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Modal Logic Without Possible Worlds: A New Semantics for Modal Logic in Simplicial Complexes

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Abstract

In this paper, we set out to give a novel semantics for modal logic in simplicial complexes. The motivation for this semantics will be first the replacement of possible worlds with the idea of an "agent perspective". After exploring some of the philosophical implications of such a move, we give a semantics based around this idea. Following this, we explore some of the more interesting consequences of such a system, in particular the soundness of an unusual axiom we call $\mathbf{NU}^*$. After giving soundness and completeness, we conclude by exploring ways to weaken this axiom in our semantics.

Contents

1 Brief Primer on Modal Logic

Modal Logic is one of the many tools in the toolkit of a computer scientist. In particular, if you need to have a model of a system with an understanding of what each agent or process "knows" in that system, and in particular, what they "know" about what other processes "know" and so on, modal logic is the ideal tool. As a result, modal logic has a storied and rich relationship with the world of distributed computing [11] [20] [24] [17] [5]. The goal of this section will be to give a brief introduction to modal logic such that the rest of paper is comprehensible.

Modal logic, like most logics, has two pieces, a syntax and a semantics. The syntax itself is composed of two pieces. The first is the rules for constructing formulas, which are the basic unit of logic. Formulas are strings of symbols with specific rules for how the strings can be constructed. You should think of them as sentences, especially in the sense that formulas in a given context can be true or false. The second half of syntax are deductive rules. These are rules for saying which formulas "entail" which others. This will make more sense in a moment.

The second half of a logical system is semantics. This provides a notion of "truth" or "meaning" for formulas, and the goal of this paper is to provide a novel semantics for the traditional syntax of modal logic. In general, syntax and semantics need to "match up" in what are called Soundness and Completeness. Explaining what these are in the context of traditional modal logic will be the goal of this section.

So first, the syntax of modal logic. A formula first consists of atomic statements of the form X, Y, P, Q, and so on. These should be read as truth-evaluable claims with no substructural features. Sentences like "Grass is green" or "The building is evacuated" and so on. These are stitched together with unary connectives like "$\neg$" and binary connectives "$\wedge$", "$\vee$", and "$\rightarrow$". If P is "The building is evacuated", then $\neg P$ is "The building is NOT evacuated". If Q is "Grass is green", then "$P \wedge Q$" is "The building is evacuated AND grass is green", while "$P \vee Q$" is "The building is evacuated OR grass is green". The last, "$P \rightarrow Q$" reads "IF the building is evacuated, THEN grass is green". Parentheses will be used as needed to make formulas more legible. The last thing we need is a special atom, falsum, namely "$\perp$". This formula simply means a contradiction, like $1 = 0$.

Modal logic is special in that it contains at least one additional unary connective, $\Box$. "$\Box P$" can have many interpretations in modal logic, though distributed computing, and thus ourselves, is invested in the interpretation "It is known that the building is evacuated". In general, if we have a set of agents A, we will have unary connectives of the form K_a where "$K_a P$" reads "a knows that the building is evacuated."

This syntax is recursive and therefore arbitrarily complex formulas can be given, such as the following:

$$K_a(K_a P \rightarrow P) \rightarrow K_a P$$

The above reads "If a knows that if a knows the building is evacuated, then the building is evacuated, then a knows that the building is evacuated." This may seem like a nonsensical sentence but sentences like it have important interpretations in proof theory, and modal logic lets us compartmentalize that information in more readable forms. For instance, if instead of knowledge, K_a meant "provable", then $K_a(K_a\bot \to \bot) \to K_a\bot$ expresses Gödel's Second Incompleteness Theorem. Of note is that the largest number of steps required to go from an atom to the formula is called the depth of that formula. In the case of $K_a(K_a\bot \to \bot) \to K_a\bot$, it is one step to go from $\bot$ to $K_a\bot$, another to then go to $K_a\bot \to \bot$, a third to get $K_a(K_a\bot \to \bot)$, and a fourth to get the final formula. So the formula has depth 4. This can be defined far more rigorously but the rough idea is all that is needed here.

In particular, sentences of the form K_aK_bP, or "a knows that b knows that the building is evacuated", are very relevant in the distributed context. In general, we will let Greek letters like φ and ψ stand in for arbitrary formulas, and then things like $K_aK_b\varphi$ mean "a knows that b knows that φ".

Now we need the second part of syntax, the deductive rules. Of note is that given sufficiently many deductive rules, certain symbols will be redundant. For instance, $A \wedge B$ is the same as $\neg(\neg A \vee \neg B)$, in the sense that each can derive the other given sufficiently many deduction rules. So, for the remainder of this paper, we will only use $\bot$, $\to$ and our modal operators for knowledge, as all of the other symbols are redundant, and will only use other symbols as shorthand.

The first deductive rules are known as "Church's Axioms" and are the following, where φ and ψ are any formula:

$$\mathbf{C1} : \varphi \to (\psi \to \varphi)$$

$$\mathbf{C2} : (\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$$

$$\mathbf{C3} : ((\varphi \to \bot) \to \bot) \to \varphi$$

The way a deduction works is as a list of formulas (this is called a Hilbert System). Those formulas are either of the form of an axiom, or follow from previous items in the list via rules of inference. We will be interested in two rules of inference for this paper, the first being Modus Ponens:

$$\mathbf{MP} : \text{ From } \varphi \text{ and } \varphi \to \psi \text{ infer } \psi$$

If a formula of the form $\varphi \to \psi$ appears in the deduction, and a formula of the form φ does as well, we are entitled to write the formula ψ in the list. So if $(P \wedge Q) \to X$ is in the deduction, and $P \wedge Q$ is as well, then we may write X. We say, for instance, that this application of **MP** is an "instance" of the rule, and in general that formulas of the form of a rule are instances of that rule.

These are the "non-modal" rules, so of course, we must give the modal rules. These are the following for any agent a[1]:

$$\mathbf{K} : K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$$

$$\mathbf{D} : K_a\bot \to \bot \text{ or equivalently } \neg K_a\bot$$

[1]One could in theory assume some of these rules hold for some agents while others don't. For most applications and our purposes, we assume that rules apply to agents equally.

$$\mathbf{T} : K_a\varphi \to \varphi$$

$$\mathbf{B} : \varphi \to K_a((K_a(\varphi \to \bot)) \to \bot)$$

$$\mathbf{4} : K_a\varphi \to K_aK_a\varphi$$

$$\mathbf{5} : \neg K_a\varphi \to K_a\neg K_a\varphi$$

Modal logic also requires an additional rule of inference, known as necessitation:

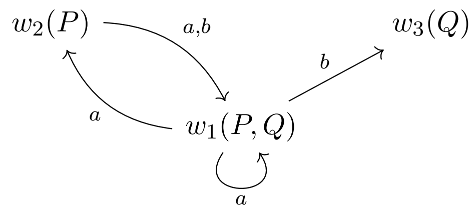
$$\mathbf{Nec} : \text{ From } \varphi \text{ infer } K_a\varphi$$

Of note is that given the presence of certain rules, some of these other rules are redundant. For instance, it is clear to see that all instances of **D** are also instances of **T**, so if we take the latter as a rule we need not take the former. More subtly, one can derive all instances of **5** if one has **Nec**, **K**, **T**, **B**, and **4**, in addition to all of the non modal rules. Modal logics are named by what modal rules are assumed. All logics traditionally considered assume all of the non-modal rules, in addition to **Nec**. Following from this, then, **KD45** is the modal logic assuming in addition to these **D**, **4**, and **5**. An important modal logic is **KT45**, or **S5**, for short. This is the logic standardly used in distributed computing contexts, as the axioms are typically associated with "perfect" reasoning. The logics we will be working with in our new system are usually either **S5** or extensions of **S5**.

If a formula φ is provable from a list of axioms $\mathcal{L}$, we say $\vdash_{\mathcal{L}} \varphi$. We can also allow for a set of assumptions in a deduction, call it Γ, consisting of formulas which may not be axioms or derivable from axioms. These appear in the deduction simply as formulas in the list. If φ is provable from assumptions Γ with axioms $\mathcal{L}$, we say $\Gamma \vDash_{\mathcal{L}} \varphi$.

