Comparative Study on a Variety of High Order Spatial Discretizations with the Entropy Split Methods for MHD

H.C. Yee, NASA Ames Research Center
(Joint work: B. Sjogreen)

Lecture Notes
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Objectives

- Present Our Recent Development of Entropy Split Methods (ES) for the Ideal MHD
  - > Two ES methods are proposed
- Comparison of 8th-Order Entropy Split Methods with Six Other 8th-Order Methods
  - > Entropy Conserving Methods
  - > Momentum Conserving Method
  - > Kinetic Energy Preserving Method
  - > Combination of Two of These Properties Methods

Motivation

- **Improvement in Nonlinear Stability (reduce aliasing errors)**
  - > More efficient than Tadmor-type entropy conserving methods of same order
  - > Methods cater to long time integration of compressible turbulence
- **Improvement in Accuracy** (for a wide spectrum of flow speeds)
  - > Efficient Nonlinear Filter methods with adaptive local flow sensors designed to minimize the use of numerical dissipation
- **Mass Conservation Issue for shock-free & flows with shocks**
  - > Entropy split method is entropy conserving but it is not in conservation form
  - > Two proposed methods to overcome the issue

Comparison Among Seven 8th-Order Methods

- **ECLOG**: Tadmor-type entropy conserving method using the Tadmor entropy function $E_L = -\rho \log(p\rho^{-\gamma})$.

- **ECLOGKP**: Tadmor-type entropy conserving method using the Tadmor entropy function with Ranocha's kinetic energy preserving modification [32].

- **ECHKP**: Entropy conserving using the Harten class of entropy functions $E_H = -\frac{\gamma+\alpha}{\gamma-1}\rho(p\rho^{-\gamma})^{\frac{1}{\alpha+\gamma}}$ [50, 23]. In its base form also satisfies Ranocha's kinetic energy preservation condition (KEP); only one variant for this method [43, 32].

- **KGP**: Kennedy-Gruber-Pirozzoli (KGP) skew-symmetric splitting of the inviscid flux derivative that is kinetic energy preserving [26, 31, 13].

Comparison Among Seven 8th-Order Methods
(Cont.)

- **ESSW**: Entropy split with Ducros et al. splitting but switch to regular central near discontinuities [46].
- **DSKP**: Ducros Split with KEP.
- **DS**: Momentum conserving Ducros et al. skew-symmetric split of the inviscid flux derivative [14].

Treatment of Discontinuities for Entropy Split Schemes

Accurate schemes developed for short time integration (or rapidly developing flows) usually SUFFER from nonlinear instability for long time integration

- Standard shock-capturing methods are too diffusive for long time integration
- Careful design of appropriate nonlinear numerical dissipations with flow sensors can improve accuracy

Yee et al. Nonlinear Filter Approach:

- High order entropy split schemes as the base scheme
- Local flow sensor to examine the regularity of the computed date after each full time step of the base scheme.
- Determine the location and amount of numerical dissipation needed
- Nonlinearly filter the computed data at these locations only

Preliminary

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}$$

Entropy function: $E(\mathbf{U})$
Entropy variables: $\mathbf{W} = \nabla_{\mathbf{U}} E$ (require $E_{\mathbf{U},\mathbf{U}} > 0$)
Entropy flux: $f = f(\mathbf{U})$
Entropy conservation if

$$\mathbf{W}^T \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \nabla_{\mathbf{U}} f$$

Take conservation law & multiply by $\mathbf{W}$ to give

$$\mathbf{W}^T \mathbf{U}_t + \mathbf{W}^T \mathbf{F}_x = 0 \quad \rightarrow \quad E_t + \mathbf{W}^T \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \mathbf{U}_x = 0 \quad \rightarrow \quad E_t + f_x = 0$$

Alternatively, if f is known, denote $\psi = \mathbf{W}^T \mathbf{F} - f$ and require

$$\psi_x = \mathbf{W}_x^T \mathbf{F}.$$

Entropy conservation shown by

$$\mathbf{W}^T \mathbf{F}_x = (\mathbf{W}^T \mathbf{F})_x - \mathbf{W}_x^T \mathbf{F} = (\mathbf{W}^T \mathbf{F})_x - (\mathbf{W}^T \mathbf{F} - f)_x = f_x$$

Entropy Conserving Discretization

Numerical flux: $\mathbf{H}_{j+1/2}$
Entropy numerical flux: $\mathbf{h}_{j+1/2}$
Semi-discretized

$$\frac{d}{dt}\mathbf{U}_j + \frac{1}{\Delta x}(\mathbf{H}_{j+1/2} - \mathbf{H}_{j-1/2}) = \mathbf{0}$$

If

$$(\mathbf{W}_{j+1} - \mathbf{W}_j)^T \mathbf{H}_{j+1/2} = \psi_{j+1} - \psi_j$$

Then can derive discrete entropy conservation law

$$\frac{d}{dt}E_j + \frac{1}{\Delta x}(h_{j+1/2} - h_{j-1/2}) = 0$$

with

$$h_{j+1/2} = \frac{1}{2}(\mathbf{W}_{j+1} + \mathbf{W}_j)^T \mathbf{H}_{j+1/2} - \frac{1}{2}(\psi_{j+1} + \psi_j)$$

[1] E.Tadmor, Math. Comput., 43(1984) pp. 369–381.

