Discontinuous Galerkin and Related Methods for ODE

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Ordinary Differential Equations (ODE)

- Mathematical modeling of physical phenomenon (e.g., for fluid flows, Navier-Stokes equations):

$$u_t + f_x + g_y + h_z = 0$$

$$u(0) = u_0$$

- Simplified to a time stepping problem or ODE

$$u_t = F(t, u)$$

$$u(0) = u_0$$

- NASA CFD Vision 2030 Report (2014): Time-stepping remains to be a bottleneck for turbulent flow simulations.

ODE (t replaced by x)

- Find $u(x)$ $\begin{cases} u'(x) = f(x, u(x)) \\ u(0) = u_0 \end{cases}$

- Example 1: quadrature

$$\begin{cases} u'(x) = f(x) \\ u(0) = u_0 \end{cases} \quad \text{Solution} \quad u(x) = u_0 + \int_0^x f(\xi)d\xi$$

- Example 2: stability and accuracy (imaginary λ for advection, real and negative λ for diffusion)

$$\begin{cases} u'(x) = \lambda u(x) \\ u(0) = 1 \end{cases} \quad \text{Solution} \quad u(x) = e^{\lambda x}$$

Outline

- Formulation of discontinuous Galerkin method for ODE (geometric and constructive point of view, different from standard algebraic and analytic view)
- Resulting implicit Runge-Kutta scheme
- Stability and Accuracy
- Conclusions and discussion

Local Frame (Coordinate)

- ODE:

$$\frac{du}{dx} = f(x, u(x)), \qquad u(0) = u_0$$

- Suppose data u_n at x_n is known; with step size h, wish to obtain solution u_{n+1} at $x_{n+1} = x_n + h$.
- Rescale so step size equals 1: with ξ on $[0, 1]$, set $x = x_n + \xi h$. Then

$$\frac{dx}{d\xi} = h \quad \text{and} \quad \frac{du}{d\xi} = \frac{du}{dx}\frac{dx}{d\xi} = h\frac{du}{dx} \ .$$

- On $[0, 1]$, solve

$$\frac{du}{d\xi} = hf(\xi, u(\xi)), \qquad u(0) = u_n$$

- Absorb h into f. On $[0, 1]$, solve

$$\frac{du}{d\xi} = f(\xi, u(\xi)), \qquad u(0) = u_n$$

Discontinuous Galerkin Formulation for ODE

- On $[0,1]$, solve $u'(\xi) = f\big(\xi, u(\xi)\big),\ u(0) = u_n$.

  The DG method seeks a polynomial u_h of degree k on $(0,1]$ such that $u_h' \approx f$ in an average sense, i.e., for $v = 1, \xi, \xi^2, \dots, \xi^k$,

  $$\int_0^1 u_h'(\xi)\, v(\xi) d\xi \ \approx \int_0^1 f\big(\xi, u_h(\xi)\big)\, v(\xi)\, d\xi$$

  and u_h can be discontinuous at x_n. To involve u_n, use integration by parts

  $$\int_0^1 u_h'(\xi) v(\xi) d\xi \ \approx u_h(1)v(1) - u_h(0)v(0) - \int_0^1 u_h(\xi) v'(\xi) d\xi.$$

- Replace $u_h(0)$ above with u_n to involve the starting data.

  The DG method seeks u_h of degree k such that for $v = 1, \xi, \xi^2, \dots, \xi^k$,

  $$u_h(1)v(1) - u_n v(0) - \int_0^1 u(\xi) v'(\xi) d\xi = \int_0^1 f\big(\xi, u(\xi)\big)\, v(\xi)\, d\xi.$$

Example

- On $[0, 1]$, find the linear DG solution for

$$u'(\xi) = 6\xi - 5, \qquad u(0) = u_n = 3.$$

- The exact solution

$$U(\xi) = 3\xi^2 - 5\xi + 3.$$

- The linear DG solution $u_h = a\xi + b$ satisfies, with $v = 1$ and $v = \xi$,

$$u_h(1)v(1) - u_n v(0) - \int_0^1 u_h(\xi)v'(\xi)d\xi = \int_0^1 f(\xi)v(\xi)d\xi$$

$v = 1, \quad a + b - 3 = 3 - 5 \qquad$ or $\qquad a + b = 1$

$v = \xi, \quad a + b - \left(\frac{a}{2} + b\right) = 2 - \frac{5}{2} \quad$ or $\quad a = -1$

Thus,

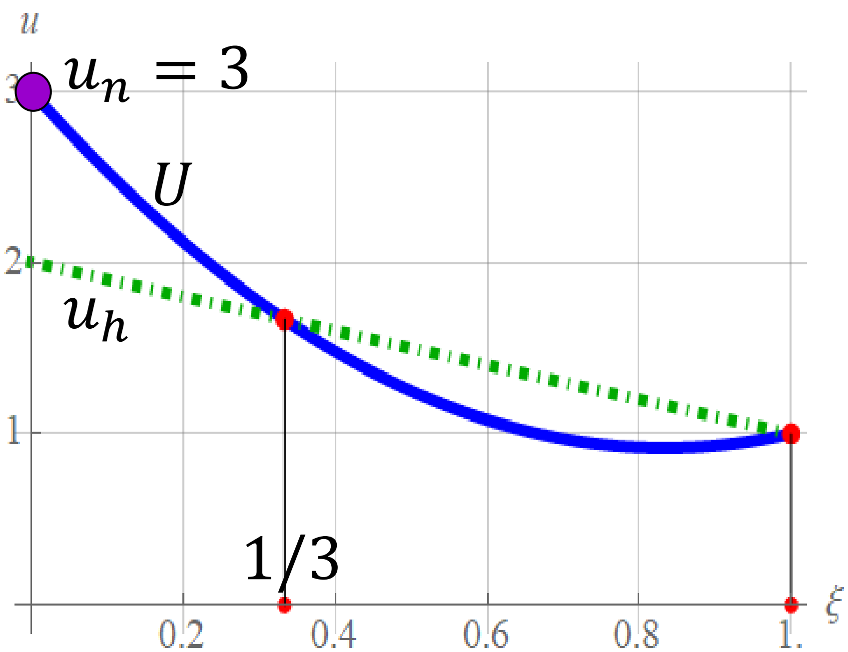
$$a = -1 \text{ and } b = 2$$

Example

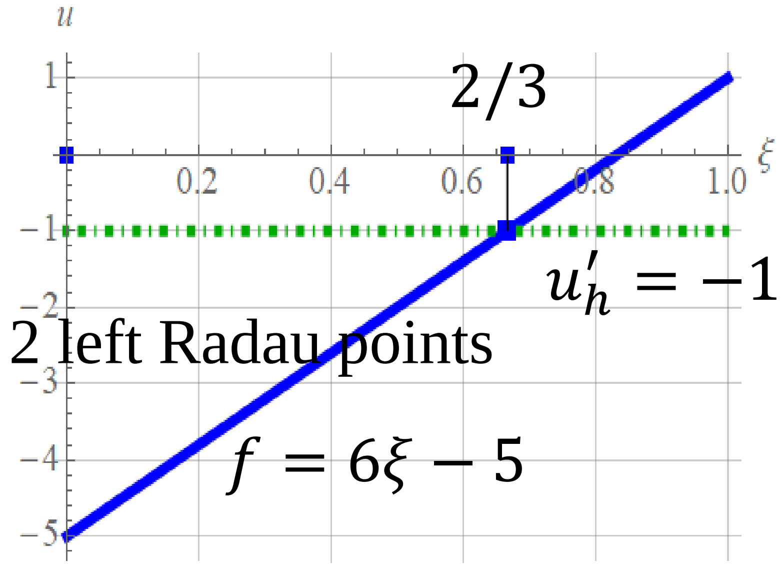
- On $[0, 1]$, find the linear DG solution for

$$u'(\xi) = 6\xi - 5, \qquad u(0) = u_n = 3.$$

- The exact solution $\quad U(\xi) = 3\xi^2 - 5\xi + 3.$
- The linear DG solution $\quad u_h(\xi) = -\xi + 2.$

