \sin k\tau + V_\theta \cos k\tau) - \frac{e_c}{c} (T_\theta \cos k\tau - V_\theta \sin k\tau) \right] \quad (31)$$

Summing all the contributions of the pitching moment coefficient, the unsteady pitching moment coefficient is obtained as

$$\tilde{c}_m = c_{l_\alpha} \theta_0 \left[F_m \sin k\tau + G_m \cos k\tau - \frac{1}{2} (G_c A_\theta + F_c B_\theta) \sin 2k\tau + \frac{1}{2} (F_c A_\theta - G_c B_\theta) \cos 2k\tau \right] \quad (32)$$

where

$$F_m = \frac{\bar{e}}{c} F_c + \frac{1}{2} k \left[\frac{k}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) T_\theta + \frac{e_c}{c} V_\theta \right] \quad (33)$$

$$G_m = \frac{\bar{e}}{c} G_c + \frac{1}{2} k \left[\frac{k}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) V_\theta - \frac{e_c}{c} T_\theta \right] \quad (34)$$

The transonic pitching moment correction functions A , B , T , and V are obtained from the Fourier transforms as

$$A_\theta = \frac{2k \left(F_c \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \cos 2k\tau d\tau - G_c \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \sin 2k\tau d\tau \right)}{n\pi c_{l\alpha} \theta_0 (F_c^2 + G_c^2)} \quad (35)$$

$$B_\theta = - \frac{2k \left(F_c \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \sin 2k\tau d\tau + G_c \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \cos 2k\tau d\tau \right)}{n\pi c_{l\alpha} \theta_0 (F_c^2 + G_c^2)} \quad (36)$$

$$T_\theta = \frac{\frac{k}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) (F_m - \frac{\bar{e}}{c} F_c) - \frac{e_c}{c} (G_m - \frac{\bar{e}}{c} G_c)}{\frac{1}{2} k \left[\left(\frac{k}{16} \right)^2 \left(1 + 32 \frac{e_m^2}{c^2} \right)^2 + \left(\frac{e_c}{c} \right)^2 \right]} \quad (37)$$

$$V_\theta = \frac{\frac{k}{16} \left(1 + 32 \frac{e_m^2}{c^2} \right) (G_m - \frac{\bar{e}}{c} G_c) + \frac{e_c}{c} (F_m - \frac{\bar{e}}{c} F_c)}{\frac{1}{2} k \left[\left(\frac{k}{16} \right)^2 \left(1 + 32 \frac{e_m^2}{c^2} \right)^2 + \left(\frac{e_c}{c} \right)^2 \right]} \quad (38)$$

B. Plunging Motion

The motion of an airfoil in a plunging creates an angle of attack on the airfoil

$$\tilde{\alpha} = \frac{\dot{h}}{V_\infty} \quad (39)$$

If the plunging motion is described by

$$h = -h_0 \cos \omega t = -h_0 \cos k\tau \quad (40)$$

Then, the airfoil sees an angle of attack

$$\alpha = \tilde{\alpha} + \frac{\dot{h}}{V_\infty} = \tilde{\alpha} + \alpha_0 \sin k\tau \quad (41)$$

where $\alpha_0 = \frac{2h_0k}{c}$.

1. Unsteady Lift Correction

The unsteady lift coefficient for the plunging motion is given by the Theodorsen's theory as

$$\tilde{c}_l^* = c_{l\alpha} \left[C(k) \frac{\dot{h}}{V_\infty} + \frac{c\ddot{h}}{4V_\infty^2} \right] = c_{l\alpha} \left[C(k) \tilde{\alpha} + \frac{1}{2} \frac{c\dot{\tilde{\alpha}}}{2V_\infty} \right] = c_{l\alpha} \left[C(k) \tilde{\alpha} + \frac{1}{2} \frac{d\tilde{\alpha}}{d\tau} \right] \quad (42)$$

Applying the transonic correction, the unsteady lift coefficient is expressed as

$$\begin{aligned} \tilde{c}_l &= (U_\alpha + iW_\alpha) c_{l\alpha} \alpha_0 \left[C(k) \sin k\tau + \frac{k}{2} \cos k\tau \right] \\ &= c_{l\alpha} \alpha_0 \left\{ F(k) U_\alpha - \left[G(k) + \frac{k}{2} \right] W_\alpha \right\} \sin k\tau + c_{l\alpha} \alpha_0 \left\{ \left[G(k) + \frac{k}{2} \right] U_\alpha + F(k) W_\alpha \right\} \cos k\tau \end{aligned} \quad (43)$$

Using the Fourier transforms, the correction functions U_α and W_α are obtained as

$$U_\alpha = \frac{k \left\{ F(k) \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \sin k\tau d\tau + \left[G(k) + \frac{k}{2} \right] \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \cos k\tau d\tau \right\}}{n\pi c_{l\alpha} \alpha_0 \left\{ F^2(k) + \left[G(k) + \frac{k}{2} \right]^2 \right\}} \quad (44)$$

$$W_\alpha = \frac{k \left\{ F(k) \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \cos k\tau d\tau - \left[G(k) + \frac{k}{2} \right] \int_0^{\frac{2n\pi}{k}} \tilde{c}_l \sin k\tau d\tau \right\}}{n\pi c_{l_\alpha} \alpha_0 \left\{ F^2(k) + \left[G(k) + \frac{k}{2} \right]^2 \right\}} \quad (45)$$

2. Unsteady Pitching Moment Correction

The unsteady pitching moment coefficient for the plunging motion is given by the Theodorsen's theory as

$$c_m^* = c_{l_\alpha} \left[\frac{\bar{e}}{c} C(k) \frac{\dot{h}}{V_\infty} - \frac{1}{4} \frac{e_m \ddot{h}}{V_\infty^2} \right] = c_{l_\alpha} \left[\frac{\bar{e}}{c} C(k) \tilde{\alpha} - \frac{1}{2} \frac{e_m}{c} \frac{d\tilde{\alpha}}{d\tau} \right] \quad (46)$$

The transonic corrected unsteady pitching moment coefficient is expressed as

$$\tilde{c}_m = \left[\frac{\bar{e}}{c} - (A_\alpha + iB_\alpha) \frac{1}{h_0 k} \frac{dh}{d\tau} \right] (U_\alpha + iW_\alpha) c_{l_\alpha} C(k) \tilde{\alpha} - (T_\alpha + iV_\alpha) \frac{1}{2} c_{l_\alpha} \frac{e_m}{c} \frac{d\tilde{\alpha}}{d\tau} \quad (47)$$

The expression for the unsteady pitching moment coefficient is expanded as

$$\begin{aligned} \tilde{c}_m &= \left[\frac{\bar{e}}{c} - (A_\alpha + iB_\alpha) \sin k\tau \right] \{ c_{l_\alpha} \alpha_0 [F(k) U_\alpha - G(k) W_\alpha] \sin k\tau + c_{l_\alpha} \alpha_0 [F(k) W_\alpha + G(k) U_\alpha] \cos k\tau \} \\ &\quad - (T_\alpha + iV_\alpha) c_{l_\alpha} \alpha_0 \frac{k}{2} \frac{e_m}{c} \cos k\tau \\ &= c_{l_\alpha} \alpha_0 \left(\frac{\bar{e}}{c} F_\alpha + \frac{1}{2} k \frac{e_m}{c} V_\alpha \right) \sin k\tau + c_{l_\alpha} \alpha_0 \left(\frac{\bar{e}}{c} G_\alpha - \frac{k}{2} \frac{e_m}{c} T_\alpha \right) \cos k\tau \\ &\quad - \frac{1}{2} c_{l_\alpha} \alpha_0 (F_\alpha A_\alpha - G_\alpha B_\alpha) + \frac{1}{2} c_{l_\alpha} \alpha_0 (F_\alpha A_\alpha - G_\alpha B_\alpha) \cos 2k\tau - \frac{1}{2} c_{l_\alpha} \alpha_0 (G_\alpha A_\alpha + F_\alpha B_\alpha) \sin 2k\tau \end{aligned} \quad (48)$$

where

$$F_\alpha = F(k) U_\alpha - G(k) W_\alpha \quad (49)$$

$$G_\alpha = G(k) U_\alpha + F(k) W_\alpha \quad (50)$$

Using the Fourier transforms, we obtain the transonic correction functions A_α , B_α , T_α , and V_α as

$$A_\alpha = \frac{2k \left(F_h \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \cos 2k\tau d\tau - G_h \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \sin 2k\tau d\tau \right)}{n\pi c_{l_\alpha} \alpha_0 (F_h^2 + G_h^2)} \quad (51)$$

$$B_\alpha = - \frac{2k \left(F_h \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \sin 2k\tau d\tau + G_h \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \cos 2k\tau d\tau \right)}{n\pi c_{l_\alpha} \alpha_0 (F_h^2 + G_h^2)} \quad (52)$$

$$T_\alpha = \frac{-\frac{k}{n\pi} \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \cos k\tau d\tau + \frac{\bar{e}}{c} G_\alpha}{\frac{k}{2} \frac{e_m}{c}} \quad (53)$$

$$V_\alpha = \frac{\frac{k}{n\pi} \int_0^{\frac{2n\pi}{k}} \tilde{c}_m \sin k\tau d\tau - \frac{\bar{e}}{c} F_\alpha}{\frac{k}{2} \frac{e_m}{c}} \quad (54)$$

C. Application of Transonic Correction in Unsteady Flow Simulations

A series of unsteady RANS CFD simulations of oscillating transonic airfoils are conducted in FUN3D. The airfoils are extracted from the Boeing Mach 0.745 Transonic Truss-Braced Wing (TTBW) aircraft configuration at four wing stations as shown in Fig. 2. The pitch center of the oscillating airfoils is at the quarter-chord location. The FUN3D mesh has about 30 thousand grid points. The mesh domain size is about 100 times the airfoil chord length. The Roe's flux-difference splitting scheme and the Spalart-Allmaras turbulence model are used in the simulations. An optimized second-order backward finite-difference scheme is used in the time integration. The simulations are conducted in

FUN3D for Mach 0.65, 0.7, 0.75, and 0.8 at four different reduced frequencies $k = 0.02, 0.1, 0.2$, and $k = 0.3$ where k is defined as

$$k = \frac{\omega \bar{c}}{2V_\infty} \quad (55)$$

where ω is the frequency of oscillations, $\bar{c}$ is the mean aerodynamic chord (MAC), and V_∞ is the freestream airspeed.

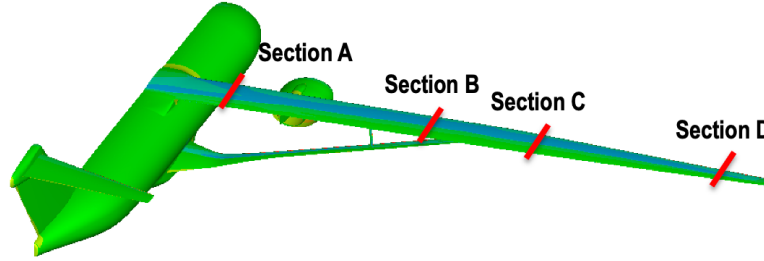


Figure 2. Mach 0.745 TTBW Wing Sections

Figures 3 (a) and (b) show the time history of the unsteady lift coefficient and the hysteresis loop in the lift curve computed by FUN3D for the MAC wing section at Mach 0.8 and the reduced frequency $k = 0.02$. The computed unsteady lift coefficient by the transonic correction method is in good agreement with the FUN3D results.

Figures 4(a) and (b) present the unsteady pitching moment coefficient for Mach 0.8 and the reduced frequency $k = 0.02$. The amplitude of pitch oscillation is 0.5° . The unsteady pitching moment coefficient exhibits a strong nonlinearity not seen in the unsteady lift coefficient in Figure 3. The transonic correction method is able to capture the nonlinear pitching moment coefficient quite accurately. There is an excellent agreement in the time histories of the unsteady pitching moment coefficients between the FUN3D results and the transonic correction. The hysteresis loop in the pitching moment curve illustrates a strong nonlinearity. The match in the hysteresis loop is generally quite good.