Local & Global Entropy Stability

Local entropy conservation:

$$\frac{d}{dt}E_j + \frac{1}{\Delta x}(h_{j+1/2} - h_{j-1/2}) = 0$$

Local entropy stability:

$$\frac{d}{dt}E_j + \frac{1}{\Delta x}(h_{j+1/2} - h_{j-1/2}) \leq 0$$

Global entropy conservation:

$$\frac{d}{dt}\sum_j \Delta x E_j = 0$$

Global entropy stability:

$$\frac{d}{dt}\sum_j \Delta x E_j \leq 0$$

Entropy Splitting of the Euler Flux Derivatives
Leads to Global Entropy Conservation (Stability) by the Energy Norm

Entropy Split Methods (ECHKP)

Entropies considered by Harten (1983)

$$E = -\rho(p\rho^{-\gamma})^{1/(\alpha+\gamma)}$$

Symmetrizable & Homogeneous property of the Euler Flux lead to

$$\mathbf{F_W W} = \beta \mathbf{F} \tag{1}$$

for $\beta = \frac{\alpha+\gamma}{1-\gamma}$. Use the split form with $\beta \neq -1$

$$\mathbf{F}_x = \frac{\beta}{\beta+1}\mathbf{F}_x + \frac{1}{\beta+1}\mathbf{F_w W}_x$$

in a semi-discrete approximation:

$$\frac{d}{dt}\mathbf{U}_j + \frac{\beta}{\beta+1}D\mathbf{F}_j + \frac{1}{\beta+1}(\mathbf{F_W})_j D\mathbf{W}_j = 0$$

where D is a SBP difference operator.

ECHKP: ***Entropy Split Methods automatically satisfies the Ranocha's kinetic energy preservation condition***

Entropy Splitting Leads to Global Entropy Conservation & Stability

Form the SBP scalar product by $\mathbf{W}_j^T$ to obtain the entropy

$$\frac{d}{dt}\sum_j \omega_j \Delta x E_j + \frac{\beta}{\beta+1}(\mathbf{W}, D\mathbf{F}) + \frac{1}{\beta+1}(\mathbf{W}, \mathbf{F_W} D\mathbf{W}) = 0$$

(ω_j are SBP norm weights) where SBP and property (1) give

$$\beta(\mathbf{W}, D\mathbf{F}) = -\beta(D\mathbf{W}, \mathbf{F}) - \beta\mathbf{W}_1^T\mathbf{F}_1 + \beta\mathbf{W}_N^T\mathbf{F}_N =$$
$$-(D\mathbf{W}, \mathbf{F_W}\mathbf{W}) - \beta\mathbf{W}_1^T\mathbf{F}_1 + \beta\mathbf{W}_N^T\mathbf{F}_N$$

and symmetry of $\mathbf{F_W}$ gives $(\mathbf{W}, \mathbf{F_W} D\mathbf{W}) = (\mathbf{F_W}\mathbf{W}, D\mathbf{W})$, so that

$$\frac{d}{dt}\sum_j \omega_j \Delta x E_j = \frac{\beta}{\beta+1}(\mathbf{W}_1^T\mathbf{F}_1 - \mathbf{W}_N^T\mathbf{F}_N)$$

i.e., global entropy conservation with boundary contributions.
Note, property (1) does not hold for $E = -\rho log(p\rho^{-\gamma})$ commonly used in the literature

Entropy Splitting of Euler Flux Derivatives

(Oliger & Olsson, Olsson, Gerittsen & Olsson, Yee et al. (1994-1999)

Entropy stable (generalized energy norm) with SBP for Central Schemes

Physical Entropy, Homogeneous of Degree One of Euler Flux Derivative F_x

$$\mathbf{F}_x = \frac{\beta}{\beta+1}\mathbf{F}_x + \frac{1}{\beta+1}\mathbf{F}_W W_x, \quad \beta \neq -1$$

$$W = [w_1, \quad w_2, \quad w_3, \quad w_3, \quad w_5]^T$$

$$= \frac{p^*}{p}\left[e + \frac{\alpha-1}{\gamma-1}p, \quad -\rho u, -\rho v, \quad \rho w, \rho\right]^T,$$

where

$$p^* = -(p\rho^{-\gamma})^{\frac{1}{\alpha+\gamma}}$$

and

$$\beta = \frac{\alpha+\gamma}{1-\gamma}, \quad \alpha > 0 \text{ or } \alpha < -\gamma.$$

Negative Beta-> unphysical

Entropy Conserving Discretization
Tadmor-type (ECLOG) with Boundaries

Entropy Split Schemes: Achieving global entropy conservation & stability automatically by standard SBP & energy norm estimate (periodic or non-periodic BC)

Entropy Conserving Schemes - Tadmor-type: Achieving global entropy conservation & stability require generalized SBP & dissipation

Domain $j = 1, 2, \ldots,$ with boundary at $j = 1$. Consider the scheme

$$\frac{d}{dt}\mathbf{U}_j + \frac{1}{\Delta x}(\mathbf{H}^{EC}_{j+1/2} - \mathbf{H}^{EC}_{j-1/2}) = \mathbf{0}$$

with the E.C. 2nd-order numerical flux $\mathbf{H}^{EC}_{j+1/2} = \mathbf{H}^{EC}(\mathbf{U}_j, \mathbf{U}_{j+1})$, defined at $j = 1, 2, \ldots,$

Define $\mathbf{H}^{EC}_{1/2} = 2\mathbf{F}_1 - \mathbf{H}^{EC}_{3/2}$, then the local entropy conservation law

$$\frac{d}{dt}E_j + \frac{1}{\Delta x}(h_{j+1/2} - h_{j-1/2}) = 0$$

holds for $j = 1, 2, \ldots,$ with

$$h_{1/2} = \mathbf{W}_1^T \mathbf{F}_1 - \psi_1$$

and

$$h_{j+1/2} = \frac{1}{2}(\mathbf{W}_{j+1} + \mathbf{W}_j)^T \mathbf{H}_{j+1/2} - \frac{1}{2}(\psi_{j+1} + \psi_j)$$

for $j \geq 1$ (as previously).

Entropy Conserving Discretization of Tadmor-type (ECLOG) with Boundaries (Cont.)

The global entropy conservation with boundaries

$$\Delta x \frac{d}{dt}\left(\frac{1}{2}E_1 + \sum_{j=2} E_j\right) + \mathbf{W}_1^T \mathbf{F}_1 - \psi_1 = 0$$

follows, where 1/2 is the SBP weight of the boundary entropy, and $f_1 = \mathbf{W}_1^T \mathbf{F}_1 - \psi_1$ is the boundary entropy flux.

For a construction that generalizes to higher order accuracy, see:

Parsani, Carpenter, Nielsen, J.Comput.Phys., 292(2015),pp. 88–113.