Now, finally, we must discuss semantics. The usual semantics for modal logic are known as "possible world" semantics or "frame" semantics. These semantics have been used fruitfully in applications, but they are burdened with some interpretive difficulty. Giving an alternative is the goal of this paper, but to explain the difficulty (and to complete some of our proofs) understanding the frame semantics is necessary. The object of study in semantics is a model, and a model consists of three elements. The first is W, a set of possible worlds. From a purely formal point of view, W is just any set. Second is a set of relations $\mathcal{R}$, consisting of a binary relation R_a for each agent a. The third piece is a function V which maps atomic propositions (the atoms of formulas, recall) to sets of possible worlds. V says, in simpler terms, at which worlds a proposition is true (and, consequently, at which worlds it is false). A triple $\mathcal{M} = \langle W, \mathcal{R}, V\rangle$ is called a "frame" model.

The following is an example. Let P and Q be our only atoms, and then let $W = \{w_1, w_2, w_3\}$. Let our agents be a and b, and $R_a = \{(w_1, w_2), (w_1, w_1), (w_2, w_1)\}$, and $R_b = \{(w_1, w_3), (w_2, w_1)\}$. Finally, $V(P) = \{w_1, w_2\}$, while $V(Q) = \{w_1, w_3\}$. We draw this model as follows:



To explain the notation, the list of formulas in parenthesis after a given world explains which formulas V assigned to be true at that world. So at world w_1, both P and Q are true, whereas only P is true at w_2. The arrows are labelled with which agent can "see" a given world from another. At w_1, a can see two worlds, namely w_1 and w_2, whereas b can only see w_3.

We have already hinted that these models provide a notion of truth for formulas when we say that V tells us which atoms are true at which worlds. In general, at any given possible world, one can ask of any formula if it is true or false at that world. Since the definition of a formula is recursive, so must our definition of truth be as well. For a model $\mathcal{M}$, a world w, and a formula φ, the expression $\mathcal{M}, w \vDash \varphi$ reads "φ is true in $\mathcal{M}$ at w". We define this expression as follows:

$$\mathcal{M}, w \vDash P \text{ iff } w \in V(P)$$

$$\mathcal{M}, w \nvDash \bot$$

$$\mathcal{M}, w \vDash \varphi \to \psi \text{ iff } \mathcal{M}, w \vDash \psi \text{ or } \mathcal{M}, w \nvDash \varphi$$

$$\mathcal{M}, w \vDash K_a\varphi \text{ iff } \mathcal{M}, v \vDash \varphi \text{ for all } v \text{ such that } (u, v) \in R_a$$

it is the last of these that requires some more explication. Indeed, what it says is that at a world w, it is true that a knows φ if in all the worlds a considers possible, φ is true. The frame tells us which worlds a "sees" at a given world, or equivalently, which worlds a considers possible at a given world. And if a does not entertain the possibility that φ is false, that is, all worlds a considers possible are such that φ is true, we say that a knows φ. The intuition is that a has ruled out all other possibilities and thus is entitled to claim φ as knowledge. That said, if the notion of a possible world feels abstract, pushing on that is indeed the aim of this paper. One last piece of notation is $\Gamma \vDash \varphi$, which says that any world in any model which makes true all formulas in the set Γ also makes true the formula φ.

The relationship between syntax and semantics, one should hope, is very close, and indeed it is. For instance, the modal logic **K** (remember this also includes **Nec** and all of the non-modal rules) is such that anything provable in this system from the rules is true at every single world in every single frame model. We say that **K** is "sound" with respect to frame models. We can abbreviate this by saying $\Gamma \vdash_{\mathbf{K}} \varphi \Rightarrow \Gamma \vDash \varphi$. However, the opposite is also true, which is far more remarkable. Anything true in every single frame at every single world is provable in **K**. This is called completeness, and is abbreviated $\Gamma \vDash \varphi \Rightarrow \Gamma \vdash_{\mathbf{K}} \varphi$.

One can also restrict the frames to get even more soundness and completeness results. Say that one only considers models where the frames are a transitive relation. Then $\Gamma \vDash_{\text{transitive}} \varphi$ means that every *transitive* model and world making true the set Γ must also make true φ. Indeed, the following is true:

$$\Gamma \vDash_{\text{transitive}} \varphi \Rightarrow \Gamma \vdash_{\mathbf{K4}} \varphi$$

So one has soundness and completeness with respect to the rule **4** and transitive frames. These connections are very powerful in general. Since we mentioned that **S5**[2] is the usual logic for doing distributed computing in, one might ask which class of frames this logic is sound and complete with respect to. It turns out that those frames which are equivalence relations for all agents is the appropriate logic. Indeed:

$$\Gamma \vDash_{\text{equivalence relation}} \varphi \Rightarrow \Gamma \vdash_{\mathbf{S5}} \varphi$$

We will not go into the proofs of either soundness or completeness in any detail. However, some comment should be given as we make use of the traditional proof of completeness later in the paper. Soundness is proven by induction on each axiom and rule of inference, showing that each individual rule is satisfied in the relevant class of models, and therefore a whole deduction will be. Completeness is a bit trickier. One builds what is called a "maximal model" which serves as a kind of universal counterexample. One builds this model first by assuming the set of worlds are the maximal consistent sets of formulas. A maximal consistent set Γ' is such that Γ' is consistent, that is, in the relevant list of axioms, one cannot prove falsum, and Γ' contains every formula or the negation of that formula. Lindenbaum's Lemma states that every consistent set of formulas Γ may be extended to a maximal consistent set Γ'. The valuation for the maximal model simply assigns an atom to a world if it is in the set of formulas. The trickiest thing are the frames for this model. For any agent a, we say that $(u, v) \in R_a$ for the maximal model if and only if for all formulas of the form $K_a\varphi \in u$, $\varphi \in v$. This is known as the "unboxing" frame.

In general, the logics and their axioms and the class of frame models with respect to which they are sound and complete is worth having on hand. For the rest of this paper, these normal modal logics and their respective frame classes in the following table are good to have in mind (all logics have the propositional Church axioms, Modus Ponens, and Necessitation):

Logic	*Axioms*	*Sound/Complete Frame Class*
K	**K**	All
T	$\mathbf{K}+\mathbf{T}$	Reflexive
D	$\mathbf{K}+\mathbf{D}$	Serial (No worlds with no exiting edge)
B	$\mathbf{K}+\mathbf{B}$	Symmetric
K4	$\mathbf{K}+\mathbf{4}$	Transitive
S4	$\mathbf{K}+\mathbf{T}+\mathbf{4}$	Reflexive and Transitive
K5	$\mathbf{K}+\mathbf{5}$	Euclidean (If one world sees two worlds those two worlds see each other)
S5	$\mathbf{K}+\mathbf{T}+\mathbf{4}+\mathbf{5}$ $=\mathbf{K}+\mathbf{T}+\mathbf{4}+\mathbf{B}$ $=\mathbf{K}+\mathbf{T}+\mathbf{5}$	Equivalence Relation

[2]Recall this is the abbreviation for **KT45**.

2 A Story

Imagine you are a modal logician. Say you want to model a situation with two agents (a and b), where both agents know P, but neither knows anyone else knows it. Since knowledge is factive, we need a world for the actual world where P is true. However, this cannot be the only world the agents have access to, else they would know the other knows P. So we need another world accessible to each agent. These worlds must also be P worlds so that knowledge is preserved. However, at each of these worlds, we need an edge for the other agent so that at these worlds, the other agent does not know P. This gives us the following model[3]:

$$w_{a,b}(\neg P) \overset{b}{\text{———}} w_a(P) \overset{a}{\text{———}} w(P) \overset{b}{\text{———}} w_b(P) \overset{a}{\text{———}} w_{b,a}(\neg P)$$

Building this model is curious because we used relations and possible worlds in a very distinct way from what is usually prescribed [16]. Usually worlds are taken to be prior instantiated - we consider those that we need - and the relations between them are a fact. This should be understood as weaker than treating the possible worlds as "actual" in a strong metaphysical sense. Take Kripke's considerably weaker position, where he says "a possible world is given by the descriptive conditions we associate with it." [16, p. 44] Kripke's idea here is that possible worlds - what they are - are constrained by a description (the relevant propositions to what we are modeling). This seems to contrast sharply with the above story. There, what constrained the possible world was a description (the propositions) but it was not our "imagination" that brought the world to our attention, so to speak. Kripke seems to be presuming that the modal logician imagines the relevant features of description (relevant propositions) and introduces worlds based on this (he does not mention anything else). By contrast, in our story, we introduced worlds indeed by such a description, but the motivation for introducing both worlds and relations was our understanding of the perspectives of the agents. Kripke makes no mention of the relations here, and seems to take it that how the worlds are related to each other is an understandable fact once we have all the worlds on the table. This again contrasts sharply with how the modal logician seems to actually build models in practice, where worlds and relations are introduced simultaneously, and each to account for the perspectives of the agents. Kripke's description should therefore be understood as post-hoc, a description of how to understand an already completed model, and even if it is successful here, it leaves the practically minded modal logician relatively clueless as to how to build such a model for a given task.