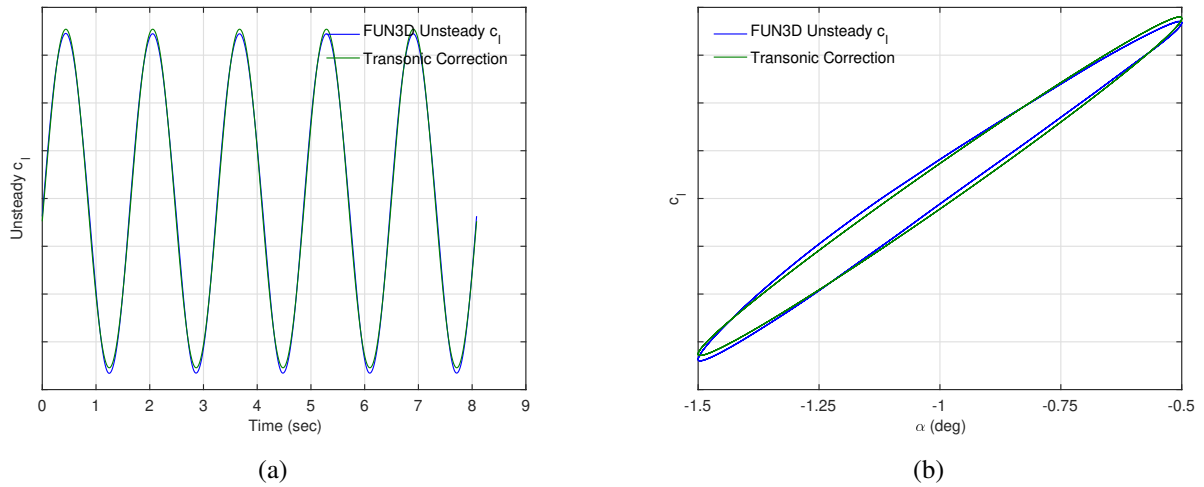


Figure 3. Mach 0.745 TTBW MAC Wing Section c_l Time History and c_l vs. α for Mach 0.8 and $k = 0.02$

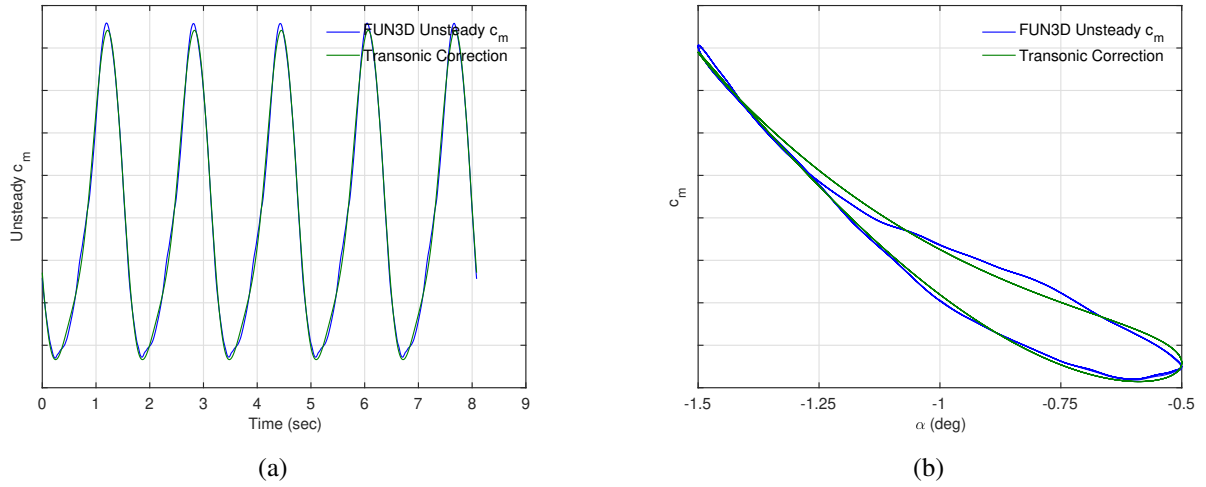


Figure 4. Mach 0.745 TTBW MAC Wing Section c_m Time History and c_m vs. α for Mach 0.8 and $k = 0.02$

The nonlinear effect in the unsteady pitching moment coefficient is postulated to be due to the unsteady motion of the aerodynamic center due to the moving shock. The inclusion of a transonic correction term to capture the unsteady motion of the aerodynamic center allows the transonic correction method to accurately model the unsteady pitching moment coefficient. Figure 5 presents the time history of the motion of the aerodynamic center about the mean aerodynamic center computed by the transonic correction method. The motion is about 2.3% of chord.

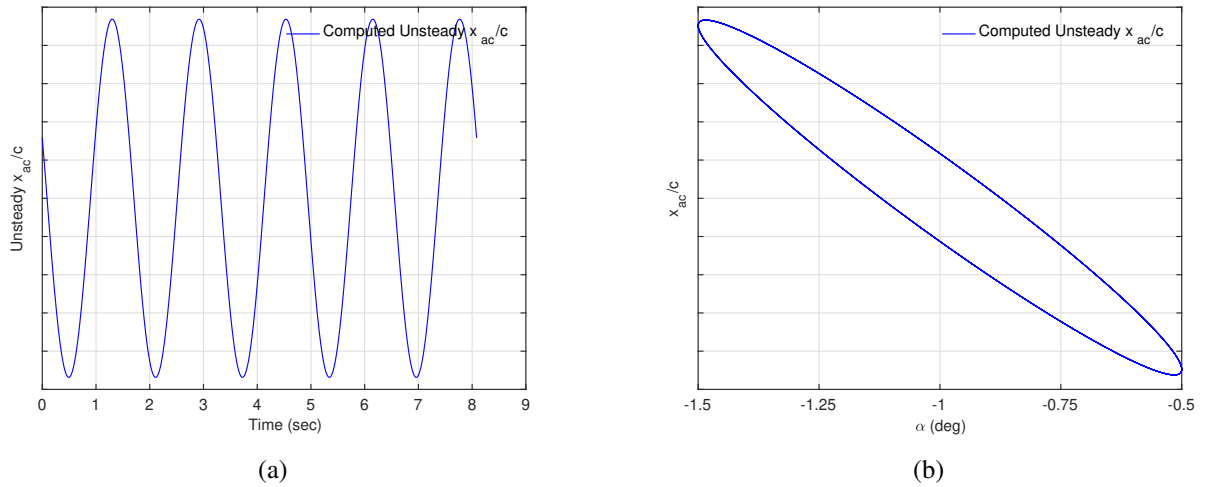


Figure 5. Mach 0.745 TTBW MAC Wing Section x_{ac} Time History and x_{ac} vs. α for Mach 0.8 and $k = 0.02$

To illustrate the effect of the reduced frequency, Figs. 6(a) and (b) present the unsteady lift coefficient for Mach 0.8 and the reduced frequency $k = 0.3$. The transonic correction method captures the unsteady lift coefficient very accurately.

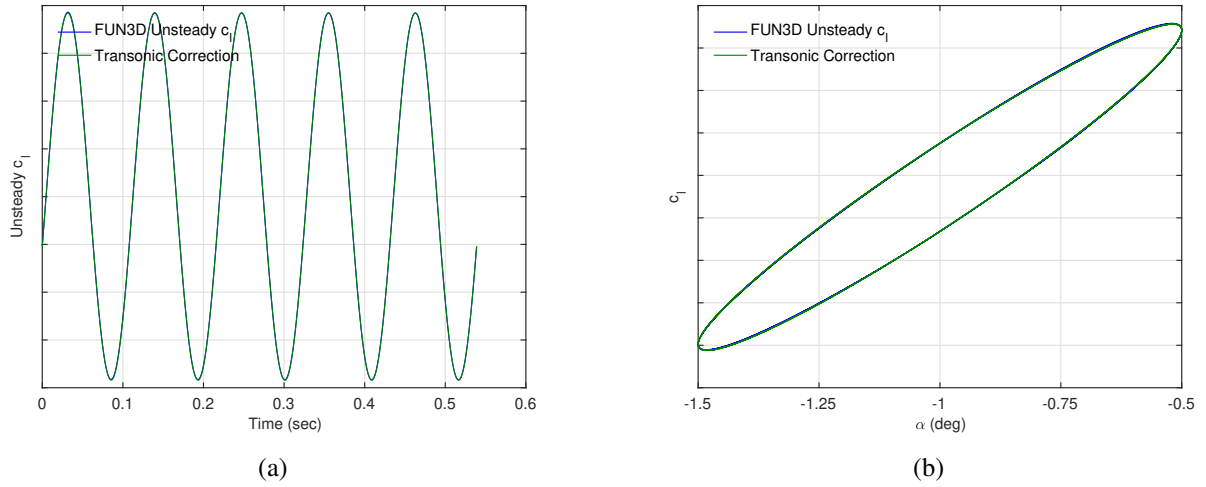


Figure 6. Mach 0.745 TTBW MAC Wing Section c_l Time History and c_l vs. α for Mach 0.8 and $k = 0.3$

Figures 7(a) and (b) show the unsteady pitching moment coefficient for Mach 0.8 and the reduced frequency $k = 0.3$. It is noted that the strong nonlinear effect in the unsteady pitching moment coefficient for the reduced frequency $k = 0.02$ is not present in the unsteady pitching moment for the reduced frequency $k = 0.3$. Nonetheless, the hysteresis loop in the unsteady pitching moment curve still indicates a slight nonlinearity in the unsteady pitching moment coefficient. A possible explanation for the reduced nonlinearity in the unsteady pitching moment coefficient as the reduced frequency increases could be due to the lift circulation not being able to fully establish at higher values of the reduced frequency. At very small values of the reduced frequency, the flow is dominated by the steady-state lift circulation which is strongly nonlinear in transonic flow. Therefore, as the flow begins to transition from a steady flow to unsteady flow, the nonlinear effect of the steady flow diminishes as the reduced frequency increases.

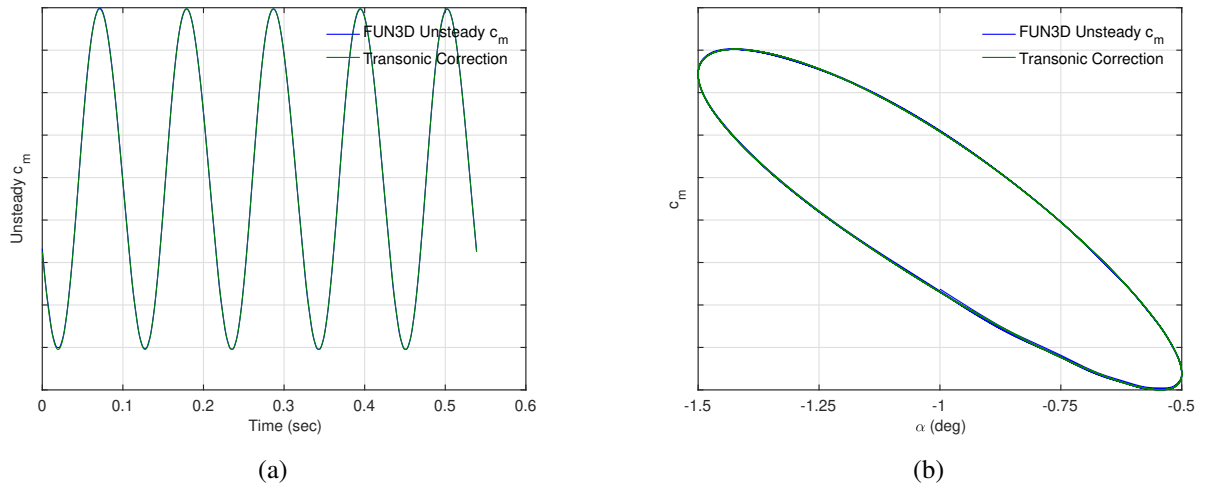


Figure 7. Mach 0.745 TTBW MAC Wing Section c_m Time History and c_m vs. α for Mach 0.8 and $k = 0.3$

Figures 8(a) and (b) show the unsteady motion of the aerodynamic center computed by the transonic correction method for Mach 0.8 and the reduced frequency $k = 0.3$. It is noted that that the amplitude of the motion of the aerodynamic center decreases as the reduced frequency increases when comparing the amplitudes for the reduced frequency $k = 0.02$ in Fig. 5. Moreover, the phase angle of the motion of the aerodynamic center also reverses.

