

Recent Advancement of High Order Entropy Split Methods for Compressible Gas Dynamics and MHD

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Outline

- Introduction
- High Order Entropy Split Methods for the Euler Equations
- High Order Tadmor-type Entropy Conserving Method for the Euler Equations
 - Employing Harten entropy functions (1983) resulting in high order entropy conserving method
- Comparison Among **Seven 8th-order Methods**
 - Accuracy & stability comparison for long time-accurate unsteady integration
 - L2 Max-norm error,
 - Mass conservation errors,
 - Entropy conservation errors
 - Kinetic energy preservation errors
- Summary

Objectives

- **Improvement in Nonlinear Stability**

(Reduce aliasing errors & minimize the use of numerical dissipation)

- > More efficient than Tadmor-type entropy conserving methods of same order
- > Methods cater to **long time integration** of compressible turbulence

- **Improvement in Accuracy** (for a wide spectrum of flow speeds)

- > **Efficient Nonlinear Filter** methods with adaptive local **flow sensors** designed to **further minimize** the use of numerical dissipation

Yee et al. (1999, 2009), Sjogreen & Yee (2004)

- **Mass Conservation Error for shock-free & flows with shocks**

- > Entropy split method is not a conservative splitting

- > **New entropy split methods** to minimize mass conservation errors for long time integrations

Key Steps in the Construction of Stable High Order Entropy Split Methods

- (I) **Skew-Symmetric Splitting of the Euler Flux Derivatives for Finite Domain**
(Perfect gas, thermally perfect gas & chemical reacting flows)
- Harten entropy functions for the Euler equations, Harten (1983)
 - Homogeneous of degree one of the Euler Flux
 - Split the Euler flux into conservative & non-conservative portions by transforming the Euler equations in term of the entropy variables
(integration-by-parts, including the boundary terms based on positivity of the Hessian entropy vector)
 - Stable L_2 energy-norm estimate
- (II) **Discrete Analogue of Integration-by-Parts L_2 Norm Estimate for Stability:**
High Order Central Spatial Differencing (Classical, DRP & Pade') with SBP Operators on the Resulting Entropy Splitting Form of the Flux Derivative → **Entropy Split Methods**
Olsson & Oliger (1994), Gerritsen & Olsson (1996), and Yee et al. (2000)

Entropy Splitting of Euler Flux Derivatives

(Oliger & Olsson, Olsson, Gerittsen & Olsson, Yee et al. (1994-1999))

Entropy stable (generalized energy norm) with SBP for Central Schemes

Physical Entropy, Homogeneous of Degree One of Euler Flux Derivative $\mathbf{F}_x$ $\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0$

$$\mathbf{F}_x = \frac{\beta}{\beta + 1} \mathbf{F}_x + \frac{1}{\beta + 1} \mathbf{F}_w W_x, \quad \beta \neq -1$$

$$\begin{aligned} W &= [w_1, w_2, w_3, w_4, w_5]^T \\ &= \frac{p^*}{p} \left[e + \frac{\alpha - 1}{\gamma - 1} p, -\rho u, -\rho v, -\rho w, \rho \right]^T, \end{aligned}$$

where

$$p^* = -(p\rho^{-\gamma})^{\frac{1}{\alpha+\gamma}}$$

and

$$\beta = \frac{\alpha + \gamma}{1 - \gamma}, \quad \alpha > 0 \text{ or } \alpha < -\gamma.$$

Entropy Split Method Parameter β

$$\boxed{U_t + F(U)_x = 0} \quad F_x = \frac{\beta}{\beta + 1} F_x + \frac{1}{\beta + 1} F_w W_x$$

- No splitting of F_x when $\beta \rightarrow \infty$
- The convexity condition of the entropy function restricts the splitting into a conservative & non-conservative portions
- Harten, Gerritsen & Olsson considered the unphysical branch $\beta < 0$
Yee et al. (2000) selected the physical branch
- When $\beta = 1$ the splitting can rewrite into a conservative splitting → same as Ducros et al. splitting
- For long time integration of certain smooth flows, the entropy split methods are **stable without added numerical dissipation** using Classical, DRP & Pade' spatial discretizations with $\beta = 1$ or 2
- For flows with discontinuities, using ESSW (switch to conservative at discontinuities) in conjunction with Yee et al. nonlinear filters, **wrong propagation speed of discontinuities are not affected**

Special Properties of Entropy Split Methods

Valid for perfect gas, thermally perfect gas & chemical reacting flows

High order methods free stream preservation for moving & deforming grids

-
- The splitting of the flux derivative using Harten entropy functions is **a skew-symmetric splitting**
 - For the **entropy splitting parameter** $\beta = 1$, **it is the Ducros et al. splitting**
 - Only a **linear difference operator** is required to define the method
 - It does **not rely on two-point numerical flux**
 - The method **conserves the entropy in a global sense**, but the method itself is **not in conservation form**
 - The **numerical boundary closure** follows directly from the **SBP operator for nonperiodic BC**
 - Harten entropy functions suitable for **the Tadmor-type entropy conservative method in a local sense**
 - **Two Proposed New Split Methods to Correct Mass Conservation Error:**
Sjogreen & Yee, “Construction of Conservative Numerical Fluxes for the Entropy Split Method”,
HONOM-2019 special CAMC volume, 2021

Comparison Among Seven 8th-Order Methods

- **ECLOG**: Tadmor-type entropy conserving method using the Tadmor entropy function $E_L = -\rho \log(p\rho^{-\gamma})$.
- **ECLOGKP**: Tadmor-type entropy conserving method using the Tadmor entropy function with Ranocha's kinetic energy preserving modification [32].
- **ECHKP**: Entropy conserving using the Harten class of entropy functions $E_H = -\frac{\gamma+\alpha}{\gamma-1}\rho(p\rho^{-\gamma})^{\frac{1}{\alpha+\gamma}}$ [50, 23]. In its base form also satisfies Ranocha's kinetic energy preservation condition (KEP); only one variant for this method [43, 32].
- **KGP**: Kennedy-Gruber-Pirozzoli (KGP) skew-symmetric splitting of the inviscid flux derivative that is kinetic energy preserving [26, 31, 13].

Comparison Among **Seven** 8th-Order Methods (*Cont.*)

- **ESSW**: Entropy split with Ducros et al. splitting but switch to regular central near discontinuities [46].
- **DSKP**: Ducros Split with KEP.
- **DS**: Momentum conserving Ducros et al. skew-symmetric split of the inviscid flux derivative [14].