Our models then would be a simplification or fragment of this fact, where we divine what worlds are relevant and appropriately draw relations between them. The exercise we did above, however, suggests this picture is, if not wrong, at least straightforwardly incomplete. In the exercise above, we introduced worlds which were not divined from some abstract idea of what worlds there might be, but were instead influenced by the constraints of our agents and the problem we were trying

[3]The lack of arrows on the edges we assume means symmetric edges.

to model, in addition to existing relations. Similarly, relations we introduced were constrained by the worlds and relations already present and the problem itself.

We can divine two things from this exercise. Firstly, worlds and relations are not independent. What relations we have influence how we create worlds and vice versa. And, secondly, constraining both of these is the problem at hand, the application to which we are applying our modal logic skill, and the perspectives of our agents. These two themes, broadly speaking, motivate our goals for this paper. We will begin by exploring the difficulties of possible worlds hinted at in the Kripke quote above (both those he identifies and those his own interpretation make salient), and develop a formalism that both gets around this and points towards the practical needs of a modal logician evident in this exercise. We will go from there towards a novel formalism where the semantics for modal logic is done in simplicial complexes, compared to the existing literature in this area. Finally we provide a novel soundness and completeness proof for this system, and use it to point towards the difficulties related systems have had in achieving completeness proofs.

3 Previous Work

Interpretations of possible worlds are fraught with difficulty. Many metaphysical concessions have been made in order for them to fit into various philosophies and arguably none of these are terribly satisfying [3] [15] [19] [21]. The motivations for this are varied, but arguably one is that philosophers in the 20th century wanted to make better sense of one of their favorite tools, modal logic, and defend it from the objections of figures like Quine, who infamously said that "Modal logic was conceived in sin" [22].

This stands in stark relief with the fact that modal logic has a storied and varied history in applications, notably in distributed computing [11] [20] [24] [17] [5]. Somehow modal logic works perfectly fine despite the lack of a detailed explanation of what a possible world is. The tools and protocols developed from modal logic models of distributed systems are widely applied in many areas [11] [20] [24] [17] [5]. We take this to be a tension worth exploring further: How is it that modal logic can be so useful as a model when the fundamental building block of that model is itself inscrutable?

I cannot take credit for the first piece of insight here but I can articulate it. The following comes from a conversation I had with Dr. Adam Bjorndahl. We were discussing the pedagogy of modal logic, and in particular, this exact question: How do we explain to students how to think about and use possible worlds without going over the deep end, so to speak? His example is as succinct as it is illuminating: "If your agents think it might be true that they are on the moon, then they consider the moon a possible world."

To be clear, this example is stated exactly correctly. The collection of propositions consistent with *the world being the moon* is indeed what they consider possible, not that they consider it possible that they are hallucinating the moon or something else. Indeed, however, the only consistent set of such propositions entails that they are hallucinating that they are on earth.

There are two takeaways from this thought experiment and they both point to the same thing. The first is that how we arrived at realizing "the moon" needed to be in our model was by thinking about our agents. In particular, it was arrived at by thinking about what our agents consider (relevantly) possible. The second takeaway is how we should understand the propositional contents of "the moon" was arrived at specifically by considering what the relevant agent was "seeing" - since they see, presumably, earth, if they think it is possible they are on the moon, the propositional content of the possible world "the moon" must include (or at least, be consistent with) the fact that the agent is hallucinating earth.

Both of these takeaways point us straight at the idea that what defines a possible world in a modal context is what we might call an "agent perspective". We only care about possible worlds in as much as they are consistent with a relevant "agent perspective" in the context we are modeling. The big question is, then, can we formally build a semantics for modal logic where the foundational object is an agent perspective and not a possible world? It turns out it can be done [8] [9] [18] [4] [10] [23]. The solution to our woes is to turn away from possible worlds frames and towards simplicial complexes.

4 Possible Worlds and the Difficulty They Bring Us

Simplicial complexes and techniques from algebraic topology are not new to distributed computing [2] [6] [1] [13] [14] [12]. What is new is directly linking them to modal logic [8] [9] [18] [4] [10] [23]. The existing attempts are quite profound in their implications. It seems that we should be able to do modal logic without ever making reference to possible worlds. *Formulas in a model are instead assessed of their truth with respect to conglomerations of agent perspectives, which in the models are "facets" of the simplicial complexes.*

The existing models do, however, make some strange choices. The authors of "Knowledge and Simplicial Complexes" do not actually give a logic for soundness and completeness, which is understandable given the scope of their system [4]. While in the paper "Epistemic Logic with Agents that May Die" the authors assign propositions directly to facets[4] (negating the advantage of moving to agential perspectives in the first place) while both "Knowledge and simplicial complexes" and "Epistemic logic for impure simplicial complexes" have atomic propositions tagged by their relevant agent in a way that makes the proposition agent specific [4] [10] [23]. Neither of these options is ideal. And none of these papers, despite using them as motivating examples, explain clearly how to build classical thought experiments like "Muddy Children" in their languages[5] [4] [10] [23]. And none of these papers explicitly take advantage of the idea that what a facet should be is a consistent union of agent perspectives. Our paper seeks to address all of this.

However, a feature of the previous models that will be carried into ours is that it is not possible to have a semantics for modal logic using simplicial complexes without satisfying **B** (sound and complete with respect to symmetrical frames).

[4]Exactly what a facet of a simplicial complex is will be explained in the next section

[5]In the case of some of these papers, this is possibly because doing so would be quite difficult.

This is odd from the traditional perspective, but makes enormous sense when you realize the philosophical advantages of looking at things through the lens of agential perspectives. Two "scenarios" or facets (the formal stand-in for possible worlds here) are "accessible" from each other only by passing through some agent perspective. Put another way, all this means is that there are two distinct consistent sets of agential perspectives, and between each of these they share a particular agential perspective. And consistency is of course agnostic about any sense of direction, hence the enforced symmetry.

This of course means that accepting **4** enforces accepting **5** and vice versa. This again is strange from the traditional modal logic perspective. However again we feel it is entirely philosophically warranted. First of all, most applications of modal logic, especially in distributed computing, use **S5** or **KD45**. The reasons are because these applications are again effectively already using the agential perspective. It is difficult to imagine what it would mean for two distinct agent perspectives to be consistent with each other but distinct - why would they not just be the same agential perspective? This scenario effectively corresponds with an underspecified model, and is also precisely what corresponds with breaking **4** or equivalently **5**.

The last thing to explore is breaking **T**, and the current literature is exploring this deeply, as in these contexts, breaking **T** behaves in a way that arguably corresponds with crash failure. That is, epistemic scenarios where agents can leave the scenario for one reason or another. While this is worth exploring, and we do show soundness of the relevant rules in this setting, it turns out completeness for the models sans **T** is really quite difficult, as anticipated by [23] [10]. So we focus on an **S5**-like logic for most of this paper.

The last axiom we consider for the system for which we prove completeness is easily our most controversial. We call it **NU** for "No Unknowables", and it is a kind of reverse factivity on atoms:

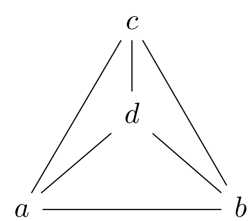
$$\mathbf{NU} : P \to \bigvee_{a \in A} K_a P$$

This axiom says that if an atom P is true then somebody knows this. At first glance such a rule feels absurd in an epistemic context. It however clearly results from the agential perspective we are taking. The only way to introduce an atom to a conglomeration of agential perspectives is for some atom to do so. And of course that perspective is one where P is true.