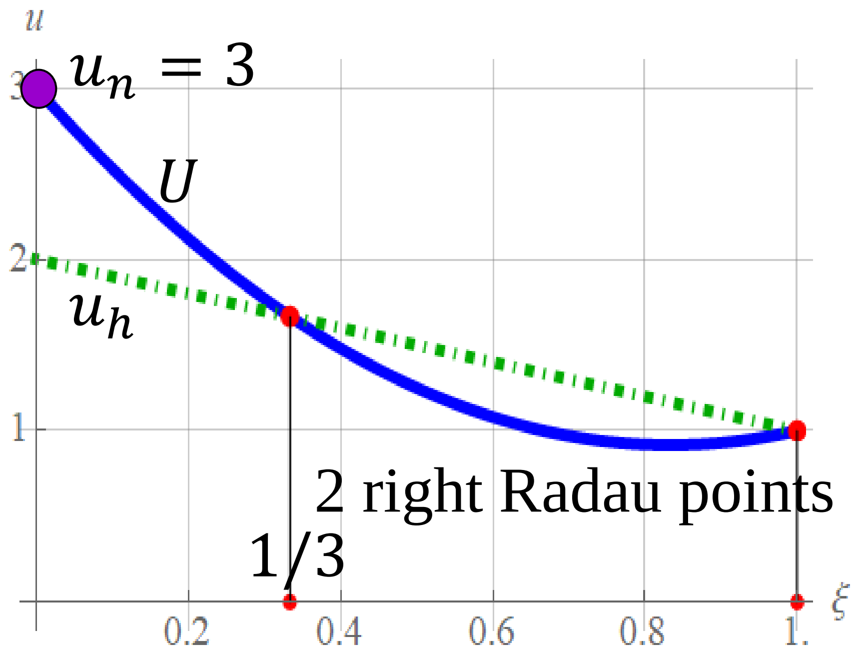


2 right Radau points



2 left Radau points

Derivative of a Function with a Jump

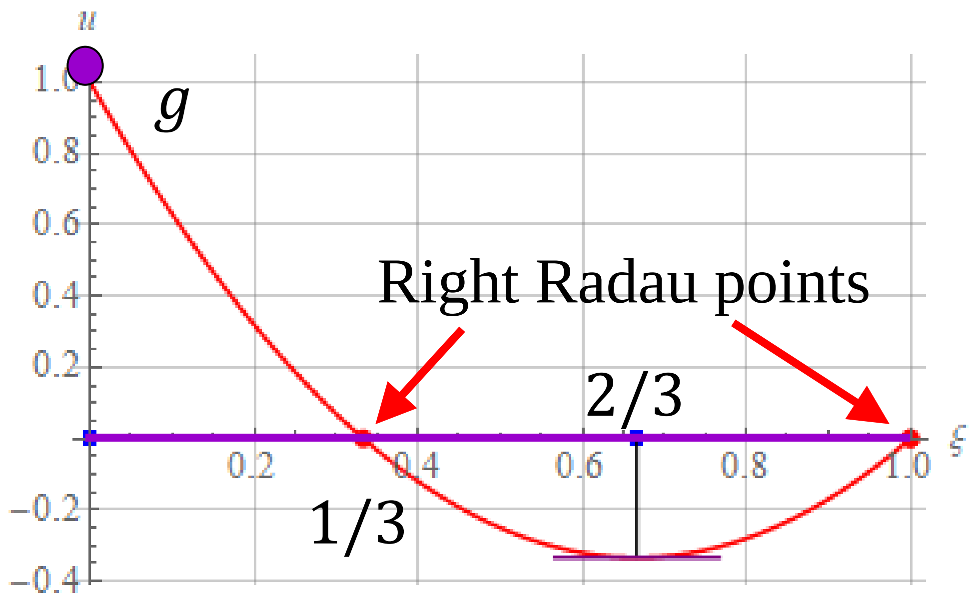


- How to calculate the derivative of a function with a jump:

$$\text{At} \quad \xi = 0,\ w(0) = 3\ ;\ \text{for}\ \ 0 < \xi \leq 1, \quad w(\xi) = -\xi + 2$$

- Obtain quadratic U that satisfies $U(0) = u_n = 3$ and U matches u_h at the 2 right Radau points.
- w' by the DG method is given by U'.

Approximating a Jump by a Polynomial



Approximating the jump from 1 at $\xi = 0$ to 0 for $0 < \xi \leq 1$ by a polynomial of degree $k+1$ defined by $k+2$ conditions:

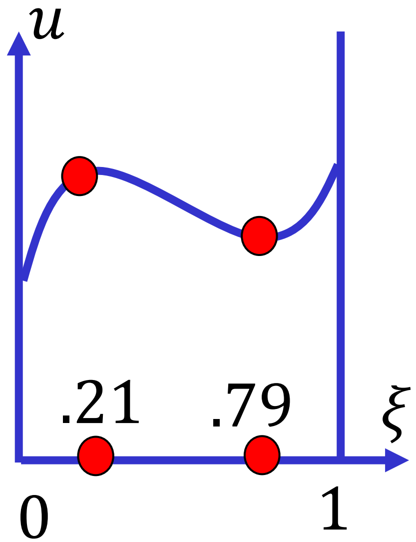
$g(0) = 1$ and g vanishes at the $k+1$ right Radau points

Then g is the right Radau polynomial $R_{R,k+1}$, and

$$U = u_h + [u_n - u_h(0)]g$$

2-Point Quadratures

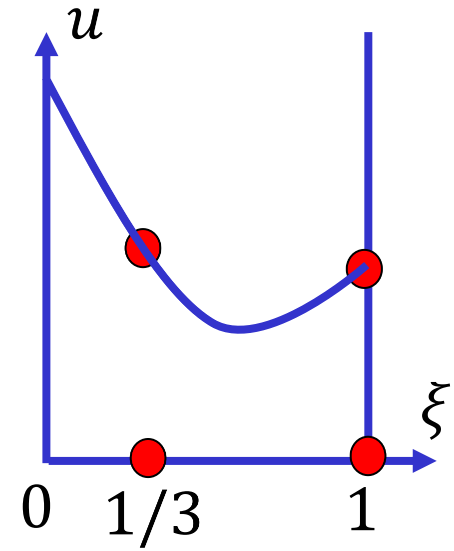
$$\int_0^1 f(\xi)d\xi \approx b_1 f(\xi_1) + b_2 f(\xi_2)$$