2D Isentropic Vortex Convection

(Inviscid, pure convection of the IC)

Computational Domain: Period BC, $0 \leq x \leq 18, 0 \leq y \leq 18$

Initial Condition:

$$\rho(x, y) = \left(\left(1 - \frac{(\gamma - 1)\beta^2}{8\gamma\pi^2} e^{1-r^2} \right)^{\frac{1}{\gamma-1}} \right)$$

$$u(x, y) = u_{\infty} - \frac{\beta(y - y_0)}{2\pi} e^{(1-r^2)/2}$$

$$v(x, y) = v_{\infty} + \frac{\beta(x - x_0)}{2\pi} e^{(1-r^2)/2}$$

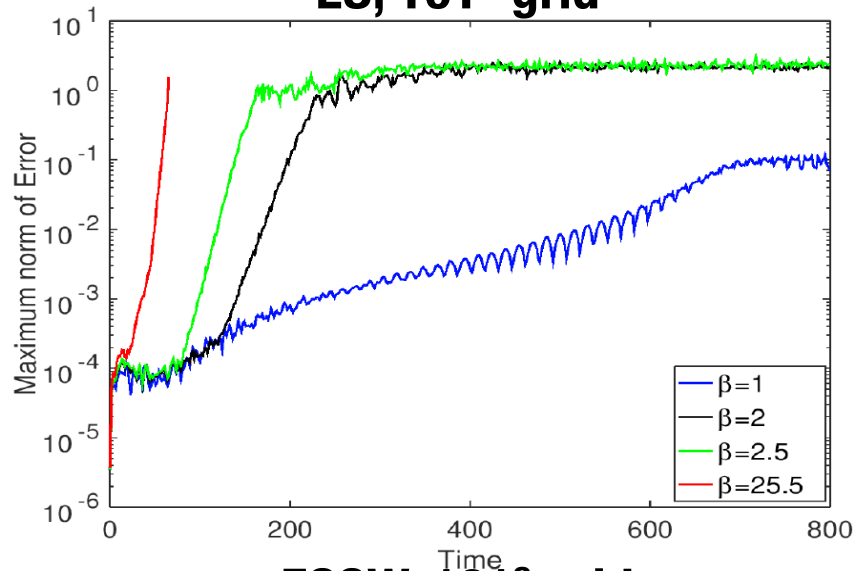
$$p(x, y) = \rho(x, y)^{\gamma},$$

$$r^2 = x^2 + y^2, \beta = 5, \gamma = 1.4, u_{\infty} = 1, v_{\infty} = 0.$$

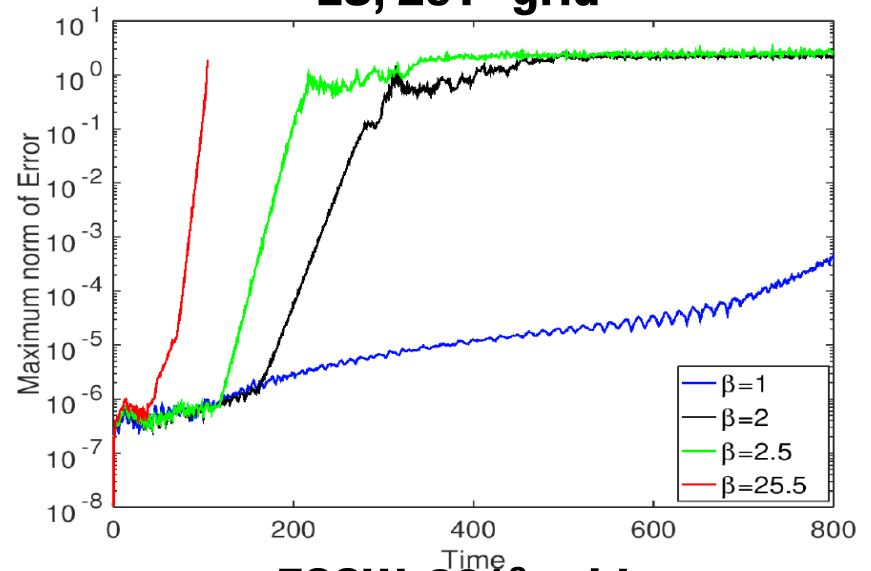