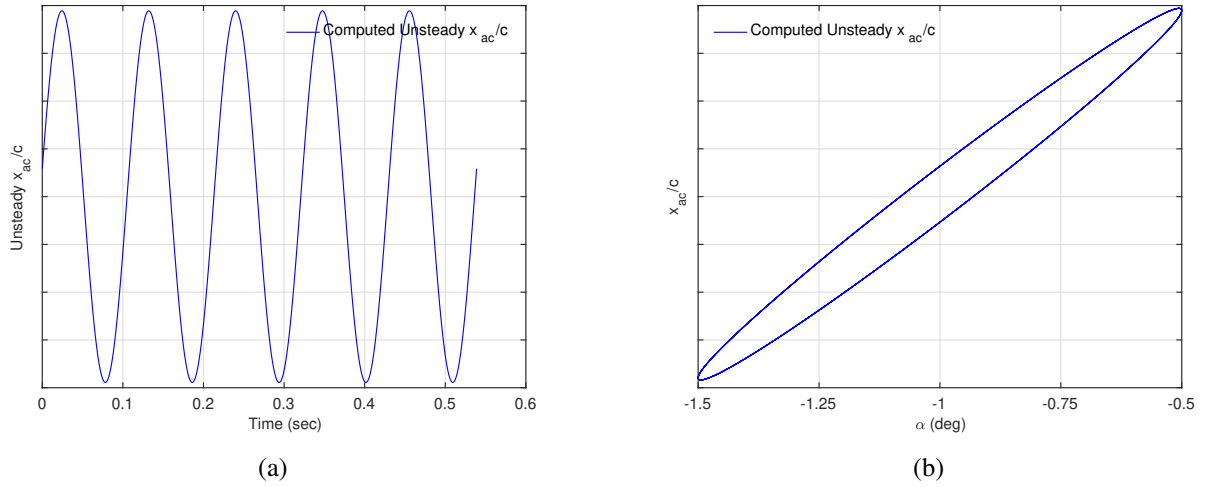


Figure 8. Mach 0.745 TTBW MAC Wing Section x_{ac} Time History and x_{ac} vs. α for Mach 0.8 and $k = 0.3$

To further illustrate the effect of the reduced frequency on the motion of the aerodynamic center, Figs. 9(a) and (b) show the pressure coefficients for the reduced frequency $k = 0.02$ and $k = 0.3$ corresponding to the aftmost shock position and mean shock position for Mach 0.8, the mean angle of attack of -1° , and the amplitude of oscillation of 0.5° . Shocks exist on both upper and lower surfaces, but the upper surface shock dominates as it causes the pressure center to shift aft. The shock movement can be seen on the surface pressure coefficient plots. The displacement of the upper surface shock position is greater for the reduced frequency $k = 0.02$ than for the reduced frequency $k = 0.3$. The displacement is estimated to be about 1.6% for the reduced frequency $k = 0.02$ and 0.8% for the reduced frequency $k = 0.3$. These estimates are close to the estimates computed from the transonic correction method. This observation thus supports the model of the motion of the aerodynamic center.

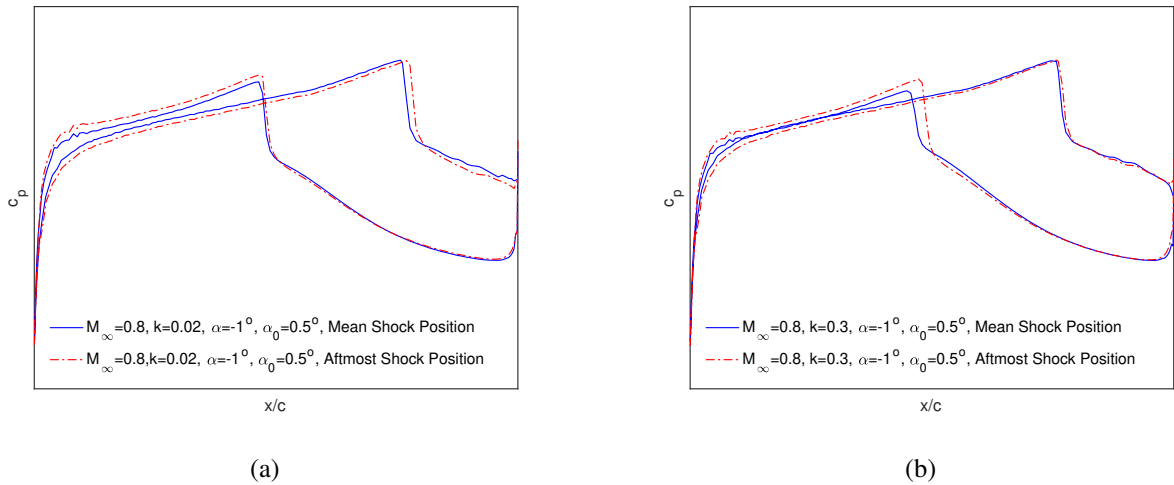


Figure 9. TTBW MAC Wing Section c_p for Mach 0.8, $k = 0.02$ and $k = 0.3$, $\bar{\alpha} = -1^\circ$, and $\alpha_0 = 0.5^\circ$

III. Dynamic Stability Derivative Analysis

The dynamic stability analysis is performed to determine the frequency responses of the stability derivatives of an aircraft as a function of the reduced frequency. The frequency responses of the dynamic stability derivatives can be obtained from the transonic correction method. In the following analysis, the wing contributions to the dynamic stability derivatives are considered. These contributions are significant with respect to the angle of attack, pitch rate,

and roll rate. The wing contributions with respect to the angle of sideslip and yaw rate are usually small.

A. Dynamic Stability Derivatives Due to Angle of Attack

The incremental lift force contribution by the wing due to the angle of attack is computed as

$$\Delta L = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} G_\alpha(\bar{s}) \left[C(\bar{s}) \alpha + \frac{1}{2} \frac{\bar{c}\dot{\alpha}}{2V_\infty} \right] c dy \quad (56)$$

where $\bar{s} = ik$ is a non-dimensional Laplace variable and $G_\alpha(\bar{s})$ is a proper transfer function obtained by a frequency-domain regression to approximate the transonic correction function $U_\alpha(\bar{s}) + iW_\alpha(\bar{s})$. $G_\alpha(\bar{s})$ represents the transonic correction for the lift coefficient and is computed from the following equation:

$$G_\alpha(\bar{s}) = \frac{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} (U_\alpha + iW_\alpha) \left[C(\bar{s}) + \frac{1}{2}\bar{s} \right] c dy}{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} \left[C(\bar{s}) + \frac{1}{2}\bar{s} \right] c dy} \approx \frac{0.4062\bar{s}^2 + 0.1777\bar{s} + 0.004045}{\bar{s}^2 + 0.2125\bar{s} + 0.004045} \quad (57)$$

Figure 10(a) shows the transonic correction function $G_\alpha(\bar{s})$. As the reduced frequency increases, the magnitude of the transonic correction on the lift coefficient decreases.

The Theodorsen's function $C(\bar{s})$ is approximated by the R. T. Jones' method and is given by

$$C(\bar{s}) \approx \frac{0.5\bar{s}^2 + 0.2808\bar{s} + 0.01365}{\bar{s}^2 + 0.3455\bar{s} + 0.01365} \quad (58)$$

The lift dynamic stability derivatives due to the angle of attack are obtained by taking the partial derivatives of the lift force with respect to α and non-dimensional $\frac{\bar{c}\dot{\alpha}}{2V_\infty}$

$$C_{L_\alpha} = \frac{1}{q_\infty S} \frac{\partial \Delta L}{\partial \alpha} = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} G_\alpha(\bar{s}) C(\bar{s}) c dy \quad (59)$$

$$C_{L_{\dot{\alpha}}} = \frac{1}{q_\infty S} \frac{\partial \Delta L}{\partial \left(\frac{\bar{c}\dot{\alpha}}{2V_\infty} \right)} = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} G_\alpha(\bar{s}) \frac{1}{2} c dy \quad (60)$$

Figures 11(a) and (b) show the plots of C_{L_α} and $C_{L_{\dot{\alpha}}}$, respectively.

The drag dynamic stability derivatives can be approximated as

$$C_{D_{\alpha}} \approx \frac{C_{D_\alpha}}{C_{L_\alpha}} C_{L_{\dot{\alpha}}} \quad (61)$$

$$C_{D_{\dot{\alpha}}} \approx \frac{C_{D_\alpha}}{C_{L_\alpha}} C_{L_{\ddot{\alpha}}} \quad (62)$$

where C_{L_α} and C_{D_α} are the steady-state stability derivatives.

The incremental pitching moment contribution by the wing due to the angle of attack is computed as

$$\Delta m = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} \left[G_\alpha(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} \alpha - H_\alpha(\bar{s}) \frac{\bar{e}_m}{2c} \frac{\bar{c}\dot{\alpha}}{2V_\infty} \right] c^2 dy \quad (63)$$

where $\bar{e}$ and $\bar{e}_m$ are defined with respect the the aircraft center of gravity (CG), and $H_\alpha(\bar{s})$ is a proper transfer function obtained by a frequency-domain regression to approximate the transonic correction function $T_\alpha(\bar{s}) + iV_\alpha(\bar{s})$. $H_\alpha(\bar{s})$ represents the transonic correction for the non-circulatory pitching moment and is computed from the following equation:

$$H_\alpha(\bar{s}) = \frac{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} (T_\alpha + iV_\alpha) \frac{\bar{e}_m}{2c} \bar{s} c dy}{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} \frac{\bar{e}_m}{2c} \bar{s} c dy} \approx \frac{0.4402\bar{s}^2 + 0.2743\bar{s} + 0.01488}{\bar{s}^2 + 0.131\bar{s} + 0.0003419} \quad (64)$$

Figure 10(b) shows the transonic correction function $H_\alpha(\bar{s})$. The magnitude of the transonic correction on the non-circulatory pitching moment coefficient decreases rapidly after small values of the reduced frequency.

The pitching moment dynamic stability derivatives due to the angle of attack are obtained as

$$C_{m_{\bar{\alpha}}} = \frac{1}{q_\infty S \bar{c}} \frac{\partial \Delta m}{\partial \alpha} = \frac{1}{S \bar{c}} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} G_\alpha(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} c^2 dy \quad (65)$$

$$C_{m_{\dot{\alpha}}} = \frac{1}{q_\infty S \bar{c}} \frac{\partial \Delta m}{\partial \left(\frac{\bar{c} \dot{\alpha}}{2V_\infty} \right)} = -\frac{1}{S \bar{c}} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} H_\alpha(\bar{s}) \frac{\bar{e}_m}{2c} c^2 dy \quad (66)$$

Figures 12(a) and (b) show the plots of $C_{m_{\bar{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$ due to the wing contributions. The steady-state stability derivative C_{m_α} due to the wing contribution is obtained to be -3.037 at zero reduced frequency. The horizontal tail has a significant influence on the pitching moment stability derivatives. The steady-state stability derivative C_{m_α} for the whole aircraft is estimated to be -5.264 from VSPAERO with steady-state transonic and integral boundary layer corrections.⁸⁻¹⁰ The difference between these two values reflects the horizontal tail contribution. To make a correction to the steady-state stability derivative C_{m_α} , we apply the following formula:

$$C_{m_{\bar{\alpha}}} = C_{m_{\bar{\alpha}}}(\bar{s}) - C_{m_{\bar{\alpha}}}(0) + C_{m_\alpha} \quad (67)$$

where $C_{m_{\bar{\alpha}}}(\bar{s})$ is from Eq. (65) and C_{m_α} is the steady-state stability derivative for the whole aircraft.