Gauss

$$\frac{1}{2}f(.21) + \frac{1}{2}f(.79)$$

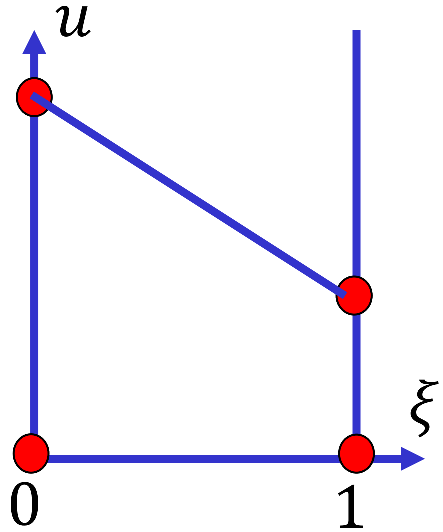
Exact for a cubic f



Right Radau

$$\frac{3}{4}f\left(\frac{1}{3}\right) + \frac{1}{4}f(1)$$

Exact for a parabolla f



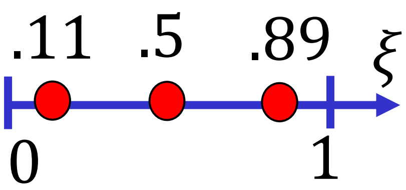
Equidistance

$$\frac{1}{2}f(0) + \frac{1}{2}f(1)$$

Exact for a linear f

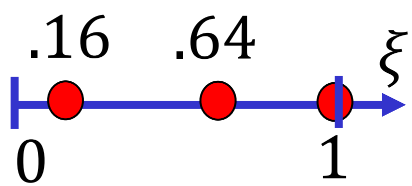
$(k+1)$-Point Quadratures

$$\int_0^1 f(\xi)d\xi \approx \sum_{i=1}^{k+1} b_i f(\xi_i)$$



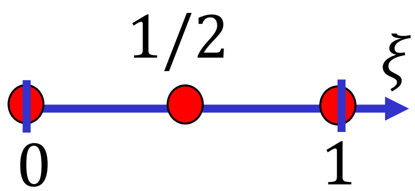
Gauss

Exact for polynomials of degree $2k+1$



Right Radau

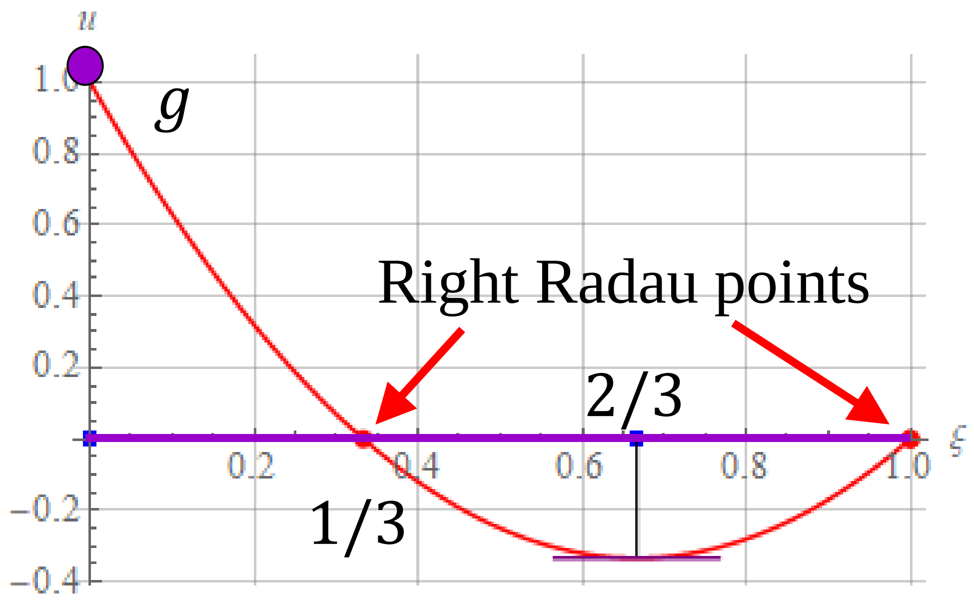
Exact for polynomials of degree $2k$



Equidistance

Exact for polynomials of degree $k+1$

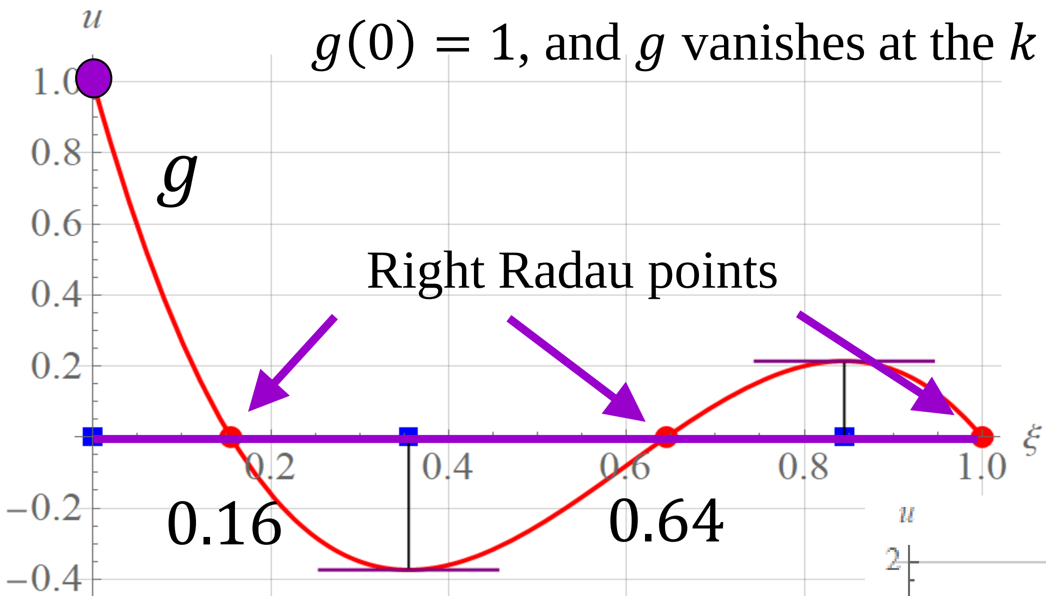
Correction Function g for $k = 1$

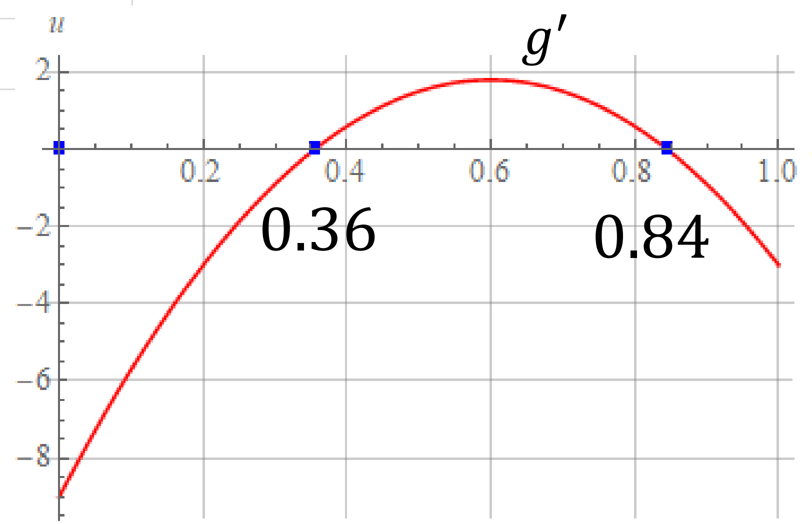


$$U = u_h + [u_n - u_h(0)]g$$

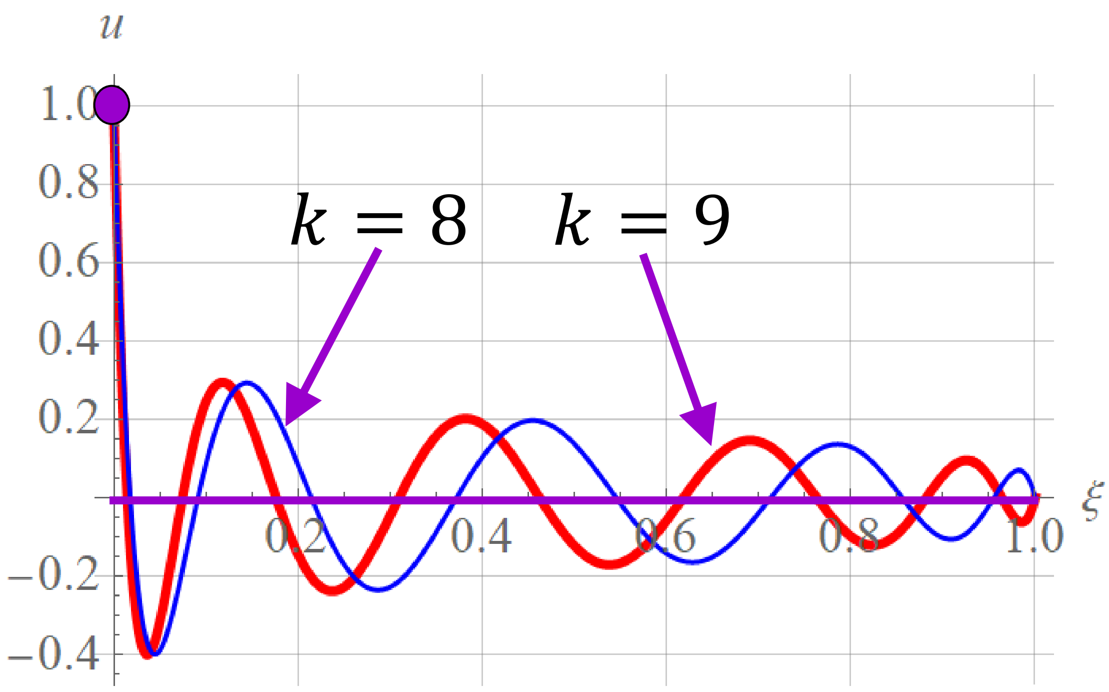
Correction Function g for $k = 2$

g is the right Radau polynomial of degree $k + 1$ defined by $g(0) = 1$, and g vanishes at the $k + 1$ right Radau points $\xi_{R,i}$.