2D Isentropic Vortex Convection

Split parameter β study by ES & ESSW methods, 100^2 & 202^2 grids

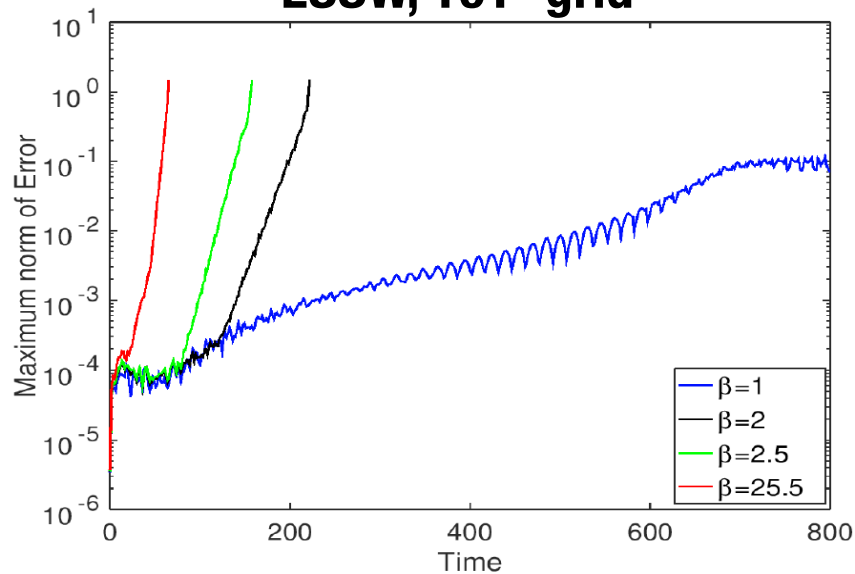
ES, 101^2 grid



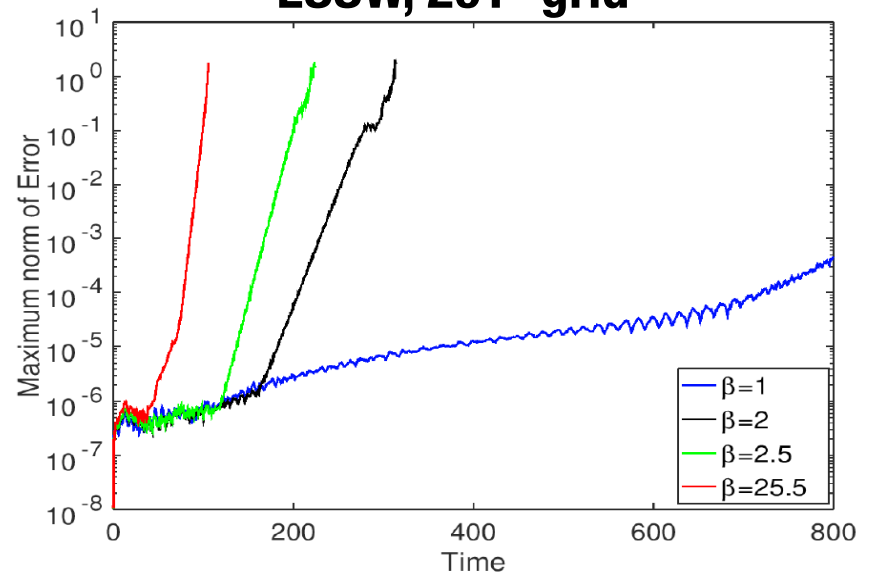
ES, 201^2 grid



ESSW, 101^2 grid



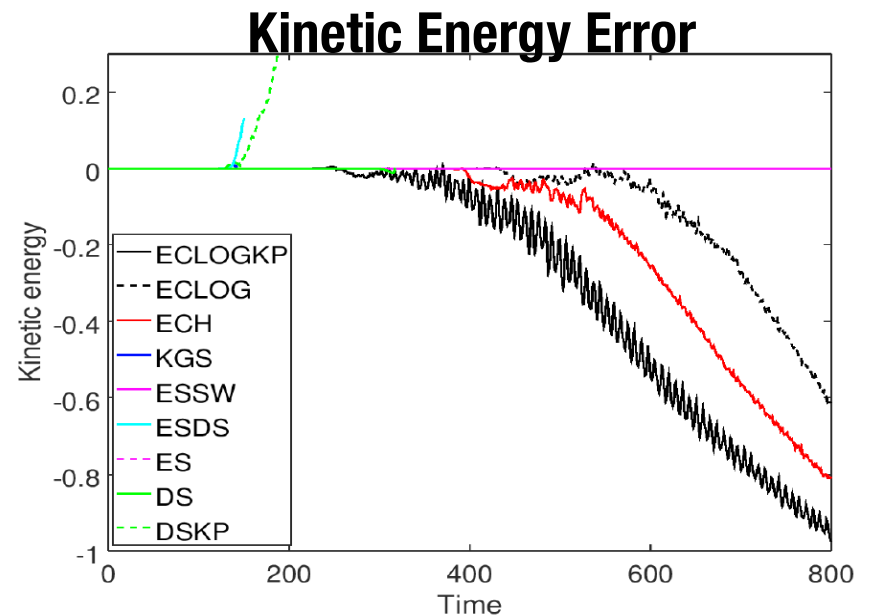
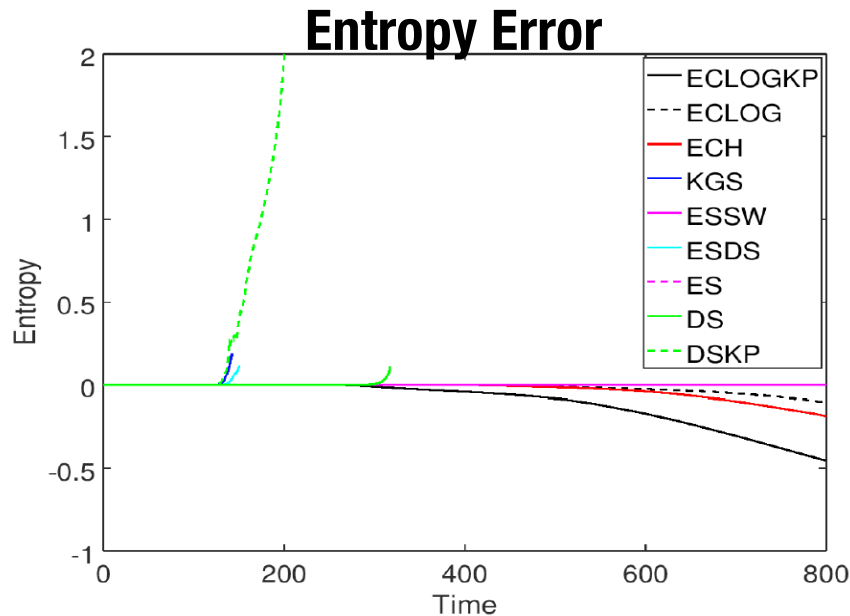
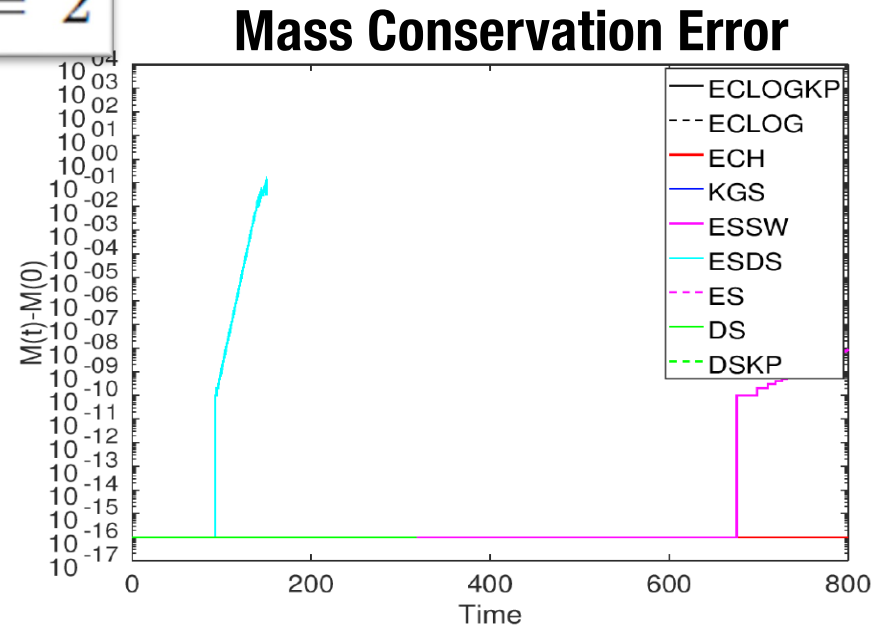
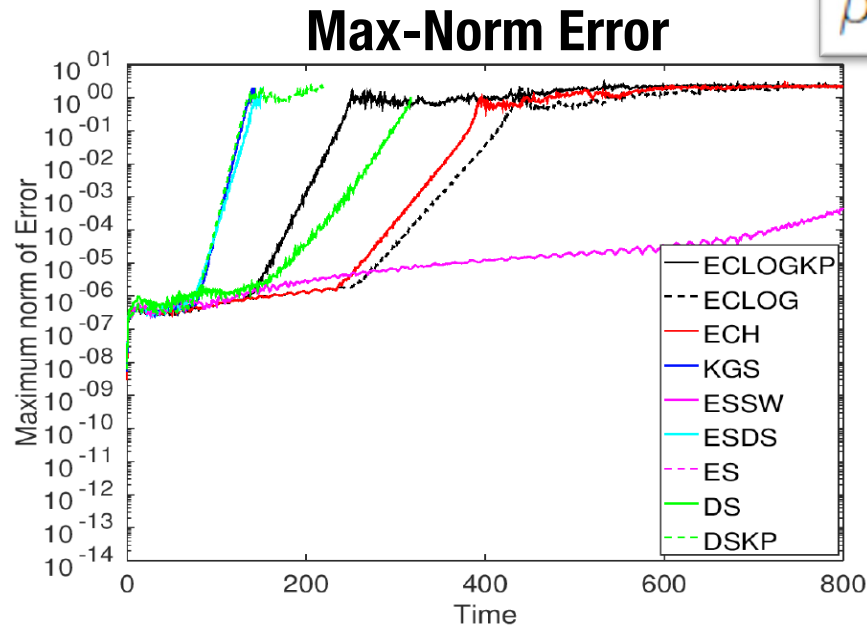
ESSW, 201^2 grid



2D Isentropic Vortex Convection

Nine 8th-order method comparison, 201^2 grid

$$\beta = 2$$



3D Taylor-Green vortex

(Inviscid & Viscous Shock-Free Turbulence)