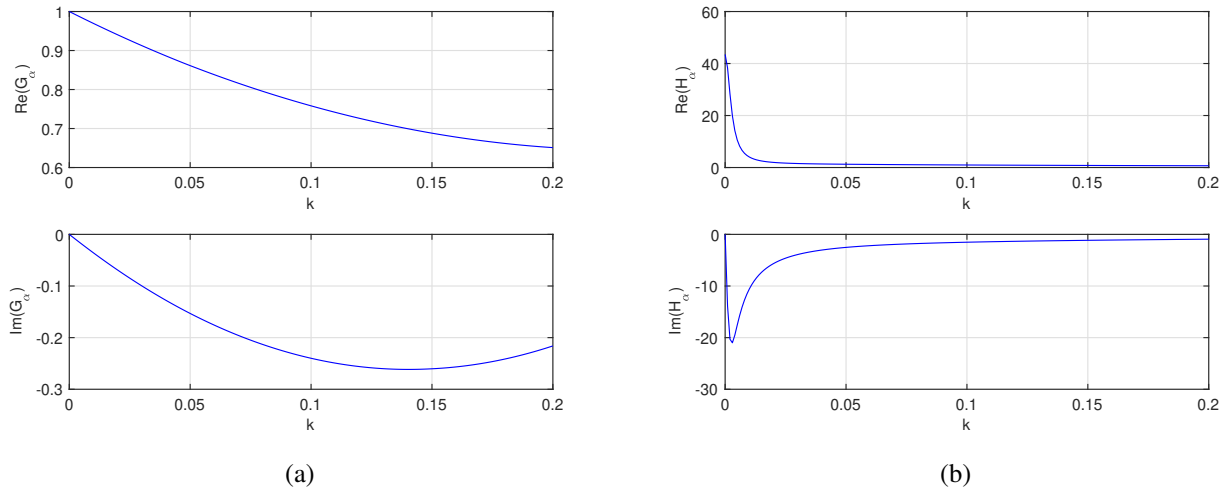


Figure 10. Transonic Correction Transfer Functions $G_\alpha(\bar{s})$ and $H_\alpha(\bar{s})$

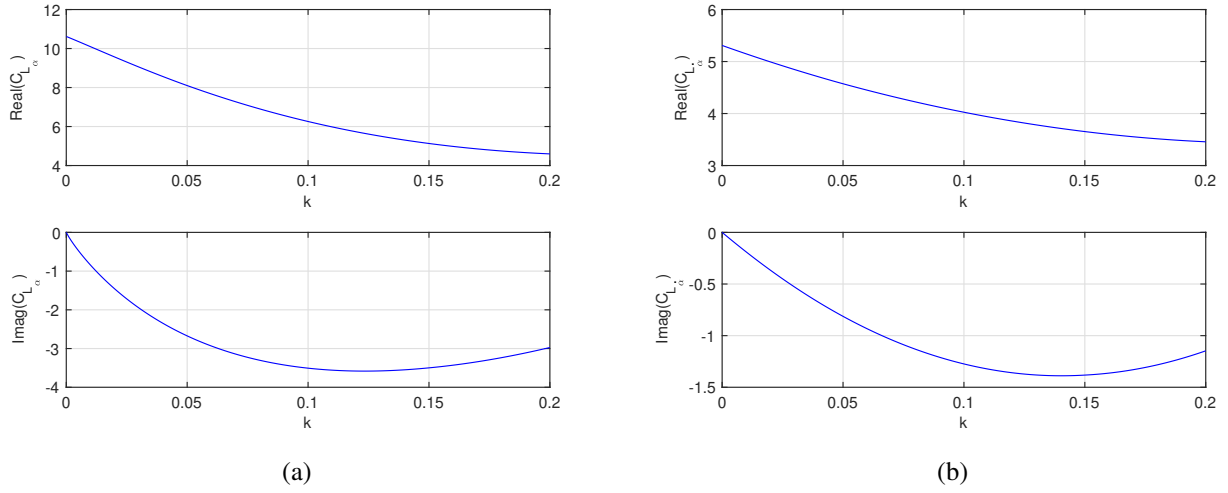


Figure 11. Lift Dynamic Stability Derivatives Due to Angle of Attack $C_{L_{\dot{\alpha}}}$ and $C_{L_{\ddot{\alpha}}}$

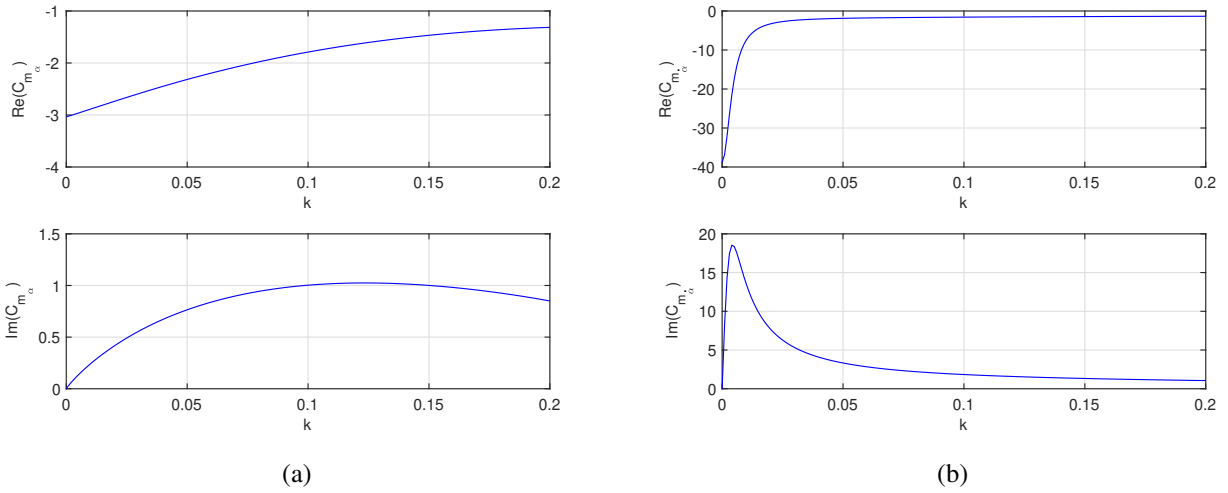


Figure 12. Pitching Moment Dynamic Stability Derivatives Due to Angle of Attack $C_{m_{\dot{\alpha}}}$ and $C_{m_{\ddot{\alpha}}}$

B. Dynamic Stability Derivatives Due to Pitch Rate

The pitch rate is the time derivative of the pitch angle. The incremental lift force contribution by the wing due to the pitch rate is computed as

$$\Delta L = q_{\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} G_{\theta}(\bar{s}) \left\{ \left[C(\bar{s}) \frac{2\bar{e}_c}{c} + \frac{1}{2} \right] \frac{\bar{c}q}{2V_{\infty}} + \frac{\bar{e}_m}{c} \frac{\bar{c}^2 \dot{q}}{4V_{\infty}^2} \right\} c dy \quad (68)$$

where $\bar{e}_c$ is defined with respect to the aircraft CG and $G_{\theta}(\bar{s})$ is a proper transfer function obtained by a frequency-domain regression to approximate the transonic correction function $U_{\theta}(\bar{s}) + iW_{\theta}(\bar{s})$. $G_{\theta}(\bar{s})$ is computed from the following equation and is plotted in Fig.13(a):

$$G_{\theta}(\bar{s}) = \frac{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} (U_{\theta} + W_{\theta}) \left\{ \left[C(\bar{s}) \frac{2\bar{e}_c}{c} + \frac{1}{2} \right] \bar{s} + \frac{\bar{e}_m}{c} \bar{s}^2 \right\} c dy}{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l\alpha} \left\{ \left[C(\bar{s}) \frac{2\bar{e}_c}{c} + \frac{1}{2} \right] \bar{s} + \frac{\bar{e}_m}{c} \bar{s}^2 \right\} c dy} \approx \frac{0.3714\bar{s}^2 + 0.1802\bar{s} + 0.003724}{\bar{s}^2 + 0.2178\bar{s} + 0.003724} \quad (69)$$

The lift dynamic stability derivatives due to the pitch rate is obtained as

$$C_{L_{\dot{q}}} = \frac{1}{q_{\infty} S} \frac{\partial \Delta L}{\partial \left(\frac{\bar{c} \dot{q}}{2V_{\infty}} \right)} = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} G_{\theta}(\bar{s}) \left[C(\bar{s}) \frac{2\bar{e}_c}{\bar{c}} + \frac{1}{2} \right] c dy \quad (70)$$

$$C_{L_{\dot{q}}} = \frac{1}{q_{\infty} S} \frac{\partial \Delta L}{\partial \left(\frac{\bar{c}^2 \dot{q}}{4V_{\infty}^2} \right)} = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} G_{\theta}(\bar{s}) \frac{\bar{e}_m}{c} c dy \quad (71)$$

Figures 14(a) and (b) show the plots of $C_{L_{\dot{q}}}$ and $C_{L_{\ddot{q}}}$. The steady-state stability derivative C_{L_q} is obtained to be 24.9 at zero reduced frequency. The steady-state stability derivative C_{L_q} for the whole aircraft computed by VSPAERO is 26.1. The small difference reflects the horizontal tail contribution. To make a correction to the steady-state stability derivative C_{L_q} , we apply the following formula:

$$C_{L_{\dot{q}}} = C_{L_{\dot{q}}}(\bar{s}) - C_{L_{\dot{q}}}(0) + C_{L_q} \quad (72)$$

where $C_{L_{\dot{q}}}(\bar{s})$ is from Eq. (73) and C_{L_q} is the steady-state stability derivative for the whole aircraft.

The drag dynamic stability derivatives are approximated as

$$C_{D_{\dot{q}}} \approx \frac{C_{D_q}}{C_{L_q}} C_{L_{\dot{q}}} \quad (73)$$

$$C_{D_{\ddot{q}}} \approx \frac{C_{D_q}}{C_{L_q}} C_{L_{\ddot{q}}} \quad (74)$$

where C_{L_q} and C_{D_q} are the steady-state stability derivatives.

The incremental pitching moment contribution by the wing due to the pitch rate is computed as

$$\Delta m = q_{\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} \left\{ G_{\theta}(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} \frac{2\bar{e}_c}{c} \frac{\bar{c} \dot{q}}{2V_{\infty}} - H_{\theta}(\bar{s}) \left[\frac{\bar{e}_c}{2c} \frac{\bar{c} \dot{q}}{2V_{\infty}} + \left(\frac{1}{32} + \frac{\bar{e}_m^2}{c^2} \right) \frac{\bar{c}^2 \dot{q}}{4V_{\infty}^2} \right] \right\} c^2 dy \quad (75)$$

where $H_{\theta}(\bar{s})$ is a proper transfer function obtained by a frequency-domain regression to approximate the transonic correction function $T_{\theta}(\bar{s}) + iV_{\theta}(\bar{s})$. $H_{\theta}(\bar{s})$ is computed from the following equation and is plotted in Fig. 13(b):

$$H_{\theta}(\bar{s}) = \frac{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} (T_{\theta} + iV_{\theta}) \left[\frac{\bar{e}_c}{2c} \bar{s} + \left(\frac{1}{32} + \frac{\bar{e}_m^2}{c^2} \right) \bar{s}^2 \right] c^2 dy}{\int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} \left[\frac{\bar{e}_c}{2c} \bar{s} + \left(\frac{1}{32} + \frac{\bar{e}_m^2}{c^2} \right) \bar{s}^2 \right] c^2 dy} \approx \frac{0.4199\bar{s}^2 + 0.1549\bar{s} + 0.009447}{\bar{s}^2 + 0.1574\bar{s} + 0.0006584} \quad (76)$$

The pitching moment dynamic stability derivatives due to the pitch rate is obtained as

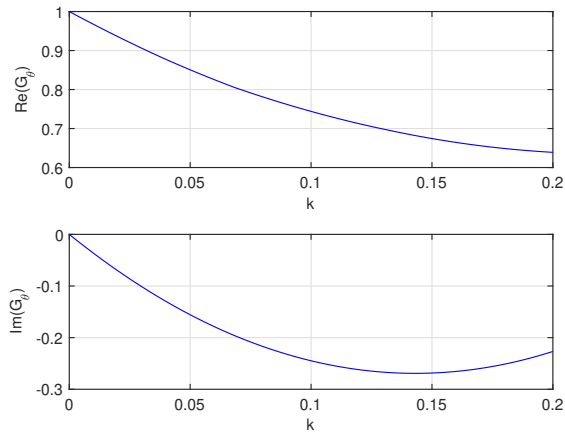
$$C_{m_{\dot{q}}} = \frac{1}{q_{\infty} S \bar{c}} \frac{\partial \Delta C_m}{\partial \left(\frac{q \bar{c}}{2V_{\infty}} \right)} = \frac{1}{S \bar{c}} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} \left[G_{\theta}(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} \frac{2\bar{e}_c}{c} - H_{\theta}(\bar{s}) \frac{\bar{e}_c}{2c} \right] c^2 dy \quad (77)$$

$$C_{m_{\ddot{q}}} = \frac{1}{q_{\infty} S \bar{c}} \frac{\partial \Delta C_m}{\partial \left(\frac{\bar{c}^2 \dot{q}}{4V_{\infty}^2} \right)} = -\frac{1}{S \bar{c}} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} H_{\theta}(\bar{s}) \left(\frac{1}{32} + \frac{\bar{e}_m^2}{c^2} \right) c^2 dy \quad (78)$$

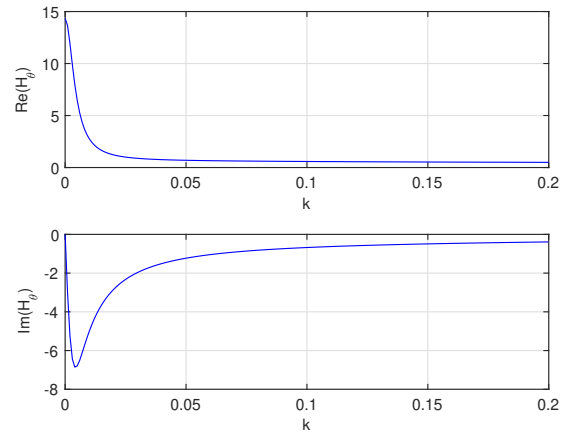
Figures 15(a) and (b) show the plots of $C_{m_{\dot{q}}}$ and $C_{m_{\ddot{q}}}$. The dynamic stability derivative $C_{m_{\dot{q}}}$ is generally small. The horizontal tail contributes largely to the pitching moment stability derivative $C_{m_{\dot{q}}}$. The wing contribution to the steady-state stability derivative C_{m_q} is obtained to be -0.8734 . The steady-state stability derivative C_{m_q} for the whole aircraft is computed to be -122.2852 . To adjust for the wing contribution to the total pitch moment stability derivative, we use the following formula

$$C_{m_{\dot{q}}} = C_{m_{\dot{q}}}(\bar{s}) - C_{m_{\dot{q}}}(0) + C_{m_q} \quad (79)$$

where $C_{m_{\dot{q}}}(\bar{s})$ is from Eq. (77) and C_{m_q} is the steady-state stability derivative for the whole aircraft.