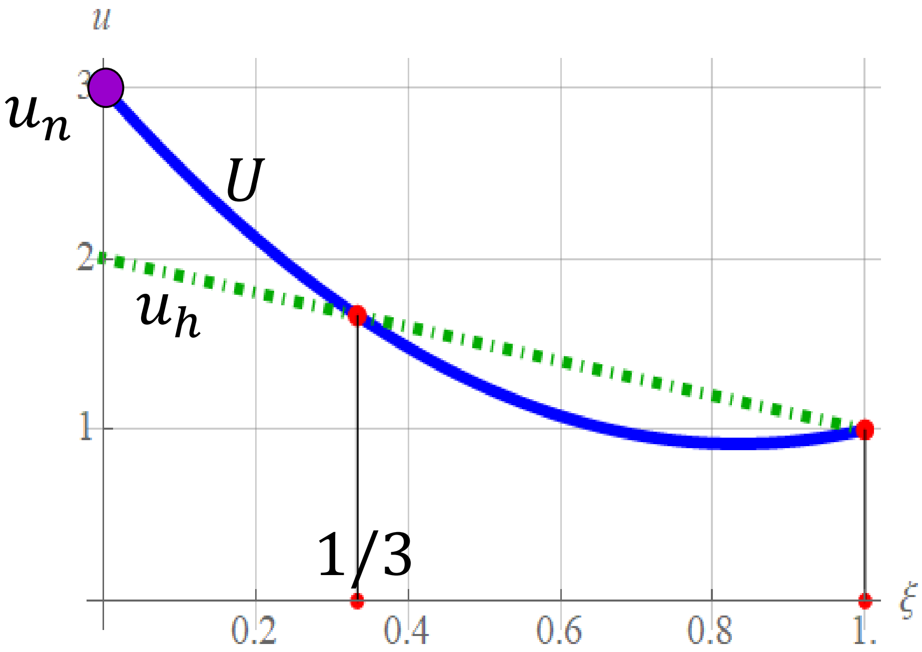




Correction Functions (Radau Polynomial) of degree k for $k = 8$ and $k = 9$



DG Solutions u_h and U



u_h is of degree k; U and g are of degree $k+1$,

$$U(\xi) = u_h + [u_n - u_h(0)]g$$

U satisfies $U(0) = u_n$ and, for $v = 1, \xi, \xi^2, \dots, \xi^k$,

$$\int_0^1 U'(\xi)\, v(\xi) d\xi = \int_0^1 f(\xi, u_h(\xi))\, v(\xi)\, d\xi$$

DG, CG, and Collocation Methods under Right Radau quadrature

$$U = u_h + [u_n - u_h(0)]g$$

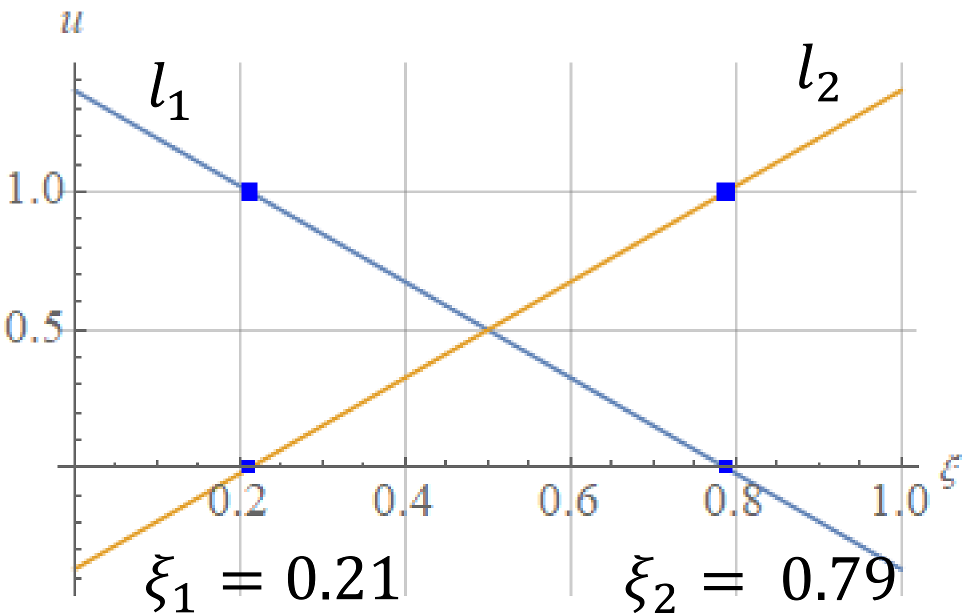
- u_h and U take on the same values at the $k + 1$ right Radau points.
- u_h is discontinuous, but U is continuous.
- Under the $k + 1$ point right Radau quadrature, the DG, CG, and collocation methods yield the same solution U.