Computational Domain: 2π square cube, 64^3 grid.
(Reference solution on 256^3 grid)

Initial condition

$$\begin{aligned}\rho &= 1, \\ p &= 100 + ([\cos(2z) + 2][\cos(2x) + \cos(2y)] - 2)/16, \\ u_x &= \sin x \cos y \cos z \\ u_y &= -\cos x \sin y \cos z \\ u_z &= 0.\end{aligned}$$

Initial turbulent Mach number: $M_{t,0} = 0.042$

Final time: $t = 10$

Viscous case

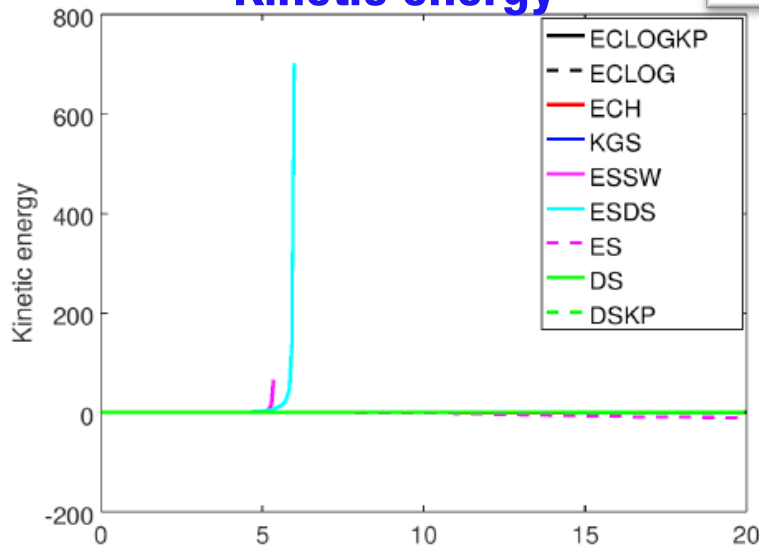
$$\begin{aligned}\mu / \mu_{ref} &= (T / T_{ref})^{3/4} \\ \mu_{ref} &= 0.005, T_{ref} = 1, Re_0 = 2040\end{aligned}$$

3D Taylor-Green Vortex

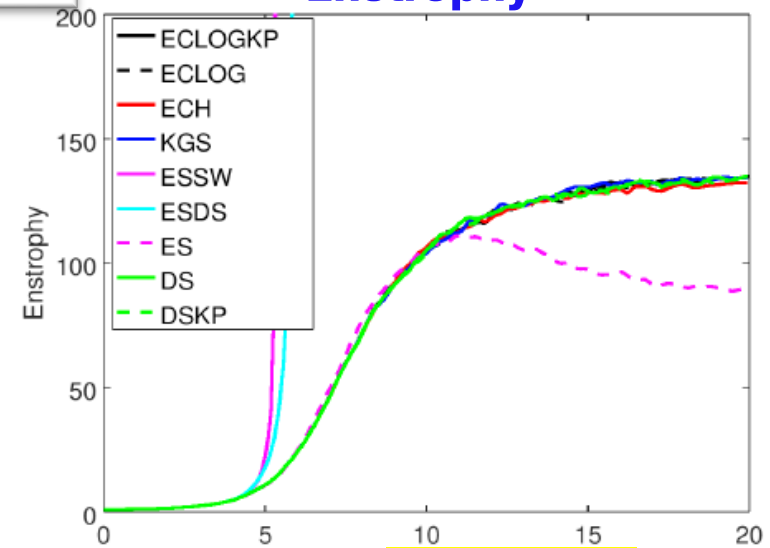
Nine 8th-order method comparison, 64³ grid

$$\beta = 2$$

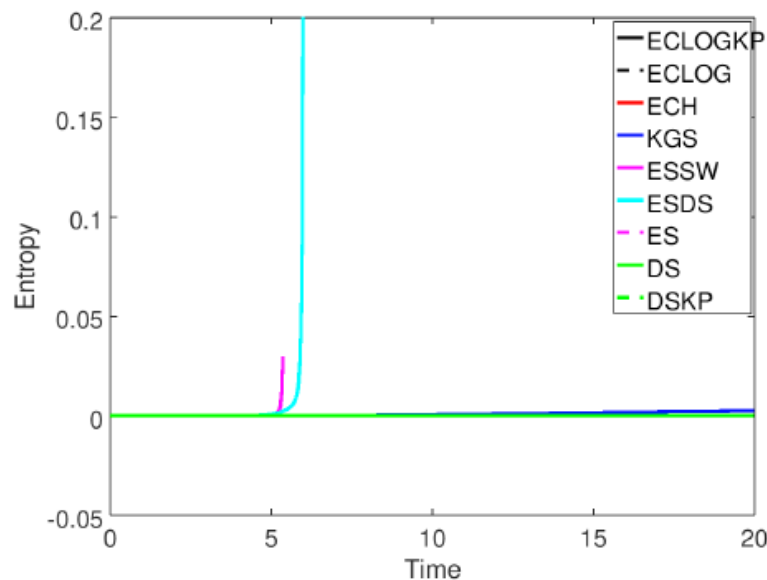
Kinetic energy



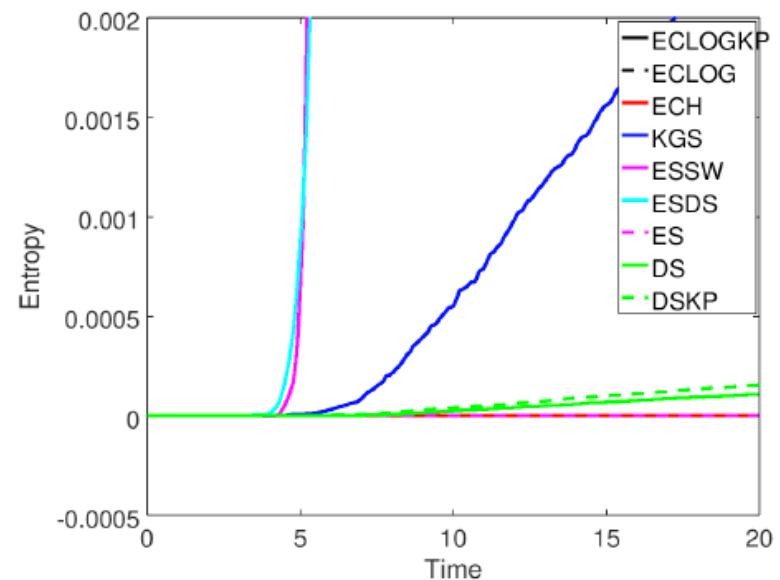
Enstrophy



Entropy



Entropy (closed up)



3-D Compressible MHD (Ideal)

$$\left(\begin{array}{c} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \\ B_x \\ B_y \\ B_z \end{array} \right)_t + \text{div} \left(\begin{array}{c} \rho \mathbf{u} \mathbf{u}^T + (p + \frac{1}{2} B^2) \mathbf{I} - \mathbf{B} \mathbf{B}^T \\ \mathbf{u} (e + p + \frac{1}{2} B^2) - \mathbf{B} (\mathbf{u}^T \mathbf{B}) \\ \mathbf{u} \mathbf{B}^T - \mathbf{B} \mathbf{u}^T \end{array} \right) = 0$$