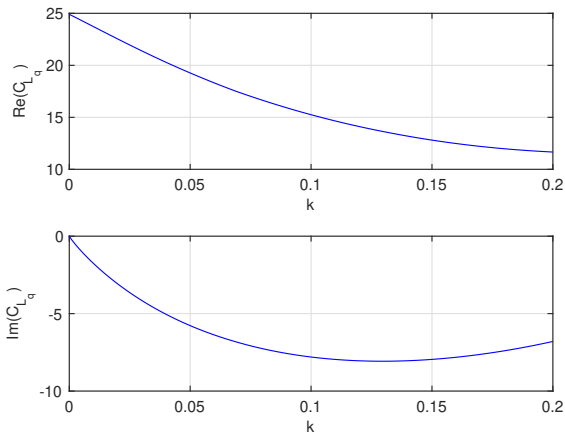


(a)

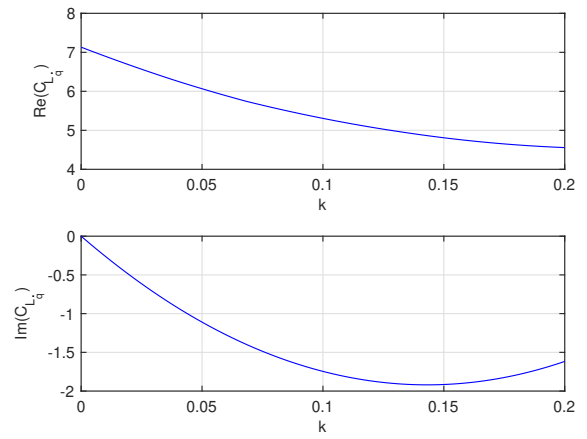


(b)

Figure 13. Transonic Correction Transfer Functions $G_\theta(\bar{s})$ and $H_\theta(\bar{s})$



(a)



(b)

Figure 14. Lift Dynamic Stability Derivatives Due to Pitch Rate $C_{L_{\dot{q}}}$ and C_{L_q}

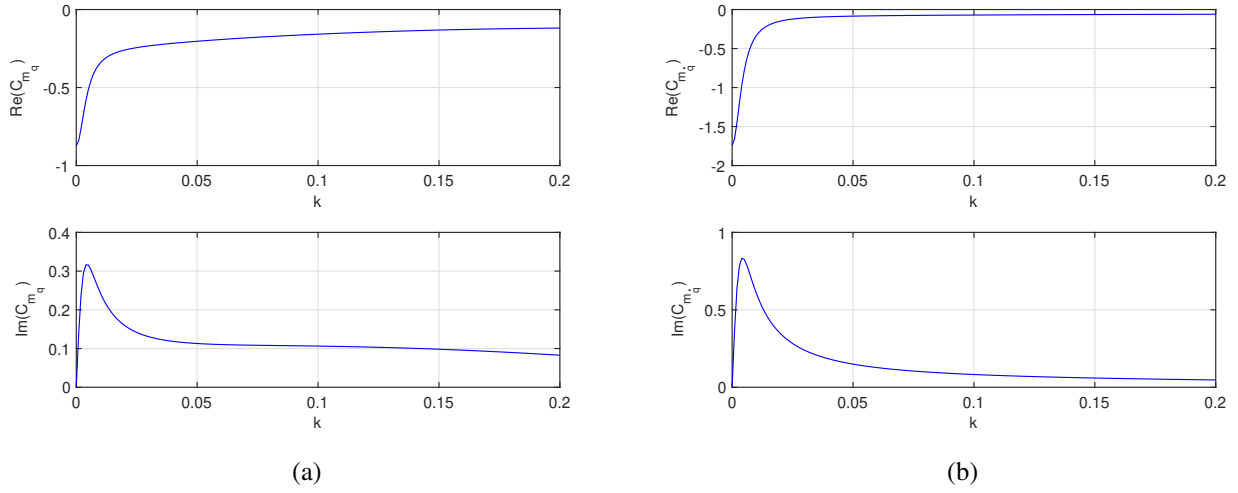


Figure 15. Pitching Moment Dynamic Stability Derivatives Due to Pitch Rate $C_{m_{\bar{q}}}$ and $C_{m_{\dot{q}}}$

C. Dynamic Stability Derivatives Due to Roll Rate

The plunge airspeed due to the roll rate is

$$\dot{h} = yp \quad (80)$$

The incremental rolling moment contribution by the wing due to the roll rate is computed as

$$\Delta l = -q_{\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} G_{\alpha}(\bar{s}) \left[C(\bar{s}) 2 \frac{y}{b} \frac{bp}{2V_{\infty}} + \frac{\bar{c}y}{b^2} \frac{b^2 \dot{p}}{4V_{\infty}^2} \right] cy dy$$

The rolling moment derivatives are evaluated as

$$C_{l_{\dot{p}}} = \frac{1}{q_{\infty} Sb} \frac{\partial \Delta l}{\partial \left(\frac{p b}{2V_{\infty}} \right)} = -\frac{1}{Sb} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} G_{\alpha}(\bar{s}) C(\bar{s}) 2 \frac{y}{b} cy dy \quad (81)$$

$$C_{l_{\ddot{p}}} = \frac{1}{q_{\infty} Sb} \frac{\partial \Delta l}{\partial \left(\frac{p b^2}{4V_{\infty}^2} \right)} = -\frac{1}{Sb} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_{\alpha}} G_{\alpha}(\bar{s}) \frac{\bar{c}y}{b^2} cy dy \quad (82)$$

Figures 16(a) and (b) show the plots of $C_{l_{\dot{p}}}$ and $C_{l_{\ddot{p}}}$. The stability derivative $C_{l_{\dot{p}}}$ is generally small.

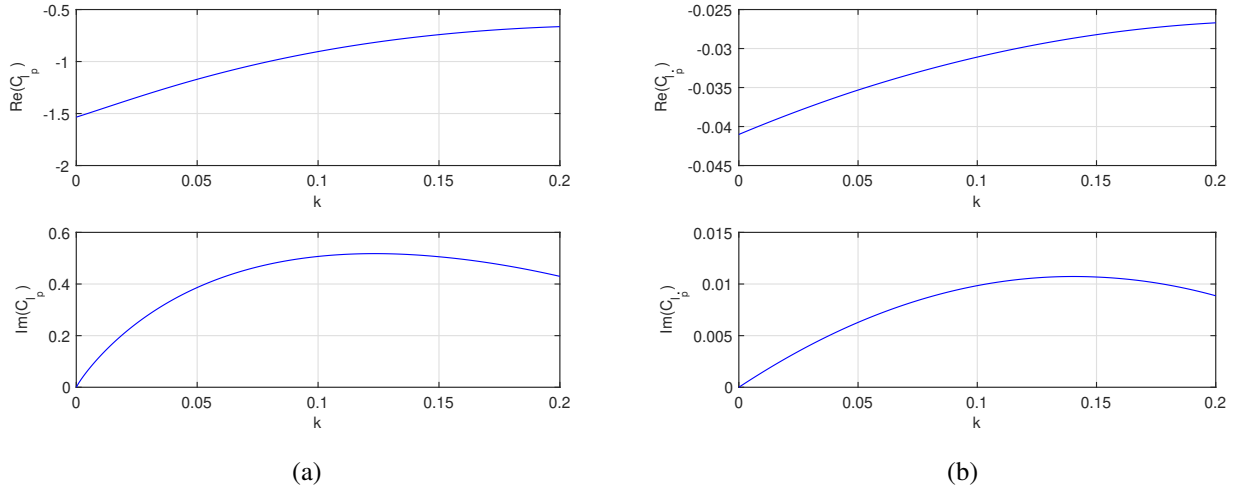


Figure 16. Rolling Moment Dynamic Stability Derivatives Due to Roll Rate C_{l_p} and $C_{l_{\dot{p}}}$

D. Flight Dynamic Stability Analysis

The steady-state stability derivatives of the Mach 0.745 TTBW are computed by VSPAERO which includes the steady-state transonic and boundary layer corrections.^{8–10} The dynamic stability derivatives due to the wing contribution are added to the flight dynamic model for the stability analysis.

The linearized equations of motion of flight dynamics are presented in a frequency-domain form as

$$\begin{bmatrix} \mathbf{M}_{lon} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{lat} \end{bmatrix} \begin{bmatrix} s\mathbf{x}_{lon} \\ s\mathbf{x}_{lat} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{lon} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{lat} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{lon} \\ \mathbf{x}_{lat} \end{bmatrix} \quad (83)$$

where $s = \frac{2V_\infty}{\bar{c}}\bar{s}$ is the Laplace variable, $\mathbf{x}_{lon} = \begin{bmatrix} q & \Delta\alpha & \frac{\Delta V}{V_\infty} & \frac{\Delta h}{h} & \Delta\theta \end{bmatrix}^\top$, $\mathbf{x}_{lat} = \begin{bmatrix} p & r & \phi & \beta \end{bmatrix}^\top$, and

$$\mathbf{M}_{lon} = \begin{bmatrix} \frac{I_{yy}}{q_\infty S \bar{c}} - \frac{C_{m_{\bar{q}}}(\bar{s})\bar{c}^2}{4V_\infty^2} & -\frac{C_{m_{\bar{\alpha}}}(\bar{s})\bar{c}}{2V_\infty} & 0 & 0 & 0 \\ \frac{C_{L_{\bar{q}}}(\bar{s})\bar{c}^2}{4V_\infty^2} & \frac{mV_\infty}{q_\infty S} + \frac{C_{L_{\bar{\alpha}}}(\bar{s})\bar{c}}{2V_\infty} & 0 & 0 & 0 \\ 0 & \frac{C_{D_{\bar{\alpha}}}(\bar{s})\bar{c}}{2V_\infty} & \frac{mV_\infty}{q_\infty S} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (84)$$

$$\mathbf{S}_{lon} = \begin{bmatrix} \frac{C_{m_{\bar{q}}}(\bar{s})\bar{c}}{2V_\infty} & C_{m_{\bar{\alpha}}}(\bar{s}) & C_{m_u} & 0 & 0 \\ \frac{mV_\infty}{q_\infty S} - \frac{C_{L_{\bar{q}}}(\bar{s})\bar{c}}{2V_\infty} & -C_{L_{\bar{\alpha}}}(\bar{s}) & -C_{L_u} - 2C_L & -C_L \frac{hd\rho}{pdh} & 0 \\ -\frac{C_{D_{\bar{q}}}(\bar{s})\bar{c}}{2V_\infty} & \frac{W}{q_\infty S} - C_{D_{\bar{\alpha}}}(\bar{s}) & -C_{D_u} - 2C_D & -C_D \frac{hd\rho}{pdh} & -\frac{W}{q_\infty S} \\ 0 & -\frac{V_\infty}{h} & 0 & 0 & \frac{V_\infty}{h} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (85)$$

$$\mathbf{M}_{lat} = \begin{bmatrix} \frac{I_{xx}}{q_\infty S b} - \frac{C_{m_{\bar{p}}}(\bar{s})b^2}{4V_\infty^2} & -\frac{I_{xz}}{q_\infty S b} & 0 & 0 \\ -\frac{I_{xz}}{q_\infty S b} & \frac{I_{zz}}{q_\infty S b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{mV_\infty}{q_\infty S} \end{bmatrix} \quad (86)$$

$$\mathbf{S}_{lat} = \begin{bmatrix} \frac{C_{l_{\bar{\beta}}}(\bar{s})b}{2V_{\infty}} & \frac{C_{l_r}b}{2V_{\infty}} & 0 & C_{l_{\beta}} \\ \frac{C_{n_{\bar{\beta}}}b}{2V_{\infty}} & \frac{C_{n_r}b}{2V_{\infty}} & 0 & C_{n_{\beta}} \\ 1 & 0 & 0 & 0 \\ \frac{mV_{\infty}\alpha}{q_{\infty}S} + \frac{C_{Y_p}b}{2V_{\infty}} & \frac{mV_{\infty}}{q_{\infty}S} + \frac{C_{Y_r}b}{2V_{\infty}} & \frac{W}{q_{\infty}S} & C_{Y_{\beta}} \end{bmatrix} \quad (87)$$

It is noted that the matrices are frequency-dependent as functions of the Laplace variable s whereupon the non-dimensional Laplace variable $\bar{s}$ has been converted to s . Table 1 presents the eigenvalues of the flight dynamic modes with and without the transonic-corrected dynamic stability derivatives at Mach 0.745, design lift coefficient $C_L = 0.73$, and an altitude of 43,606 ft.

