Non-Hermitian gyroscopes: Can exceptional points increase sensor precision?

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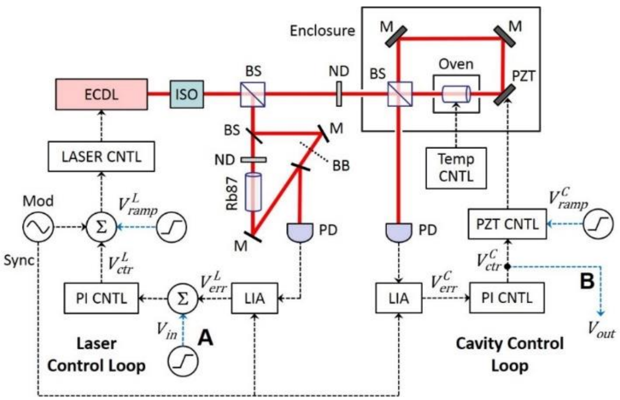
Outline

- Scale-factor sensitivity enhancement, *S*
  - Passive and active fast light (FL) gyros
  - Exceptional points (EPs) in coupled resonators (CRs)
- Excess noise
  - Linear theory – Petermann factor, *K*
  - Nonlinear approaches
- Practical limitations

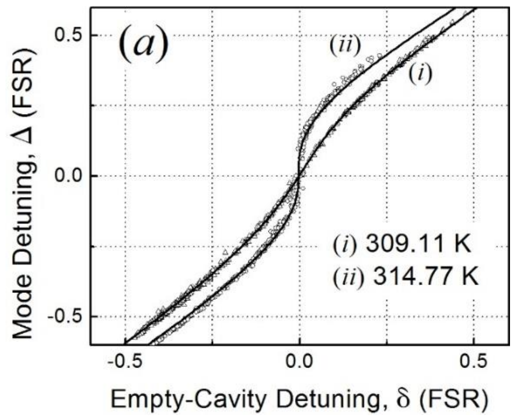
EP / FL Sensitivity Enhancement

Passive FL Cavity

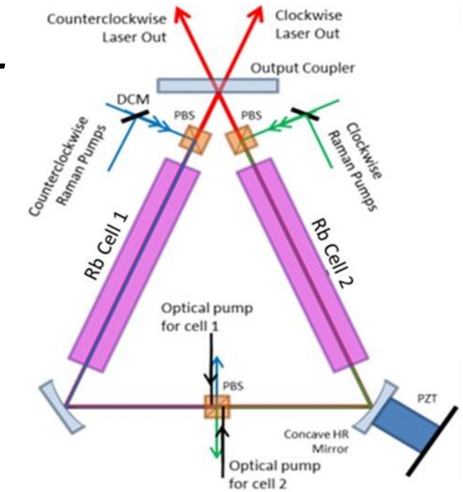


D. Smith et al., *Opt. Expr.* **26**, 14905, (2018).

$$S \sim 1/n_g$$

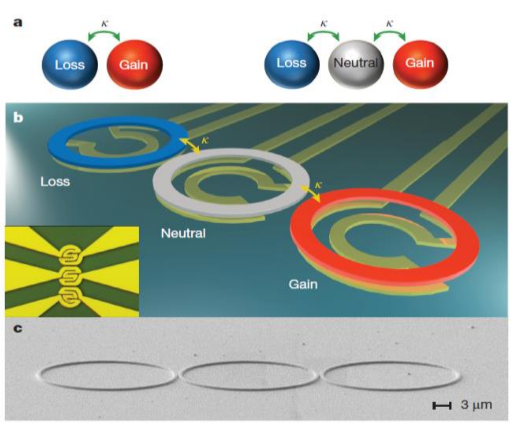


Active FL Gyro



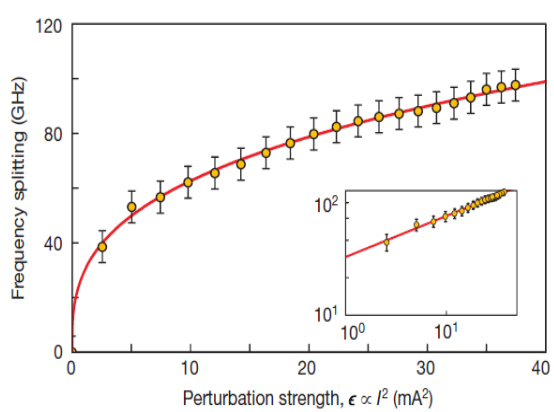
Z. Zhou et al., *Opt. Expr.* **27**, 29738, (2019).

PT-Symmetric

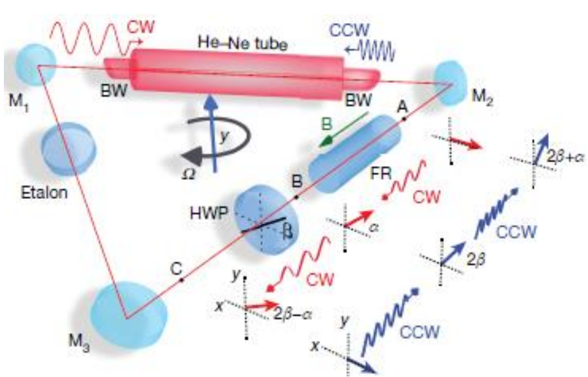


H. Hodaei et al., *Nature* **548**, 187 (2017).

$$S \sim 1/\sqrt{\delta}$$



Non-Hermitian He-Ne

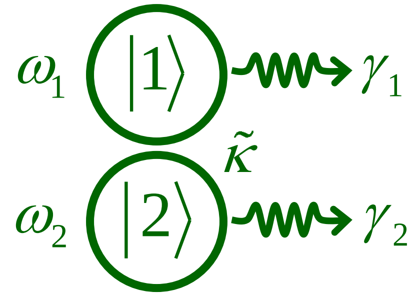


Hokmabadi et al., Nature **576**, 70 (2019).

Coupled Resonators = Two Level Atom

$$\left|\psi(t)\right\rangle = E_1(t)\left|1\right\rangle + E_2(t)\left|2\right\rangle$$

Coherent Superposition



$$\delta = \omega_1 - \omega_2$$

$$\gamma = \gamma_1 - \gamma_2$$

Non-Hermitian Hamiltonian:

$$i\hbar\left|\dot{\psi}(t)\right\rangle = -\frac{\hbar}{2}\begin{pmatrix} -2\tilde{\omega}_1 & \tilde{\kappa} \\ \tilde{\kappa} & -2\tilde{\omega}_2 \end{pmatrix}\left|\psi(t)\right\rangle$$

$$= \tilde{H}\left|\psi(t)\right\rangle$$

$$\tilde{\omega}_{1,2} = \omega_{1,2} - i\,\gamma_{1,2}/2$$

$$\tilde{\delta} = \tilde{\omega}_1 - \tilde{\omega}_2 = \delta - i\,\frac{\gamma}{2}$$

$$\tilde{\kappa} = \kappa' + i\kappa''$$

conservative *dissipative*

Complex Eigenvalues:

$$\boxed{\tilde{\omega}_{\pm} = \tilde{\omega}_{avg} \pm \frac{\tilde{\Omega}}{2}}$$