Conservative

$$\left(\begin{array}{c} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \\ B_x \\ B_y \\ B_z \end{array} \right)_t + \text{div} \left(\begin{array}{c} \rho \mathbf{u} \mathbf{u}^T + (p + \frac{1}{2} B^2) \mathbf{I} - \mathbf{B} \mathbf{B}^T \\ \mathbf{u} (e + p + \frac{1}{2} B^2) - \mathbf{B} (\mathbf{u}^T \mathbf{B}) \\ \mathbf{u} \mathbf{B}^T - \mathbf{B} \mathbf{u}^T \end{array} \right) = -(\nabla \cdot \mathbf{B}) \left(\begin{array}{c} 0 \\ B_x \\ B_y \\ B_z \\ \mathbf{u}^T \mathbf{B} \\ u \\ v \\ w \end{array} \right)$$

**Non-conservative
(Symmetrizable -
Godunov, Powell)**

$$\mathbf{u} = (u, v, w)^T$$

$$\mathbf{B} = (B_x, B_y, B_z)^T$$

$$B^2 = B_x^2 + B_y^2 + B_z^2$$

$$p = (\gamma - 1) \left(e - \frac{1}{2} \rho (u^2 + v^2 + w^2) - \frac{1}{2} (B_x^2 + B_y^2 + B_z^2) \right)$$

Conservative form is not symmetrizable

Two Entropy Split forms (ES & ES2) including Godunov terms are considered

Entropy Split Methods (ES) for MHD

$$\mathbf{q}_t + \mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)} + \mathbf{e} \operatorname{div} \mathbf{B} = \mathbf{0}.$$

The entropy splitting on; e.g., the x -flux derivative of the form

$$\mathbf{f}_x^{(x)} = \frac{\beta}{\beta + 1} \mathbf{f}_x^{(x)} + \frac{1}{\beta + 1} \mathbf{f}_{\mathbf{v}}^{(x)} \mathbf{v}_x,$$

and similarly for the other directions, to obtain

$$\mathbf{q}_t + \frac{\beta}{\beta + 1} (\mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)} + \mathbf{e} \operatorname{div} \mathbf{B}) + \frac{1}{\beta + 1} (\mathbf{f}_{\mathbf{v}}^{(x)} \mathbf{v}_x + \mathbf{f}_{\mathbf{v}}^{(y)} \mathbf{v}_y + \mathbf{f}_{\mathbf{v}}^{(z)} \mathbf{v}_z + \mathbf{e} \operatorname{div} \mathbf{B}) = \mathbf{0}.$$

The Jacobians with respect to the entropy variables, $\mathbf{f}_{\mathbf{v}}$, are not symmetric, but thanks to the presence of the source term

$$\mathbf{f}_{\mathbf{v}}^{(x)} \mathbf{v}_x + \mathbf{e} \mathbf{B}_x = A^{(x)} \mathbf{v}_x,$$

where $A^{(x)}$ is symmetric. The y - and z - directions are similar. Hence, the split form of the equations

$$\textbf{(ES)} \mathbf{q}_t + \frac{\beta}{\beta + 1} (\mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)}) + \frac{1}{\beta + 1} (A^{(x)} \mathbf{v}_x + A^{(y)} \mathbf{v}_y + A^{(z)} \mathbf{v}_z) + \frac{\beta}{\beta + 1} \mathbf{e} \operatorname{div} \mathbf{B} = \mathbf{0},$$

2D Alfven Wave Test Case

$$\rho = 1 \quad p = 1 \quad \mathbf{v} = A(-\sin \alpha \sin \phi, \cos \alpha \sin \phi, \cos \phi)$$

$$\mathbf{B} = (\cos \alpha - A \sin \alpha \sin \phi, \sin \alpha + A \cos \alpha \sin \phi, A \cos \phi) \quad .$$

$$\phi = 2\pi(x \cos \alpha + y \sin \alpha + t)$$

where $\alpha = 30^\circ$ and $A = 0.1$ are chosen with periodic boundary conditions

The computational domain is 1.3546×2.1915

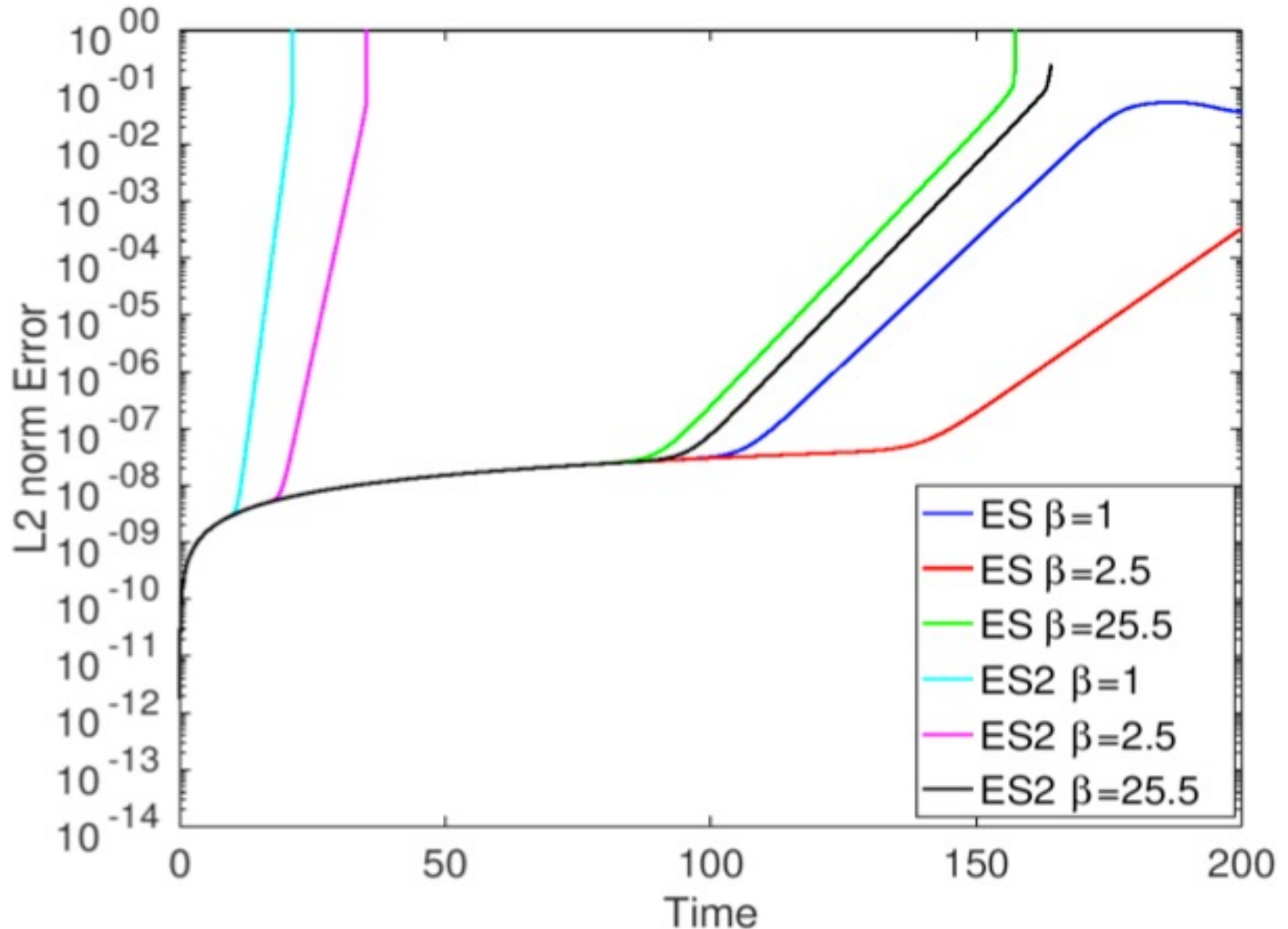
The flow is solved up to time 300 until all methods eventually diverge

