Supporting Information for "Modeling the influence of moulin shape on subglacial hydrology"

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Introduction The supporting information provides additional information about the simulations and methods. **Text S1** shows the derivation for the change in head with cylindrical and non-cylindrical moulins. **Text S2** describes our moulin shape parameterization for constant meltwater input. **Text S3** describes the 1D discretized version of the single-channel model used to test assumptions made in the 0D model. **Figure S1** and

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Figure S2 provide visualizations of the damping and oscillation timescales. **Figure S3** is an additional figure showing how an abrupt change in radius impacts the equilibration timescales. **Figure S4** demonstrates how the equilibrium head changes across the ice sheet. **Figure S5** shows the parametrization of moulin shape for an oscillating meltwater input. **Figures S6 and S7** compare the single-channel model with a discretized subglacial channel model. **Figure S8** compares several damping moulin head timeseries at different positions on the ice sheet. **Tables S1 to S6** summarize the input and fitting parameters for all the simulations and figures in the paper.

Text S1. Here we derive the moulin radius as a function of elevation

Case for a cylinder:

The continuity equation says that for Δt, the change in storage, ΔV, equals the input meltwater, Q_{in}, minus the discharge out of the channel, Q_{out}, times Δt, or

$$\frac{\Delta V}{\Delta t} = Q_{in} - Q_{out} \tag{1}$$

For each time-step, the **storage of water** $\Delta V = A_{\mathrm{r}}\Delta h$. If we plug in this relationship to Equation 1, then we get:

$$\frac{\Delta(hA_{\mathrm{r}})}{\Delta t} = Q_{in} - Q_{out} \tag{2}$$

If we rearrange then we obtain:

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{A_{\mathrm{r}}} \tag{3}$$

If $\Delta t \longrightarrow 0$ then:

$$\lim_{\Delta t \to 0}[\frac{\Delta h}{\Delta t}] = \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{Q_{in} - Q_{out}}{A_{\mathrm{r}}} \tag{4}$$

Case for a conical frustum:

If we use Equation 1 and plug in the volume of a frustum $\Delta V = \frac{1}{3}\pi(r_{\text{top}}^2 + r_{\text{top}}r_{\text{base}} + r_{\text{base}}^2) * \Delta h$. We obtain

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{\frac{1}{3}\pi(r_{\text{top}}^2 + r_{\text{top}}r_{\text{base}} + r_{\text{base}}^2)} \tag{5}$$

We then define $r_{\text{top}}, r_{\text{base}}$ w.r.t. Δh or Δt. The slope $m = \frac{\Delta h}{\Delta r}$ and the change between the radius $\Delta r = r_{\text{base}} - r_{\text{top}} = \frac{\Delta h}{m}$. Therefore, we can express $r_{\text{base}} = r_{\text{top}} + \frac{\Delta h}{m}$ and replace r_{base} in Equation 5, giving

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{\frac{\pi}{3}[r_{\text{top}}^2 + r_{\text{top}}(r_{\text{top}} + \frac{\Delta h}{m}) + (r_{\text{top}} + \frac{\Delta h}{m})^2]} \tag{6}$$

We distribute and reorganize the denominator and get

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{\pi r_{\text{top}}^2 + \pi r_{\text{top}}\frac{\Delta h}{m} + \frac{\pi}{3}\frac{\Delta h^2}{m^2}} \tag{7}$$

If $\Delta t \longrightarrow 0$, $\Delta h \longrightarrow 0$, then $\frac{2r_{\text{top}}\Delta h}{m}$ and $\frac{\Delta h^2}{m^2} \longrightarrow 0$, we are left with πr^2 at the denominator and we recover the continuity equation (1) for a cylindrical moulin:

$$\frac{dh}{dt} = \frac{Q_{in} - Q_{out}}{\pi r^2} = \frac{Q_{in} - Q_{out}}{A_{\text{r}}} \tag{8}$$

Text S2. Here we describe our moulin shape parameterization for constant meltwater input. We use a cone-shaped moulin with various wall slopes and the radius fixed at a certain depth. To explore equilibration timescales, we use a conical frustum where r_{base}/r_{top} can be greater than or less than 0.

$$A_{\text{r}}(z) = \pi(mz + r_{base})^2 \tag{9}$$

To fix the radius at the middle of the ice thickness, we define the base radius to be

$$r_{base} = r_{h_{\text{eq}}} - m(H/2) \tag{10}$$

. To fix the radius at equilibrium head, we define the base radius to be

$$r_{base} = r_{h_{\text{eq}}} - m h_{\text{eq}} \tag{11}$$

Text S3.

