A generalized, compactly-supported correlation function for data assimilation applications

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Outline

1. Present a new application of forecasting (evolving) estimation error covariances during data assimilation,
2. Introduce a new correlation function that generalizes the Gaspari-Cohn compactly-supported approximation to a Gaussian,
3. Apply this generalized correlation function for evolving covariances.

Motivation: Covariance Evolution in Data Assimilation

At the heart of modern data assimilation is *covariance evolution*, *e.g.*,

$$\boldsymbol{P}_{k+1} = \boldsymbol{M}_{k+1,k} \boldsymbol{P}_k \boldsymbol{M}_{k+1,k}^T + \boldsymbol{Q}_k.$$

Common Issues with Covariance Evolution:

- Computational Expense
- Variance Loss

Covariance Evolution in Data Assimilation

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Common Issues with Covariance Evolution:

- ~~Computational Expense~~ $\rightarrow$ Computationally efficient
- ~~Variance Loss~~ $\rightarrow$ Evolves variance directly

Alternative: evolve the variance and correlation length, then reconstruct $\boldsymbol{P}$ using a parametric correlation function.

Local Covariance Evolution

Consider covariances $P = P(x_1, x_2, t)$ associated with states $q = q(x, t)$ on the unit circle (S_1^1),

$$q_t + vq_x + v_x q = 0, \qquad P_t + v_1 P_{x_1} + v_2 P_{x_2} + (v_{x_1} + v_{x_2})P = 0,$$
$$q(x, t_0) = q_0(x) \qquad P(x_1, x_2, t_0) = P_0(x_1, x_2)$$

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Variance Equation:	Correlation Length Equation:
$\sigma_t^2 + v\sigma_x^2 + 2v_x\sigma^2 = 0,$	$L_t + vL_x - v_x L = 0,$
$\sigma^2(x, t_0) = \sigma_0^2(x)$	$L(x, t_0) = L_0(x)$

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1. Evolve σ^2 and L from initial condition $P_0(x_1, x_2) = \sigma_0(x_1)C_0(x_1, x_2)\sigma_0(x_2)$,
2. Approximate $P(x_1, x_2, t) = \sigma(x_1, t)C(x_1, x_2, t)\sigma(x, t)$ with evolved σ^2 and L using a parametric correlation function.

The Gaspari and Cohn (1999) Correlation Function

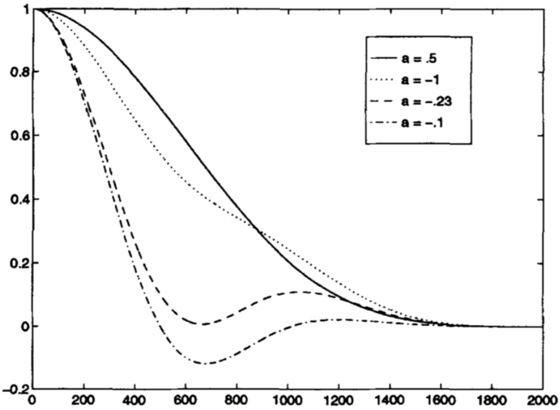

Figure 7. The function $C_0(z, a, c)$ of the example in section 4(c) for $c = 1000$ km and various values of a. See text for explanation.

Figure 1: Figure 7 from Gaspari and Cohn (1999). The function $C_0(z, a, c)$ is the general form of the compactly-supported, fifth-order, piecewise rational correlation function derived in Sec. 4(c). Typically, $a = 1/2$ (solid black).

The Generalized Gaspari-Cohn (GenGC) Correlation Function

Correlation length for the compactly-supported, piecewise rational:

$$L = c\left(\frac{3(22a^2+3a+1)}{40(8a^2-2a+1)}\right)^{1/2}, \quad a = 1/2 \Rightarrow L = \sqrt{0.3}c \qquad (1)$$

Need to generalize this correlation function to allow for *variable* L.

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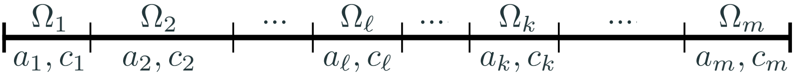
Generalized Gaspari-Cohn

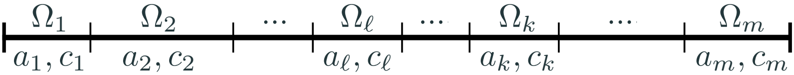
Allow $a = a_k$ and $c = c_k$ to vary over the spatial index k.

Now $L = L_k$ can vary!

The GenGC Correlation Function, cont.



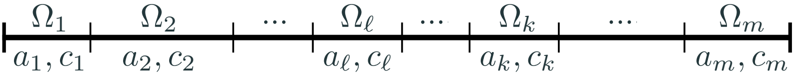
The GenGC Correlation Function, cont.



For $r \in \Omega_k$, $s \in \Omega_\ell$, and each fixed $k, \ell = 1, 2, ..., m$,

GenGC: $C_{k\ell}(r, s; a_k, a_\ell, c_k, c_\ell)$

The GenGC Correlation Function, cont.