$$\Delta x = \Delta y = 0.022$$

$$\Delta x = \Delta y = 0.011$$

Alfven Wave Beta Study (L2-Norm Error vs. Time)

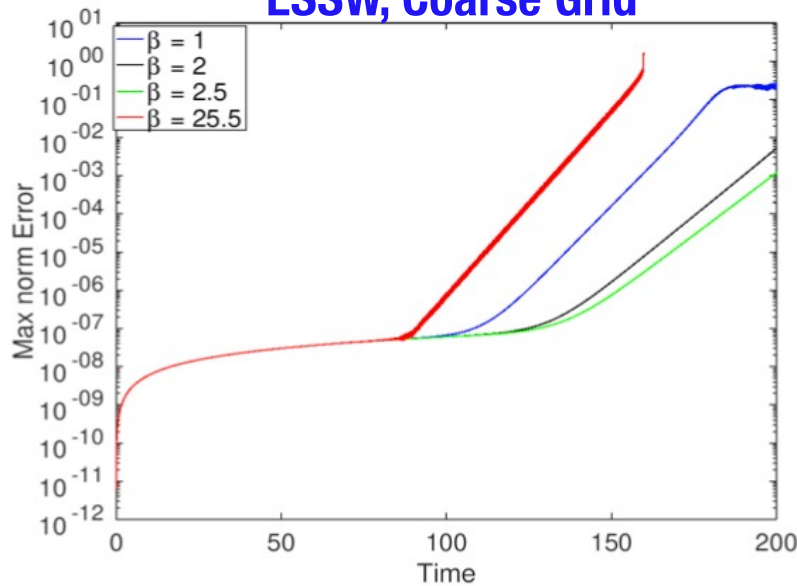
(Comparison between ES & ES2 by 8th-order Entropy Split Method)



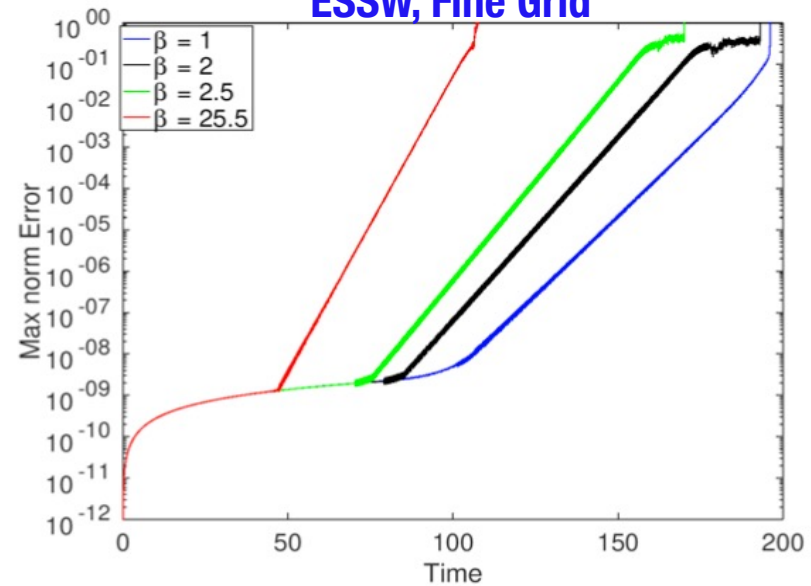
Alfven Wave (Max-Norm Error vs. Time)

(Beta Parameter Study: 8th-order Entropy Split Method)

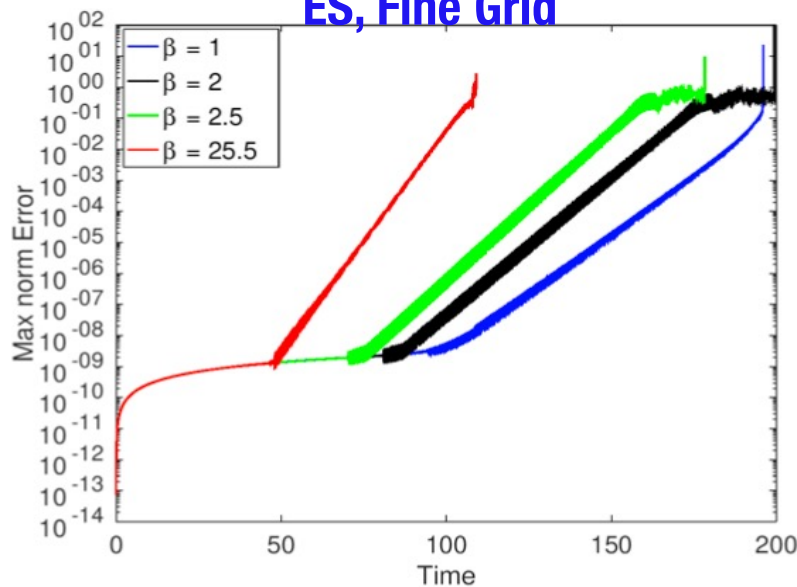
ESSW, Coarse Grid



ESSW, Fine Grid



ES, Fine Grid

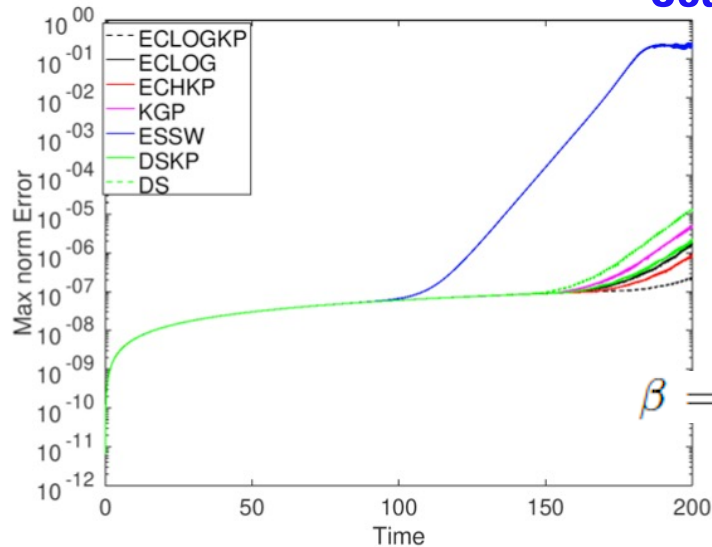


Alfven Wave (Max-Norm Error vs. Time)