Here we describe the discretized subglacial channel model used as a comparison in Supplemental Figures S6 and S7. The model uses Equations 1-3 from the main text, which are the same as for the 0D model. However, instead of calculating the effective pressure at the moulin only, we use a one dimensional grid with nodes spaced evenly along the channel length (in this case: 76 nodes, every 400 m). The model begins with initial head values at the upstream and downstream end of the channel. We solve for discharge by noting that:

$$\Delta h_{\mathrm{TOT}} = \sum_{i=1}^{n} \Delta h_i, \tag{12}$$

where Δh_{TOT} is the total head loss over the channel length, and Δh_i is the head loss over an individual segment, with n total segments. For fixed node spacing, Δx, substituting the Darcy-Weisbach equation for each channel segment (Equation 2 in the main text) into Equation 12 produces

$$\Delta h_{\mathrm{TOT}} = \frac{Q^2 \Delta x}{C_3^2 \rho_w g} \sum_{i=1}^{n} \frac{1}{S_i^{5/2}}, \tag{13}$$

where S_i is the cross-sectional area of the ith channel segment. This equation can be rearranged to solve for the discharge in the channel, Q, and we then use Q to calculate the head loss within each individual segment. Once heads and discharge are known, the melt and creep are calculated within each individual segment and segment cross-sections, S_i are updated. Finally, moulin head is updated using the input discharge, channel discharge and Equation 1 from the main text. This procedure is repeated for every timestep.

At any given timestep, one can calculate the equivalent uniform channel cross-sectional area, S_{equiv}, for a pipe of total length, L_{TOT}, that would produce the calculated discharge and the current head gradient using

$$\Delta h_{\mathrm{TOT}} = \frac{L_{\mathrm{TOT}} Q^2}{c_3^2 \rho_w g S_{\mathrm{equiv}}^{5/2}}, \tag{14}$$

in combination with Equation 12 and the Darcy-Weisbach Equation for each channel segment. This leads to the relation

$$S_{\mathrm{equiv}} = \left[\frac{n}{\sum_{i=1}^{n} S_i^{-5/2}} \right]^{2/5}. \tag{15}$$

This is the equivalent cross-sectional area needed in our 0D model to reproduce the moulin head that is predicted in the 1D model. Figure S7 shows that, for margin distances between 5 and 25 km, and discharges of 1 and 10 $\mathrm{m^3 s^{-1}}$, our 0D model reproduces this equivalent cross-section to within a few percent.

The ice thickness is calculated with the square-root glacier function from equation 5 in Section 2, to enable testing of the assumptions underlying our 0D model. The subglacial channel pressure at the last node on the margin is set to zero, and runs presented here use a timestep of 500 seconds and divide the channel into 50 segments.

The code “onedim_channel_model.py” for the 1D model is in the Github repository of the project (Trunz & Covington, 2022) in the folder “Compare_0D_1D”.

References

Meierbachtol, T., Harper, J., & Humphrey, N. (2013). Basal Drainage System Response to Increasing Surface Melt on the Greenland Ice Sheet. *Science*, *341*(6147), 777–779. doi: 10.1126/science.1235905

Röthlisberger, H. (1972). Water Pressure in Intra- and Subglacial Channels. *Journal of Glaciology*, *11*(62), 177–203. doi: 10.1017/S0022143000022188

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trunz/ModelRepo_moulinshapestoragepaper. Zenodo. (type: Computational-Notebook) doi: 10.5281/zenodo.6574988