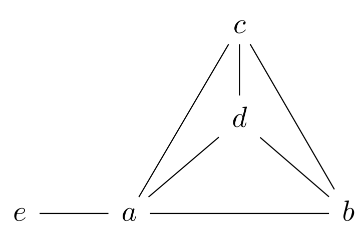
For modelling purposes, there is a way around this. It is very easy for $\neg P$ to be true and for no agent to know $\neg P$, so if this scenario is important for modelling, one can simply reverse the truth of P (take the negation as the atom). Of interest is that all of the classical examples, such as Muddy Children, satisfy this axiom anyways. We will also weaken this assumption in a later section. Additionally, in the final section, we will explain how to eliminate this assumption (though doing so introduces some philosophical and modeling baggage).

5 Brief Primer on Topology

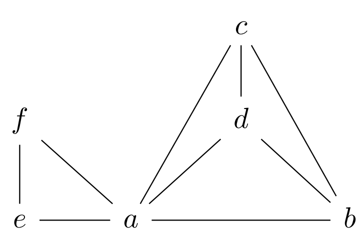
Topological semantics for modal logic is almost as old as modal logic itself, and predates the frame semantics. However, this will not concern us directly here. Instead we need the notions of a "simplicial complex". Formally, a simplex is a triangle. That is, it is a collection of nodes where each node is connected to each other. So, if one has 3 nodes, one is left with the usual triangle. 4 gives a tetrahedron, and in general, n many nodes is an $n-1$-dimensional triangle. Triangles of arbitrary dimensions are called simplexes, and any subset of the nodes of the simplex is called a "face" of the simplex. Consider the following example:



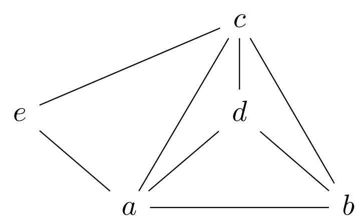
We can use this to get a sense of faces of a simplex. For a tetrahedron with nodes $\{a, b, c, d\}$, the subset $\{a, b, c\}$ is a triangle and a face of the simplex, as is the edge $\{a, b\}$ and the singleton $\{a\}$. In general a simplicial complex is a stitching together of triangles of arbitrary dimension - one can imagine a tetrahedron and a fifth node e, and a single edge from a to e.



One could also imagine that instead of a linear tail the tail was shaped like a triangle:



One could also imagine that triangular tail connected at two points instead of one:



These are all examples of simplicial complexes interpreted geometrically. Formally, a simplicial complex is simply a set N of nodes, and a subset of 2^N, the powerset of N, closed under subsets. That is, if $X \in 2^N$ and $Y \subseteq X$, then $Y \in 2^N$. Elements of the simplicial complex are called simplexes or faces, and faces not a proper subset of some other face are called maximal faces. Our semantics will make heavy use of maximal faces. Contra much of the literature, we will sometimes refer to maximal faces as facets. One can see in the above examples that a subset of any face is itself a face, justifying the formal definition.

6 Primary Definitions

It is now time to formally define our new semantics. Fix a set of agents A. We define our language as follows, where $P \in \mathfrak{P}$ and $a \in A$:

$$\varphi ::= P | \bot | \varphi \to \psi | K_a \varphi$$

Let $\mathfrak{P}^*$ be the propositional fragment of this language (that is, all formulas without a modal component)

For each $a \in A$ let $V(a)$ be a set of agential nodes. Let $L(a) : V(a) \to C(2^{\mathfrak{P}^*})$, where $C(2^{\mathfrak{P}^*})$ is the collection of sets of propositionally consistent formulas in $\mathfrak{P}^*$. For convenience, assume that the collection of $L(a)(a_i)$ for each $a_i \in V(a)$ are pairwise inconsistent (this will help us with soundness and completeness later and related to **4**, and, because as we shall see we also have **B**, also **5**). For intuition, note that V and L are each themselves a part of the valuation. In frame models, the valuation is handled by a single function, but in this context, we need two functions. V assigns a set of nodes, which we can think of as perspectives, to each agent. L then assigns propositions (non-modal formulas) to these perspectives, very much so how the original valuation function from the frame model behaves.

We say a simplicial model is a tuple $\mathcal{M} = \langle V, L, S \rangle$, where S is a subcomplex of S', which itself is the resultant maximal simplicial complex:

$$S' = \{X \subseteq \bigcup_{a \in A} V(a) | \bigcup_{x \in X} L(x) \text{ is consistent}\}$$

We show this is a simplicial complex as follows. Suppose $Y \subseteq X \in S'$. Then $X \subseteq \bigcup_{a \in A} V(a)$ and $\bigcup_{x \in X} L(x)$ is consistent. Since $Y \subseteq X$, we have both that $Y \subseteq \bigcup_{a \in A} V(a)$ and $\bigcup_{x \in Y} L(x)$ is consistent (it is a subset of a consistent set). So, $Y \in S'$, as desired.

We say the following, where $\mathcal{M} = \langle V, L, S \rangle$ is a simplicial model and X a facet (maximal face):

$$\mathcal{M}, X \vDash P \text{ iff } \bigcup_{x \in X} L(x) \vdash P \text{ where } \vdash \text{ is classical propositional entailment}$$

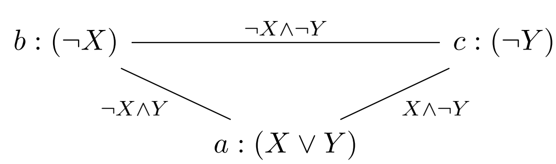
$$\mathcal{M}, X \nvDash \bot$$

$$\mathcal{M}, X \vDash \varphi \to \psi \text{ iff, if } \mathcal{M}, X \vDash \varphi \text{ then } \mathcal{M}, X \vDash \psi$$

$$\mathcal{M}, X \vDash K_a\varphi \text{ iff } \forall Y \text{ such that } V(a) \cap (X \cap Y) \neq \emptyset, \text{ then } \mathcal{M}, Y \vDash \varphi$$

This is a two valued logic but we have made a particular assumption - agential perspectives, when agnostic about the truth of some claim, can be stitched together both with perspectives where the claim is true and those where it is false. However, once the simplicial complex is built from the agential perspectives, worlds that are agnostic about a given proposition are said to force it to be false. This follows from our semantics for $\to$, as is useful when proving soundness. Philosophically, I also think this is okay - facets are ”truth minimal” - if the model doesn't consider a possibility for making something true, the model assumes its false. Nothing should be lost practically by this assumption.

At this stage it might be helpful to look at a specific example. The following I call the “Where should we go to dinner” model. Let $A = \{a, b, c\}$ and X and Y be propositions corresponding to “We should eat at X” and “We should eat at Y” respectively. Then each agent gets one node, so simply say $V(a) = \{a\}$, and say that $L(a) = X \vee Y$, $L(b) = \{\neg X\}$, $L(c) = \{\neg Y\}$, and $S = S'$. This gives us the following model:



The faces (edges) are labeled with the truth values of the literals (atoms and negated atoms). What is interesting is that any two of our agents can agree on where to go to dinner (b and c presumably agreeing to stay in) but the three of them cannot agree. This corresponds with the fact that the two simplex $\{a, b, c\}$ is inconsistent.

Of note is that there is a special class of assumptions. Define a maximal face in a simplicial complex to be a face not contained in any face except itself. A facet is a face of maximal dimension. Simplicial complexes of dimension $|A|$ where every maximal face is a facet and every maximal face X is such that $|X \cap V(A)| = 1$ will be the class for which we prove soundness and completeness. Call these perfect complexes. Note that “Where should we go to dinner” is not perfect, as each maximal face (the edges) is short one agent.