Mode	Eigenvalue w/o Dynamic Stability	Eigenvalue w/ Dynamic Stability
Short-Period	$-0.3500 \pm 0.3514i$	$-0.3789 \pm 0.3817i$
Phugoid	$-0.008208 \pm 0.07703i$	$-0.008331 \pm 0.07692i$
Plunge	-0.008621	-0.008621
Dutch-Roll	$-0.005294 \pm 0.2266i$	$-0.006793 \pm 0.2280i$
Roll	-0.4950	-0.5264
Spiral	-0.001945	-0.001945

Table 1. Eigenvalues of Flight Dynamic Modes of Mach 0.745 TTBW

IV. Generalized Aerodynamic Force Analysis

A NASTRAN finite-element model (FEM) of the Mach 0.745 TTBW is available from Boeing. The NASTRAN FEM comprises a detail structural model of the wing and struts, a shell structural model of the fuselage section that joins with the wing, and a beam-stick model of the rest of the fuselage and the tail empennage. This is shown in Figure 17.

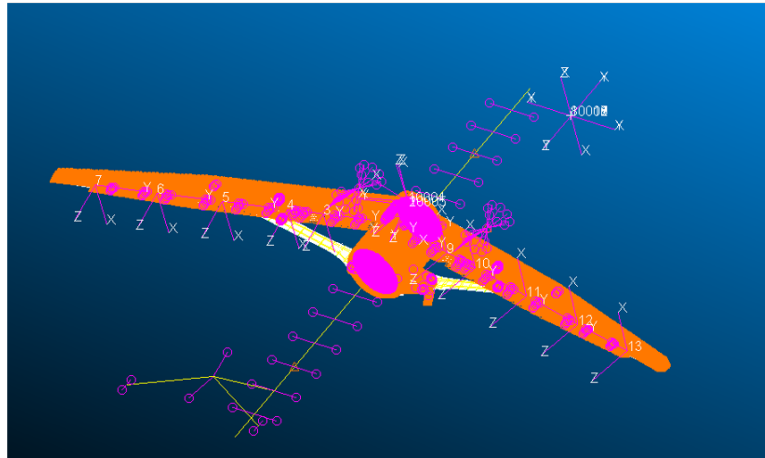


Figure 17. Boeing Mach 0.745 NASTRAN Finite-Element Model

The frequencies of the structural dynamic modes are shown in Table 2. The mode shapes of the first six modes are shown in Figure 18.

MODE	EIGENVALUE	RADIANS	CYCLES	GENERALIZED MASS	GENERALIZED STIFFNESS
1	7.602844E+01	8.719429E+00	1.387740E+00	1.616053E+01	1.228660E+03
2	9.013939E+01	9.494176E+00	1.511045E+00	7.625986E+02	6.874017E+04
3	1.265952E+02	1.125145E+01	1.790725E+00	4.345584E+00	5.501301E+02
4	1.482324E+02	1.217507E+01	1.937723E+00	3.393707E+00	5.030572E+02
5	2.192438E+02	1.480688E+01	2.356589E+00	1.485700E+01	3.257305E+03
6	2.208044E+02	1.485949E+01	2.364961E+00	9.364201E+02	2.067657E+05
7	3.154186E+02	1.776003E+01	2.826596E+00	1.629658E+01	5.140246E+03
8	3.190327E+02	1.786149E+01	2.842744E+00	2.370701E+01	7.563310E+03
9	3.409872E+02	1.846584E+01	2.938930E+00	7.999038E+01	2.727570E+04
10	4.343747E+02	2.084166E+01	3.317053E+00	1.057838E+01	4.594981E+03
11	5.990791E+02	2.447609E+01	3.895491E+00	1.172665E+01	7.025192E+03
12	7.103877E+02	2.665310E+01	4.241972E+00	1.196001E+01	8.496244E+03
13	8.052690E+02	2.837726E+01	4.516381E+00	1.076582E+01	8.669378E+03
14	1.417630E+03	3.765142E+01	5.992410E+00	2.830640E+00	4.012799E+03
15	1.871930E+03	4.326581E+01	6.885967E+00	1.704194E+00	3.190132E+03
16	2.234587E+03	4.727141E+01	7.523479E+00	5.157762E+00	1.152547E+04
17	3.321114E+03	5.762910E+01	9.171957E+00	2.065572E-02	6.859999E+01
18	3.484454E+03	5.902927E+01	9.394800E+00	1.436693E-01	5.006093E+02
19	4.005732E+03	6.329085E+01	1.007305E+01	7.076377E+00	2.834607E+04
20	4.159567E+03	6.449471E+01	1.026465E+01	6.857036E+00	2.852230E+04

Table 2. Frequencies of Mach 0.745 Transonic Truss Braced-Wing

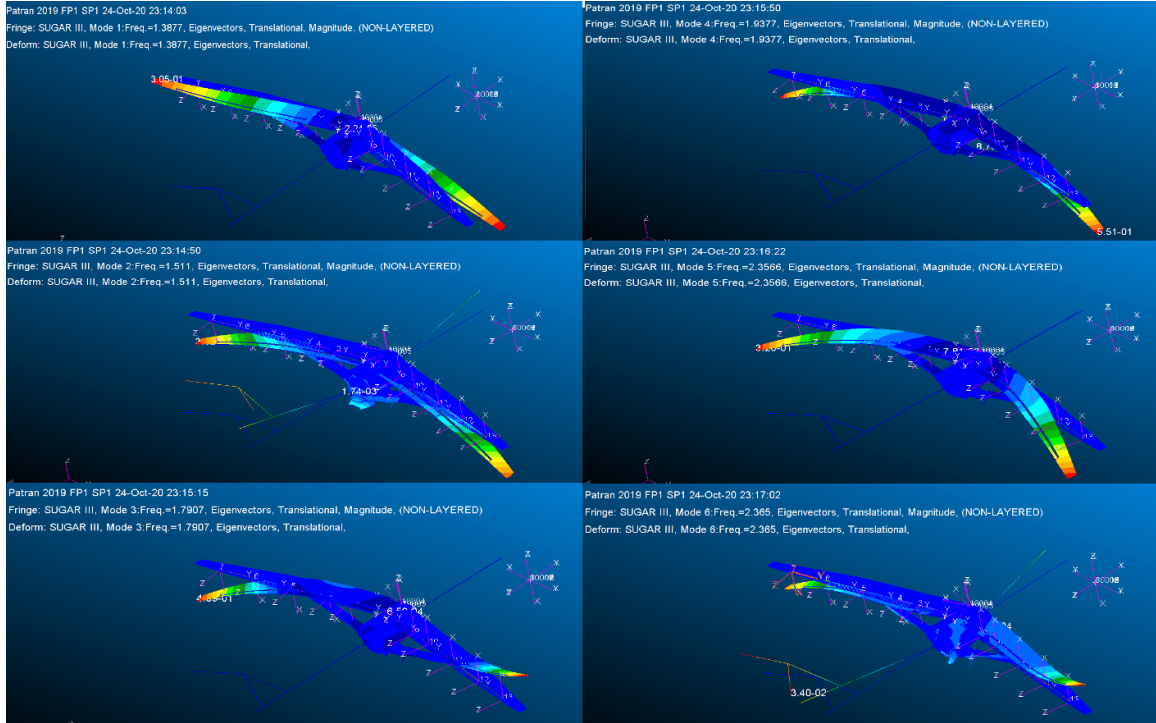


Figure 18. Boeing Mach 0.745 NASTRAN Mode Shapes

A. Generalized Aerodynamic Force Due to Generalized Displacements

The generalized aerodynamic force is computed as

$$\mathbf{F}_e = \int_{-\frac{b}{2}}^{\frac{b}{2}} \Phi_z^\top l_e dy + \int_{-\frac{b}{2}}^{\frac{b}{2}} \Psi_y^\top m_e dy \quad (88)$$

where Φ_z and Ψ_y are the vertical displacement matrix (positive upward) and pitch rotation matrix (positive leading edge up) of the modes, respectively. The lift force and pitching moment due to the generalized displacements are

expressed as

$$l_e = q_\infty c c_{l\alpha} G_\theta(\bar{s}) \left[C(\bar{s}) \left(\mathbf{q} + \frac{e_c \dot{\mathbf{q}}}{V_\infty} \right) + \frac{c \dot{\mathbf{q}}}{4V_\infty} + \frac{c e_m \ddot{\mathbf{q}}}{4V_\infty^2} \right] \Psi_y - q_\infty c c_{l\alpha} G_\alpha(\bar{s}) \left[C(\bar{s}) \frac{\dot{\mathbf{q}}}{V_\infty} + \frac{c \ddot{\mathbf{q}}}{4V_\infty^2} \right] \Phi_z \quad (89)$$

$$m_e = q_\infty c^2 c_{l\alpha} \left\{ G_\theta(\bar{s}) C(\bar{s}) \frac{e}{c} \left(\mathbf{q} + \frac{e_c \dot{\mathbf{q}}}{V_\infty} \right) - H_\theta(\bar{s}) \left[\frac{e_c \dot{\mathbf{q}}}{4V_\infty} + \frac{(c^2 + 32e_m^2) \ddot{\mathbf{q}}}{128V_\infty^2} \right] \right\} \Psi_y \\ - q_\infty c^2 c_{l\alpha} \left[G_\alpha(\bar{s}) C(\bar{s}) \frac{e}{c} \frac{\dot{\mathbf{q}}}{V_\infty} - H_\alpha(\bar{s}) \frac{e_m \ddot{\mathbf{q}}}{4V_\infty^2} \right] \Phi_z \quad (90)$$

where $\mathbf{q}$ is the generalized displacement vector of the modes, and the quantities e , e_m , and e_c are defined with respect to the elastic axis. The generalized force is expressed as

$$\mathbf{F}_e = -\mathbf{K}_a \mathbf{q} - \mathbf{C}_a \dot{\mathbf{q}} - \mathbf{M}_a \ddot{\mathbf{q}} \quad (91)$$

where $\mathbf{K}_a$, $\mathbf{C}_a$, and $\mathbf{M}_a$ are the aerodynamic stiffness, damping, and mass matrices, respectively and computed as

$$\mathbf{K}_a = -q_\infty G_\theta(\bar{s}) C(\bar{s}) \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\Phi_z^\top + \Psi_y^\top e \right) c_{l\alpha} \Psi_y c dy \quad (92)$$

$$\mathbf{C}_a = -q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Phi_z^\top c_{l\alpha} \left\{ G_\theta(\bar{s}) \left[C(\bar{s}) \frac{e_c}{V_\infty} + \frac{c}{4V_\infty} \right] \Psi_y - G_\alpha(\bar{s}) C(\bar{s}) \frac{\Phi_z}{V_\infty} \right\} c dy \\ - q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Psi_y^\top c_{l\alpha} \left\{ G_\theta(\bar{s}) C(\bar{s}) \frac{e}{c} \left[\frac{e_c}{V_\infty} - H_\theta(\bar{s}) \frac{e_c}{4V_\infty} \right] \Psi_y - G_\alpha(\bar{s}) C(\bar{s}) \frac{e}{c} \frac{\Phi_z}{V_\infty} \right\} c^2 dy \quad (93)$$

$$\mathbf{M}_a = -q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Phi_z^\top c_{l\alpha} \left[G_\theta(\bar{s}) \frac{c e_m \Psi_y}{4V_\infty^2} - G_\alpha(\bar{s}) \frac{c \Phi_z}{4V_\infty^2} \right] c dy \\ - q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Psi_y^\top c_{l\alpha} \left[-H_\theta(\bar{s}) \frac{(c^2 + 32e_m^2) \Psi_y}{128V_\infty^2} + H_\alpha(\bar{s}) \frac{e_m \Phi_z}{4V_\infty^2} \right] c^2 dy \quad (94)$$