Advantage of Entropy Split Method over Standard Tadmor-Type Entropy Conserving Methods

Tadmor-type entropy conserving method (logarithmic entropy or physical entropy) **requires 2 times CPU than entropy split method**

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Both Approaches Exhibit Similar Accuracy & Stability
(Same order of central discretization)

Mass Conservation issue by Entropy Split Methods

Fact: **Entropy splitting of the Euler inviscid flux derivatives is not in conservation form**

> ES method in its original form is entropy conservative but is not a conservative method

> Mass conservation is not a problem on smooth flow test cases

> Mass conservation error increases as a function of the duration of time integration for flows with discontinuities

Two Proposed New Split Methods to Correct Mass Conservation Error:

Sjogreen & Yee, "Construction of Conservative Numerical Fluxes for the Entropy Split Method", HONOM-2019 special CAMC volume, 2022

Test Cases on the new approach with improve stability and accuracy:

> 2D Vortex Convection
> 3D Taylor-Green Vortex
> 3D Isotropic Turbulence with Shocklets
> 1D Shu Osher Shock-Turbulence
> 2D Alfven Wave
> Orzag-Tang MHD Vortex
> Brio-Wu MHD Shock-Tube

Kennedy & Gruber Splitting (2008)

Split the derivative of triple products, kinetic energy preservation

$$\frac{\partial}{\partial x}(abc) = \alpha\frac{\partial}{\partial x}(abc) + \beta\left[a\frac{\partial}{\partial x}(bc) + bc\frac{\partial}{\partial x}(a)\right] + \kappa\left[b\frac{\partial}{\partial x}(ac) + ac\frac{\partial}{\partial x}(b)\right] + \delta\left[c\frac{\partial}{\partial x}(ab) + ab\frac{\partial}{\partial x}(c)\right]$$
$$+ \epsilon\left[bc\frac{\partial}{\partial x}(a) + ac\frac{\partial}{\partial x}(b) + ab\frac{\partial}{\partial x}(c)\right],$$

$$\text{where } \epsilon = 1 - \alpha - \beta - \kappa - \delta$$

Common parameter:

$$\alpha = \beta = \kappa = \delta = \tfrac{1}{4} \text{ and } \epsilon = 0,$$

$$(a\,b\,c)_x = \frac{1}{4}(a\,b\,c)_x + \frac{1}{4}(a_x\,(b\,c) + b_x\,(a\,c) + c_x\,(a\,b)) + \frac{1}{4}(a\,(b\,c)_x + b\,(a\,c)_x + c\,(a\,b)_x)$$

Dispersion Relation-Preserving (DRP) Schemes

Three DRP schemes in CAA: DRP4S7 (Tam & Webb), DRP4S9, ST09 (Bogey & Bailly)

$$Du_j = \frac{1}{h}\sum_{k=1}^{q} a_k (u_{j+k} - u_{j-k})$$

DRP4S7

Coefficients of DRP4S7, optimized over [0, 1.1].

k	a_k
1	0.77088238051822552
2	−0.16670590441458047
3	0.02084314277031176

DRP4S9

Coefficients of DRP4S9, optimized over $[\pi/16, \pi/2]$.

k	a_k
1	0.846863763009931
2	−0.251240526849904
3	0.063181723773749
4	−0.008481970157843

ST09

Coefficients of ST09, optimized over $[\pi/16, \pi/2]$, from [18].

k	a_k
1	0.841570216389881
2	−0.244678789340406
3	0.059463699920073
4	−0.007650934367322

3-D Compressible MHD (Ideal)

$$\boxed{\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \\ B_x \\ B_y \\ B_z \end{pmatrix}_t + \operatorname{div} \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u}\mathbf{u}^T + (p + \frac{1}{2}B^2)I - \mathbf{B}\mathbf{B}^T \\ \mathbf{u}(e + p + \frac{1}{2}B^2) - \mathbf{B}(\mathbf{u}^T\mathbf{B}) \\ \mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T \end{pmatrix} = 0}$$

Conservative

$$\boxed{\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \\ B_x \\ B_y \\ B_z \end{pmatrix}_t + \operatorname{div} \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u}\mathbf{u}^T + (p + \frac{1}{2}B^2)I - \mathbf{B}\mathbf{B}^T \\ \mathbf{u}(e + p + \frac{1}{2}B^2) - \mathbf{B}(\mathbf{u}^T\mathbf{B}) \\ \mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T \end{pmatrix} = -(\nabla \cdot \mathbf{B}) \begin{pmatrix} 0 \\ B_x \\ B_y \\ B_z \\ \mathbf{u}^T\mathbf{B} \\ u \\ v \\ w \end{pmatrix}}$$

Non-conservative (Symmetrizable - *Godunov, Powell*)

$$\mathbf{u} = (u, v, w)^T$$

$$\mathbf{B} = (B_x, B_y, B_z)^T$$

$$B^2 = B_x^2 + B_y^2 + B_z^2$$

$$p = (\gamma - 1)(e - \frac{1}{2}\rho(u^2 + v^2 + w^2) - \frac{1}{2}(B_x^2 + B_y^2 + B_z^2))$$

Conservative form is not symmetrizable

Two Entropy Split forms (ES & ES2) include the Godunov term are considered

Entropy Split Methods (ES) for MHD

$$\mathbf{q}_t + \mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)} + \mathbf{e}\,\mathrm{div}\,\mathbf{B} = \mathbf{0}.$$

The entropy splitting on; e.g., the x-flux derivative of the form

$$\mathbf{f}_x^{(x)} = \frac{\beta}{\beta+1}\mathbf{f}_x^{(x)} + \frac{1}{\beta+1}\mathbf{f}_{\mathbf{v}}^{(x)}\mathbf{v}_x,$$

and similarly for the other directions, to obtain

$$\mathbf{q}_t + \frac{\beta}{\beta+1}(\mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)} + \mathbf{e}\,\mathrm{div}\,\mathbf{B}) + \frac{1}{\beta+1}(\mathbf{f}_{\mathbf{v}}^{(x)}\mathbf{v}_x + \mathbf{f}_{\mathbf{v}}^{(y)}\mathbf{v}_y + \mathbf{f}_{\mathbf{v}}^{(z)}\mathbf{v}_z + \mathbf{e}\,\mathrm{div}\,\mathbf{B}) = \mathbf{0}.$$