DG under Gauss Quadrature

$$U' = f_h, \quad U(0) = u_n;$$

$$U(\xi) = u_n + \int_0^{\xi} f_h d\eta; \quad U = u_h + [u_n - u_h(0)]g$$

2 Gauss points; linear $f_h = f_1 l_1 + f_2 l_2$

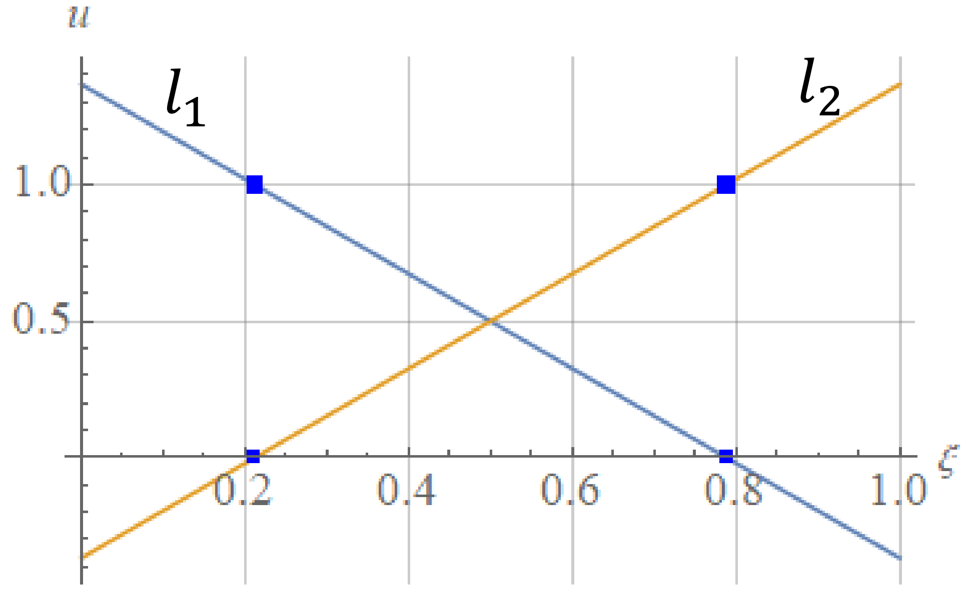


$$\int_0^{\xi} f_h d\eta =$$

$$f_1 \int_0^{\xi} l_1(\eta) d\eta +$$

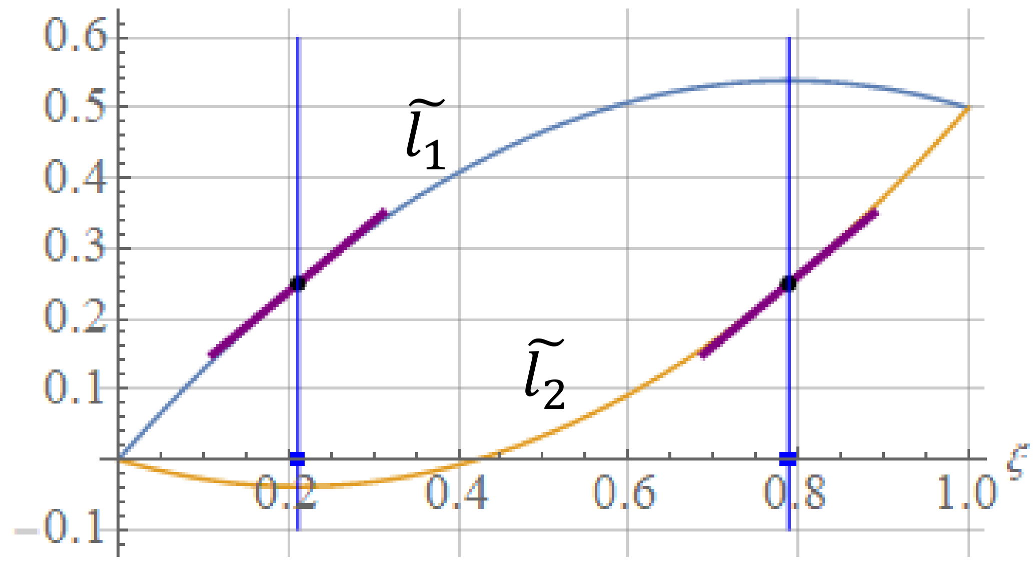
$$f_2 \int_0^{\xi} l_2(\eta) d\eta$$

DG with Gauss Quadrature

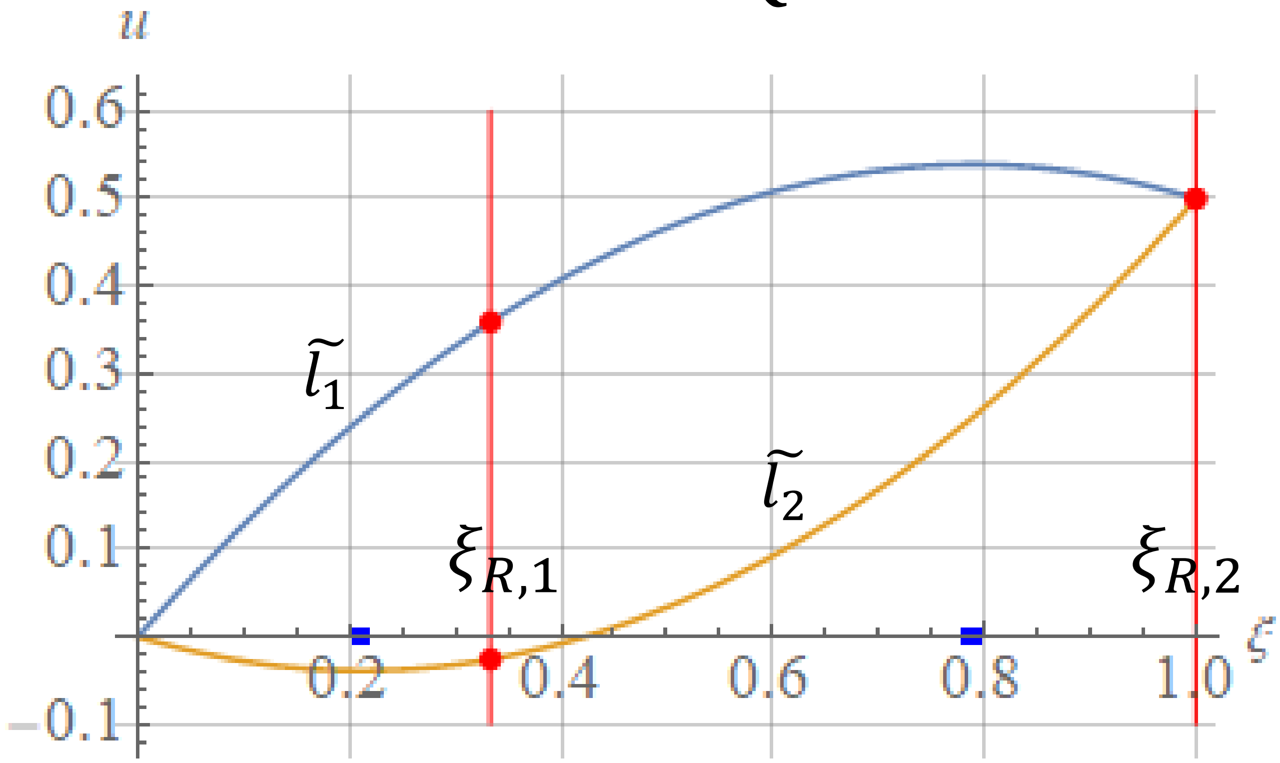


$$\widetilde{l_1}(\xi) = \int_0^{\xi} l_1(\eta)d\eta$$

$$\widetilde{l_2}(\xi) = \int_0^{\xi} l_2(\eta)d\eta$$



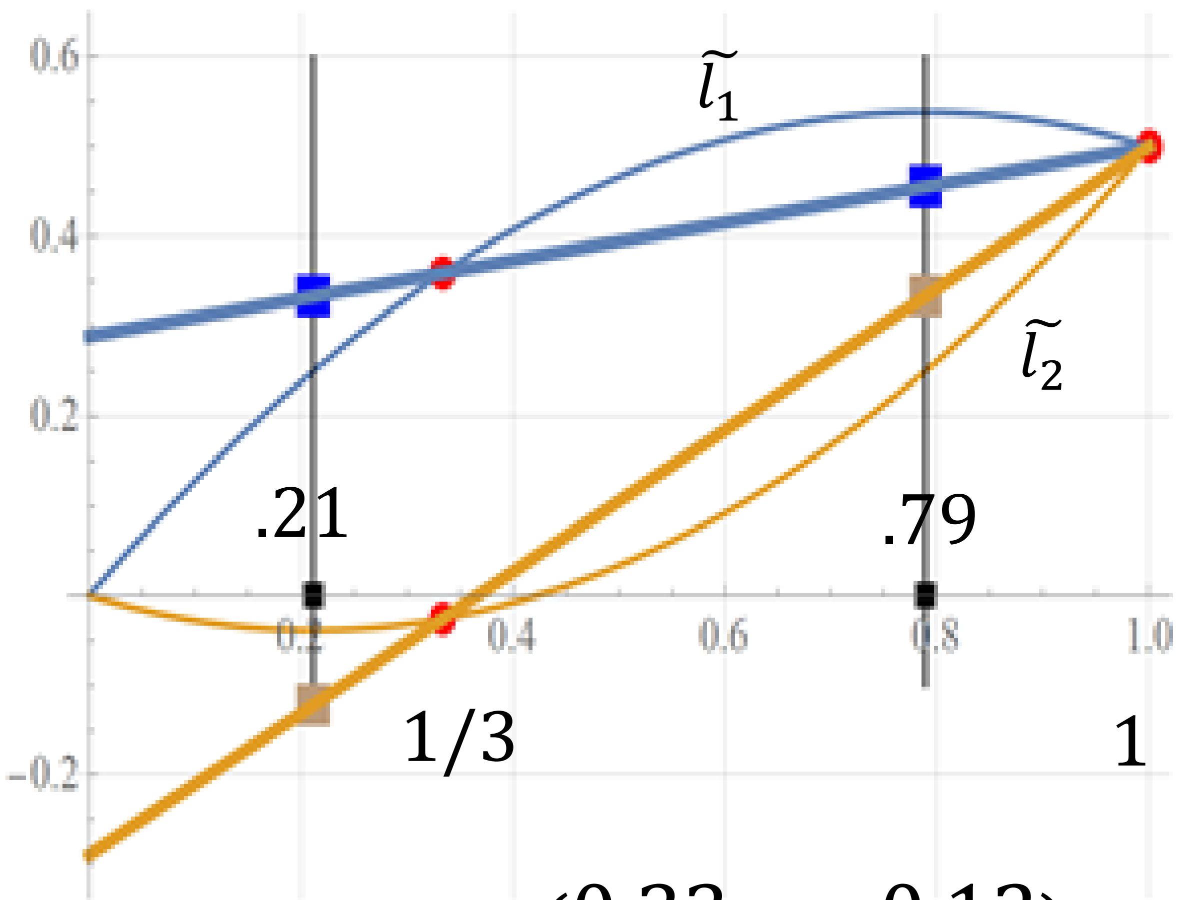
DG with Gauss Quadrature



$$U(\xi) = u_n + f_1\widetilde{l_1}(\xi) + f_2\widetilde{l_2}(\xi); \qquad U = u_h + [u_n - u_h(0)]g$$