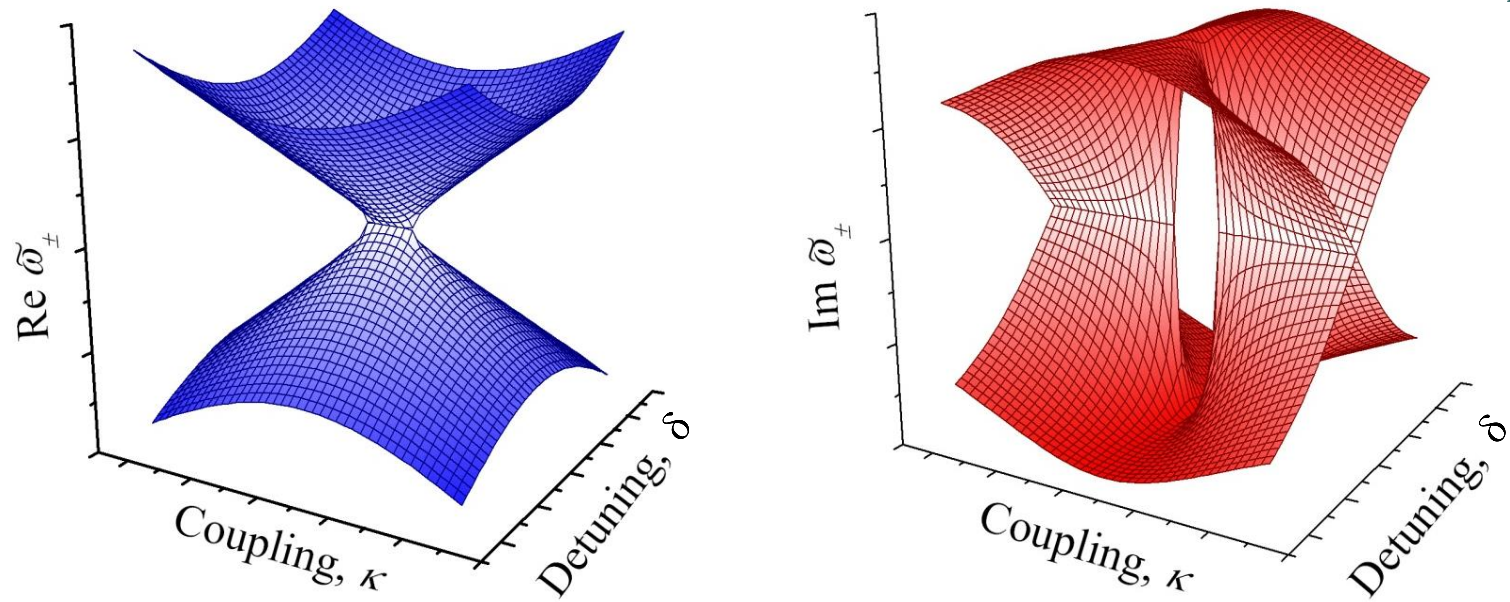
$$\tilde{\Omega} = \left[\tilde{\delta}^2 + \tilde{\kappa}^2\right]^{1/2}$$

Generalized Rabi Freq.

Complex Eigenvalues

$$\tilde{\omega}_{\pm} = \tilde{\omega}_{avg} \pm \frac{\tilde{\Omega}}{2} = \omega_{\pm} - i\,\gamma_{\pm}/2$$

Freq. Width



Sub-exceptional

$\kappa < |\gamma / 2|$

Frequency Crossing
& Width Anti-Crossing

Exceptional Point (EP)

$\kappa = |\gamma / 2|$

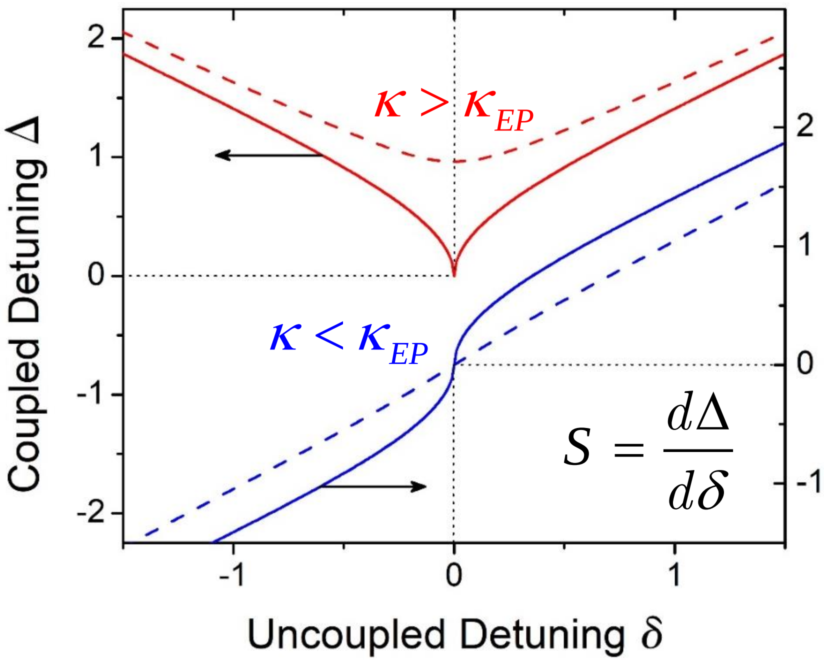
Frequency
& Width Crossing

Super-exceptional

$\kappa > |\gamma / 2|$

Frequency Anti-Crossing
& Width Crossing

Coupled-Resonator (CR) Gyros



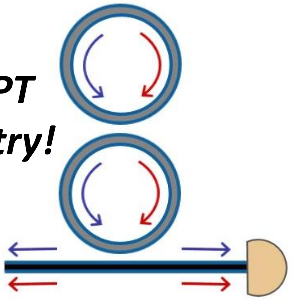
Beat Freq: $\Delta = \omega_+ - \omega_- = \Omega'$

at EP: $\Delta \sim \sqrt{\delta} \;\; \Rightarrow \boxed{S(0) \to \infty}$

Eigenvalues: $\tilde{\omega}_\pm = \omega_{avg} - i\frac{\gamma_{avg}}{2} \pm \frac{\tilde{\Omega}}{2}$

Case I: $\tilde{\Omega}$ *real* ($\kappa \geq \kappa_{EP}$ & $\delta = 0$)

Exact PT Symmetry!

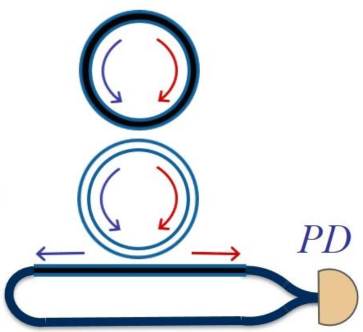


$\gamma_{avg} = 0$

Usual lasing condition

Both modes lase

Case II: $\tilde{\Omega}$ *complex* ($\kappa < \kappa_{EP}$ *or* $\delta \neq 0$)



$-\gamma_{avg} < 0$

Lasing w/o Gain (LWG)

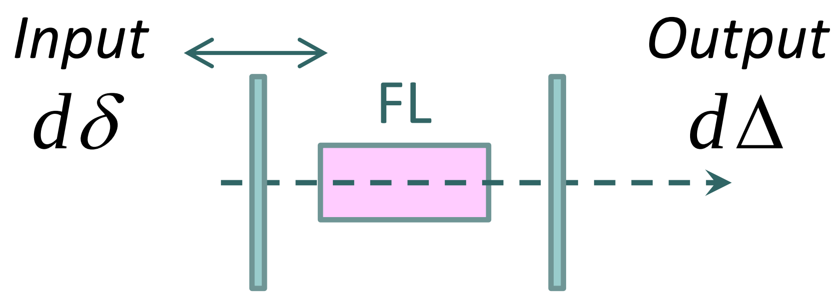
One mode lases

Nonzero detunings result in LWG and only one mode lasing!