The Jacobians with respect to the entropy variables, $\mathbf{f}_{\mathbf{v}}$, are not symmetric, but thanks to the presence of the source term

$$\mathbf{f}_{\mathbf{v}}^{(x)}\mathbf{v}_x + \mathbf{e}\mathbf{B}_x = A^{(x)}\mathbf{v}_x,$$

where $A^{(x)}$ is symmetric. The y- and $z-$ directions are similar. Hence, the split form of the equations

$$\textbf{(ES)}\ \mathbf{q}_t + \frac{\beta}{\beta+1}(\mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)}) + \frac{1}{\beta+1}(A^{(x)}\mathbf{v}_x + A^{(y)}\mathbf{v}_y + A^{(z)}\mathbf{v}_z) + \frac{\beta}{\beta+1}\mathbf{e}\,\mathrm{div}\,\mathbf{B} = \mathbf{0},$$

Entropy Split Methods (ES2) for MHD

A different form of the equations is obtained by replacing the weight $\beta/(1+\beta)$ on the Godunov term in (46) by 1,

$$\mathbf{q}_t+\frac{\beta}{\beta+1}(\mathbf{f}_x^{(x)}+\mathbf{f}_y^{(y)}+\mathbf{f}_z^{(z)})+\frac{1}{\beta+1}(A^{(x)}\mathbf{v}_x+A^{(y)}\mathbf{v}_y+A^{(z)}\mathbf{v}_z)+\mathbf{e}\,\mathrm{div}\,\mathbf{B}=\mathbf{0}. \tag{47}$$

We refer to this alternative form as ES2.

It is noted that by looking at the non-split form of the equations it is seen that ES2 is equivalent with

$$\mathbf{q}_t+\mathbf{f}_x^{(x)}+\mathbf{f}_y^{(y)}+\mathbf{f}_z^{(z)}+\frac{2+\beta}{1+\beta}\mathbf{e}\,\mathrm{div}\,\mathbf{B}=\mathbf{0}.$$

The weight in front of the Godunov term varies between 2 and 1 when β varies between 0 and ∞.

Semi Discrete Entropy Split Approximation

$$\frac{d}{dt}\mathbf{q}_j + \frac{\beta}{\beta+1} D\mathbf{f}_j + \frac{1}{\beta+1} A_j D\mathbf{v}_j + \frac{\beta}{\beta+1} \mathbf{e}_j DB_1 = 0, \quad j = 1, \ldots, N, \quad (48)$$

where D is a linear finite difference operator; e.g., $\mathbf{f}_x(x_j) \approx Df_j = \frac{1}{\Delta x} \sum_{k=1}^{p} \alpha_k (\mathbf{f}_{j+k} - \mathbf{f}_{j-k})$ and $\beta > 0$ is a parameter related to α by $\beta = (\alpha + \gamma)/(1 - \gamma)$.

Although this entropy split method is formally defined in the same way as in the case of the gas dynamics equations, because the MHD flux function is not homogeneous, entropy conservation cannot be proved in the same way

Non-uniqueness of Ducros et al. Splitting for MHD

(Minimize the use of numerical dissipation for high order central schemes)

- **MHD inviscid (ideal) flux derivatives consist of triple products of conservative variables & their derivatives**
- **Splitting such triple products is not unique**
  (See Sjogreen & Yee, ICOSAHOM-2016 for the chosen forms)
- **Splitting all 8 flux derivatives, partial or just the gas dynamic portion**
- **Solving the Conservative MHD**
- **Solving the Godunov/Powell symmetrizable MHD (Non-conservative)**

The above consists of 8 combinations

2D Alfven Wave Test Case

$$\rho = 1 \;\; p = 1 \;\; \mathbf{v} = A(-\sin\alpha\sin\phi,\; \cos\alpha\sin\phi,\; cos\phi)$$

$$\mathbf{B} = (\cos\alpha - A\sin\alpha\sin\phi,\; \sin\alpha + A\cos\alpha\sin\phi,\; Acos\phi) \;\; .$$

$$\phi = 2\pi(x\cos\alpha + y\sin\alpha + t)$$

where $\alpha = 30°$ and $A = 0.1$ are chosen with periodic boundary conditions

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The computational domain is 1.3546×2.1915