$$i = 1, 2, \;\; u_h(\xi_{R,\,i}) = U(\xi_{R,\,i}) = u_n + f_1\widetilde{l_1}(\xi_{R,\,i}) + f_2\widetilde{l_2}(\xi_{R,\,i})$$

DG with Gauss Quadrature



$$\begin{pmatrix} 0.33 & -0.12 \\ 0.46 & 0.33 \end{pmatrix}$$

Implicit Runge-Kutta Method DG-Gauss

$$\begin{pmatrix} .33 & -.12 \\ .46 & .33 \end{pmatrix}$$

Butcher Tableau

$$\begin{array}{c|cc} .21 & .33 & -.12 \\ .79 & .46 & .33 \\ \hline * & .5 & .5 \end{array}$$

$$u_{n,1} = u_n + h\,[\,.33 f\left(x_n + .21h, u_{n,1}\right) - .12 f\left(x_n + .79h, u_{n,2}\right)]$$

$$u_{n,2} = u_n + h\,[\,.46 f\left(x_n + .21h, u_{n,1}\right) + .33 f\left(x_n + .79h, u_{n,2}\right)]$$

$$u_{n+1} = u_n + h\,[\,.5 f\left(x_n + .21h, u_{n,1}\right) + .5\, f\left(x_n + .79h, u_{n,2}\right)]$$