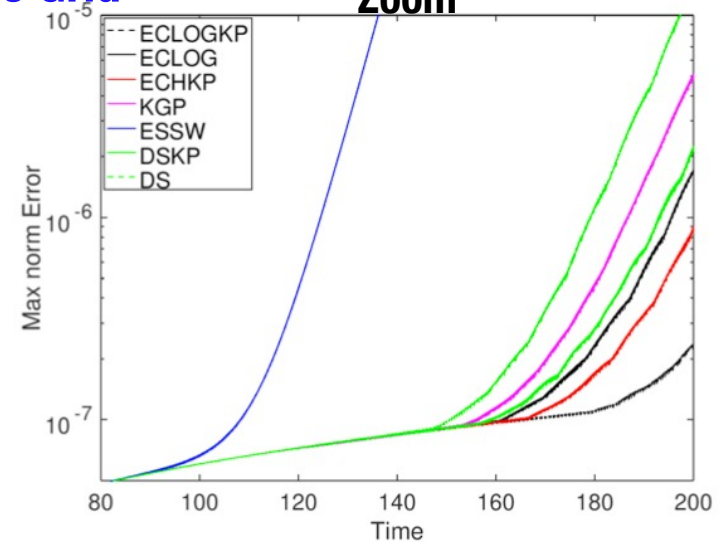
(Comparison Among Seven 8th-order Methods)

$$\beta = 1$$

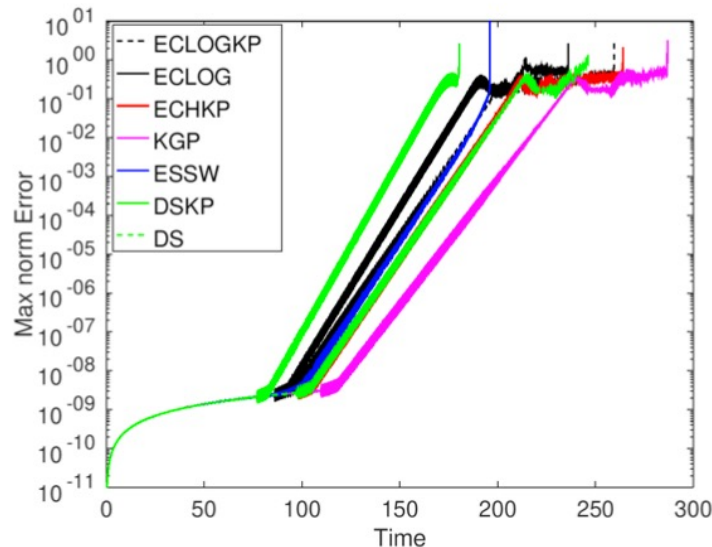
Coarse Grid



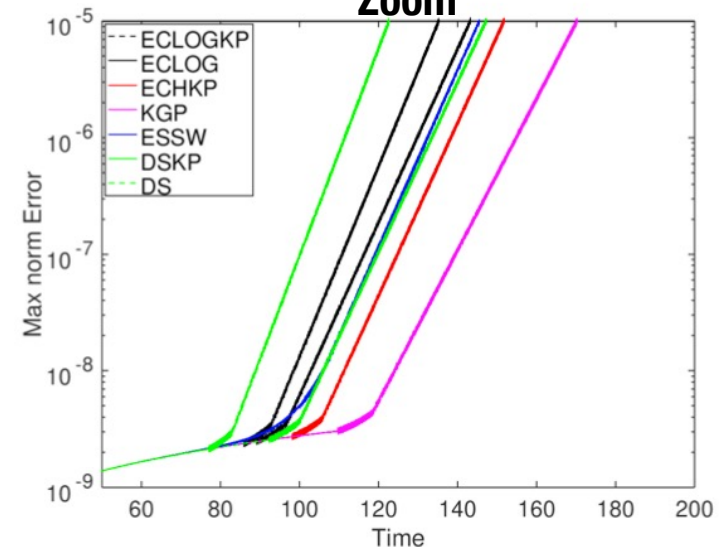
Zoom



Fine Grid



Zoom



Brio-Wu MHD Shock Tube Problem

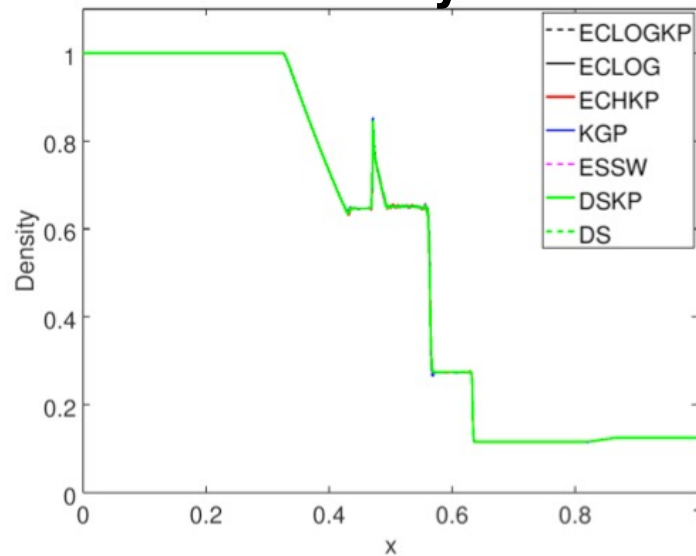
$$(\rho_L, u_L, p_L, B^{(x)}, B^{(y)}, B^{(z)}) = (1, 0, 1, 0.75, 1, 0)$$

$$(\rho_R, u_R, p_R, B^{(x)}, B^{(y)}, B^{(z)}) = (0.125, 0, 0.1, 0.75, -1, 0).$$

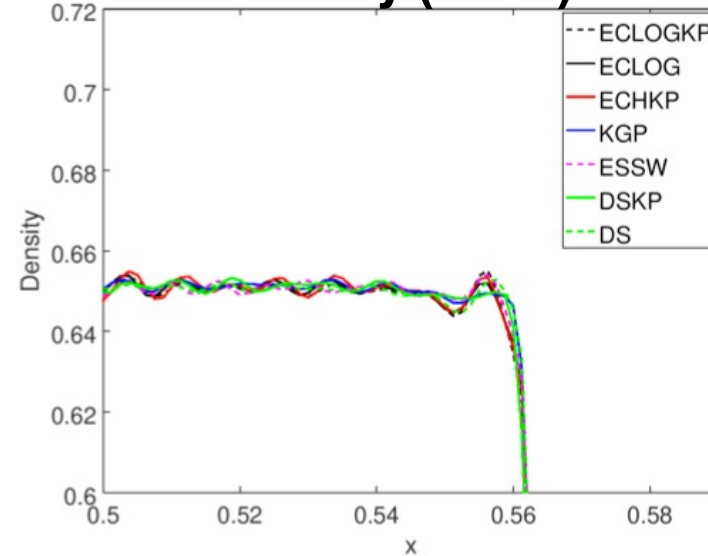
Brio-Wu MHD Shock Tube Problem

(Comparsion Among Seven Methods + 7th-order nonlinear filter)