The flow is solved up to time 300 until all methods eventually diverge

$\Delta x = \Delta y = 0.022$

$\Delta x = \Delta y = 0.011$

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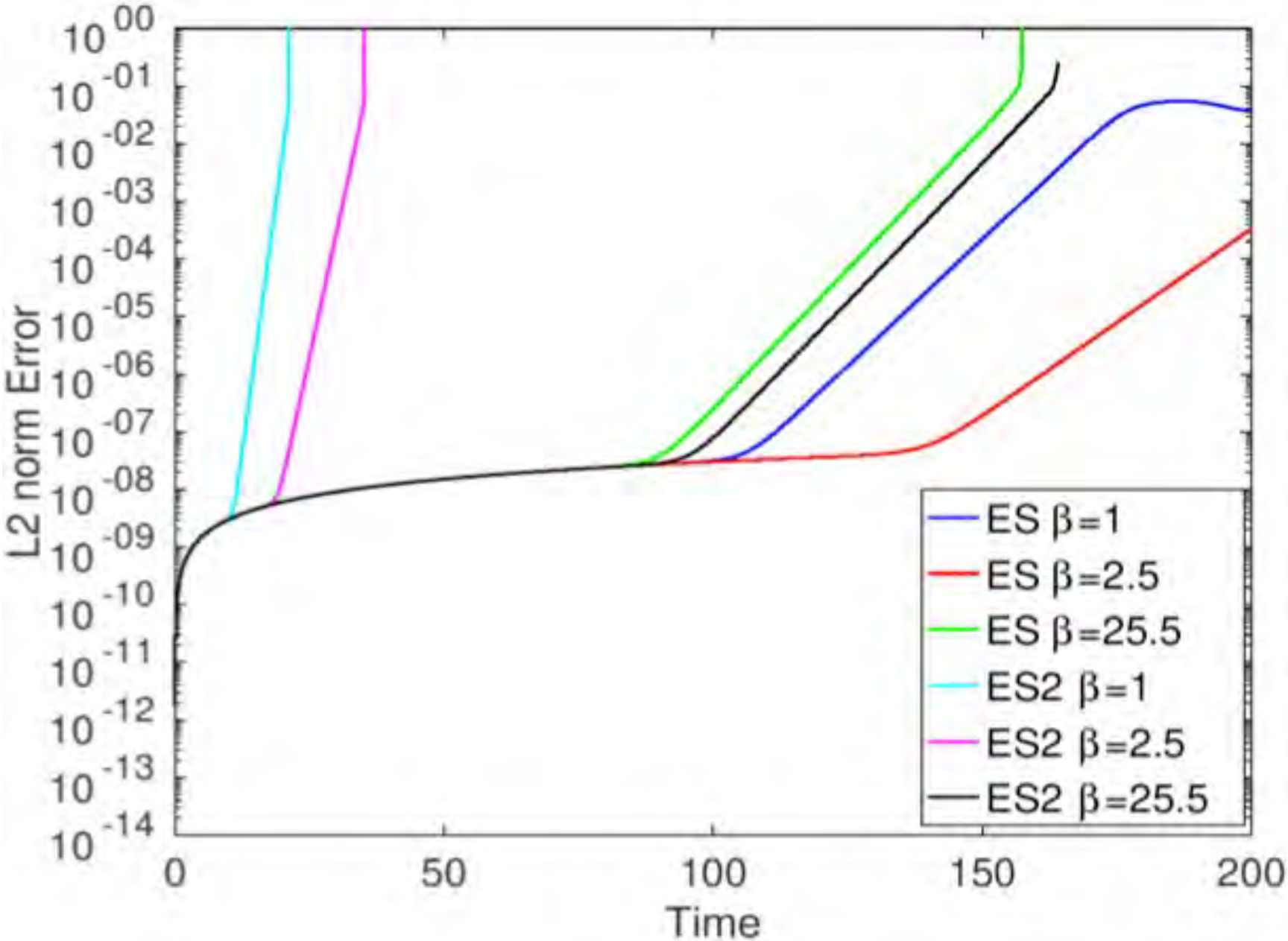
Alfven Wave Beta Study (L2-Norm Error vs. Time)

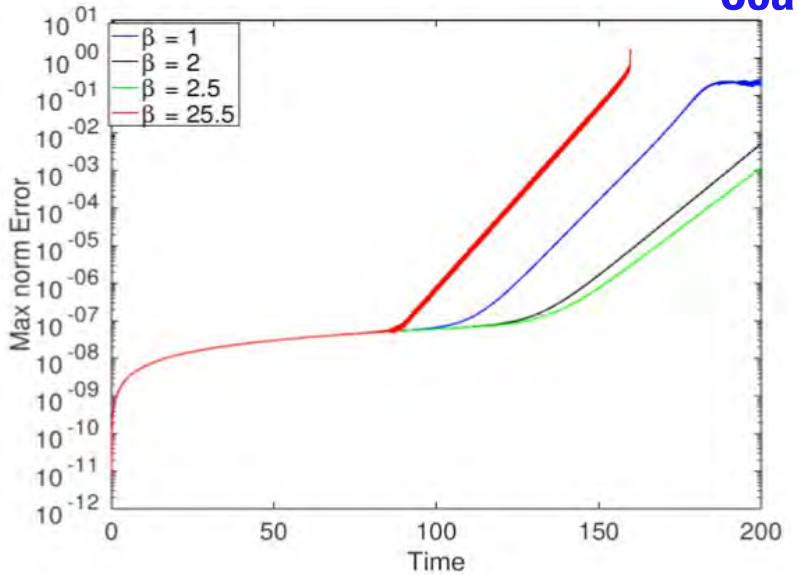
(Comparison between ES & ES2 by 8th-order Entropy Split Method)



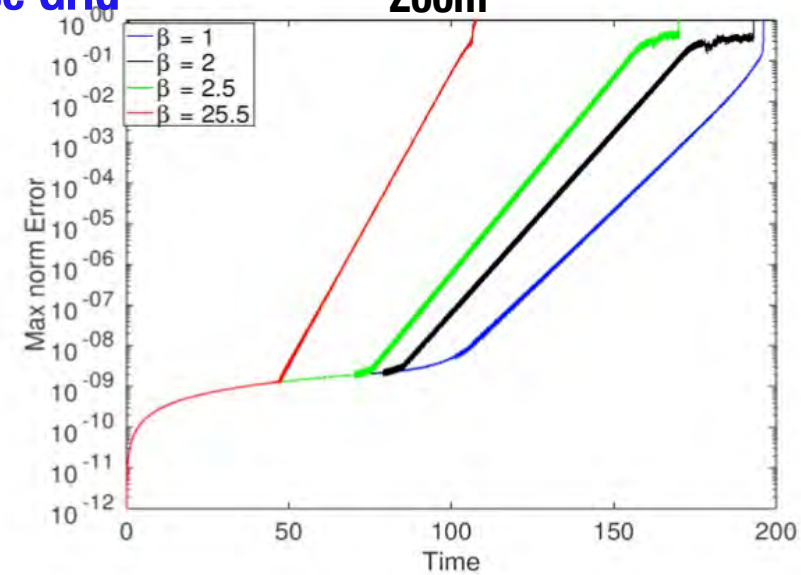
Alfven Wave (Max-Norm Error vs.Time)

(Beta Parameter Study: 8th-order Entropy Split Method)