Enhancement in Precision?

Enhancement in precision:

$$\zeta = \frac{d\Delta / d\delta}{\sigma_\Delta / \sigma_\delta} = \frac{S}{\varepsilon}$$



Uncertainty for a laser:

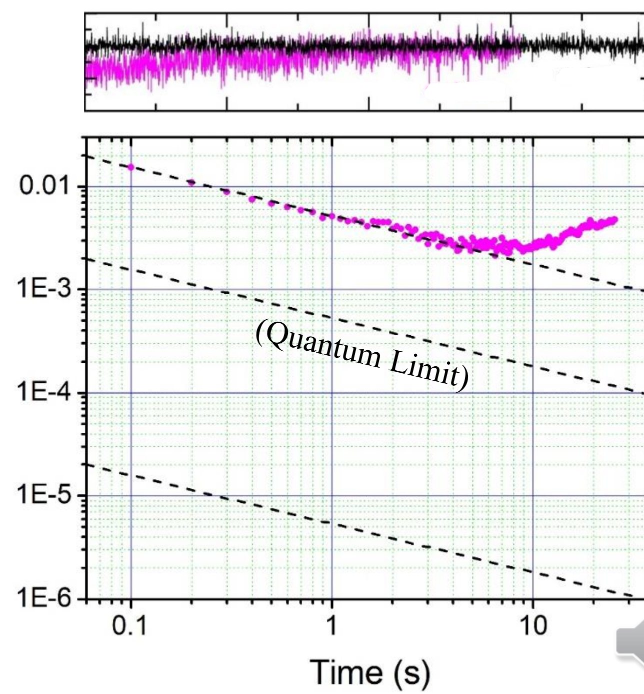
$$\sigma_\nu(\tau) = \sqrt{\frac{\Delta\nu}{2\pi\tau}}$$

Allan deviation, white noise (short τ) limit

In quantum limit:

$$\zeta = \frac{S}{\sqrt{\Delta\nu_{ST} / \Delta\nu_{ST}^{e}}}$$

Schawlow-Townes



Excess Noise (Petermann Factor)

Eigenmodes: $$|e_\pm\rangle = N_\pm \begin{pmatrix} \tilde{\kappa} \\ \tilde{\delta} \mp \tilde{\Omega} \end{pmatrix}$$ $\langle e_+ | e_- \rangle \neq 0$ *Not orthogonal*

- noise sources correlated
- Einstein A $\neq$ B
- 1 photon / mode $\rightarrow$ *K* photons / mode

$$\boxed{K = \frac{1}{1 - \left| \langle e_+ | e_- \rangle \right|^2}}$$

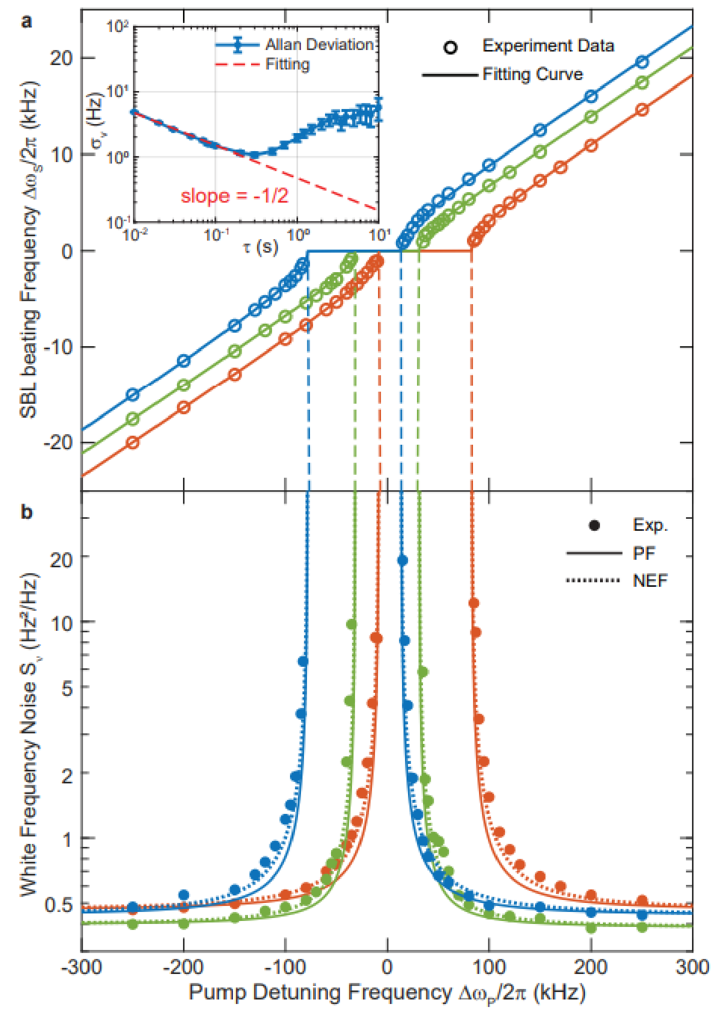
Petermann Excess-Noise (EN) Factor

ST linewidth increases by $K \quad \Rightarrow \quad \zeta = \dfrac{S}{\sqrt{K}}$

Data for Dissipative Coupling EP

- Measured S and K near EP at deadband edge of SBS laser gyroscope.
- Petermann factor worked pretty well, even though it's linear
- Observed **no increase** in $S / K^{1/2}$

Will this be true near other types of EPs, i.e., for conservative coupling?



Wang et al., Nat. Comm **11**, 1610 (2020).

Linear theory: general case (any EP)

For conservative coupling ($\kappa'' = 0$) : $\dfrac{|S|}{K} = \left|\dfrac{\delta}{\Omega'}\right|$

For partially dissipative coupling:

$$\Rightarrow \frac{|S|}{K} = \left|\frac{\delta_R}{\Omega'_R}\right|$$

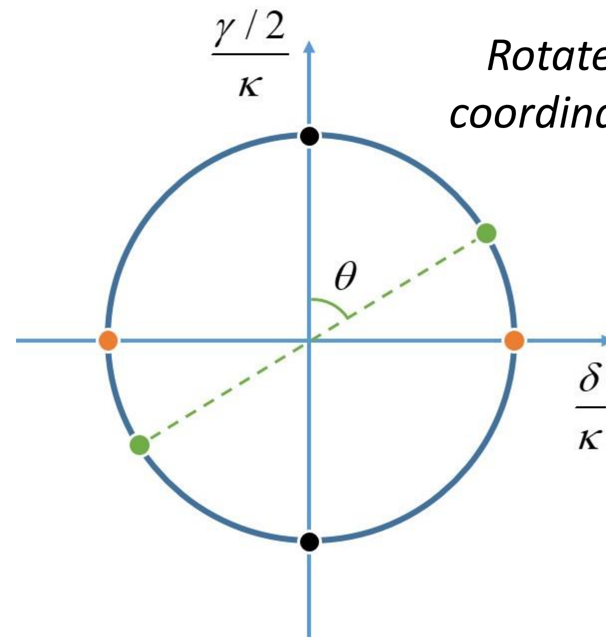
General formula for S in terms of K

where $\Omega'_R = \mathrm{Re}\left\{\sqrt{(\delta_R - i\gamma_R/2)^2 + \kappa^2}\right\}$