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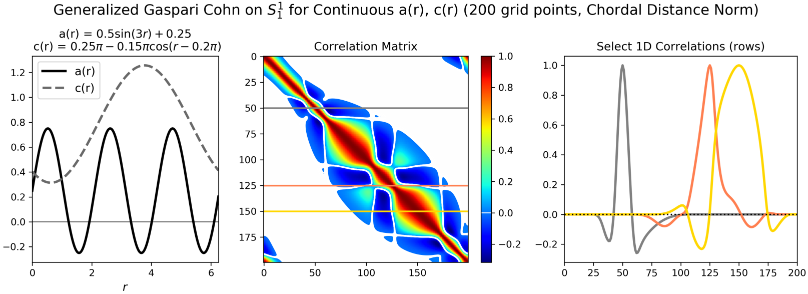

Figure 2: GenGC on the unit circle (S_1^1) for continuous a and c (left), correlation matrix (middle), and selected correlations (right). White regions in the correlation matrix correspond to correlations between -0.003 and 0.003.

GenGC in 2D

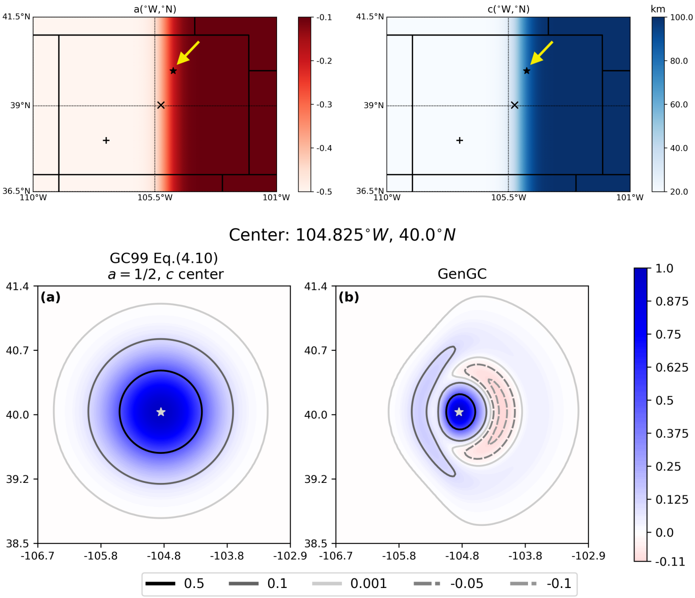

Figure 3: Top: Background a and c fields as functions of longitude and latitude over Colorado. Bottom: 2D correlations with respect to $\star$ for the given a and c fields, plotted using a Mercator projection.

Local Covariance Evolution: Correlation Reconstruction with GenGC

Methodology:

1. Define the initial correlation length from standard Gaspari-Cohn correlation ($a_0 = 1/2$, $c = c_0$), $L_0(x) = \sqrt{0.3}c_0$

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2. Evolve $L(x,t)$ forward in time by solving its PDE
3. At fixed time t, use $L(x,t)$ and $a = a_0$ fixed to compute $c(x,t)$, then reconstruct the correlations with GenGC: for $x_1 \in \Omega_k, x_2 \in \Omega_\ell$

$$C(x_1, x_2, t) = C_{k\ell}\big(x_1, x_2; a_0, a_0, c(x_1, t), c(x_2, t)\big)$$

Correlation Reconstruction with GenGC: Preliminary Results

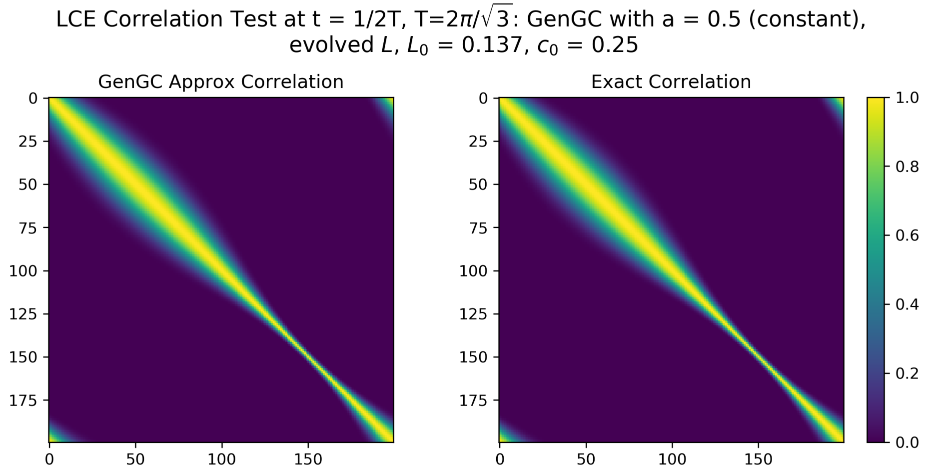

Figure 3: Comparisons of the correlation matrix approximated with GenGC using evolved correlation lengths L and $a = 1/2$ constant (left) with the exact correlation matrix (right). $1D$ experiments on S_1^1 with $v(x) = \sin(x) + 2$ and $c_0 = 0.25$. Errors in the GenGC approximation are between -0.0125 and 0.0125.