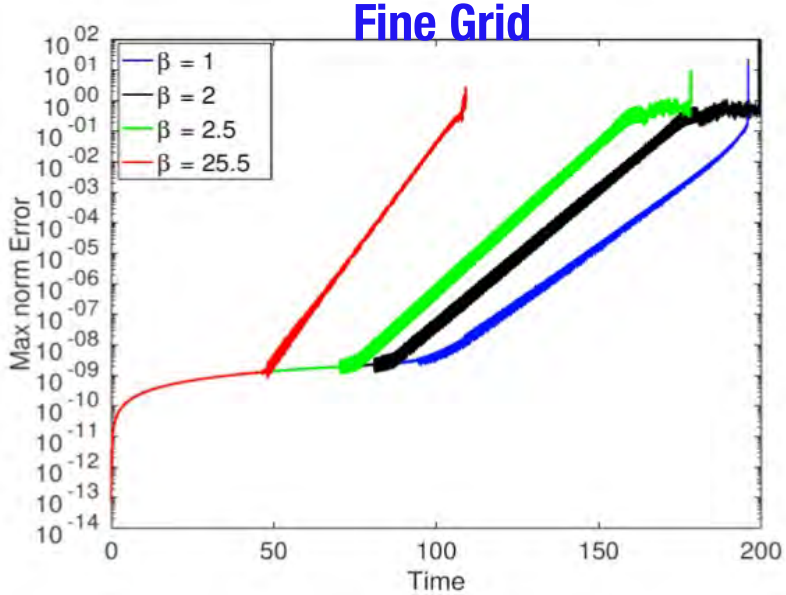
Coarse Grid



Zoom



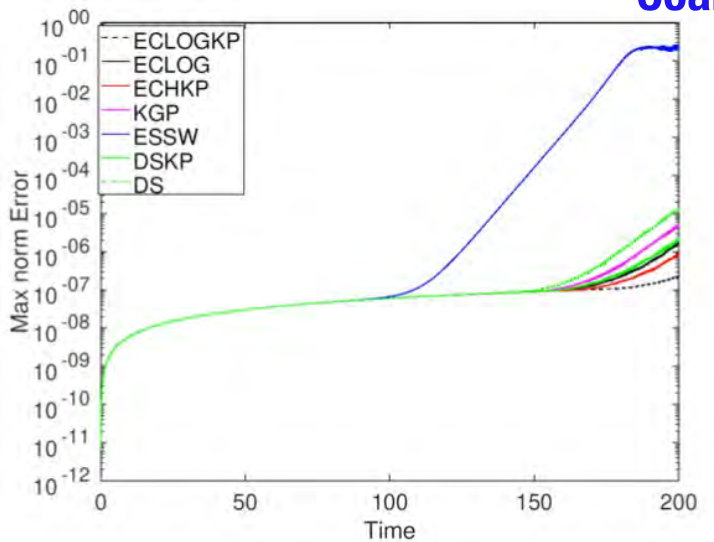
Fine Grid



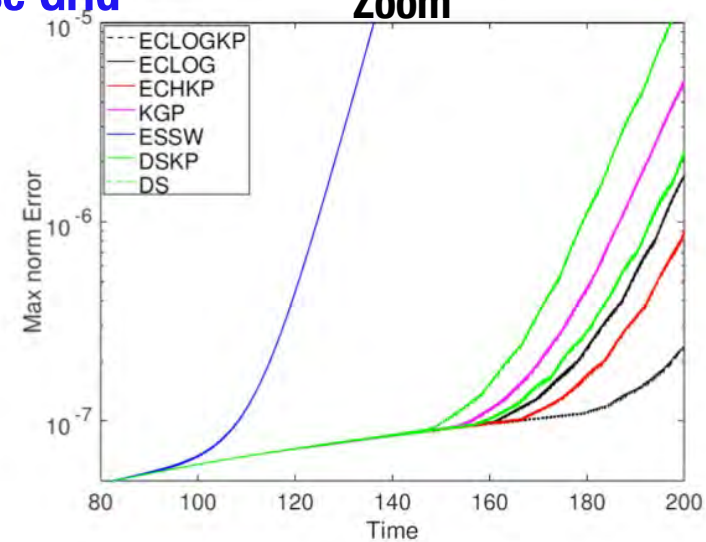
Alfven Wave (Max-Norm Error vs. Time)

(Comparison Among Seven 8th-order Methods)

Coarse Grid



Zoom



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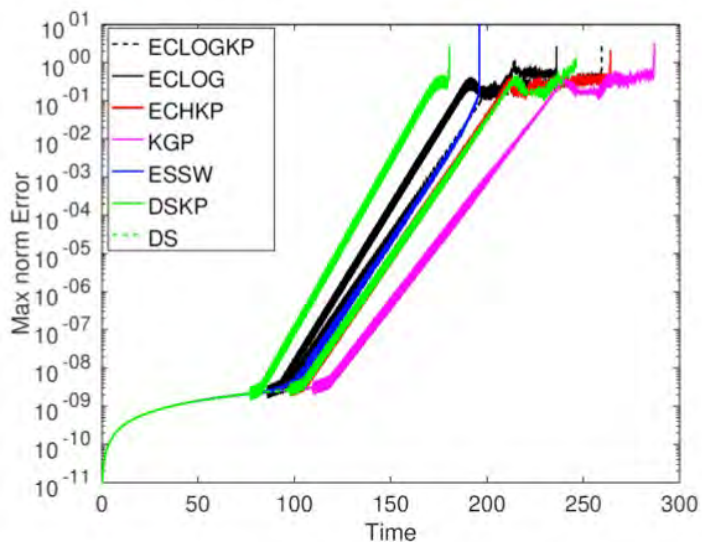
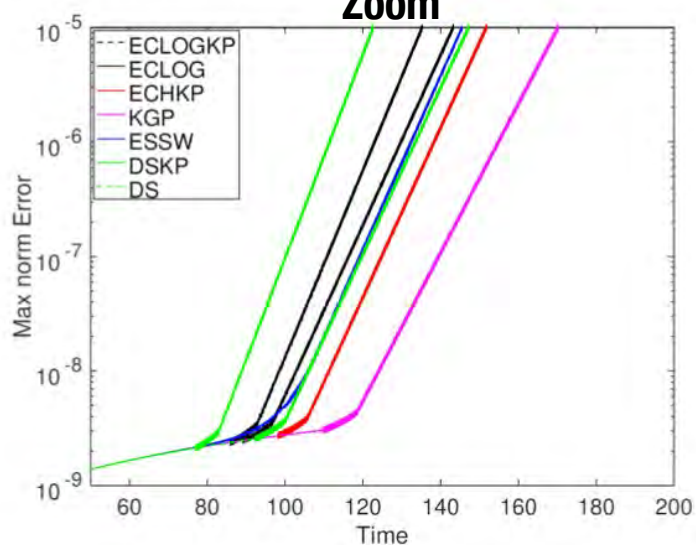