Of note is that we can define muddy children in this context, and it is a perfect model. Let $A = a, b, c$, and $M_a, M_b, M_c \in \mathfrak{P}$ denote that the respective child is in fact muddy. Then

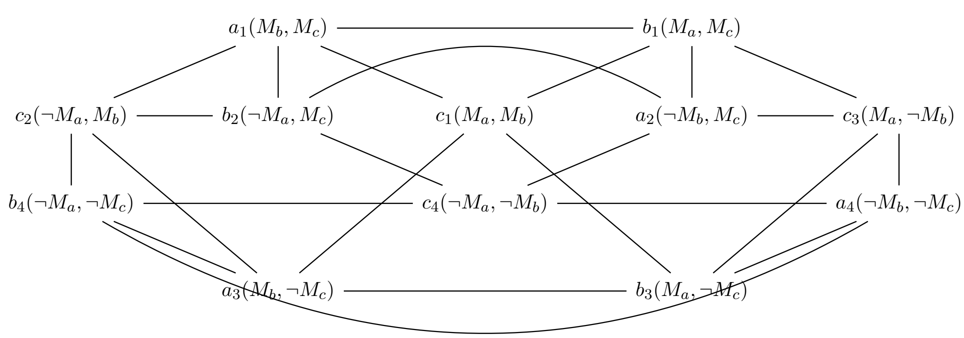
$$V(a) = \{a_1, a_2, a_3, a_4\}$$

$$V(b) = \{b_1, b_2, b_3, b_4\}$$
$$V(c) = \{c_1, c_2, c_3, c_4\}$$

We then say

$$L(a)(a_1) = \{M_b, M_c\}$$
$$L(a)(a_2) = \{(M_b \to \bot), M_c\}$$
$$L(a)(a_3) = \{M_b, (M_c \to \bot)\}$$
$$L(a)(a_4) = \{(M_b \to \bot), (M_c \to \bot)\}$$

$L(b)$ and $L(c)$ are defined analogously. Taking $S = S'$, this gives us the usual simplicial model for 3 muddy children, with triangular faces corresponding with the 8 possible arrangements of children. For instance, $\{a_1, b_1, c_1\}$ is consistent and therefore is a face because the union of their interpretations is consistent (all children clean). Note that this is also a perfect complex. We can draw it below:



A fair question to ask at this point is "how should one go about reading this?". The key idea is to center yourself on triangular faces. Consider $\{a_1, b_1, c_1\}$, call it A. At this face, note that three faces are accessible, namely $\{a_1, b_2, c_2\}$, $\{a_2, b_1, c_3\}$, and $\{a_3, b_3, c_1\}$, call these A_a, A_b, and A_c respectively. The first is accessible through the subface (in this case node) $\{a_1\}$, the second through $\{b_1\}$, and the third through $\{c_1\}$. So, at A, we can consider, for instance, all the faces consistent with A from a_1's perspective. These would be the faces whose intersection with A contains a nodes, so in this case, are the faces A and A_a. At both of these, M_b and M_c are true, so we can say that a knows that M_b and also knows that M_c. The face $\{a_3, b_4, c_2\}$, call it B, is accessible from A_a via c_2. Moreover, one can observe that $\neg M_c$ is true at B. So a considers it possible that c considers it possible that c is not muddy (and of course, they are in fact muddy, and a knows this). Following, we can observe facts like "a considers it possible that b considers it possible that c considers it possible that b is not muddy".

A key intuition here is the idea that knowledge is paths. Drawing a path between two simplexes through a shared face which is "colored" using the agents (we call a face with an a-node a-colored) is how we determine possibility. By tracing these paths, we determine what agents consider possible. So, starting at A, we pass to

A_a via an a-colored face. We can then pass to B via a c-colored face. So, this path starts with the color a and then c, and we can use this path to represent knowledge in the complex.

Now let's imagine instead the same situation, except that a is uncertain about b's state. a's perspectives reduce from 4 in count to 2. We'll say $a_1 = \{M_c\}$ and $a_2 = \{\neg M_c\}$. Let's imagine first that the other agents are also aware of a's limitation. In this scenario, the desired graph is still the maximal one. So if $A = \{a_1, b_1, c_1\}$, Then we can see that this is consistent with $\{a_1, b_1, c_3\}$. So at A, a considers it possible that b is muddy. A similar construction shows that a considers it possible b is not muddy. Additionally, at A, one can show that b knows that a and c are both muddy.

In this setting, something remarkable happens. In the possible frames setting, a similar situation would require the creation of more edges on the same set of worlds. This is because in the frame setting, uncertainty is captured by agents being unable to distinguish distinct worlds. In the simplicial setting, by contrast, that is achieved by instead *shrinking* the number of relevant perspectives. The model builds in new "edges", i.e. connections, simply by shrinking the number of nodes. There's also something extremely intuitive about how we went about defining this shrinking. In the possible worlds setting, we would say something like "a now considers it possible that b is muddy, even if they are not" and add in edges accordingly. Here, the change was remarkably slick. We simply restricted a's perspective to not include facts about b and generated the model as before. This is in line with the fact that our goal was a model where a was now uncertain about b's muddiness. Indeed, this is equivalent to saying that a's perspective does not contain b-facts, and generating the model was just as simple. Where in frame models, we increase uncertainty by adding worlds, in simplicial models, we increase uncertainty by limiting perspectives.

Now imagine a more complicated setting. In fact, everyone is certain about the muddiness of the other children. However, c is uncertain whether a is certain about the muddiness of b. Again, assume this state of affairs is something all the children are aware of. Let M_1 be the original muddy children problem and M_2 the model where a is uncertain about the muddiness of b. We build this new model by stitching M_1 and M_2 together at consistent c-nodes, that is, we are able to pass between the models via c-colored paths. Formally this would mean we have 6 a-nodes, 8 b-nodes, and 4 c-nodes. The a nodes are the 4 M_1 nodes and 2 M_2 nodes. The b-nodes are two copies of the 4 M_1 nodes, since they have the same M_2 nodes. So at A in M_1, we can say that c considers it possible they are in A in M_2. Making more and more elaborate versions of this story would similarly involve stitching together more models, as in the fame case.

Moving on from muddy children, one might ask what collection of axioms will be sound and complete with respect to these models (not just perfect models). The axioms I expect the general definition (not perfect complexes) to be sound and complete with respect to are **KB4** in addition to the following axioms (this is as of yet unproven):

$$\mathbf{NE} : \bigvee_{a \in A} ((K_a \bot) \to \bot)$$

$$\mathbf{SA_a} : (((K_a\bot) \to \bot) \wedge \bigwedge_{b\neq a} K_b\bot) \to K_a \bigwedge_{b\neq a} K_b\bot$$

NE stands for "Non-Emptyness" and $\mathbf{SA_a}$ stands for "Single Agent" as given in "Epistemic Logic with Agents that May Die" [10]. The intuition for **NE** is that no maximal face has no points. That is to say, the empty set is not a maximal face except in the empty model. The intuition for $\mathbf{SA_a}$ is that if a facet has a single point, it can't be attached to any other face, so it must be isolated, and the agential perspective of that face is that they are alone.

Of note is that "Epistemic Logic with Agents that May Die" and "Epistemic logic for impure simplicial complexes" each interpret $K_a\bot$ as a is 'dead' - a crash failure. This makes sense from the perspective of the other agents, as for $K_a\bot$ to be known to them, they would have to be entirely on facets without a nodes.

While we suggested that "Knowledge and Simplicial Complexes" used "agentially tagged propositions", more should be said about that to contrast our models. [4] In this setting, truth can be assessed against any face, and faces need not have consistent propositions around them, as the a node may bring p while the b node brings $\neg p$. Knowledge is, however, more or less defined the same way.

We now see there is a better way to model our initial story. We need the two perspectives for each agent, one where they know P and one where they do not. Call these a_1, a_2 and b_1, b_2, where $L(x)(x_1) = \{P\}$ and $L(x)(x_2) = \emptyset$. Then taking $S = S'$, we get the correct model.