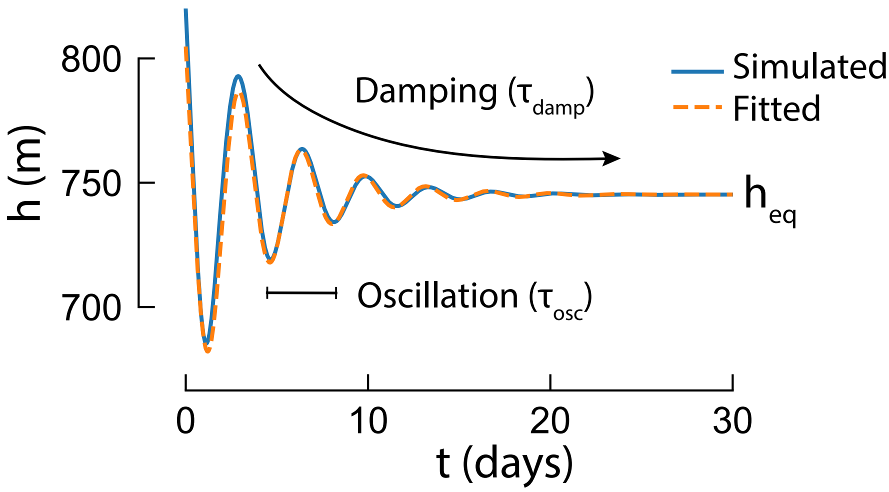

Figure S1. Oscillation of head during equilibration. The solid line shows the full numerical result, and the dashed line shows the fit of an idealized solution for a damped harmonic oscillator. The simulation is for a cylindrical moulin, with $Q_{\mathrm{in}} = 3\,\mathrm{m^3 s^{-1}}$ and $r = 10\,\mathrm{m}$.