$$\begin{pmatrix} \delta_R \\ \gamma_R/2 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} \delta \\ \gamma/2 \end{pmatrix}$$

$$\theta = \mathrm{atan}(\kappa''/\kappa')$$

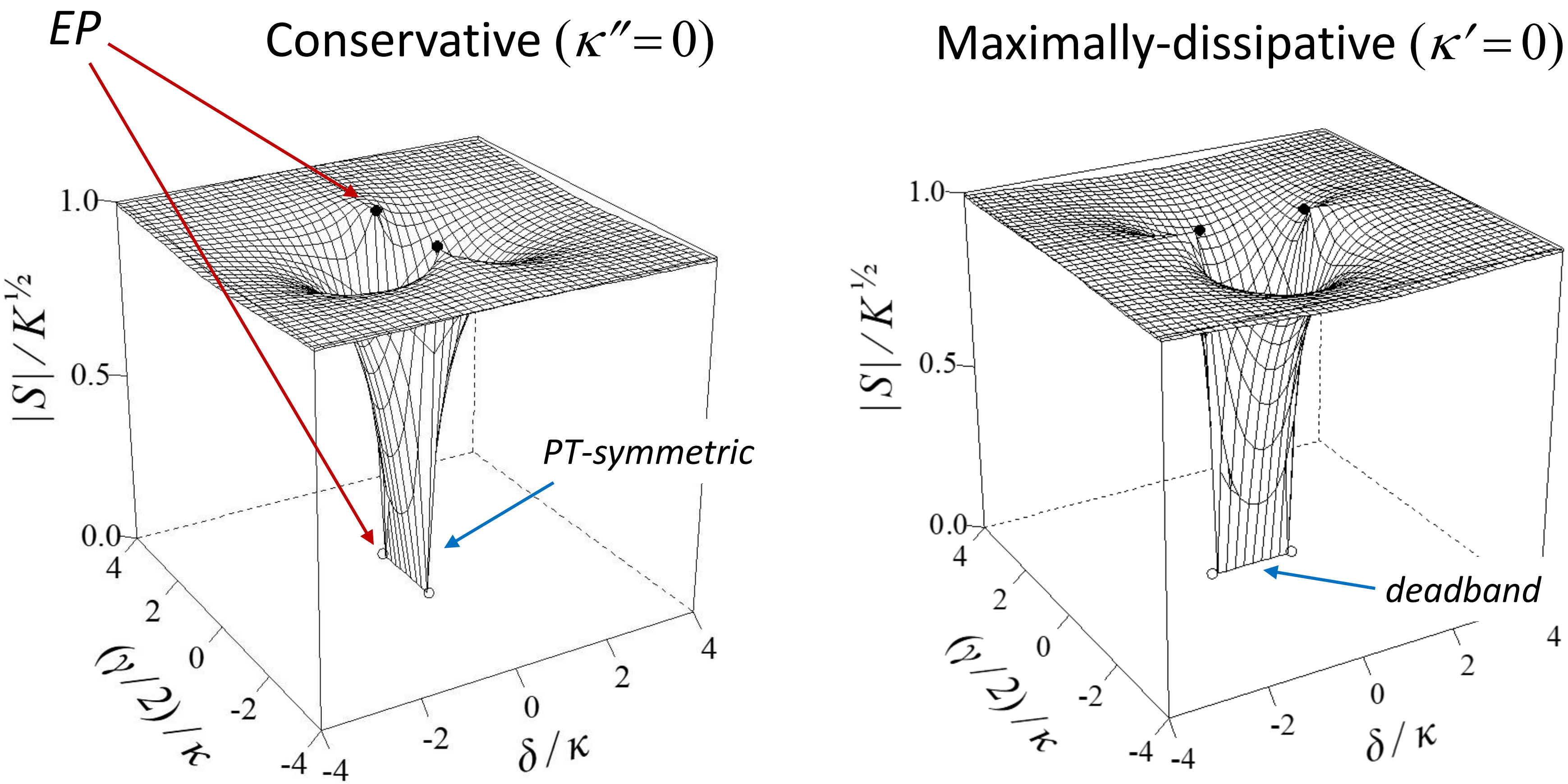
Rotated coordinates



It can be shown that $|\Omega'| \geq |S\delta| \Rightarrow$ $\boxed{\dfrac{|S|}{K^{1/2}} < 1}$

For ANY set of parameters!

Hole of reduced precision



$\Rightarrow$ *EPs are discontinuous transitions to PT-symmetry and deadband regions of <u>zero precision</u>!*

Quasi-linear theory: gain saturation

Previously, assumed independent variables in $\tilde{\Omega}(\delta,\gamma,\kappa)$.

But … $\gamma(\delta)$ $\Rightarrow$ $S = d\Omega' / d\delta$ is modified!

$$S = \frac{d\Omega'}{d\delta} = \frac{\partial\Omega'}{\partial\delta} + \frac{\partial\Omega'}{\partial\gamma}\frac{\partial\gamma}{\partial\delta}$$

extra term

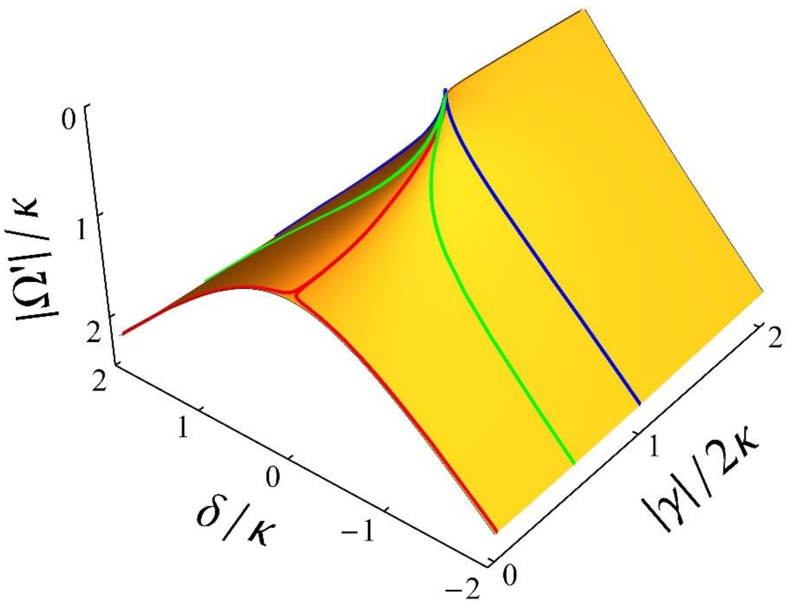
$$\Rightarrow \frac{|S|}{K} = \left|\frac{\delta_R}{\Omega'_R}\right| + \left|\frac{\delta}{\Omega'}\right| \frac{(\Omega')^2 - \delta^2 + 2\kappa'\kappa''\delta/\gamma}{(\Omega')^2 + (\gamma/2)^2 + (\kappa'')^2}\left|\frac{\gamma}{2\delta}\right| \left|\frac{\partial_\delta\gamma}{2}\right| \xrightarrow[EP]{} \frac{1}{|2S|}\left(1 + \left|\frac{\gamma}{2\delta}\right|\psi_{EP}\right)$$

where $\psi_{EP} = \left|\partial_\delta\gamma/2\right|_{EP}$ *saturation imbalance*

$$\boxed{\left.\frac{S^2}{K}\right|_{EP} = \frac{1}{2}\left(1 + \left|\frac{\kappa'}{\kappa''}\right|\psi_{EP}\right)}$$