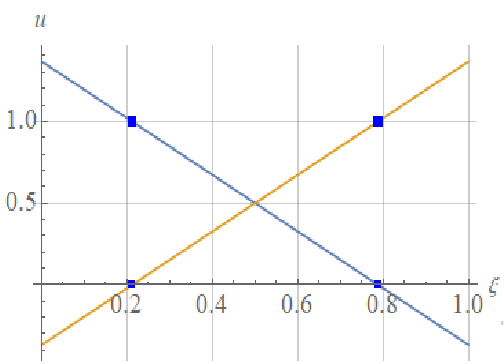
DG-Gauss and Gauss Collocation Methods

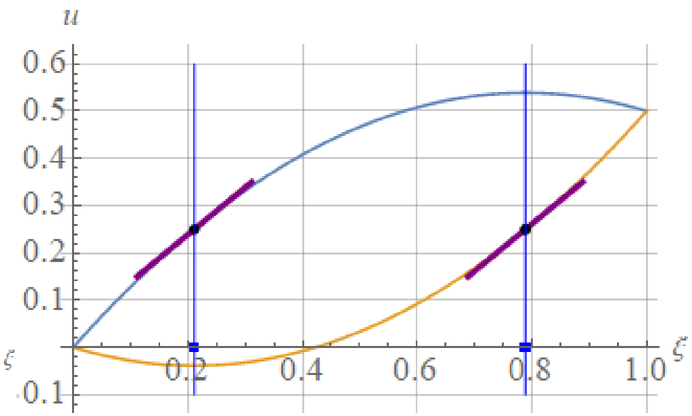
$$\begin{array}{c|cc} .21 & .33 & -.12 \\ .79 & .46 & .33 \\ \hline * & .5 & .5 \end{array} \qquad \begin{array}{c|cc} .21 & .25 & -.04 \\ .79 & .54 & .25 \\ \hline * & .5 & .5 \end{array}$$

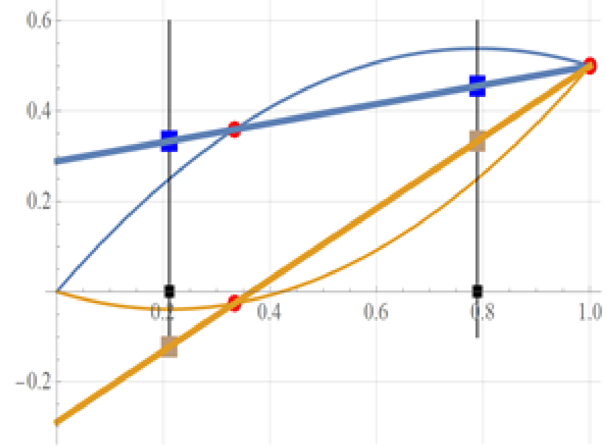
DG-Gauss	Gauss-Collocation
3rd-order accurate	4rd-order accurate
L-stable	Not L-stable