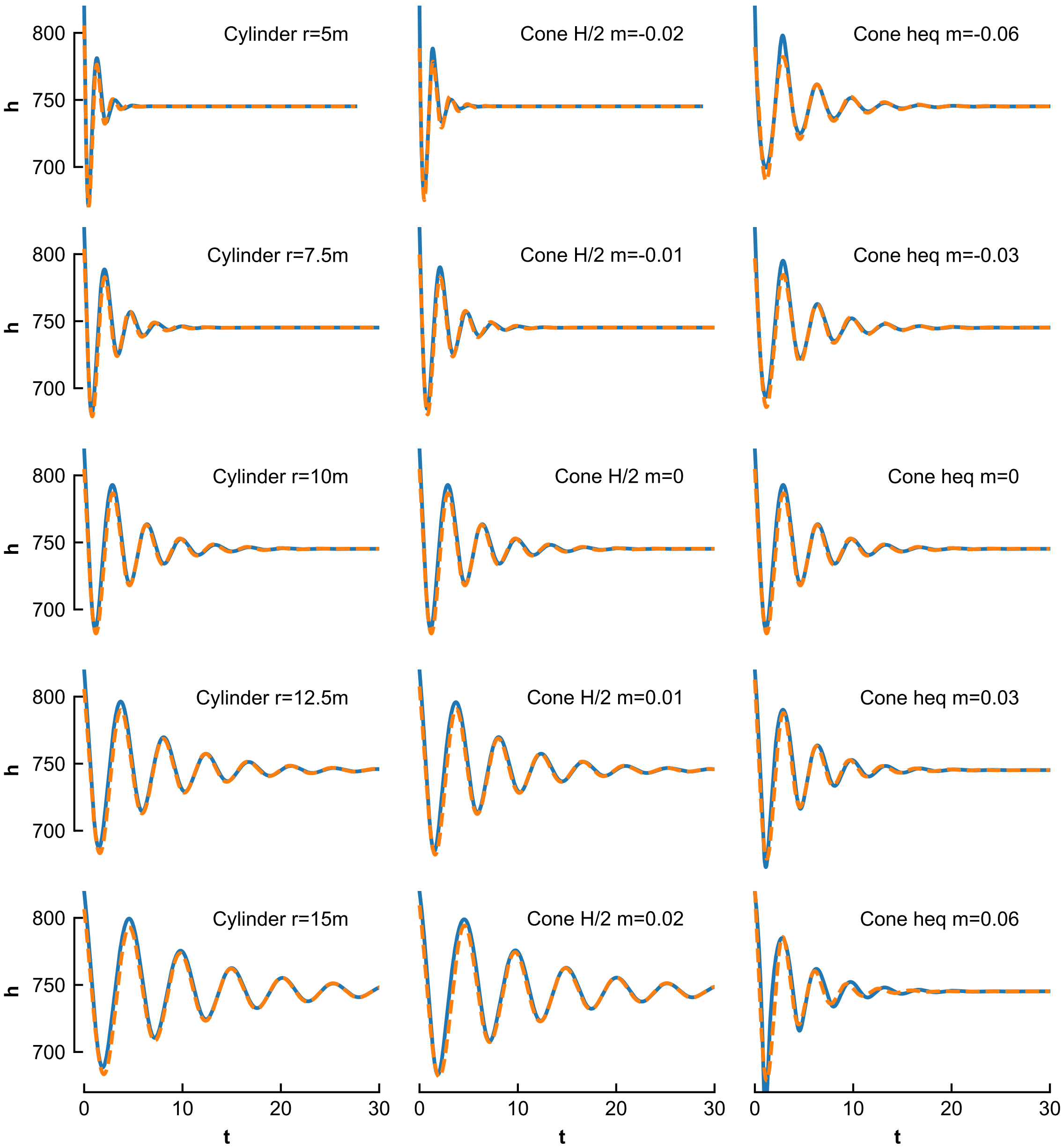

Figure S2. Comparison between simulated and fitted oscillations for different cases. The initial head and initial subglacial channel sectional area, are set to 110% of the equilibrium values, and are the same for all the simulations.

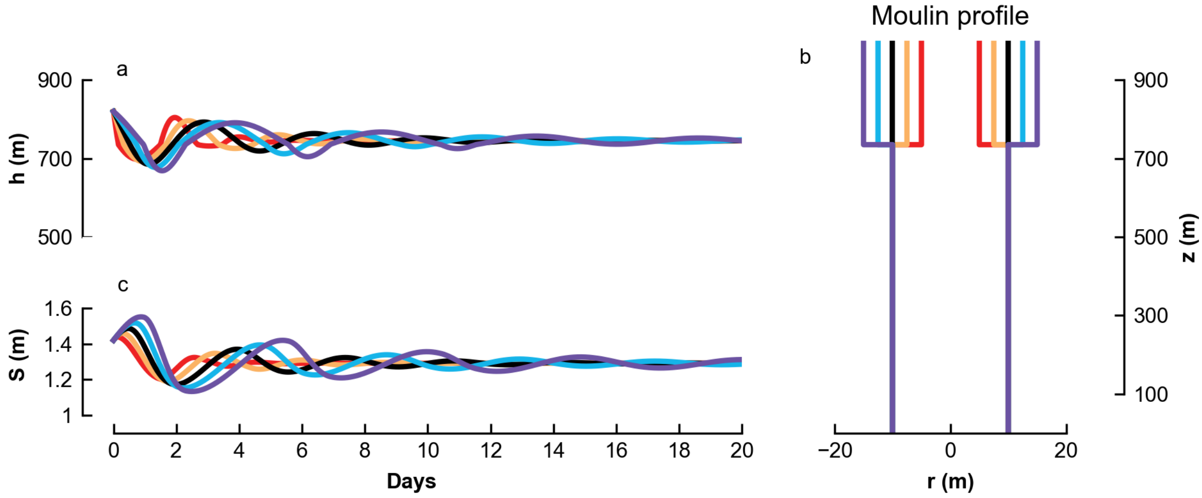

Figure S3. Timeseries of **head (h)** and **channel cross-sectional area (S)** for a fixed meltwater input Q_{in} for bottle-shaped moulins (red and yellow), a cylindrical moulin (black) and goblet-shaped moulins (blue and purple)

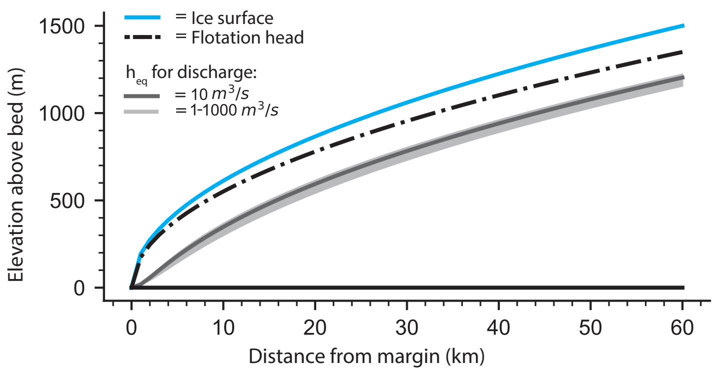

Figure S4. Equilibrium head (h_{eq}) along an ice sheet profile for a wide range of Q_{in}. Equilibrium head (h_{eq}) calculated with the model depends on channel length, ice thickness, and Q_{in}. Meierbachtol et al. (2013); Röthlisberger (1972) described similar profiles.