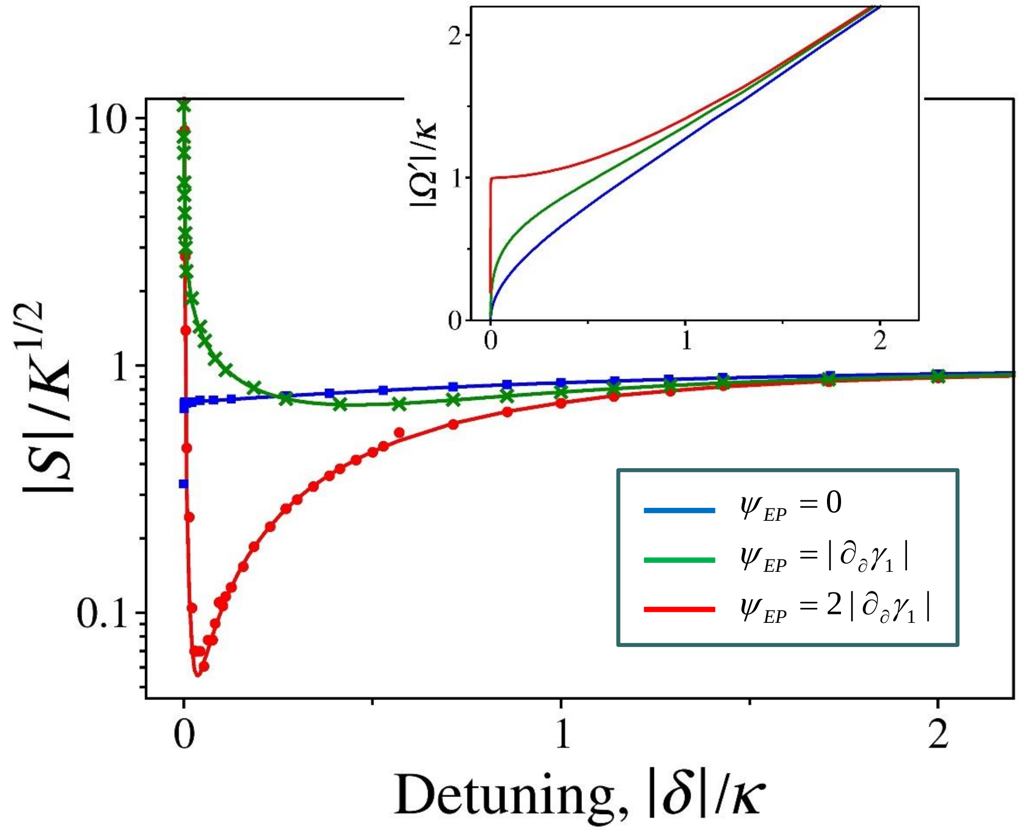
So … can $\zeta > 1$?

Effect of Gain Saturation



Solution now restricted
(to threshold)

Solve numerically and compare with analytical solutions:



$\Rightarrow$ *Larger ψ_{EP} results in higher ζ, but over narrower bandwidth*

Nonlinear theory

Stochastic (with Langevin terms) nonlinear coupled eqns:

$$\frac{d}{dt}\left|e\right\rangle = M\left|e\right\rangle + \left|f\right\rangle \qquad \left\langle f_i(t)^* f_j(t')\right\rangle = R_i^{sp}\delta_{ij}\delta(t-t') \qquad \{i=1,2\}$$

$$M = \frac{i}{2}\begin{bmatrix} -\delta + i\gamma_1(I_1) & \tilde{\kappa} \\ \tilde{\kappa} & \delta + i\gamma_2(I_2) \end{bmatrix}$$

saturable gain / loss

$$\gamma_i(I_i) \approx \hat{\gamma}_i^u - \hat{\beta}_i I_i \;\; (i=1,2)$$

Transform to new basis: $e_i = \mathcal{E}_i \exp(i\phi_i)$

$$2\theta = \phi_1 - \phi_2$$

$$2\phi = \phi_1 + \phi_2$$

$$\tan\chi = (\mathcal{E}_1 - \mathcal{E}_2)/(\mathcal{E}_1 + \mathcal{E}_2)$$

$$I = \mathcal{E}_1^2 + \mathcal{E}_2^2$$

$$\Rightarrow$$

$$\dot{\theta} = (\delta/2) - (\kappa'/2)\tan 2\chi \cos 2\theta + f_\theta$$

$$\dot{\phi} = (\kappa'/2)\cos 2\theta / \cos 2\chi + f_\phi$$

$$\dot{\chi} = (\kappa'/2)\sin 2\theta - (\gamma/4)\cos 2\chi + f_\chi$$

$$\dot{I}/2I = -(\gamma_{avg}/2) - (\gamma/4)\sin 2\chi + f_I$$

Nonlinear Theory: Linearization

Steady state solns $\dot{\chi} = \dot{I} = \dot{\theta} = 0$

$$(\kappa' \sin 2\theta_0)^2 = (\gamma_0 / 2)^2 - (\gamma_{avg,0})^2 = -\gamma_{1,0}\,\gamma_{2,0}$$
$$\gamma_{1,0} I_{1,0} = -\gamma_{2,0} I_{2,0}$$

threshold condition

only exist when : $|\kappa' \sin 2\theta| < |\gamma_0 / 2|$ (single mode LWG Regime)

Linearize around steady state: $\theta = \theta_0 + \Delta\theta,\ \ \chi = \chi_0 + \Delta\chi,\ \ I = I_0(1 + 2\Delta I)$

$$\Delta\dot{\theta} = \gamma_{avg,0}\,\Delta\theta + p\Delta\chi + f_\theta$$
$$\dot{\phi} = \gamma_0\,\Delta\theta / 2 + q\Delta\chi + f_\phi$$

1-way coupled

$$\Delta\dot{\chi} = A\Delta\chi + B\Delta I + f_\chi$$
$$\Delta\dot{I} = C\Delta\chi + D\Delta I + f_I$$

mutually coupled

where $p, q \propto \delta$

Noise Power Spectra

Take F.T. to obtain power spectra of fluctuations for each variable:

$$\left\langle \left| \Delta\theta(\omega) \right| \right\rangle^2 = \frac{1}{\omega^2 + \gamma_{avg,0}^2} \left[\mathcal{D}_\theta + p^2 \left\langle \left| \Delta\chi(\omega) \right| \right\rangle^2 \right]$$

$$\left\langle \left| \phi(\omega) \right| \right\rangle^2 = \frac{1}{\omega^2} \left[\frac{(\gamma_0 / 2)^2}{\omega^2 + \gamma_{avg,0}^2} \left(\mathcal{D}_\theta + p^2 \left\langle \left| \Delta\chi(\omega) \right| \right\rangle^2 \right) + \mathcal{D}_\phi + q^2 \left\langle \left| \Delta\chi(\omega) \right| \right\rangle^2 \right]$$

$$\left\langle \left| \Delta\chi(\omega) \right| \right\rangle^2 = \frac{(D^2 + \omega^2)\mathcal{D}_\chi + B^2 \mathcal{D}_I}{(AD - BC - \omega^2)^2 + (A + D)^2 \omega^2}$$

$$\left\langle \left| \Delta I(\omega) \right| \right\rangle^2 = \frac{(A^2 + \omega^2)\mathcal{D}_I + C^2 \mathcal{D}_\chi}{(AD - BC - \omega^2)^2 + (A + D)^2 \omega^2}.$$

