High order entropy split methods employing compact central spatial discretizations

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Outline

- Compact schemes: accuracy, stability, and SBP
- Entropy splitting for MHD with compact schemes
- Numerical results

Compact (implicit) difference operators

- One-dimensional setup: Grid $x_j = jh$, $j = 1, \ldots, N$,
- Approximations $u(x_j) \approx u_j$, $(Du)_j \approx u_x(x_j)$
- Explicit: D is a band matrix, $2p+1$ diagonals
- Compact: $D = P^{-1}Q$, P and Q band matrices
- Examples (4th-order):
  - Explicit: $(Du)_j = (-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2})/(12h)$
  - Compact: $((Du)_{j+1} + 4(Du)_j + (Du)_{j-1})/6 = (u_{j+1} - u_{j-1})/2h$
- Compact have smaller truncation error for same order of accuracy
- Compact have more high wave number content

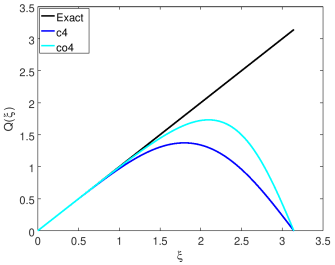
Compact faster error growth at non-linear instabilities:

Yee and Sjogreen, Computers Fluids 37(2008), pp. 593–619.

Properties of implicit operators

Truncation error (order p): $Du_j - u_x(x_j) = c_p h^P + \mathcal{O}(h^{p+1})$

p	$c^{(e)}$	$c^{(i)}$	$c^{(e)}/c^{(i)}$
4	$1/30$	$1/210$	7
6	$1/140$	$1/2100$	15
8	$1/630$	$1/44100$	70



Fourier symbol, $Q(\omega h)$. 4th-order explicit(blue), Compact(cyan)

Summation by parts for implicit operators

Find norm, H, and boundary modifications of P and Q ($D = P^{-1}Q$) s.t.

$$HD + D^T H^T = B$$

($B = 0$, except $b_{11} = -1$, $b_{NN} = 1$).

- Boundary influence on truncation error ?
- Global error advantage kept with boundary modification ?
- Full-norm operators $\rightarrow$ impractical ?

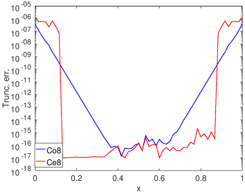
Summation by parts, example

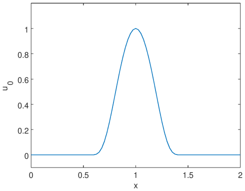
Implicit 8th order with 4th order SBP boundary op (uses $P = H$) from:

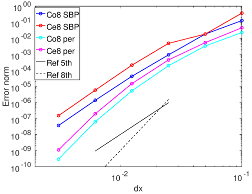
Chang, C.-C., Chen, M.-H., and Teng, C.-T.: Presentation, url

`https://www.math.ncku.edu.tw/ mhchen/HOFD/HOSBP_slide.pdf` .

Comparison 8th-order explicit (4th-order SBP on bndry, diagonal norm)
Tests: Local truncation error and linear advection







Truncation error ($u = x^5$) Initial data Global errors, one period

Entropy split scheme

Conservation law: $\mathbf{u}_t + \mathbf{f}_x = \mathbf{0}$
Rewrite flux derivative $\mathbf{f}_x = w\mathbf{f}_x + (1-w)A\mathbf{v}_x$ before disc.

$$\frac{d\mathbf{u}_j}{dt} + \frac{\beta}{\beta+1}D\mathbf{f}_j + \frac{1}{\beta+1}A_j\,D\mathbf{v}_j = \mathbf{0}$$

with β a parameter and $A = \partial\mathbf{f}/\partial\mathbf{v}$.

- $\mathbf{u}$ conserved variables; $\mathbf{v} = E_{\mathbf{u}}$ entropy variables (E entropy)
- D explicit or implicit difference operator
- Conserves total entropy if:
  - $\mathbf{f}(\theta\mathbf{v}) = \theta^{\beta}\mathbf{f}(\mathbf{v})$ (e.g., Compressible gas dynamics)
  - A symmetric
- Straightforward to define
- Not conservative
- Not kinetic energy preserving (KEP)

Entropy splitting for MHD

MHD with Godunov div($\mathbf{B}$) term: $\mathbf{u}_t + \mathbf{f}_x = \mathbf{e}\,\mathrm{div}\,\mathbf{B}$

- Gas dynamics E is an entropy also for MHD
- Entropy splitting defined as

$$\mathbf{q}_t + \frac{\beta}{\beta+1}(\mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)}) + \frac{1}{\beta+1}(A^{(x)}\mathbf{v}_x + A^{(y)}\mathbf{v}_y + A^{(z)}\mathbf{v}_z) + \frac{\beta}{\beta+1}\mathbf{e}\,\mathrm{div}\,\mathbf{B} = \mathbf{0},$$

with symmetric matrices $A^{(x)} = \mathbf{f}_{\mathbf{v}}^{(x)} + C^{(x)}$, where

$$\mathbf{e}\,\mathrm{div}\,\mathbf{B} = C^{(x)}\mathbf{v}_x + C^{(y)}\mathbf{v}_y + C^{(z)}\mathbf{v}_z$$

- Flux not homogeneous in $\mathbf{v}$ $\rightarrow$ no proof of entropy cons.