Zoom



Brio-Wu MHD Shock Tube Problem

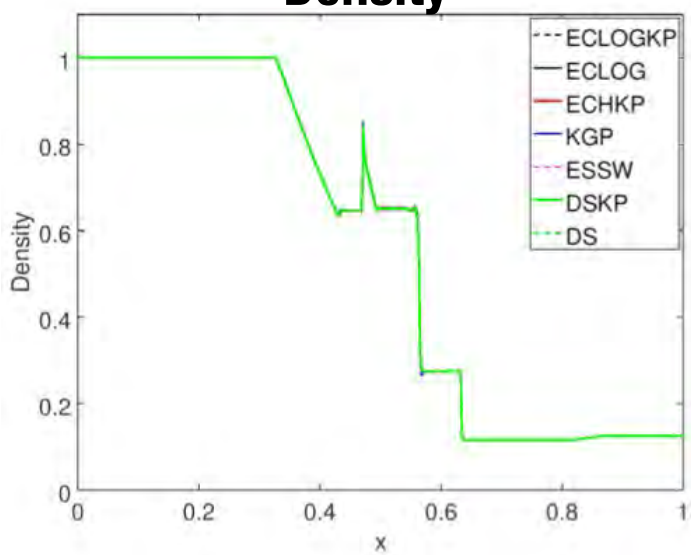
$$(\rho_L,\, u_L,\, p_L,\, B^{(x)},\, B^{(y)},\, B^{(z)}) = (1,\, 0,\, 1,\, 0.75,\, 1,\, 0)$$

$$(\rho_R,\, u_R,\, p_R,\, B^{(x)},\, B^{(y)},\, B^{(z)}) = (0.125,\, 0,\, 0.1,\, 0.75,\, -1,\, 0).$$

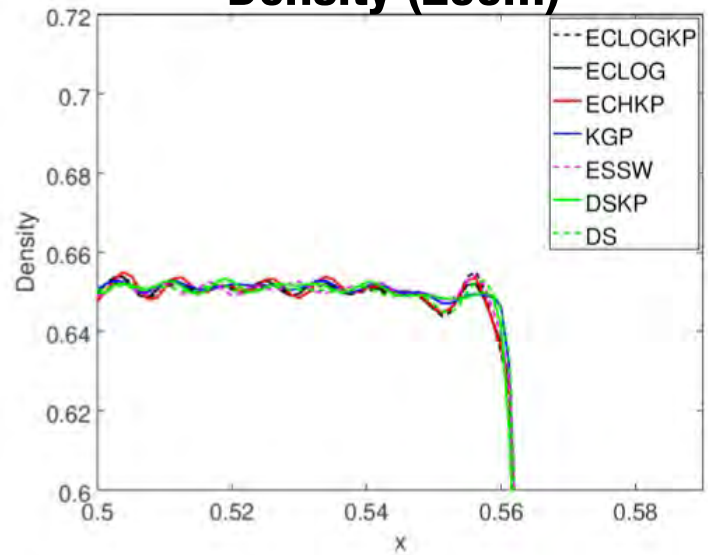
Brio-Wu MHD Shock Tube Problem

(Comparsion Among Seven Methods + 7th-order nonlinear filter)

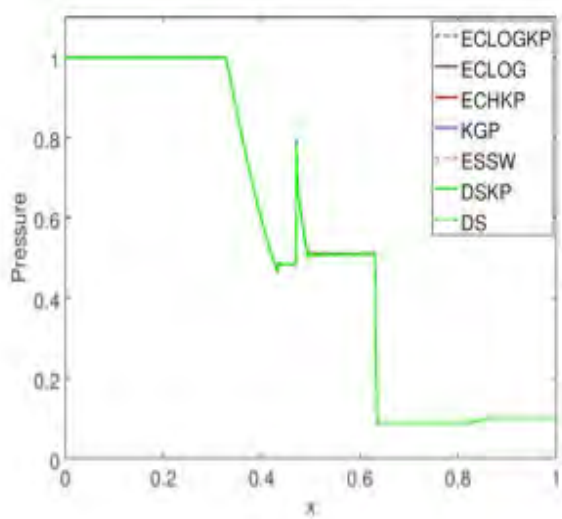
Density



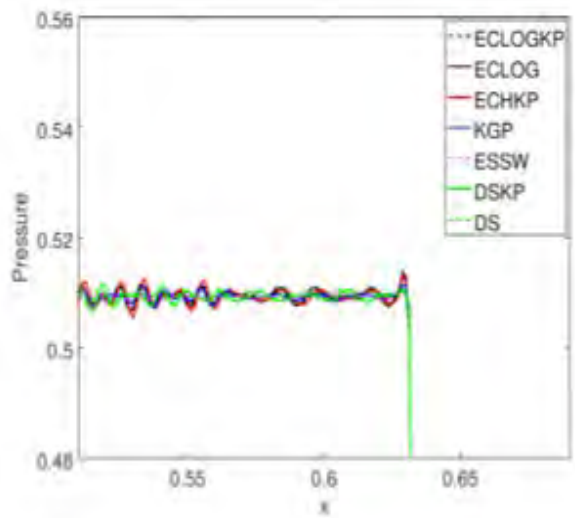
Density (Zoom)