Excess Noise Factor - Phase

Normalize by result at $\kappa' = 0$: $$K_\theta(\omega) = \frac{\left\langle \left| \Delta\theta(\omega) \right| \right\rangle^2}{\left\langle \left| \Delta\theta(\omega) \right| \right\rangle^2 \Big|_{\kappa'=0}}$$ *Phase EN factor*

At $\delta = 0$: $$K_\theta(\omega) = \frac{\mathcal{D}_\theta}{\mathcal{D}_\theta\big|_{\kappa'=0}} \left[\frac{\omega^2 + (\gamma_0/2)^2_{\kappa'=0}}{\omega^2 + \gamma^2_{avg,0}} \right] = \eta_\theta K_\omega$$

frequency dependent

noise amplification factor

At $\omega = 0$: $$K_\theta = \eta_\theta \left(\frac{(\gamma_0/2)^2_{\kappa'=0}}{(\gamma_0/2)^2} \right) \left(\frac{(\gamma_0/2)^2}{\gamma^2_{avg,0}} \right) = (\eta_\theta \Gamma) K$$

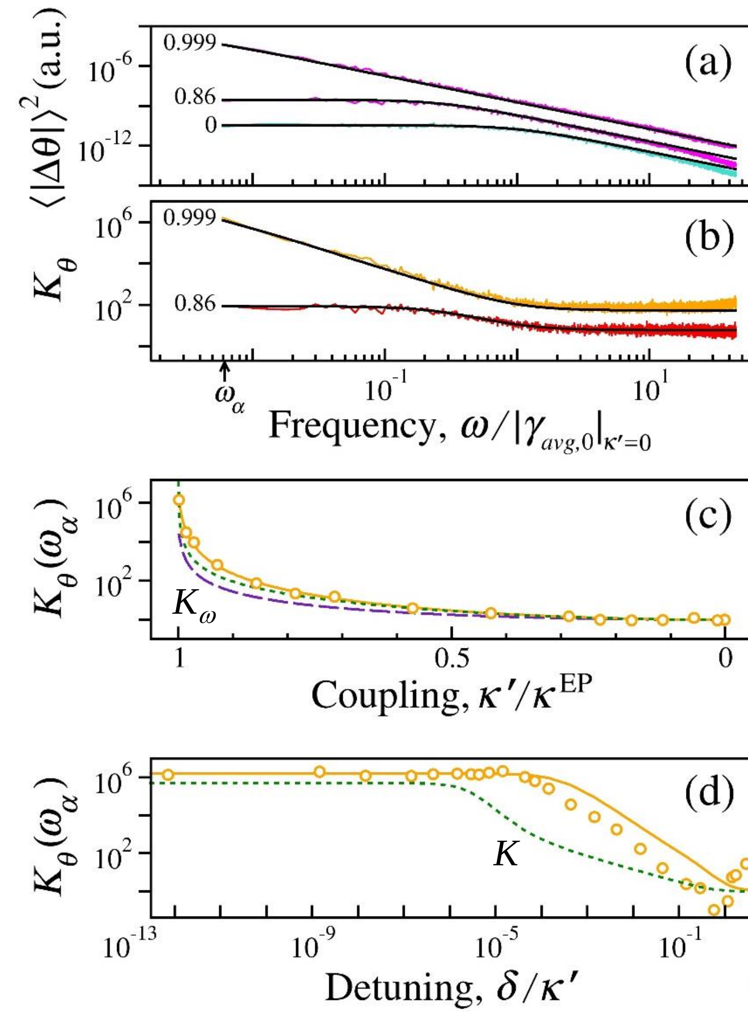
Quasilinear result (Petermann factor with saturated coeffs)

- Deviates from K due to different threshold levels and noise coloring.
- At $\delta = 0$, $\omega = 0$, & strong pumping: $K_\theta^{EP} \approx K$

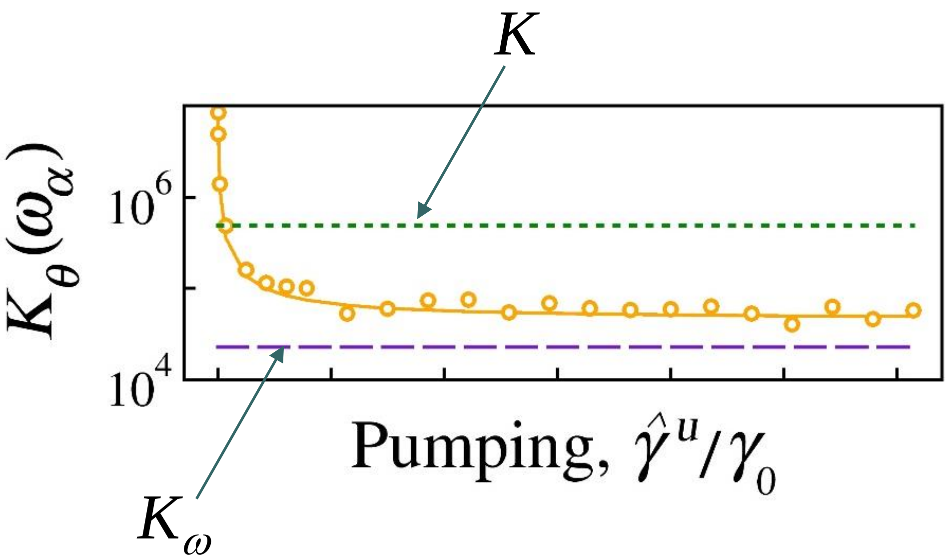
Numerical Noise Analysis

- Data matches nonlinear model
- EN is colored and diverges at EP when $\omega = 0$.
- EN is different than QL prediction by factors $\eta_\theta \Gamma$ and due to coloring (data is at nonzero frequency).
- Data taken just above threshold.

Phase



Versus Pumping



strong pumping limit

- K_θ below K

  $(K_\theta^{EP} \to K/2$ at $\omega = 0)$

<u>Explanation</u>: gain saturation suppresses amplitude noise, but no effect on phase noise due to lack of amp-phase coupling.

$\Rightarrow$ different from dissipative coupling case

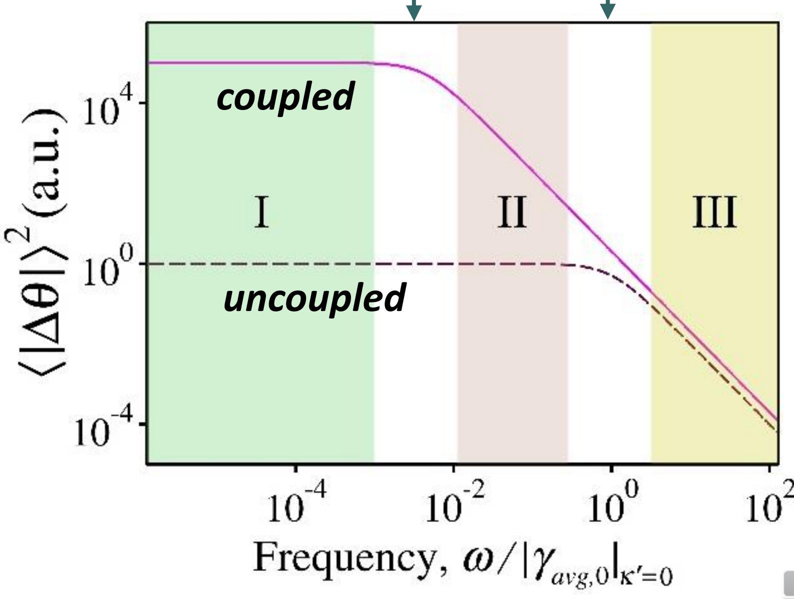
Precision for Colored Noise

Metric $\zeta = S / K^{1/2}$ assumes freq. noise in coupled and uncoupled systems is white, but noise is colored! ✓ still valid provided K_θ is used.