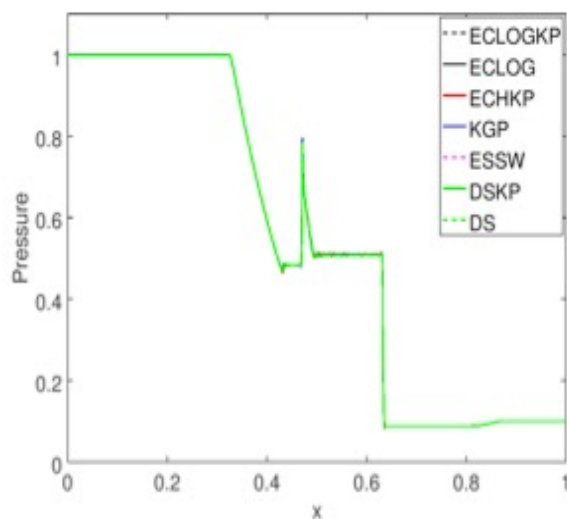
Density



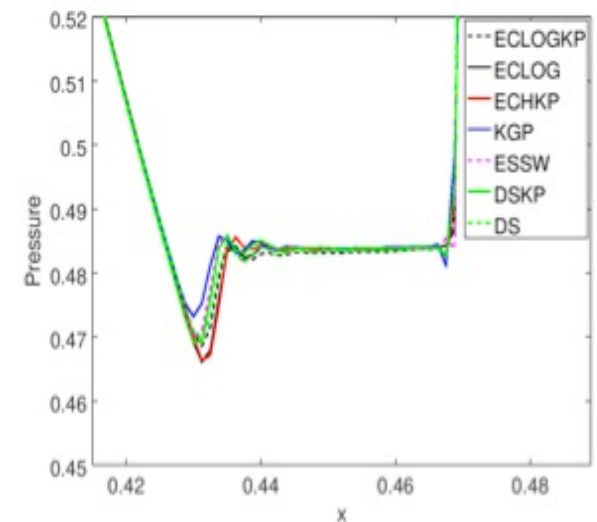
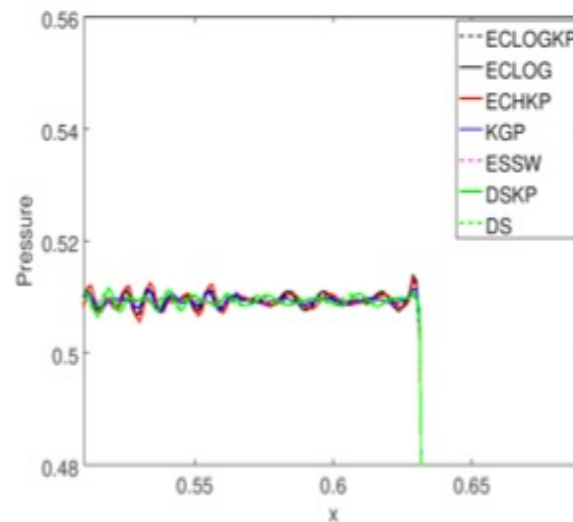
Density (Zoom)



Pressure



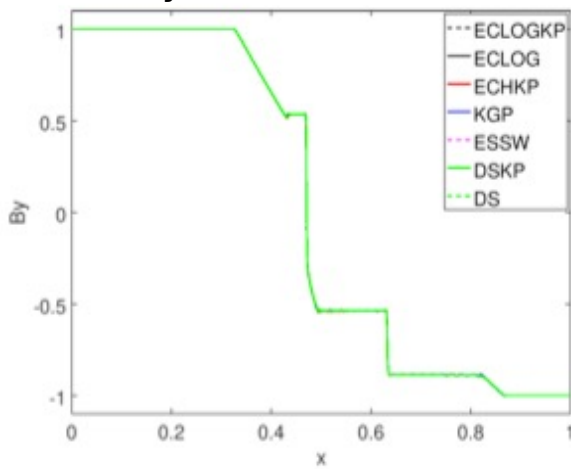
Pressure (Zoom)



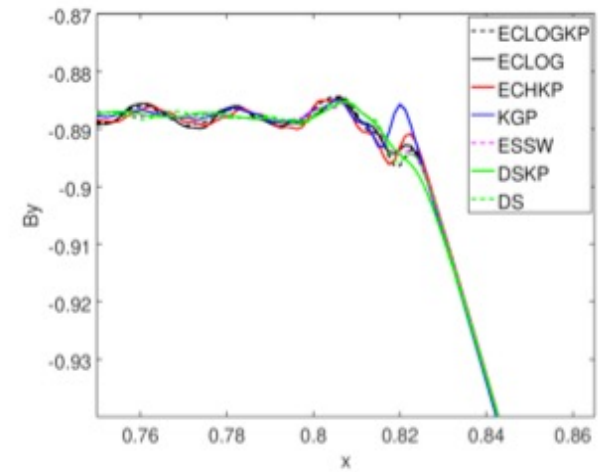
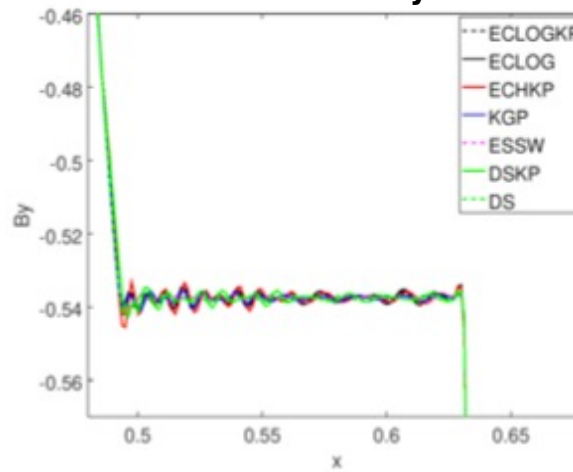
Brio-Wu MHD Shock Tube Problem

(Comparsion Among Seven Methods + 7th-order nonlinear filter)

B_y magnetic field



B_y magnetic field (Zooms)



Brief Summary

GAS dynamics: Classical, DRP and Pade' Spatial discretizations

- All split methods can improve nonlinear stability for **smooth flows**
(Without added numerical dissipation in certain flows)
- All nonlinear filter versions of split methods can improve stability & accuracy for DNS & LES
- All split methods provide similar stability & accuracy improvement
(Pade' methods are most CPU intensive & not friendly in parallelization for multi-D flows)
- All split methods are **less CPU intensive** than the **Tadmor-type entropy conserving** methods

Plasma: Classical & Pade' spatial discretizations

- Both split methods can improve nonlinear stability in general for **smooth flows**
but are MHD equations dependent (Pade' methods are most CPU intensive & not friendly in parallelization for multi-D flows)
- Nonlinear filter version of split schemes can improve stability & accuracy for flows with discontinuities **but are MHD equations dependent**
- High order entropy conserving methods (centered or nonlinear filter version) can provide **different stability & accuracy improvement**, depending on the **forms of the MHD equations & the choice of entropy fluxes**

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