In Summary

The added flexibility of GenGC while remaining compactly supported can be a useful tool for the data assimilation community.

Applications of GenGC in Data Assimilation:

- Local covariance evolution, an alternative means of mitigating problems associated with covariance evolution,
- Covariance modeling and tapering (localization) in current data assimilation schemes.

Acknowledgements and Questions

For more information or further discussion, contact Shay at

shay.gilpin@colorado.edu

Look out for our paper,

A generalized, compactly-supported correlation function for data assimilation applications

coming soon to the Quarterly Journal of the Royal Meteorological Society.

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Extra slides

1D Example

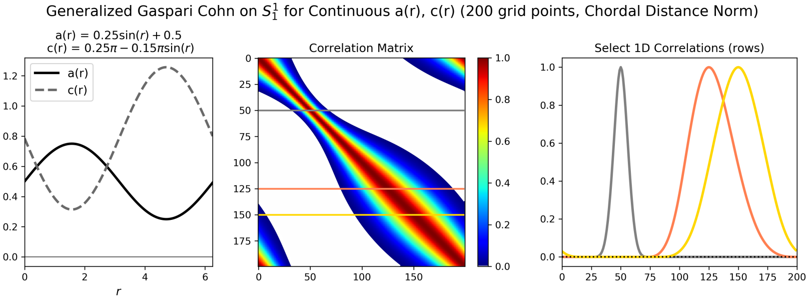

Figure 4: GenGC on the unit circle for continuous a and c (left), correlation matrix (middle), and selected correlations (right). White regions in the correlation matrix correspond to correlations between -0.003 and 0.003.

Correlation Reconstruction with GenGC: Preliminary Results 2

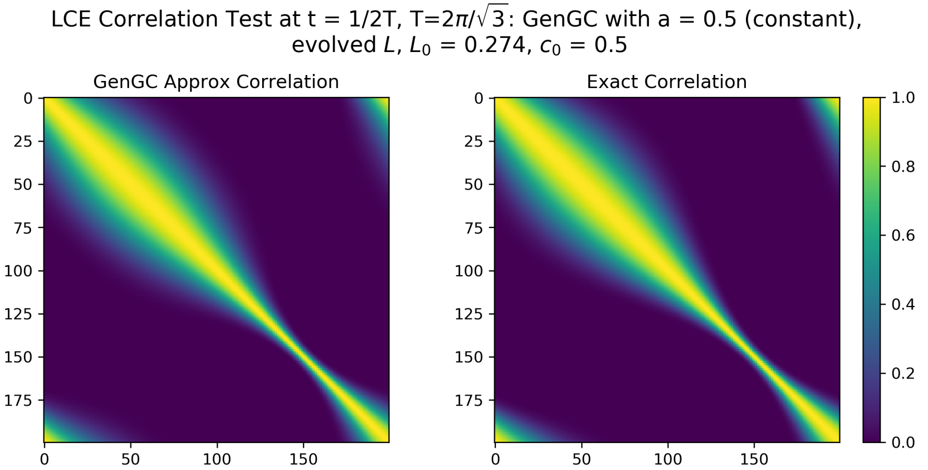

Figure 5: Comparisons of the correlation matrix approximated with GenGC using evolved correlation lengths L and $a = 1/2$ constant (left) with the exact correlation matrix (right). $1D$ experiments on S^1_1 with $v(x) = \sin(x) + 2$ and $c_0 = 0.5$. Errors in the GenGC approximation are between -0.05 and 0.05.

Continuity

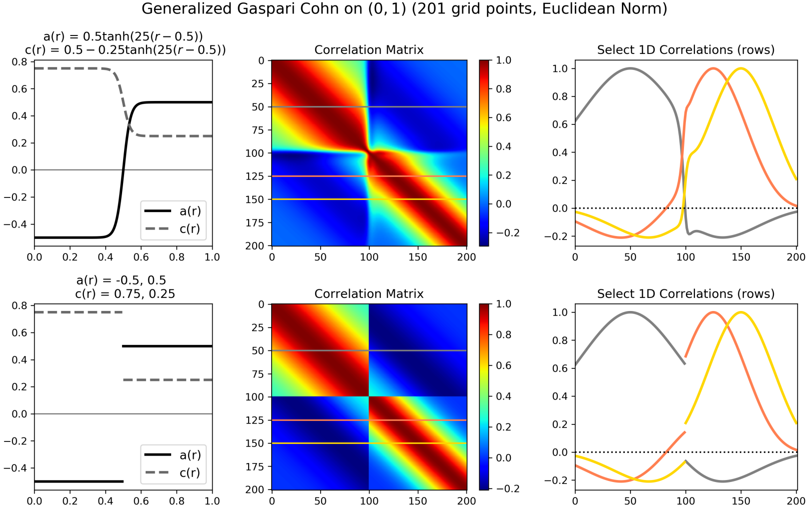

Figure 6: Example of GenGC on $(0, 1)$ comparing when each grid cell is its own subregion vs. the case where $(0, 1)$ is split into two subregions $(0, 1/2), (1/2, 1)$ (bottom row)