At $\delta = 0$: $\langle |\Delta\theta(\omega)| \rangle^2 = \dfrac{\mathcal{D}_\theta}{\omega^2 + \gamma_{avg,0}^2}$ ***Gauss-Markov noise***

different decay rates

$|\gamma_{avg,0}|$ $|\gamma_{avg,0}|_{\kappa'=0}$



- Region I vanishes at EP

  $\Rightarrow$ ***System operates in Region II***

- K_θ decreases with increasing ω

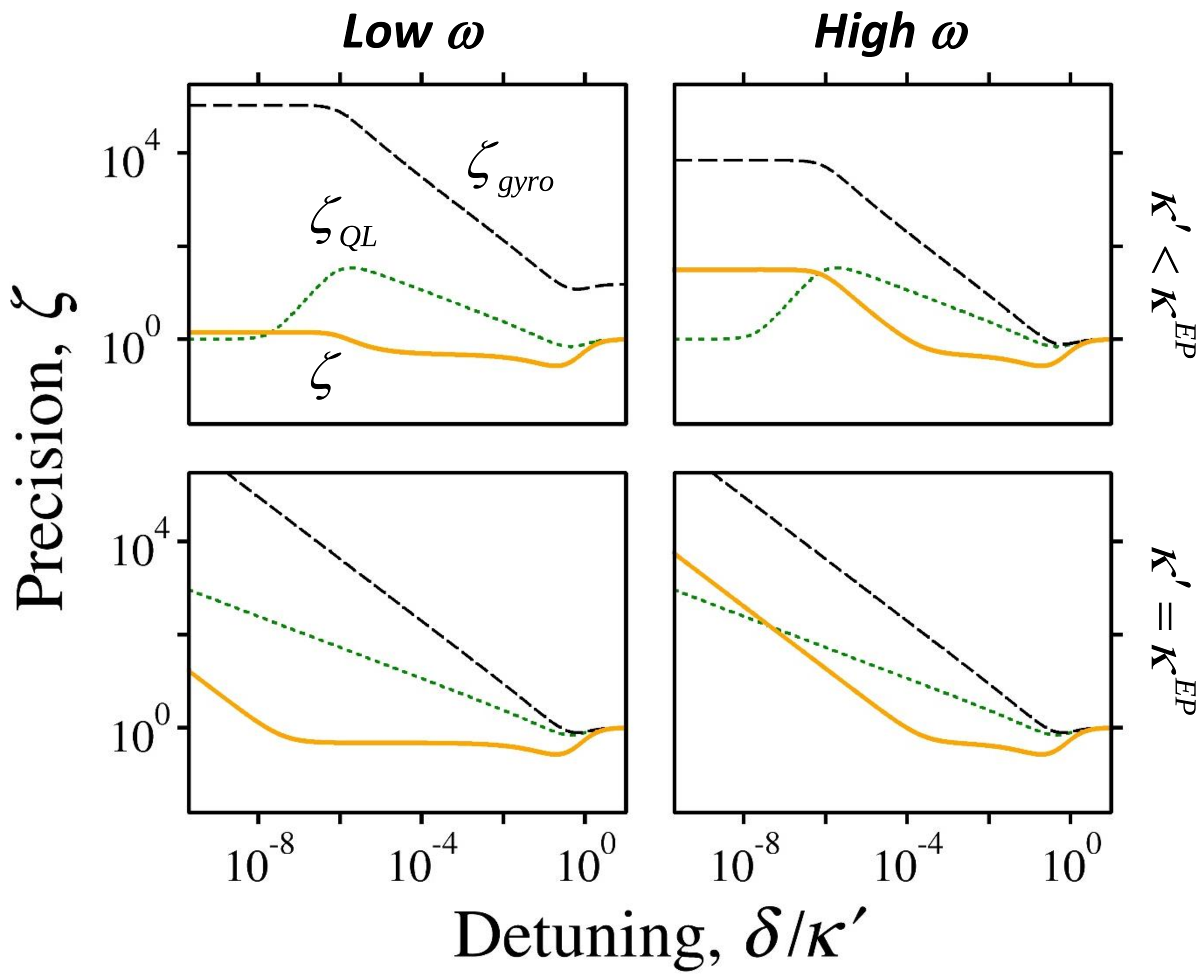
  $\Rightarrow$ ***Enables enhanced precision***

Precision for Colored Noise

$$K_\omega = \frac{\omega^2 + (\gamma_{avg,0})^2_{\kappa'=0}}{\omega^2 + (\gamma_{avg,0})^2}$$

$K_\omega = 1$

- In high ω limit.
- Relative to ideal gyro (equal decay rates)