The dynamic aeroelastic equation is given by

$$(\mathbf{M}_s + \mathbf{M}_a) s^2 \mathbf{q} + (\mathbf{C}_s + \mathbf{C}_a) s \mathbf{q} + (\mathbf{K}_s + \mathbf{K}_a) \mathbf{q} = \mathbf{F}_a \quad (95)$$

where $\mathbf{M}_s$, $\mathbf{C}_s$, and $\mathbf{K}_s$ are the structural stiffness, damping, and mass matrices, respectively, and $\mathbf{F}_a$ is the generalized aerodynamic force due to the aircraft states $\mathbf{x}_{lon}$ and $\mathbf{x}_{lat}$. The eigenvalues of the dynamic aeroelastic equation are determined from

$$\mathbf{H}(s) = (\mathbf{M}_s + \mathbf{M}_a) s^2 + (\mathbf{C}_s + \mathbf{C}_a) s + (\mathbf{K}_s + \mathbf{K}_a) = \mathbf{0} \quad (96)$$

Figure 19 shows the root locus of the eigenvalues of the first 17 aeroelastic modes. The structural damping for all the structural modes is assumed to be 2%. All the first 17 aeroelastic modes are stable. The transonic correction causes a reduction in the aerodynamic damping from the ideal values computed by Theodorsen's theory.

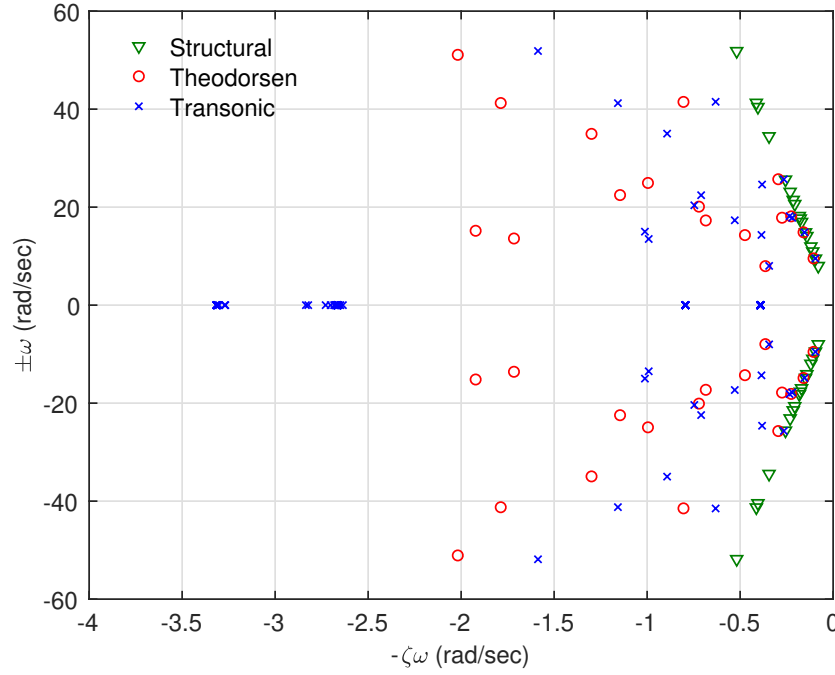


Figure 19. Eigenvalues of Aeroelastic Modes of Mach 0.745 TTBW

B. Generalized Aerodynamic Force Due to Aircraft States

The lift force and pitching moment due to the aircraft states are computed as

$$l_r = q_\infty c c_{l_\alpha} G_\alpha(\bar{s}) \left[C(\bar{s}) \left(\alpha + \frac{yp}{V_\infty} \right) + \frac{\bar{c}}{4V_\infty} \left(\dot{\alpha} + \frac{y\dot{p}}{V_\infty} \right) \right] + q_\infty c c_{l_\alpha} G_\theta(\bar{s}) \left\{ \left[C(\bar{s}) \frac{\bar{e}_c}{V_\infty} + \frac{c}{2V_\infty} \right] q + \frac{c\bar{e}_m\dot{q}}{4V_\infty^2} \right\} + q_\infty c (c_{l_u} + 2\bar{c}_l) \frac{\Delta V}{V_\infty} + q_\infty c \bar{c}_l \frac{\rho d\rho}{hdh} \frac{\Delta h}{h} \quad (97)$$

$$m_r = q_\infty c^2 c_{l_\alpha} \left[G_\alpha(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} \left(\alpha + \frac{yp}{V_\infty} \right) - H_\alpha(\bar{s}) \frac{\bar{e}_m}{4V_\infty} \left(\dot{\alpha} + \frac{y\dot{p}}{V_\infty} \right) \right] + q_\infty c^2 c_{l_\alpha} \left\{ G_\theta(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} \frac{\bar{e}_c q}{V_\infty} - H_\theta(\bar{s}) \left[\frac{\bar{e}_c q}{4V_\infty} + \frac{(c^2 + 32\bar{e}_m^2)\dot{q}}{128V_\infty^2} \right] \right\} + q_\infty c^2 (c_{m_u} + 2\bar{c}_m) \frac{\Delta V}{V_\infty} + q_\infty c^2 \bar{c}_m \frac{\rho d\rho}{hdh} \frac{\Delta h}{h} \quad (98)$$

The generalized aerodynamic force due to the aircraft states is computed

$$\mathbf{F}_a = \mathbf{F}_r \mathbf{x}_r + \mathbf{G}_r \dot{\mathbf{x}}_r \quad (99)$$

where $\mathbf{x}_r = \begin{bmatrix} \mathbf{x}_{lon}^\top & \mathbf{x}_{lat}^\top \end{bmatrix}^\top$ is the aircraft state vector, $\mathbf{F}_r = \begin{bmatrix} \mathbf{F}_q & \mathbf{F}_\alpha & \mathbf{F}_V & \mathbf{F}_h & \mathbf{0} & \mathbf{F}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, and $\mathbf{G}_r = \begin{bmatrix} \mathbf{G}_{\dot{q}} & \mathbf{G}_{\dot{\alpha}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\dot{p}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

These generalized forces are computed as

$$\mathbf{F}_q = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \boldsymbol{\Phi}_z^\top c_{l_\alpha} G_\theta(\bar{s}) \left[C(\bar{s}) \frac{2\bar{e}_c}{c} + \frac{1}{2} \right] \frac{\bar{c}}{2V_\infty} c dy + q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \boldsymbol{\Psi}_y^\top c_{l_\alpha} \left\{ G_\theta(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} \frac{2\bar{e}_c}{c} - H_\theta(\bar{s}) \frac{\bar{e}_c}{2c} \right\} \frac{\bar{c}}{2V_\infty} c^2 dy \quad (100)$$

$$\mathbf{F}_\alpha = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Phi}_z^\top c_{l_\alpha} G_\alpha(\bar{s}) C(\bar{s}) c dy + q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Psi}_y^\top c_{l_\alpha} G_\alpha(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} c^2 dy \quad (101)$$

$$\mathbf{F}_V = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Phi}_z^\top (c_{l_u} + 2\bar{c}_l) c dy + q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Psi}_y^\top (c_{m_u} + 2\bar{c}_m) c^2 dy \quad (102)$$

$$\mathbf{F}_h = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Phi}_z^\top \bar{c}_l \frac{\rho d\rho}{h dh} c dy + \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Psi}_y^\top q_\infty \bar{c}_m \frac{\rho d\rho}{h dh} c^2 dy \quad (103)$$

$$\mathbf{F}_p = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Phi}_z^\top q_\infty c_{l_\alpha} G_\alpha(\bar{s}) C(\bar{s}) \frac{2y}{b} \frac{b}{2V_\infty} c dy + q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Psi}_y^\top c_{l_\alpha} G_\alpha(\bar{s}) C(\bar{s}) \frac{\bar{e}}{c} \frac{2y}{b} \frac{b}{2V_\infty} c^2 dy \quad (104)$$

$$\mathbf{G}_q = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Phi}_z^\top c_{l_\alpha} G_\theta(\bar{s}) \frac{\bar{e}_m}{c} \frac{\bar{c}^2}{4V_\infty^2} c dy - q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Psi}_y^\top c_{l_\alpha} H_\theta(\bar{s}) \left(\frac{1}{32} + \frac{\bar{e}_m^2}{c^2} \right) \frac{\bar{c}^2}{4V_\infty^2} c^2 dy \quad (105)$$

$$\mathbf{G}_{\dot{\alpha}} = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Phi}_z^\top c_{l_\alpha} G_\alpha(\bar{s}) \frac{1}{2} \frac{\bar{c}}{2V_\infty} c dy - q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Psi}_y^\top c_{l_\alpha} H_\alpha(\bar{s}) \frac{\bar{e}_m}{2c} \frac{\bar{c}}{2V_\infty} c^2 dy \quad (106)$$

$$\mathbf{G}_{\dot{p}} = q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Phi}_z^\top c_{l_\alpha} G_\alpha(\bar{s}) \frac{\bar{c}y}{b^2} \frac{b^2}{4V_\infty^2} c dy - q_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{\Psi}_y^\top c_{l_\alpha} H_\alpha(\bar{s}) \frac{\bar{e}_m y}{b^2} \frac{b^2}{4V_\infty^2} c^2 dy \quad (107)$$

C. Aerodynamic Coefficients Due to Generalized Displacements

The generalized displacements create a lift force and a pitching moment on the wing. The aerodynamic stability derivatives due to the generalized displacements are computed as

$$C_{L_q} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} l_e dy \right)}{q_\infty S \partial \mathbf{q}} = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} G_\theta(\bar{s}) C(\bar{s}) \mathbf{\Psi}_y c dy \quad (108)$$

$$C_{L_{\dot{q}}} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} l_e dy \right)}{q_\infty S \partial \left(\frac{\bar{c}\dot{\mathbf{q}}}{2V_\infty} \right)} = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} \left\{ G_\theta(\bar{s}) \left[C(\bar{s}) \frac{2e_c}{c} + \frac{1}{2} \right] \mathbf{\Psi}_y - G_\alpha(\bar{s}) C(\bar{s}) \frac{2\mathbf{\Phi}_z}{c} \right\} c dy \quad (109)$$

$$C_{L_{\ddot{q}}} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} l_e dy \right)}{q_\infty S \partial \left(\frac{\bar{c}^2 \ddot{\mathbf{q}}}{4V_\infty^2} \right)} = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} \left[G_\theta(\bar{s}) \frac{e_m}{c} \mathbf{\Psi}_y - G_\alpha(\bar{s}) \frac{\mathbf{\Phi}_z}{c} \right] c dy \quad (110)$$

$$C_{m_q} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} m_e dy \right)}{q_\infty S \bar{c} \partial \mathbf{q}} = \frac{1}{S \bar{c}} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} G_\theta(\bar{s}) C(\bar{s}) \frac{e}{c} \mathbf{\Psi}_y c^2 dy \quad (111)$$

$$C_{m_{\dot{q}}} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} m_e dy \right)}{q_\infty S \bar{c} \partial \left(\frac{\bar{c}\dot{\mathbf{q}}}{2V_\infty} \right)} = \frac{1}{S \bar{c}} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} \left\{ \left[G_\theta(\bar{s}) C(\bar{s}) \frac{e}{c} \frac{2e_c}{c} - H_\theta(\bar{s}) \frac{e_c}{2c} \right] \mathbf{\Psi}_y - G_\alpha(\bar{s}) C(\bar{s}) \frac{e}{c} \frac{2\mathbf{\Phi}_z}{c} \right\} c^2 dy \quad (112)$$