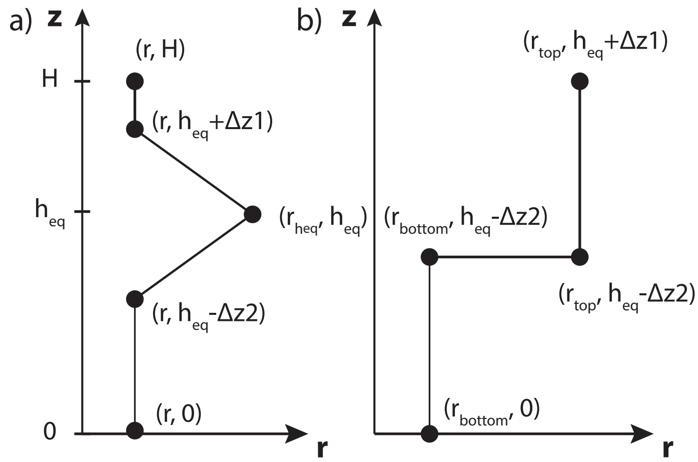

Figure S5. Cartesian coordinates of the moulin shape used for the simulation with oscillating meltwater input. Parameterization for hourglass, diamond and superposed-cylinder shaped moulin. To explore oscillating meltwater input, we define the shape by interpolating the radius defined in the cartesian coordinate system, with r in the x axis, and z in the y axis. Shape coordinates are displayed in Figure S6. The radius (r) is interpolated every meter along the axis z. (a) Hourglass and diamond shaped moulins are defined by five points. (b) Goblet and bottle-shaped moulins defined by four points.