- *Can be even better than QL result and broadband!*

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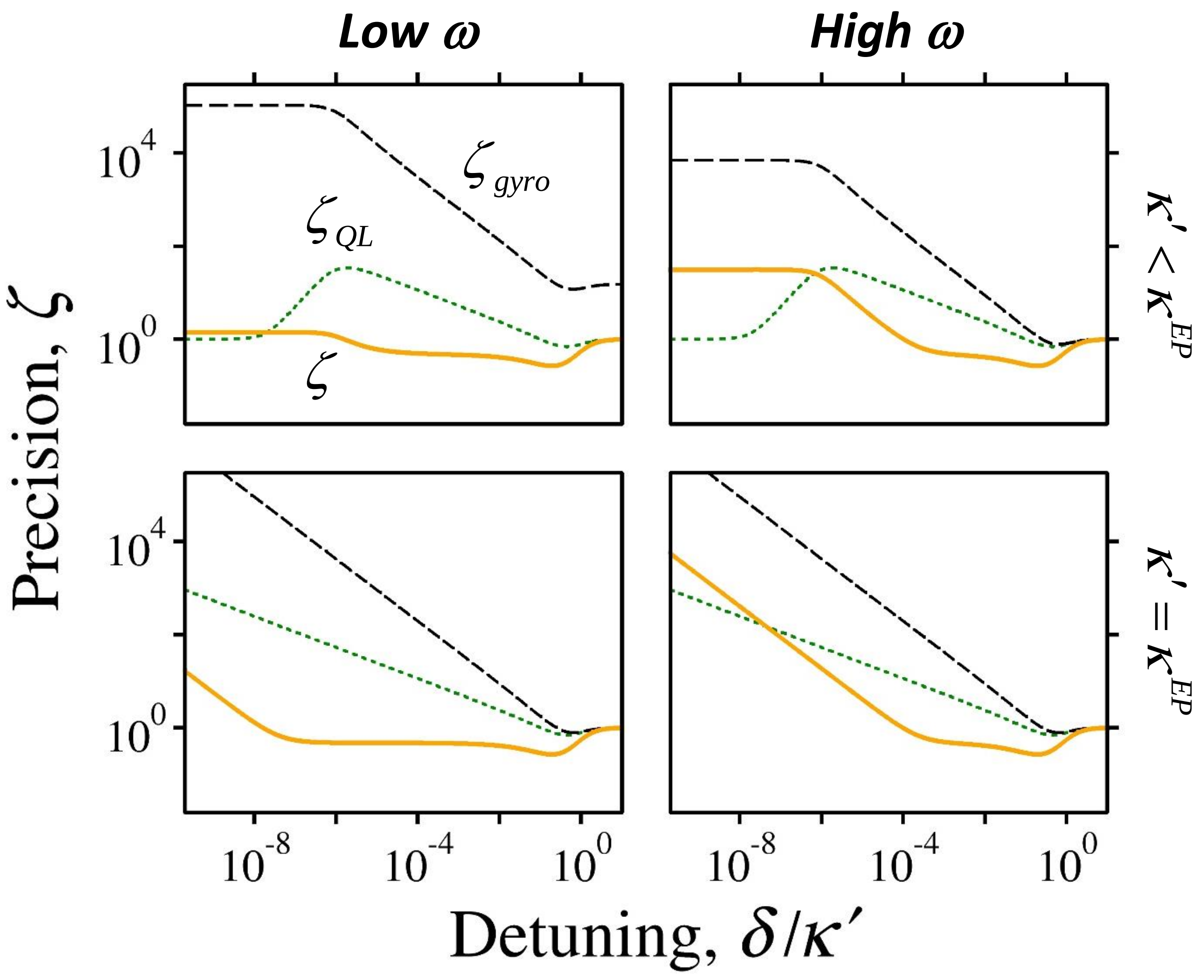
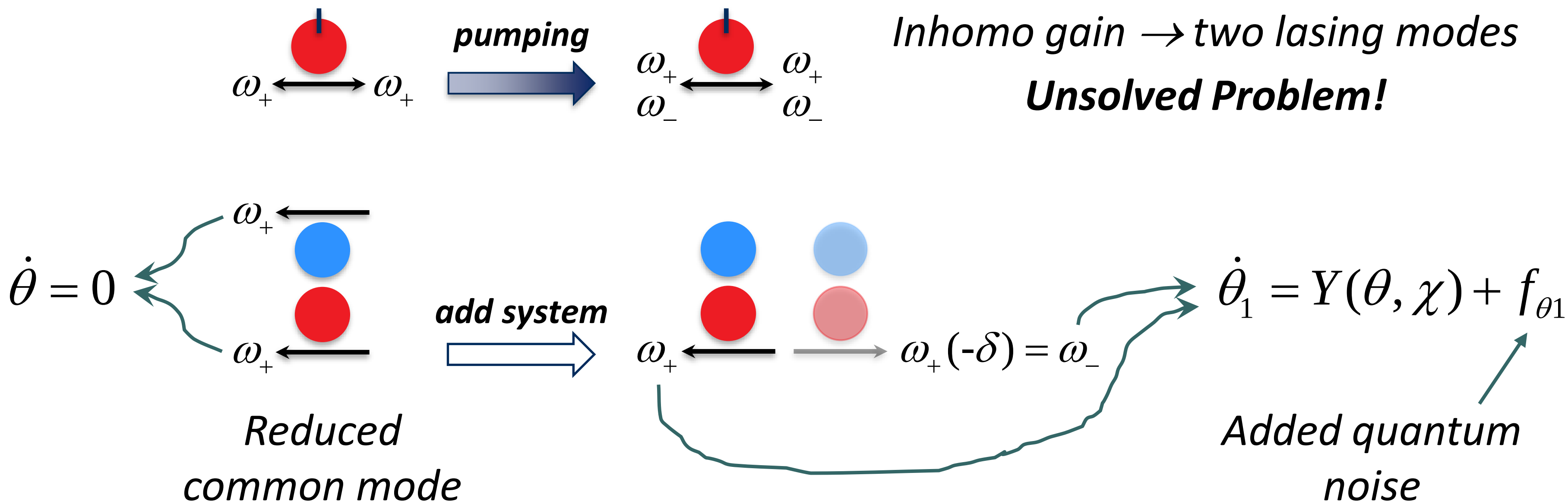


- *Can be even better than QL result and broadband!*

Problem: Zero Beat Frequency

Solution: *Recover beat by* ***adding an uncoupled counterpropagating direction*** *or* ***increase pumping to obtain two lasing modes*** *(requires inhomogeneous gain).*

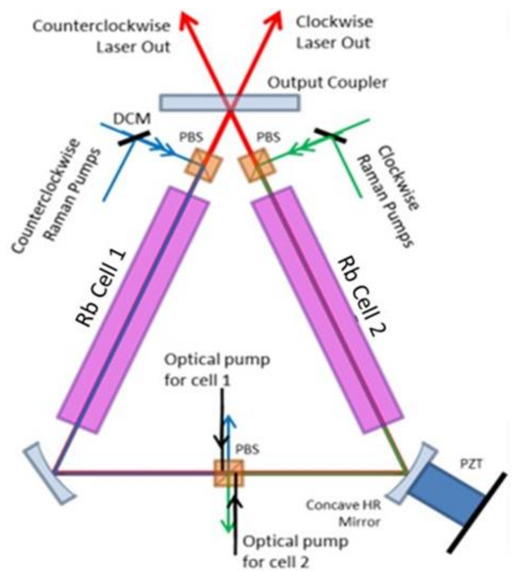


- ➢ Common mode requires two lasing modes in a single resonator, but this is currently an unsolved problem!

Summary: Key Ingredients

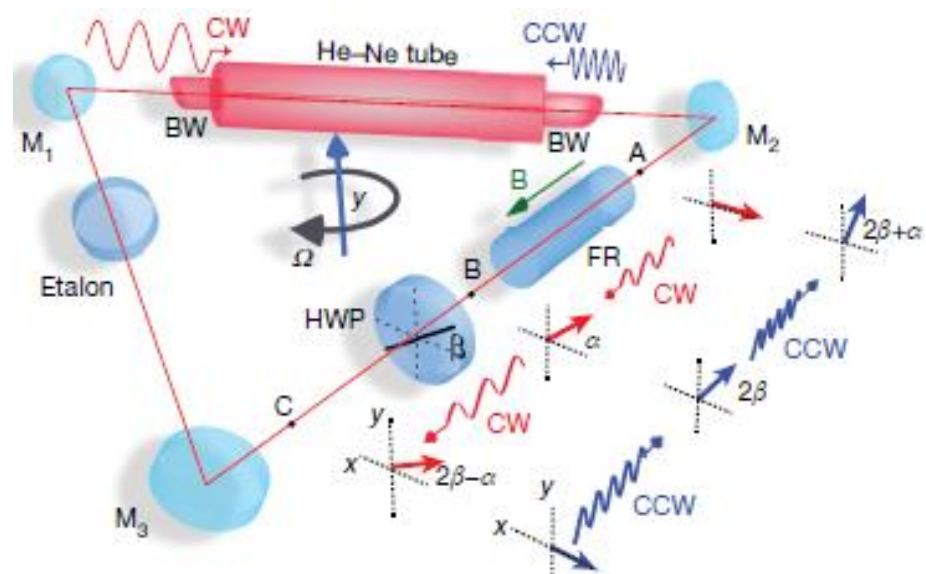
- Gain saturation
- Conservative coupling
- Beat note recovery (with minimum added noise)

Active FL Gyro (single eigenmode):



Saturation imbalance
Reduced common mode

Non-Hermitian He-Ne Gyro (two eigenmode):



No saturation imbalance,
Better common mode

$\Rightarrow$ *Need: simultaneous S and K measurements*