7 Soundness

Our axioms will be the following for any formulas φ, ψ, and γ and any agent a:

Church's axioms:

$$\mathbf{C1} : \varphi \to (\psi \to \varphi)$$

$$\mathbf{C2} : (\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$$

$$\mathbf{C3} : ((\varphi \to \bot) \to \bot) \to \varphi$$

Modal logic:

$$\mathbf{K} : K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$$

$$\mathbf{T} : K_a\varphi \to \varphi$$

$$\mathbf{B} : \varphi \to K_a((K_a(\varphi \to \bot)) \to \bot)$$

$$\mathbf{4} : K_a\varphi \to K_aK_a\varphi$$

[6]

$$\mathbf{NE} : \bigvee_{a\in A} ((K_a\bot) \to \bot)$$

$$\mathbf{SA_a} : (((K_a\bot) \to \bot) \wedge \bigwedge_{b\neq a} K_b\bot) \to K_a \bigwedge_{b\neq a} K_b\bot$$

[6] Because we list **B** and **4** there is no need to list **5** as well because it is derivable.

$$\mathbf{NU}: P \to \bigvee_{a\in A} K_a P$$

And our rules of inference will be the following:

$$\mathbf{MP}: \text{ From } \varphi \text{ and } \varphi \to \psi \text{ infer } \psi$$

$$\mathbf{Nec}: \text{ From } \varphi \text{ infer } K_a\varphi$$

Our completeness proof will focus on models where each maximal face has $|A|$ nodes and a node for each agent. These axioms will be the usual **S5** axioms plus **NU**, but we prove soundness somewhat more generally. With that said, now some soundness lemmas. Let $\mathcal{M}$ be any simplicial model and X be any facet of said model. Then we have the following lemmas:

Lemma 7.1. *Soundness of **C1***

Proof. Suppose that $\mathcal{M}, X \vDash \varphi$. Then, if $\mathcal{M}, X \vDash \psi$, it does follow that $\mathcal{M}, X \vDash \varphi$, as desired. □

Lemma 7.2. *Soundness of **C2***

Proof. Suppose that $\mathcal{M}, X \vDash \varphi \to (\psi \to \gamma)$. Suppose further that $\mathcal{M}, X \vDash \varphi \to \psi$. Suppose further that $\mathcal{M}, X \vDash \varphi$. Then it follows that $\mathcal{M}, X \vDash \psi \to \gamma$ and $\mathcal{M}, X \vDash \psi$. So, it follows that $\mathcal{M}, X \vDash \gamma$, as desired. □

Lemma 7.3. *Soundness of **C3***

Proof. Suppose that $\mathcal{M}, X \vDash (\varphi \to \bot) \to \bot$. Suppose $\mathcal{M}, X \nvDash \varphi$. Then, if $\mathcal{M}, X \vDash \varphi$, we have a contradiction, and so $\mathcal{M}, X \vDash \bot$. It follows that $\mathcal{M}, X \vDash \varphi \to \bot$. So, $\mathcal{M}, X \vDash \bot$, a contradiction, and so $\mathcal{M}, X \vDash \varphi$, as desired. □

Lemma 7.4. *Soundness of **K***

Proof. Suppose that $\mathcal{M}, X \vDash K_a(\varphi \to \psi)$. Suppose further that $\mathcal{M}, X \vDash K_a\varphi$. Then for any facets Y such that $V(a) \cap (X \cap Y) \neq \emptyset$, we have that $\mathcal{M}, Y \vDash \varphi \to \psi$ and $\mathcal{M}, Y \vDash \varphi$. So, for any such Y, we have that $\mathcal{M}, Y \vDash \psi$. So, $\mathcal{M}, X \vDash K_a\varphi$, as desired. □

Lemma 7.5. *Soundness of **T***

Proof. Assume $\mathcal{M}$ is perfect. Suppose that $\mathcal{M}, X \vDash K_a\varphi$. Then for all Y such that $V(a) \cap (X \cap Y) \neq \emptyset$, we have that $\mathcal{M}, Y \vDash \varphi$. Take $Y = X$, because $X \cap (V(a)) \neq \emptyset$ by perfection. Then $\mathcal{M}, X \vDash \varphi$, as desired. □

Lemma 7.6. *Soundness of **B***

Proof. Suppose that $\mathcal{M}, X \vDash \varphi$. Let Y be a facet such that $V(a) \cap (X \cap Y) \neq \emptyset$. Suppose for a contradiction there is some such Y such that $\mathcal{M}, Y \vDash K_a(\varphi \to \bot)$. Then for all Z such that $V(a) \cap (Y \cap Z) \neq \emptyset$, we have that $\mathcal{M}, Z \vDash \varphi \to \bot$. Taking $Z = X$, we get that $\mathcal{M}, X \vDash \bot$, a contradiction. So, for all such Y, we have that $\mathcal{M}, Y \nvDash K_a(\varphi \to \bot)$. It follows that $\mathcal{M}, Y \vDash (K_a(\varphi \to \bot)) \to \bot$, and so $\mathcal{M}, X \vDash K_a((K_a(\varphi \to \bot)) \to \bot)$, as desired. □

Lemma 7.7. *Soundness of 4*

Proof. Assume $\mathcal{M}$ is perfect. Suppose that $\mathcal{M}, X \vDash K_a\varphi$. Then for all Y such that $V(a) \cap (X \cap Y) \neq \emptyset$, we have that $\mathcal{M}, Y \vDash \varphi$. Suppose now that Z is a face such that $V(a) \cap (Y \cap Z) \neq \emptyset$. By perfection, Y can only have one element of $V(a)$, so $V(a) \cap (X \cap Z) \neq \emptyset$. So, $\mathcal{M}, Z \vDash \varphi$. So, $\mathcal{M}, Y \vDash K_a\varphi$, and therefore $\mathcal{M}, X \vDash K_aK_a\varphi$, as desired. $\square$

Lemma 7.8. *Soundness of* ***NE***

Proof. Suppose for a contradiction that $\mathcal{M}, X \vDash K_a\bot$ for all $a \in A$. Then for any $a \in A$ and any facet Y, if $V(a) \cap X \cap Y \neq \emptyset$, then $\mathcal{M}, Y \vDash \bot$. This is a contradiction, so there is $a \in A$ such that $\mathcal{M}, X \nvDash K_a\bot$. So, $\mathcal{M}, X \vDash (K_a\bot) \to \bot$, as desired. $\square$

Lemma 7.9. *Soundness of SA_a*

Proof. Suppose $\mathcal{M}, X \vDash ((K_a\bot) \to \bot) \wedge \bigwedge_{b \neq a} K_b\bot$. Suppose Y is a facet such that $V(a) \cap (X \cap Y) \neq \emptyset$. Suppose for a contradiction Z is a facet such that for some $b \neq a$, $V(b) \cap (Y \cap Z) \neq \emptyset$. Fix a and b to be witnesses of these nonemptinesses respectively. Then b and a are consistent, so $b \in X$, which contradicts that $\mathcal{M}, X \vDash K_b\bot$. So no such Z exists. So, $\mathcal{M}, Y \vDash K_b\bot$ for all $b \neq a$. So, $\mathcal{M}, X \vDash K_a \bigwedge_{b \neq a} K_b\bot$, as desired. $\square$

Lemma 7.10. *Soundness of* ***NU***

Proof. Assume $L(x_a)$ consists only of atomic propositions for each $a \in A$ and $x_a \in V(a)$. Suppose $\mathcal{M}, X \vDash P$. So the union of all the agential perspectives proves P. Since these agential perspectives consist entirely of atoms, then the only possible proof is that some perspective contains P. In specific, there is $a \in A$ such that $X \cap V(a) = \{x_a\}$ and $P \in L(x_a)$. Let Y be a facet such that $V(a) \cap (X \cap Y) \neq \emptyset$. Then by properness, $x_a \in Y$, and so $\mathcal{M}, Y \vDash P$. Therefore, $\mathcal{M}, X \vDash K_aP$, as desired. $\square$

Lemma 7.11. *Soundness of* ***MP***

Proof. Suppose for any model $\mathcal{M}$ and any facet of that model X we have that $\mathcal{M}, X \vDash \varphi$ and $\mathcal{M}, X \vDash \varphi \to \psi$. Then $\mathcal{M}, X \vDash \psi$, as desired. $\square$

Lemma 7.12. *Soundness of* ***NEC***

Proof. Suppose for any model $\mathcal{M}$ and any facet of that model X we have that $\mathcal{M}, X \vDash \varphi$. Let Y be a facet such that $V(a) \cap (X \cap Y) \neq \emptyset$. If no such Y exists, then vacuously, $\mathcal{M}, X \vDash K_a\varphi$. If such a Y exists, then $\mathcal{M}, Y \vDash \varphi$, and then again, $\mathcal{M}, X \vDash K_a\varphi$, giving the desired result. $\square$

Say that $\Gamma \vdash_{\mathbf{S5+NU}} \varphi$ if and only if there is a deduction of φ from assumptions Γ using only the **S5** and **NU** rules, and $\Gamma \vDash \varphi$ if and only if $\mathcal{M}, X \vDash \Gamma$ then $\mathcal{M}, X \vDash \varphi$.