Example1: Alfvén wave

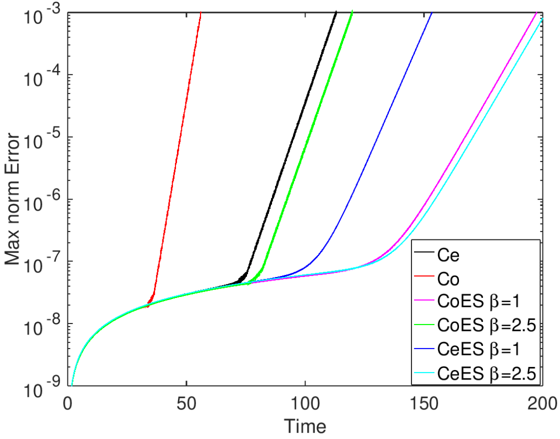
We select an Alfvén wave (a traveling transverse wave) with phase $\phi = 2\pi(x\cos\alpha + y\sin\alpha + t)$,

$$\rho = 1 \ \ p = 1 \ \ \mathbf{v} = A(-\sin\alpha\sin\phi,\ \cos\alpha\sin\phi,\ cos\phi)$$
$$\mathbf{B} = (\cos\alpha - A\sin\alpha\sin\phi,\ \sin\alpha + A\cos\alpha\sin\phi,\ Acos\phi)\ \ .$$

- Two-dimensional wave traveling in direction $\mathbf{n} = (\cos\alpha,\ \sin\alpha,\ 0)$
- $\alpha = 30^\circ$, $A = 0.1$, periodic boundary conditions.
- Domain 1.3546×2.1915, grid spacing 0.022 (coarse), 0.011 (fine).
- Solved up to time 200; 70,000 time steps (coarse).

Alfvén wave, results

Maximum norm of error vs. time



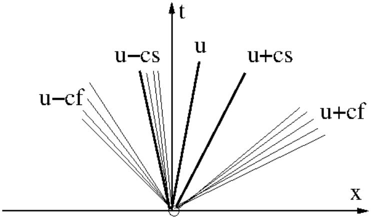
Explicit black, ES($\beta = 1$) blue, ES($\beta = 2.5$) cyan
Implicit red, ES($\beta = 1$) magenta, ES($\beta = 2.5$) green

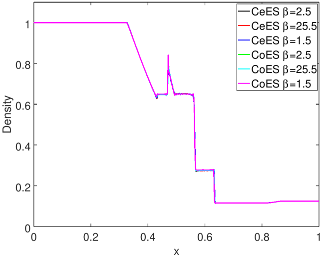
Example2: Brio-Wu Riemann problem

Domain $0 < x < 1$, initial states:

$$(\rho_L,\ u_L,\ p_L,\ B_L^{(x)},\ B_L^{(y)},\ B_L^{(z)}) = (1,\ 0,\ 1,\ 0.75,\ 1,\ 0) \qquad x < 0.5$$

$$(\rho_R,\ u_R,\ p_R,\ B_R^{(x)},\ B_R^{(y)},\ B_R^{(z)}) = (0.125.\ 0.\ 0.1.\ 0.75.\ -1.\ 0) \quad x > 0.5$$

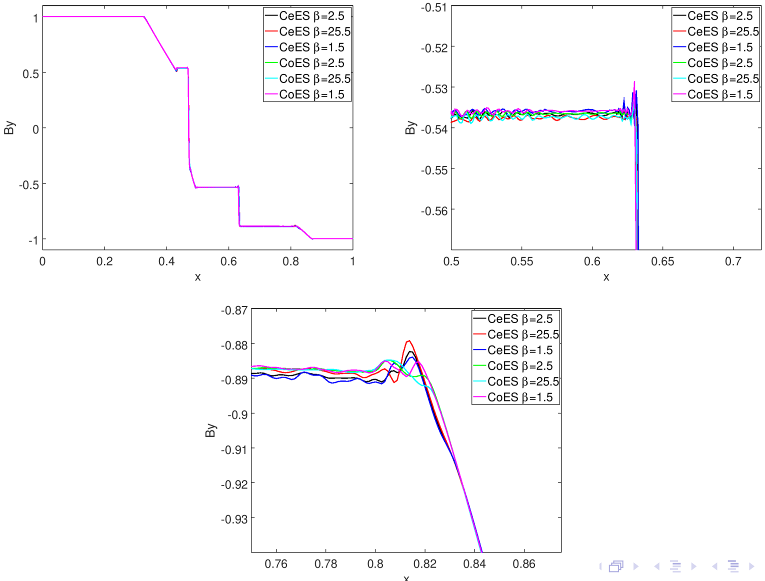




- 800 Grid points, 1500 time steps, Dirichlet bndry. cond.
- Entropy split approx. with explicit or implicit f.d. operator
- Non-linear shock capturing filter after each time step.

Brio-Wu Riemann problem, results

B_y component at time 0.1



Conclusions

- Compact schemes can be made to work in entropy split discretizations
- Accuracy and stability comparable with explicit schemes for two test problems

Future directions:

- More extensive testing on small scale dominated flows
- Try to develop diagonal-norm SBP compact discretizations