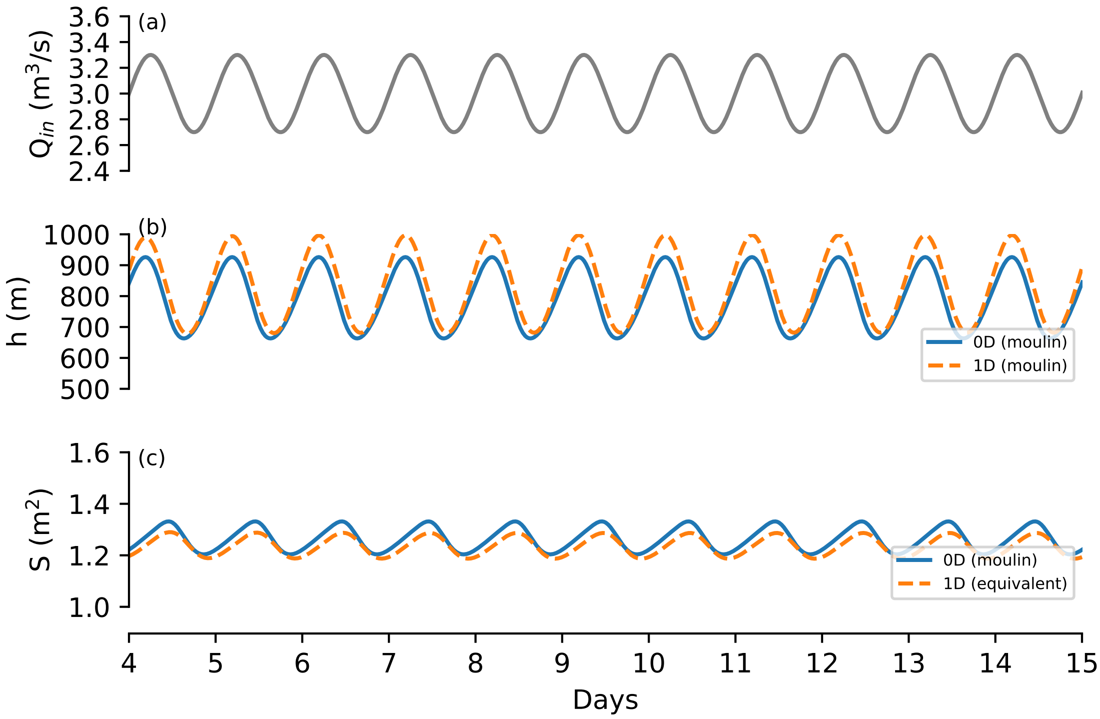

Figure S6. Comparison between the 0D single-channel model used in the paper and a similar 1D single-channel model, where each node along the channel has a different effective pressure and cross-sectional area, using an idealized square-root glacier (see equation 5 in Section 2). (a) Meltwater input (Q_{in}): we use the same sinusoidal meltwater input as for the simulations in the paper: $Q_{\mathrm{mean}} = 3\,\mathrm{m}^3\mathrm{s}^{-1}$, $Q_{\mathrm{a}} = 0.3\,\mathrm{m}^3\mathrm{s}^{-1}$; (b) Hydraulic head and (c) subglacial channel cross-sectional area (S) at the moulin (where the ice thickness is 1000m) for the 0D and the 1D model.

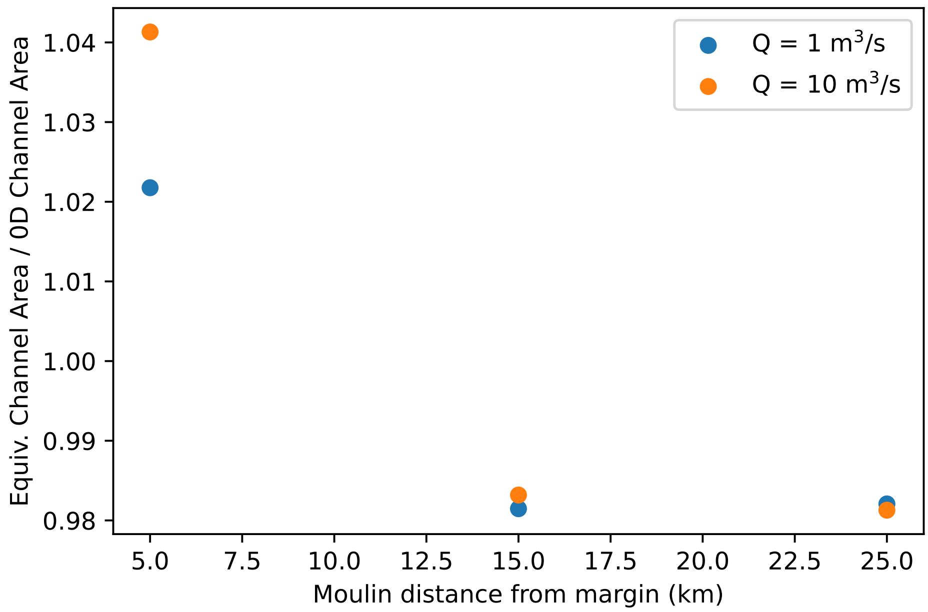

Figure S7. Ratio of equivalent channel cross-sectional areas calculated from 1D model runs and the channel cross-sections obtained in 0D simulations. Here we ran six simulations to equilibrium using discharges of 1 and 10 $m^3 s^{-1}$ and channel lengths of 5, 15, and 25 km. For these cases the 0D model comes within a few percent of reproducing the equivalent cross-sectional areas, S_{equiv}.