We can now prove Soundness:

Theorem 7.13 (Soundness)**.** *If $\Gamma \vdash_{\mathbf{S5+NU}} \varphi$, then $\Gamma \vDash \varphi$.*

Proof. Suppose $\Gamma \vdash \varphi$. We prove this by induction on the depth of the proof. If the proof has depth 1, φ is an axiom, and so our soundness lemmas cover it. Assume the obvious Inductive Hypothesis. If φ is introduced via a rule of inference from proven things, the IH shows that the model satisfies those proven things, and our soundness lemmas for rules of inference imply the model makes true φ. In all cases we get the desired result. $\square$

8 Completeness

We work with the logic **S5** + **NU** for each agent. We know this is sound with respect to perfect models. We will now show it is complete. For this section we consider only perfect simplicial models.

Theorem 8.1 (Completeness)**.** *If* $\Gamma \vDash \varphi$*, then* $\Gamma \vdash_{\mathbf{S5+NU}} \varphi$.

Proof. Let W be the collection of **S5** + **NU** maximal sets. Build the usual maximal frame model with W, that is, for $w_1, w_2 \in W$, $w_1 R_a w_2$ if and only if for all φ if $K_a\varphi \in w_1$ then $\varphi \in w_2$. We need to show this is an equivalence relation. Consider any φ. Suppose $K_a\varphi \in w_1$. Then by **T** $\varphi \in w_1$, so $w_1 R_a w_1$.

Suppose $w_1 R_a w_2$. Now assume $K_a\varphi \in w_2$. Now for a contradiction assume that $\neg K_a\varphi \in w_1$[7]. By **5**, $K_a\neg K_a\varphi \in w_1$. So, $\neg K_a\varphi \in w_2$, a contradiction. So,$K_a\varphi \in w_1$. By T, we have that $\varphi \in w_1$, and so $w_2 R_a w_1$, as desired.

Suppose $w_1 R_a w_2$ and $w_2 R_a w_3$. Suppose further $K_a\varphi \in w_1$. By **4**, we have that $K_a K_a\varphi \in w_1$. So, $K_a\varphi \in w_2$. So, $\varphi \in w_3$. This shows that $w_1 R_a w_3$, and we are done.

This lemma is a consequence of the usual completeness proofs for frame models.

Lemma 8.2. *: If* $\varphi \in w'$ *for all* w' *such that* $wR_a w'$*, then* $K_a\varphi \in w$.

Proof. Suppose $\varphi \in w'$ for all w' such that $wR_a w'$. Suppose for a contradiction that $K_a\varphi \notin w$. Then $\neg K_a\varphi \in w$ by maximal consistency. Therefore, by propositional logic and **K**, we have that $\neg K_a\neg\neg\varphi \in w$. By the Existence Lemma[8] there is a w' such that $wR_a w'$ and $\neg\varphi \in w'$. This of course is a contradiction because w' is consistent, and so $K_a\varphi \in w$, as desired. $\square$

So now we have our frame model, where given a world, the a-accessible worlds form an equivalence relation. So we have to imagine collapsing the frame into a point (the a-perspective) and expanding the worlds into facets. However, to be able to define this, we need another lemma.

Lemma 8.3. *Proof.* First, suppose $wR_a w'$. We need to show that $K_a\varphi \in w$ if and only if $K_a\varphi \in w'$. Suppose the $K_a\varphi \in w$. Then $\varphi \in w'$ for all w' such that $wR_a w'$. Suppose $w' R_a w''$. By the equivalence relation, $wR_a w''$, and so $\varphi \in w''$, so $K_a\varphi \in w'$ by the above lemma. Suppose $K_a\varphi \in w'$. By symmetry, $w' R_a w$, so $\varphi \in w$. If $wR_a w''$, then $w' R_a w''$, so $\varphi \in w'$. Again, by the previous lemma, $K_a\varphi \in w$, as desired. $\square$

[7] $\neg\varphi$ is the usual abbreviation for $\varphi \to \bot$

[8] Theorem 3.8 in http://math.uchicago.edu/ may/REU2020/REUPapers/Hebert.pdf

Because we have this new lemma, we can now let $a_w = \{\varphi \in \mathfrak{P}^* | K_a\varphi \in w \text{ and } \varphi \text{ is atomic}\}$ for each $a \in A$. If wR_aw', then $a_w = a_{w'}$. For each $a \in A$, say that $V(a) = \{a_w \cup \{w'|wR_aw'\}|w \in W\}$, call this $f_{a,w}$, and $L(f_{a,w}) = a_w$. Our facets will be $\{f_{a,w}|a \in A\}$, and for ease of notation we will refer to this facet as w. Indeed, this is well formed. Suppose for a contradiction that $\varphi \in L(f_{a,w}) = a_w$ and $\neg\varphi \in L(f_{b,w}) = b_w$. Since $L(f_{a,w})$ consists solely of atoms, this is already a contradiction. We can also say that the agents have no inconsistent knowledge with each other by **T** - the proof is trivial. Moreover, each facet has precisely one node for each agent. So these facets are $|A|$ size consistent unions.

We need to show that the a-accessible worlds from w are precisely those that share $f_{a,w}$. Suppose wR_aw'. As above, then $a_w = a_{w'}$. Moreover, because we have an equivalence relation, $\{w''|wR_aw''\} = \{w''|w'R_aw''\}$. This means that $f_{a,w} = f_{a,w'}$, so the two facets share a node. Suppose $w \cap w' \neq \emptyset$. Fix a such that $V(a) \cap (w \cap w') \neq \emptyset$. This tells us that $f_{a,w} = f_{a,w'}$, because facets have one and only one node for each agent. So, $\{w''|wR_aw''\} = \{w''|w'R_aw''\}$. Because we have an equivalence relation, this tells us that wR_aw', as desired. With this shown, call this maximal simplicial model $\mathcal{M}$.

We now need to show that $\mathcal{M}, w \vDash \varphi$ if and only if $\varphi \in w$. We do this by induction. Suppose $P \in w$. Fix a such that $K_aP \in w$ (**NU**). Then $P \in a_w$. So, $\mathcal{M}, w \vDash P$. Suppose $\mathcal{M}, w \vDash P$. Then there is $a \in A$ such that $P \in a_w$. This means $K_aP \in w$, and so by **T**, $P \in w$. The falsum case is vacuous.

Assume the inductive hypothesis, that is, for any formula φ of depth n, then $\varphi \in w$ if and only if $\mathcal{M}, w \vDash \varphi$. Now let φ and ψ be arbitrary formulas of depth n.

Suppose $\varphi \to \psi \in w$. There are two possibilities. Suppose $\psi \in w$. By the IH, $\mathcal{M}, w \vDash \psi$, and so $\mathcal{M}, w \vDash \varphi \to \psi$. Suppose $\varphi \notin w$. By maximal consistency, $\neg\varphi \in w$. Suppose for a contradiction that $\mathcal{M}, w \vDash \varphi$. Then by the IH, $\varphi \in w$, and so w is inconsistent, a contradiction. So $\mathcal{M}, w \nvDash \varphi$, and so $\mathcal{M}, w \vDash \varphi \to \psi$, as desired. Suppose $\mathcal{M}, w \vDash \varphi \to \psi$. If $\varphi \notin w$, we are done. Suppose $\varphi \in w$. Suppose, for a contradiction, that $\psi \notin w$. By the IH, we have that $\mathcal{M}, w \nvDash \psi$ and $\mathcal{M}, w \vDash \varphi$. Therefore $\mathcal{M}, w \vDash \psi$, a contradiction. So $\psi \in w$ and therefore $\varphi \to \psi \in w$, as desired.