We can adjust numerical dissipation by blending these methods

DG with Left Radau Quadrature, $k = 2$

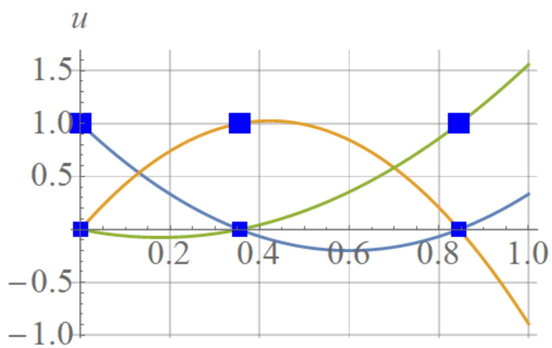


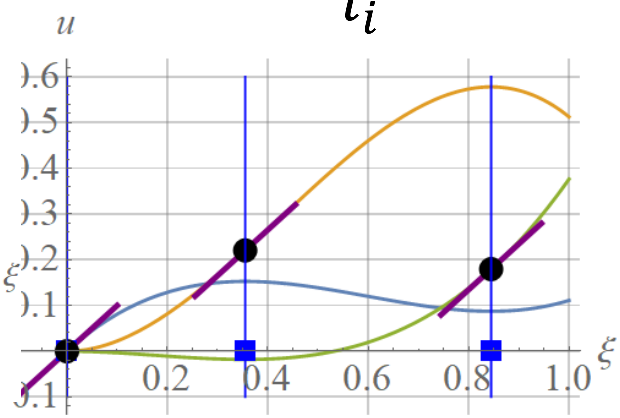


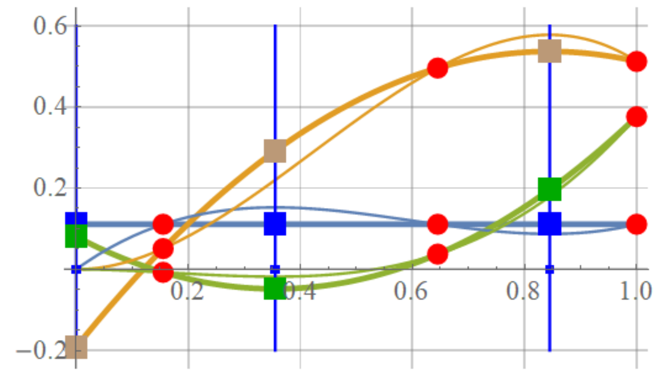


l_i

$\tilde{l}_i$

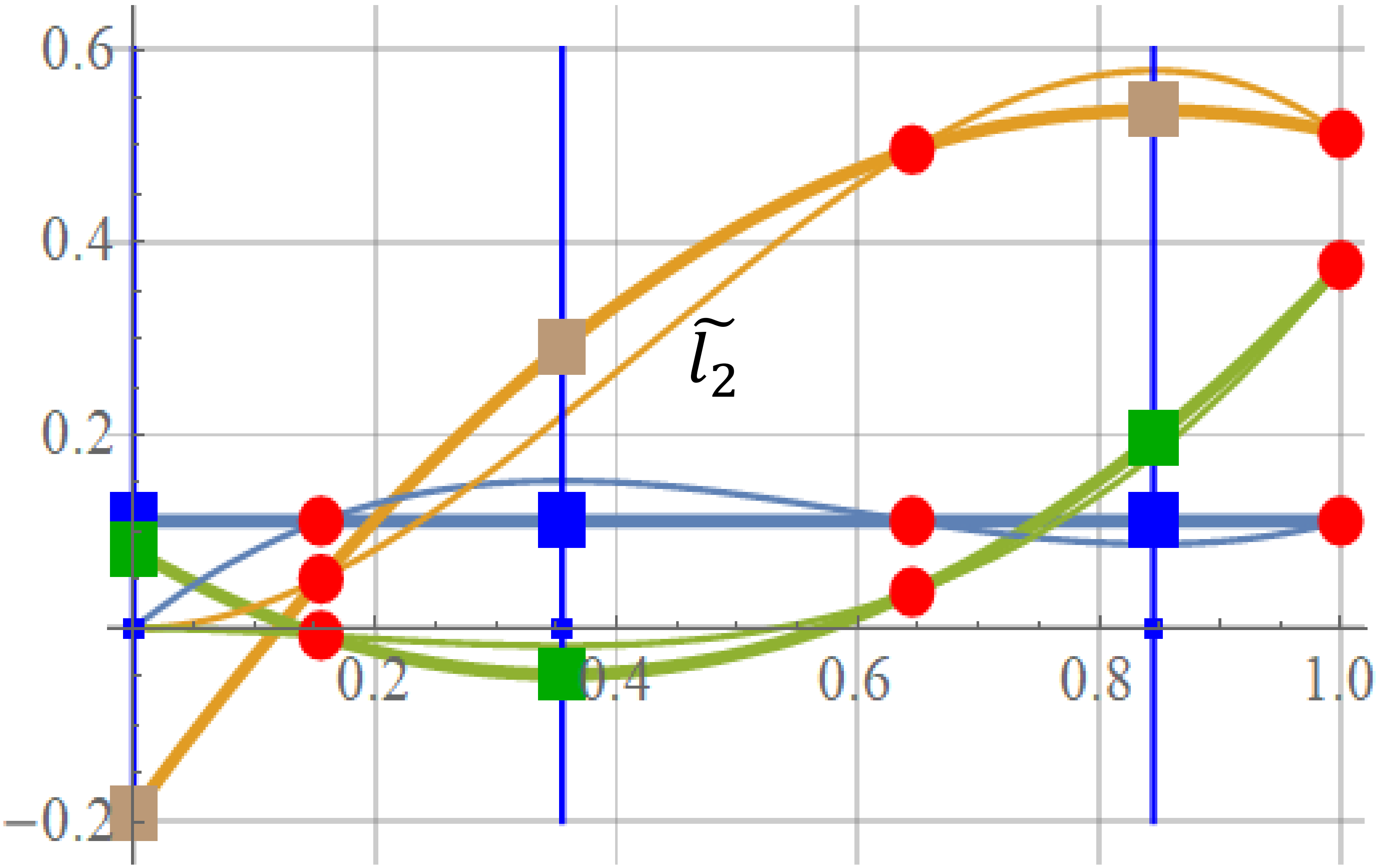






Radau IA Method

DG with Left Radau Quadrature



IRK-DG

DG	**IRK Counterpart**
Left Radau Quadrature	Radau IA
Right Radau Quadrature	Radau IIA
Gauss Quadrature	DG-Gauss

IRK-DG

(a) Radau IA (left Radau)

$$
\begin{array}{c|cc}
0 & \frac{1}{4} & -\frac{1}{4} \\
\frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\
\hline
 & \frac{1}{4} & \frac{3}{4}
\end{array}
$$

(c) Radau IIA (right Radau)

$$
\begin{array}{c|cc}
\frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\
1 & \frac{3}{4} & \frac{1}{4} \\
\hline
 & \frac{3}{4} & \frac{1}{4}
\end{array}
$$

(b) DG-Gauss

$$
\begin{array}{c|cc}
\frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{3} & \frac{1-\sqrt{3}}{6} \\
\frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1+\sqrt{3}}{6} & \frac{1}{3} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Example

On $[0, 1]$, with $\lambda = 2\pi i/3$ and $\lambda = \pi i/3$, find linear DG solution u_h and quadratic solution U for

$$u' = \lambda u$$

$$u(0) = 1$$

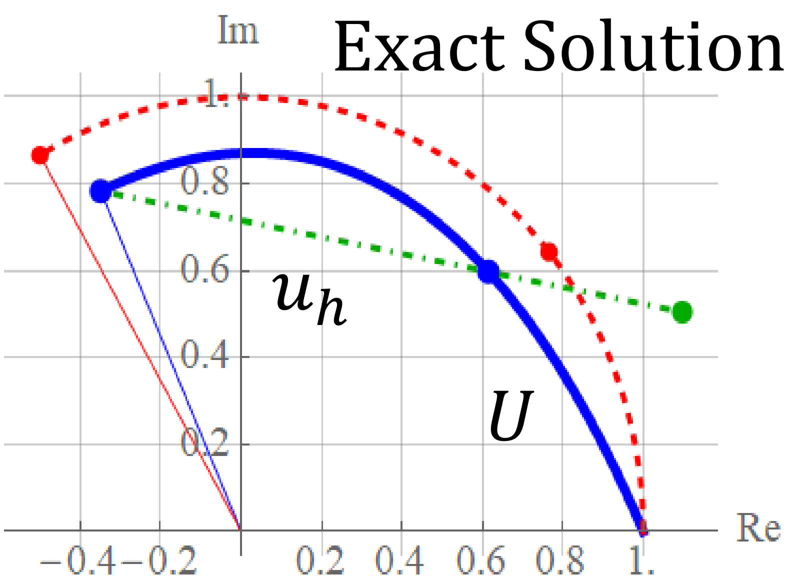
Exact solution:

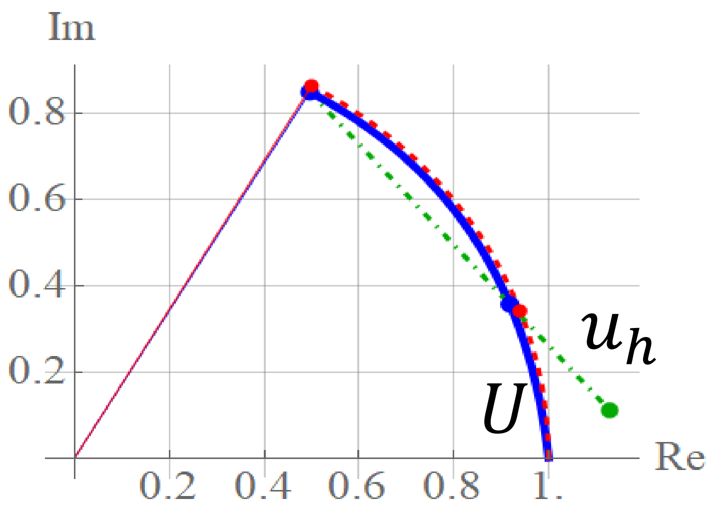
$$u_{\text{Exact}}(\xi) = e^{\lambda\,\xi}$$

After one step of size $h = 1$, the exact solution is

$$u_{\text{Exact}}(1) = e^{\lambda}$$

Example of DG Solutions





$\lambda = 2\pi i/3$ $\lambda = \pi i/3$

$\mathrm{Er} = |u_{\mathrm{Exact}}(1) - u_h(1)|$

$\mathrm{Er}_1 \approx 0.17;$ $\mathrm{Er}_2 \approx 0.015;$

$\mathrm{Er}_1/\mathrm{Er}_2 \approx 11.2$ then 15 and 15.7; third-order accuracy

Linear DG Solution

$$u_{\text{Exact}}(1) = e^z$$

$$R_1(z) = \frac{2z + 6}{z^2 - 4z + 6}$$

$$E_1 = e^z - R_1(z) = \frac{z^4}{72} + \frac{19z^5}{1080} + \cdots$$

Conclusions and Discussion

- DG method for ODE was formulated from a constructive and geometric and point of view by using the correction function, which is a polynomial approximating the jump.
- Derived IRK-DG methods, namely, Radau IA, Radau IIA, and DG-Gauss.
- The approach provides intuitions on DG for ODE, show relations between continuous and discontinuous solutions, as well as clarifies relations among CG, DG, and collocation methods.
- **An effective iteration procedure for these IRK methods remains to be found.**

Thank you.