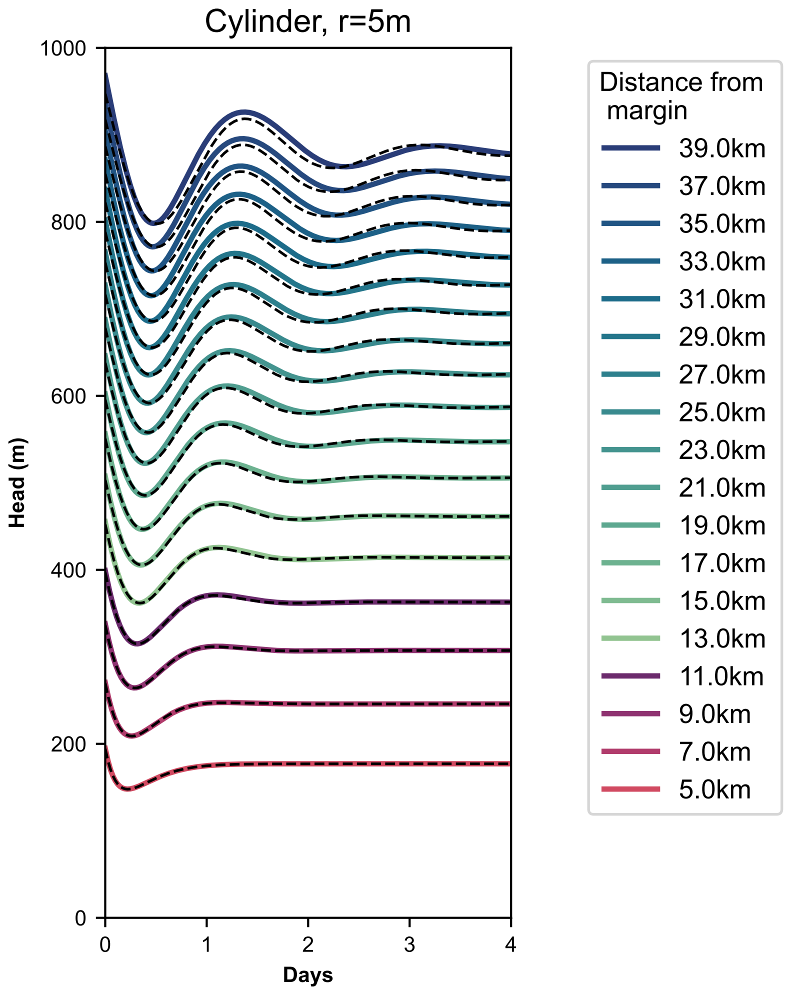

Figure S8. Comparison between simulated moulin head oscillations (solid lines) and oscillations reproduced with the damped harmonic oscillator function (dashed black lines) from Equation 6 in the main text. The graph displays the first four days of the head timeseries simulated at different positions on the icesheet for a moulin of radius 5m (Figure 3e, red line). Blue lines represent timeseries with at least one apparent full oscillation cycle, and purple lines mark the timeseries without a complete oscillation period (represented by a dashed line in Figure 3).

Table S1. Constants and model parameters used in the simulations.

Symbol	**Value**	**Description**
ρ_w	1000 kgm^{-3}	Water density
ρ_i	910 kgm^{-3}	Ice density
g	9.8 $\mathrm{m/s}^2$	Gravitational acceleration
f	0.1	Darcy-Weissbach friction factor
L_f	3.32e5 J/kg	Latent heat of fusion
B	6e-24 $1/Pa^3s$	Glen's law fluidity coefficient (Basal softness)
n	3	Glen's law exponent
C_1	$1/(\rho_\mathrm{i} * L_f)$	Melt opening parameter
C_2	$2Bn^{-n}$	Closure parameter
C_3	$2^{5/4}\sqrt{\pi}/(\pi^{1/4}\sqrt{\pi+2}\sqrt{\rho_\mathrm{w} f})$	Flux parameter

Table S2. Model parameters for simulations with constant Q_in in Figure 2 (main text). For the simulations with this parameters, the equilibrium head $h_\mathrm{eq} = 745$, and equilibrium subglacial channel cross-section area $S_\mathrm{eq} = 1.3$

Parameter	Value	Unit	Description
Q_in	3	$\mathrm{m^3s^{-1}}$	Constant meltwater input
t_0	0	d	Initial time
t_f	100	d	Final time
H_0	6	m	Ice thickness
h_0	$1.1h_\mathrm{eq}$	m	Initial head
S_0	$1.1S_\mathrm{eq}$	m	Initial subglacial channel cross-section

Table S3. Moulin shape parameters for simulations with constant Q_{in} in Figure 2 (main text). The radius (r) is in meters and the slope (m) is given in percent (%) and degrees (°) from the vertical axis.