Suppose $K_a\varphi \in w$. Then if w' is a-accessible from w we have that $\varphi \in w'$. By the above, the a-accessible worlds from w are those worlds that share a_w, and so φ is in all Y such that $V(a) \cap (w \cap Y) = \{a_w\} \neq \emptyset$. By the IH, $\mathcal{M}, Y \vDash \varphi$. So, $\mathcal{M}, w \vDash K_a\varphi$. Suppose $\mathcal{M}, w \vDash K_a\varphi$. Then for all facets Y such that $V(a) \cap (w \cap Y) \neq \emptyset$, $\mathcal{M}, Y \vDash \varphi$. Then by the IH, $\varphi \in Y$. As before, these Y are precisely those that are a-accessible from w, so by our Lemma, $K_a\varphi \in w$, as desired.

Suppose $\Gamma \nvDash \varphi$. Then there is a maximal consistent set X such that $\Gamma \subseteq X$ and $\varphi \notin X$. Breaking X into its agential portions we can find a facet X in the maximal model. This model makes Γ true and φ false, as desired. Completeness is done!

□

9 Distributed Knowledge

Extend the language with a predicate D for distributed knowledge[9]. We interpret this as follows, where $\mathcal{M}$ is a perfect complex and a' is the unique a-node in X:

$$\mathcal{M}, X \vDash D\varphi \text{ iff } \mathcal{M}, Y \vDash \varphi \text{ for all } Y \text{ such that } \forall a \in A(a' \in Y)$$

This definition is modified from "Knowledge and Simplicial Complexes" [4]. The idea is that distributed knowledge is true precisely when given a facet accessible by all the agents, it is a φ facet. However, on perfect complexes, this has a curious feature! All facets have exactly 1 node for each agent. If one specifies a node for each agent, that uniquely determined the facet. So the only facet accessible from X by all agents is X itself! This gives us the following lemma:

Lemma 9.1. *If $\mathcal{M}$ is perfect and X is a facet, then $\mathcal{M}, X \vDash D\varphi$ iff $\mathcal{M}, X \vDash \varphi$*

Proof. Suppose $\mathcal{M}, X \vDash D\varphi$. Then for any facet Y and $a \in A$ such that $\pi_a(X) \in Y$ we have that $\mathcal{M}, Y \vDash \varphi$. Taking $Y = X$ gives the desired result. Suppose, conversely, $\mathcal{M}, X \vDash \varphi$. Consider any Y such that for all $a \in A$, $\pi_a(X) \in Y$. If $Y \neq X$, then they disagree on a node, so there is $a \in A$ such that $\pi_a(X) \neq \pi_a(Y)$. Since the a-node is unique on a facet, it follows that $\pi_a(X) \notin Y$, a contradiction. So $Y = X$. It follows that $\mathcal{M}, X \vDash D\varphi$. □

It's clear, then, we can modify our semantic condition for distributed knowledge:

$$\mathcal{M}, X \vDash D\varphi \text{ iff } \mathcal{M}, X \vDash \varphi$$

This is extremely strong! And indeed it means that $\varphi \leftrightarrow D\varphi$ will be valid in our logical system. Call this rule **V** for vacuity. It follows that if one could expand $\mathbf{S5} + \mathbf{NU}$ with a series of axioms capable of proving **V**, this list of axioms would be complete with respect to perfect models in the expanded language. The obvious such list of axioms is simply the singleton including that fact! To see why this is complete, Consider our canonical model construction from the previous section, only this time with maximal consistent sets who are $\mathbf{S5} + \mathbf{NU} + \mathbf{V}$ sets. Call this model $\mathcal{M}_D$. Given a set of formulas w, let $\mathbf{D}(w)$ be the set $\{D\varphi | \varphi \in w\}$. It's clear that $\mathbf{D}$ gives a bijection from $\mathbf{S5} + \mathbf{NU}$ maximal consistent sets to $\mathbf{S5} + \mathbf{NU} + \mathbf{V}$ maximal consistent sets, as for any w which is $\mathbf{S5} + \mathbf{NU} + \mathbf{V}$ maximal consistent, $D\varphi \in w$ if and only if $\varphi \in w$. Thus, if $\Gamma \nvdash_{S5+NU+V} \varphi$, we can find in $\mathcal{M}_D$ a facet f_w corresponding to a maximal consistent set w such that $\Gamma \subset w$ and $\varphi \notin w$. One can see, then, that **NU** trivializes distributed knowledge. It's worth noting that since **NU** satisfies **V**, so should local variables as well. Indeed, in "Knowledge and Simplicial Complexes", they note this specifically. The takeaway is, then, while our system is a strict weakening of the local variables system, as **NU** is a strictly weaker axiom, it cannot avoid making distributed knowledge vacuous.

[9] We will not worry about distributed knowledge for groups smaller than the whole

10 Completeness Without NU

There is a way to get rid of **NU** entirely. Expand our set of agents A to include a new agent, n, which we think of as nature. We claim the logic is sound and complete with respect to just **S5**. Assume that perspectives can have any formulas in them. Again, the only step that needs to be changed is the atomic step. w_n is the set of formulas φ such that $\varphi \in w$ and there does not exist $a \in A$ such that $K_a\varphi \in w$.

Assume $P \in w$. If there exists a such that $K_a \in w$, then the proof proceeds as before. Otherwise, it follows that $P \in w_n$, and so $\mathcal{M}, w \vDash P$. Suppose $\mathcal{M}, w \vDash P$. Then $\bigcup_{x \in w} L(x) \vdash P$. By T, we know that $L(x) \subseteq w$ for all $x \in w$. It follows that $w \vdash P$, and so by consistency, $P \in w$.

Of note here is that it does not actually matter if we assume that perspectives are solely atomic or not. Such is the power of the nature node. To see this in action, take one agent a, and their perspective is empty. Two nature nodes, one with P, one with Q. Then there are two faces in the complex, one where P is true, the other where Q is true. Then of course $P \vee Q$ is true on both faces which are connected by the a node, and so a knows $P \vee Q$ but does not know either P or Q. The other node doesn't HAVE to be a nature node, but of course, if it weren't, then THAT agent would know P or would know Q, which is why we would otherwise need the **NU** rule. Here everything was done with atoms where before we would have required a's perspective consist of the formula $P \vee Q$.

Nevertheless, the nature nodes are in some sense too powerful. One can first see this formally - in the above model, the nature nodes handles ALL of the propositions. Indeed, one could assign atoms solely to the nature nodes and stitch empty agent perspectives around them as desired. This is equivalent to an **S5** Kripke frame, where the empty agent perspectives are the edges and the nature nodes are the possible worlds. As a result, the nature nodes in fact are so strong they can "take over" the model, so to speak. One can work entirely with them and ignore agent perspectives, and if one does this, the nature nodes become no different than possible worlds, which we initially sought to do away with. This is of course philosophically unsatisfying. For the modeler who wishes to take advantage of the intuitions outlined at the beginning of this paper, the advice is to introduce nature nodes as needed, solely to fill in when one needs propositions to be true that no agent knows (or that is not known distributively).

11 Conclusion and Future Work

This paper sought, broadly, to take tools from distributed computing to build a sound and complete semantics for modal logic without possible worlds. We did ultimately do this, with varying degrees of success. The presence of the "No Uncertainties" assumption is inversely correlated with the degree to which we allow for propositions to be true regardless of what each and all of the agents know. Nevertheless, we believe the desirability of such an understanding of modal logic, and prevalence of the **NU** assumption in most practical consequences speaks favorably for this system.

For future projects there are numerous things that can be tackled. Generalizing the proof techniques from completeness here, it seems possible to tackle completeness in the absence of **T**, though this remains to be worked out in detail. Brittany Gelb has also found numerous results using tools and techniques from Algebraic Topology to give algebraic characterizations of bisimulations [7]. We are both optimistic that extending these observations, possibly by connecting them to spectral graph theory, will prove fruitful. Additionally, I hope to expand much of this work beyond the epistemic contexts, and use similar semantics to address problems like counterfactuals and other settings in which possible worlds have previously proven ideal.

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14. ABSTRACT

In this paper, we set out to give a novel semantics for modal logic in simplicial complexes. The motivation for this semantics will be first the replacement of possible worlds with the idea of an "agent perspective". After exploring some of the philosophical implications of such a move, we give a semantics based around this idea. Following this, we explore some of the more interesting consequences of such a system, in particular the soundness of an unusual axiom we call $\mathbf{NU}^*$. After giving soundness and completeness, we conclude by exploring ways to weaken this axiom in our semantics.

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