$$C_{m_{\ddot{q}}} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} m_e dy \right)}{q_\infty S \bar{c} \partial \left(\frac{\bar{c}^2 \ddot{\mathbf{q}}}{4V_\infty^2} \right)} = \frac{1}{S \bar{c}} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} \left[-H_\theta(\bar{s}) \left(\frac{1}{32} + \frac{e_m^2}{c^2} \right) \mathbf{\Psi}_y + H_\alpha(\bar{s}) \frac{e_m}{c} \frac{\mathbf{\Phi}_z}{c} \right] c^2 dy \quad (113)$$

$$C_{l_q} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} l_e y dy \right)}{q_\infty S b \partial \mathbf{q}} = \frac{1}{S b} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} G_\theta(\bar{s}) C(\bar{s}) \mathbf{\Psi}_y c y dy \quad (114)$$

$$C_{l_q} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} l_e y dy \right)}{q_\infty S b \partial \left(\frac{b \dot{q}}{2V_\infty} \right)} = \frac{1}{Sb} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} \left\{ G_\theta(\bar{s}) \left[C(\bar{s}) \frac{2e_c}{c} + \frac{1}{2} \right] \Psi_y - G_\alpha(\bar{s}) C(\bar{s}) \frac{2\Phi_z}{c} \right\} \frac{\bar{c}}{b} c y dy \quad (115)$$

$$C_{l_q} = \frac{\partial \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} l_e y dy \right)}{q_\infty S b \partial \left(\frac{b^2 \ddot{q}}{4V_\infty^2} \right)} = \frac{1}{Sb} \int_{-\frac{b}{2}}^{\frac{b}{2}} c_{l_\alpha} \left[G_\theta(\bar{s}) \frac{e_m}{c} \Psi_y - G_\alpha(\bar{s}) \frac{\Phi_z}{c} \right] \frac{\bar{c}^2}{b^2} c y dy \quad (116)$$

The coupled aircraft equation of motion is expressed as

$$\mathbf{M}_r s \mathbf{x}_r = \mathbf{S}_r \mathbf{x}_r + \mathbf{P} s^2 \mathbf{q} + \mathbf{Q} s \mathbf{q} + \mathbf{R} \mathbf{q} \quad (117)$$

where

$$\mathbf{P} = \begin{bmatrix} \frac{C_{m_q}(\bar{s}) \bar{c}^2}{4V_\infty^2} \\ \frac{C_{L_q}(\bar{s}) \bar{c}^2}{4V_\infty^2} \\ -\frac{C_{D_q}(\bar{s}) \bar{c}^2}{4V_\infty^2} \\ 0 \\ 0 \\ \frac{C_{l_q}(\bar{s}) b^2}{4V_\infty^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} \frac{C_{m_q}(\bar{s}) \bar{c}}{2V_\infty} \\ \frac{C_{L_q}(\bar{s}) \bar{c}}{2V_\infty} \\ -\frac{C_{D_q}(\bar{s}) \bar{c}}{2V_\infty} \\ 0 \\ 0 \\ \frac{C_{l_q}(\bar{s}) b}{2V_\infty} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} C_{m_q}(\bar{s}) \\ -C_{L_q}(\bar{s}) \\ -C_{D_q}(\bar{s}) \\ 0 \\ 0 \\ C_{l_q}(\bar{s}) \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (118)$$

V. Coupled Dynamic Aeroelastic Flight Dynamics

The coupled dynamic aeroelastic flight dynamic equations of motion of the aircraft are expressed in a frequency-domain form by

$$\begin{bmatrix} \mathbf{M}_r s - \mathbf{S}_r & -\mathbf{P} s^2 - \mathbf{Q} s - \mathbf{R} \\ -\mathbf{G}_r s - \mathbf{F}_r & (\mathbf{M}_s + \mathbf{M}_a) s^2 + (\mathbf{C}_s + \mathbf{C}_a) s + (\mathbf{K}_s + \mathbf{K}_a) \end{bmatrix} \begin{bmatrix} \mathbf{x}_r \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (119)$$

It is noted that all the matrices are frequency-dependent. That is, they are functions of the Laplace variable s . The eigenvalues of the coupled equations of motion are computed. There are a total of 2355 eigenvalues all of which are stable. Table 3 presents the eigenvalues of the dynamic aeroelastic flight dynamic modes of the Mach 0.745 TTBW. For comparison, the eigenvalues of pure flight dynamic modes from Table 1 are shown. The dynamic aeroelasticity causes the damping of the short-period and Dutch-roll modes to decrease while the damping of the Phugoid mode to increase. The roll mode becomes more stable but the spiral mode becomes less stable. The plunge mode also becomes slightly more stable.

Mode	Eigenvalue w/ Dynamic Aeroelasticity	Eigenvalue w/o Dynamic Aeroelasticity
Short-Period	$-0.1539 \pm 0.5453i$	$-0.3789 \pm 0.3817i$
Phugoid	$-0.01447 \pm 0.06150i$	$-0.008331 \pm 0.07692i$
Plunge	-0.008828	-0.008621
Dutch-Roll	$-0.02488 \pm 0.1754i$	$-0.006793 \pm 0.2280i$
Roll	-0.6192	-0.5264
Spiral	-0.001249	-0.001945

Table 3. Eigenvalues of Dynamic Aeroelastic Flight Dynamic Modes of Mach 0.745 TTBW

Figures 20(a) and (b) show the partial eigenvalues of the longitudinal flight dynamic modes coupled to symmetric structural dynamic modes and eigenvalues of the lateral-directional flight dynamic modes coupled to anti-symmetric

structural dynamic modes. The effect of transonic aerodynamics can be seen in the two plots, especially in the plot of the lateral-directional eigenvalues, as the eigenvalues of the dynamic aeroelastic flight dynamic modes shift to the right, thereby indicating that the damping of the modes generally decreases due to the unsteady transonic aerodynamic effect accounted in the model.

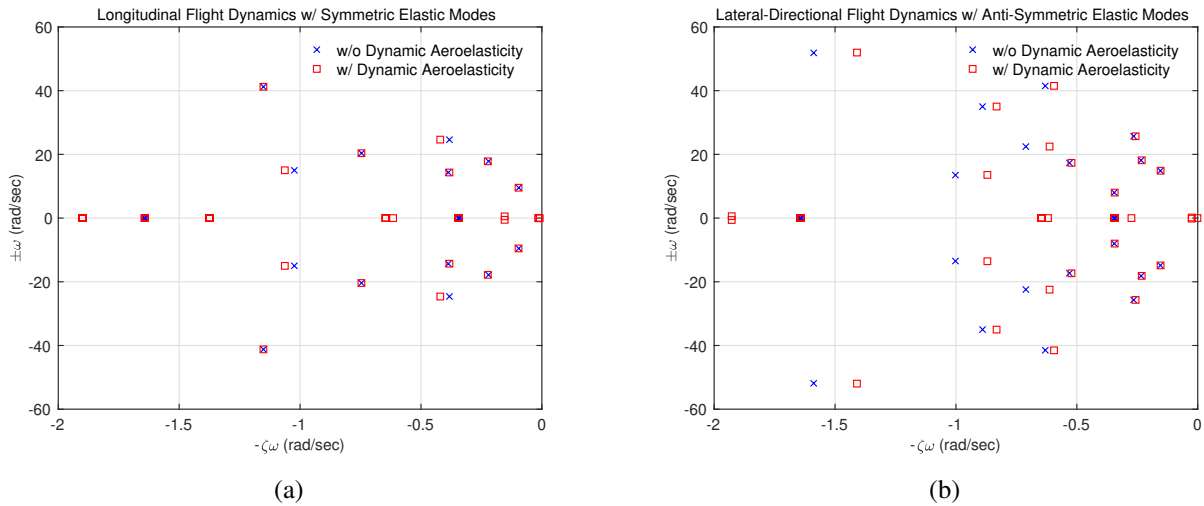


Figure 20. Eigenvalues of Longitudinal and Lateral-Directional Flight Dynamic Modes with Elastic Modes

VI. Conclusion

This paper presents a modeling approach for dynamic aeroelastic flight dynamic analysis of the Mach 0.745 Transonic Truss-Braced Wing. The modeling approach is based on a transonic correction method to correct the Theodorsen's theory for transonic flow. CFD unsteady Reynolds-averaged Navier-Stokes equations (RANS) simulations are conducted using FUN3D solver for a series of wing sections from the Mach 0.745 Transonic Truss-Braced Wing in pitch and plunge oscillations. Unsteady lift and pitching moment coefficients are obtained and used to develop the correction terms in the Theodorsen's theory to account for transonic aerodynamics. The unsteady lift and pitching moment derivatives with respect to the unsteady angle of attack are obtained as functions of the reduced frequency. These derivatives are used to compute the unsteady lift and pitching moment contributions by the angle of attack, pitch rate, and roll rate. They are then approximated using a frequency-domain regression to obtain the dynamic stability derivatives for the Mach 0.745 Transonic Truss-Braced Wing. The structural dynamic mode shapes of the Mach 0.745 TTBW are extracted from a NASTRAN finite-element model. These mode shapes are used to compute the generalized unsteady aerodynamic forces. The aerodynamic mass, damping, and stiffness and the aerodynamic lag states are constructed to couple the dynamic aeroelastic contribution to the flight dynamic model of the Mach 0.745 Transonic Truss-Braced Wing. The coupled dynamic aeroelastic flight dynamic equations of motion are formulated. The eigenvalues of the coupled system are computed. All the flight dynamic modes and structural dynamic modes are stable at Match 0.745. The effect of transonic aerodynamics generally causes all the dynamic modes to have lower damping values.

Acknowledgment

The authors wish to acknowledge NASA Advanced Air Transport Technology project for the funding support of this work. The authors also acknowledge Boeing Research and Technology and in particular Christopher Droney, Neal Harrison, Michael Beyar, Eric Dickey, and Anthony Sclafani, along with the NASA technical POC, Gregory Gatlin, for their research conducted under the NASA BAART contracts NNL10AA05B and NNL16AA04B. The research published in this paper is made possible by the technical data and wind tunnel test data furnished under these BAART contracts.

References

- ¹Bradley, M. K. and Droney, C. K., “Subsonic Ultra Green Aircraft Research: Phase I Final Report,” NASA Contractor Report NASA/CR-2011-216847, Boeing Research and Technology, April 2011.
- ²Bradley, M. K., Droney, C. K., and Allen, T. J., “Subsonic Ultra Green Aircraft Research Phase II: N+4 Advanced Concept Development,” NASA Contractor Report NASA/CR-2012-217556, Boeing Research and Technology, May 2012.
- ³Theodorsen, T., “General Theory of Aerodynamic Instability and the mechanism of Flutter”, NACA Report No. 496, 1949.
- ⁴Nguyen, N., Fugate, J., Kaul, U., and Xiong, J., “Flutter Analysis of the Transonic Truss-Braced Wing Aircraft Using Transonic Correction,” AIAA Structural Dynamics Conference, AIAA-2019-0217, January 2019.
- ⁵Kaul, U. and Nguyen, N., “RANS Simulations of a Pitching and Plunging VCCTEF Airfoil Toward a Transonic Flutter Model,” AIAA Structural Dynamics Conference, AIAA-2019-2037, January 2019.
- ⁶Kaul, U. and Nguyen, N., “Extending A Correction Method for Unsteady Transonic Aerodynamics as Applied to Variable Camber Continuous Trailing Edge Flap,” AIAA Applied Aerodynamic Conference, AIAA-2019-3157, June 2019.
- ⁷Nguyen, N. and Xiong, J., “Unsteady Transonic Corrections to Theodorsen’s Theory Toward Transonic Flutter,” AIAA Structural Dynamics Conference, January 2021.
- ⁸Chaparro, D., Fujiwara, G. E., Ting, E., and Nguyen, N. T., “Transonic and Viscous Potential Flow Method Applied to Flexible Wing Transport Aircraft,” 35th AIAA Applied Aerodynamics Conference, AIAA-2017-4221, June 2017.
- ⁹Stahara, S. S., “Operational Manual for Two-Dimensional Transonic Code TSFOIL,” NASA Contractor Report 3064, December, 1978.
- ¹⁰Fujiwara, G., Chaparro, D., and Nguyen, N., “An Integral Boundary Layer Direct Method Applied to 2D Transonic Small-Disturbance Equations,” 34th AIAA Applied Aerodynamics Conference, AIAA-2016-3160, June 2016.