plot color	red	yellow	black	blue	purple
Cylinder					
m	0	0	0	0	0
r	5	7.5	10	12.5	15
Cone $H/2$					
$m\%$	-2	-1	0	1	2
$m°$	-1.15	-0.57	0	0.57	1.15
r_{middle}	10	10	10	10	10
r_{heq}	5	7.5	10	12.5	15
r_{base}	20	15	10	5	0
r_{top}	0	5	10	15	20
Cone h_{eq}					
$m\%$	-6	-3	0	3	6
$m°$	-3.43	-1.72	0	1.72	3.43
r_{heq}	10	10	10	10	10
r_{base}	25	17.5	10	2.5	-5
r_{top}	5	7.5	10	12.5	15
Diamond-Hourglass h_{eq}					
$m\%$	-6	-3	0	3	6
$m°$	-3.43	-1.72	0	1.72	3.43
r_{heq}	10	10	10	10	10

Table S4. Fitting parameters for simulations in Figure 2 (main text). The damping timescale (τ_{damp}), the period of oscillation (τ_{osc}), the amplitude (a) in meters, and the phase shift (ϕ) in days. A visual comparison between simulations and fits is provided in Figure S2.

Cylinder	radius	τ_{damp}	τ_{osc}	a	ϕ
red	5.0	0.94	1.64	0.14	2.65
yellow	7.5	2.23	2.53	0.11	2.49
black	10.0	4.08	3.42	0.09	2.37
blue	12.5	6.61	4.30	0.09	2.28
purple	15.0	10.00	5.18	0.08	2.20

Cone $H/2$	slope	τ_{damp}	τ_{osc}	a	ϕ
red	-0.02	1.18	1.75	0.12	2.73
yellow	-0.01	2.34	2.56	0.10	2.52
black	0.00	4.08	3.42	0.09	2.37
blue	0.01	6.53	4.28	0.09	2.26
purple	0.02	9.87	5.14	0.09	2.18

Cone h_{eq}	slope	τ_{damp}	τ_{osc}	a	ϕ
red	-0.06	4.22	3.44	0.08	2.49
yellow	-0.03	4.18	3.43	0.09	2.43
black	0.00	4.08	3.42	0.09	2.37
blue	0.03	3.86	3.37	0.10	2.31
purple	0.06	3.29	3.25	0.11	2.24

Diamond-Hourglass h_{eq}	slope	τ_{damp}	τ_{osc}	a	ϕ
red	-0.06	2.71	3.03	0.09	2.59
yellow	-0.03	3.40	3.23	0.09	2.48
black	0.00	4.08	3.42	0.09	2.37
blue	0.03	4.75	3.61	0.10	2.25
purple	0.06	5.39	3.81	0.10	2.14

Table S5. Model parameters from graphs for oscillating Q_{in}, Figure 4 and 5 (main text).

Parameter	Value	Unit	Description
Q_{mean}	3	m^3s^{-1}	Mean meltwater input
Q_{a}	0.4	m^3s^{-1}	Amplitude of oscillation of the meltwater input
Q_{period}	1	d	Period of oscillation of meltwater input
t_0	0	d	Initial time
t_{f}	50	d	Final time
H	1000	m	Ice thickness
L	30000	m	Subglacial channel length

Table S6. Moulin shape parameters from graphs for oscillating Q_{in}, Figure 4 and 5 (main text). The radius (r) is in meters.

cylinder	red	yellow	black	blue	purple
r (m)	1	3.5	5	8	15

Hourglass-Diamond 1	red	yellow	black	blue	purple
r	1	2	5	10	19
r_{heq}	5	5	5	5	5

Hourglass-Diamond 2	red	yellow	black	blue	purple
r	5	5	5	5	5
r_{heq}	1	3.5	5	8	15

Diamond	red	yellow	black	blue	purple
r	1	1.5	2	4	10
r_{heq}	5	5.5	6	8	14

Hourglass	red	yellow	black	blue	purple
r	5	6.5	8	10	18
r_{heq}	1	2.5	4	6	14

Bottle-Goblet 1	red	yellow	black	blue	purple
r_{top}	3	4	5	6	10
r_{base}	5	5	5	5	5

Bottle-Goblet 2	red	yellow	black	blue	purple
r_{top}	5	5	5	5	5
r_{base}	1	2	4	6	12