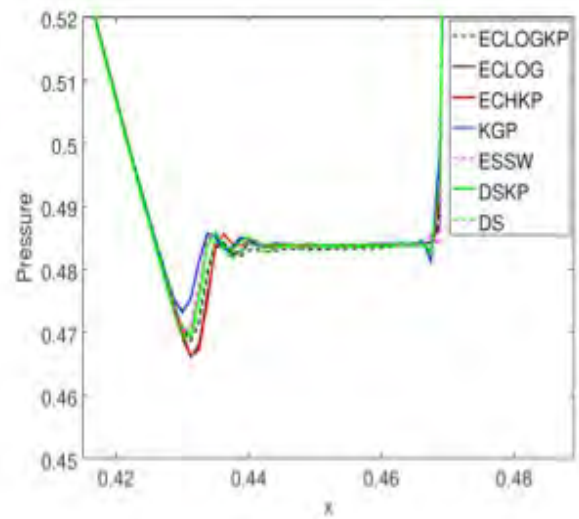


Pressure



Pressure (Zoom)

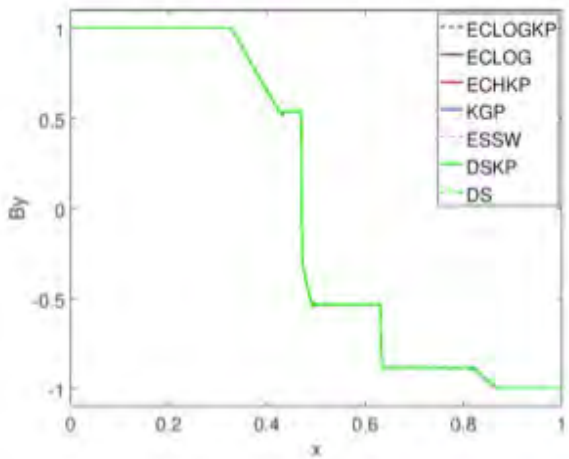




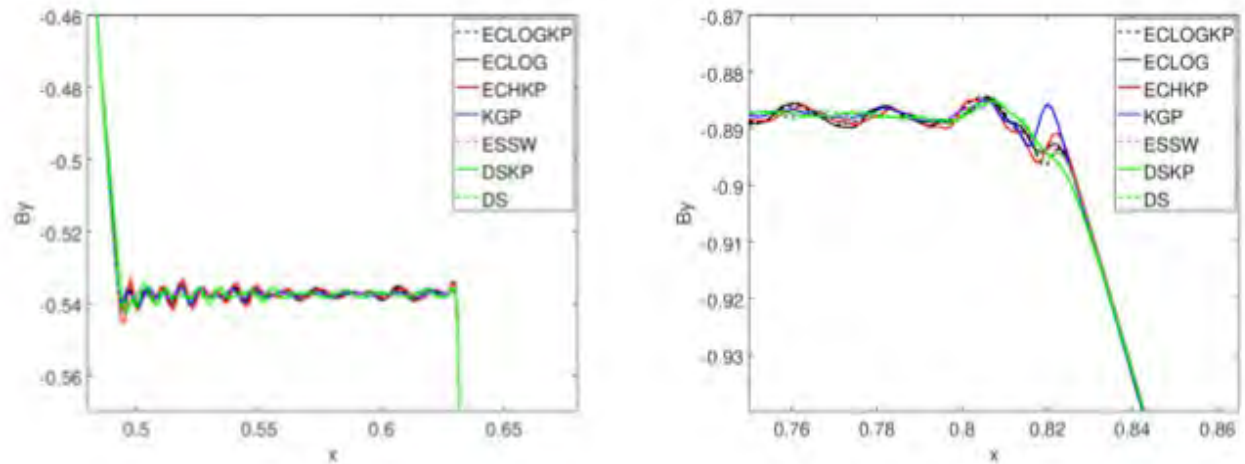
Brio-Wu MHD Shock Tube Problem

(Comparsion Among Seven Methods + 7th-order nonlinear filter)

B_y magnetic field



B_y magnetic field (Zoom)



Summary

Entropy Split (ES) Schemes & Entropy Conservative (EC) Methods

GAS dynamics: Classical Central, DRP Central & Pade (Compact) Spatial Schemes

- ES can improve nonlinear stability for smooth flows in general without added numerical dissipation
- Nonlinear filter version of ES schemes can improve stability & accuracy for DNS & LES with shocks
- High order entropy conserving methods (centered or nonlinear filter version) ES & EC provide similar stability & accuracy improvement.
- Tadmor-type EC methods are approximately twice the CPU than ES

Plasma:

- ES can improve nonlinear stability in general for smooth flows but MHD equations dependent
- Nonlinear filter version of split schemes can improve stability & accuracy for flows with discontinuities but MHD equations dependent
- High order entropy conserving methods (centered or nonlinear filter version) can provide different stability & accuracy improvement, depending on the forms of the MHD equations & the choice of entropy fluxes

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Skew-Symmetric Splitting the Derivative of Products

(Minimize the use of numerical dissipation for high order central schemes)

Splitting of the derivative of a product in conservative & non-conservative part:

$$(ab)_x = \frac{1}{2}(ab)_x + \frac{1}{2}ab_x + \frac{1}{2}a_x b$$

Entropy splitting of Euler flux, Yee et a

Split approximation can be written into conservative form:

$$\frac{1}{2}D_0(ab)_j + \frac{1}{2}a_j D_0 b_j + \frac{1}{2}b_j D_0 a_j = \frac{1}{4}D_+(a_j + a_{j-1})(b_j + b_{j-1})$$

D_0: 2nd-order central, $D_+u_j = (u_{j+1} - u_j)/\Delta x$

Can be generalized to 2pth order accurate: *Ducros et al. 2000*

$$D_{0p}u_j = \sum_{k=1}^{p} \alpha_k^{(p)} D_0(k)u_j \qquad D_0(k)u_j = (u_{j+k} - u_{j-k})/(2k\Delta x)$$

$$\sum_{k=1}^{p} \alpha_k^{(p)} = 1 \qquad \sum_{k=1}^{p} \alpha_k^{(p)} k^{2n} = 0, \quad n = 1, \ldots, p-1$$