Numerical Methods for Jet Noise Predictions Using the Generalized Acoustic Analogy

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Outline

- Application: Jet noise prediction
- Acoustic Analogy approach
- Simplification of the problem for numerical solution
- Numerical methods for two classes of jet flows
- Sample results

Jet Noise Predictions

- Jet noise is the sound produced by aircraft engine exhaust
- Can be the dominate source of sound produced by aircraft
- Prediction methods are needed to understand and develop noise-reduction technologies
- Options for Jet Noise Prediction
  1. Direct Simulation
     a) Not practical over all scales
  2. Large-Eddy Simulation/Sound Propagation
     a) Increasingly used
     b) Still expensive (high-end computing systems and long run time)
  3. Reduced-order models
     a) Lower computation requirements and faster run time while retaining main physics
     b) Based on Acoustic Analogy
  4. Empirical Methods
     1. Very fast, can be implemented into optimization tools for design trades
     2. Limited to cases ‘close’ to those for which the models are developed

Acoustic Analogy

- Acoustic analogies are often used to develop physics-based reduced-order noise prediction methods
- An acoustic analogy is a rearrangement of the Navier-Stokes equations to obtain a linear wave operator acting on a set of fluctuating quantities with all non-linearities treated as 'source terms'
- Example: Lighthill (1954, 1955)

Continuity and Momentum Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y_i}\rho v_i = 0$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial}{\partial y_j} v_i\right) = -\frac{\partial p}{\partial y_i} + \frac{\partial e_{ij}}{\partial y_j}$$

Wave equation operator with non-linear 'sources'

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0 \frac{\partial^2 \rho'}{\partial y_i \partial y_i} = \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}$$

Model Sources -- T_{ij}

Compute solution of linear wave equation

Generalized Acoustic Analogy -- Goldstein (2003)

- Include more physical effects in operator

  --> Propagation through a non-uniform mean flow

  - Reduce burden on source model
  - More complication equation for propagation –> Numerical solution required

- Third-order 'wave' equation with variable coefficients

$$\frac{D}{D\tau}\left(\frac{D^2p'}{D\tau^2} - \frac{\partial}{\partial y_i} c^2(\boldsymbol{y_T}) \frac{\partial p'}{\partial y_i}\right) + 2c^2(\boldsymbol{y_T}) \frac{dU(\boldsymbol{y_T})}{dy_i} \frac{\partial^2 p'}{\partial y_1 \partial y_i} = S \;\; ; \mathrm{i} = 1,2,3$$

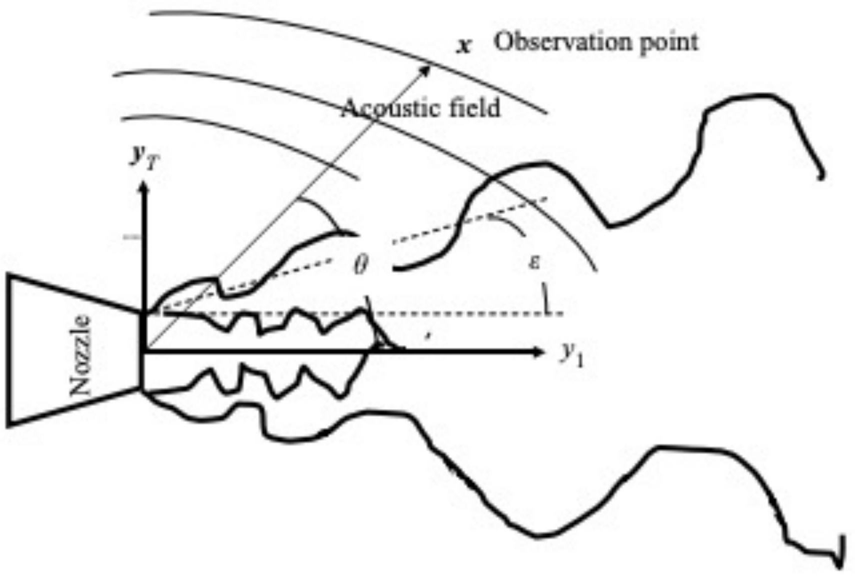
$$\frac{D}{D\tau} = \frac{\partial}{\partial \tau} + U(\boldsymbol{y_T}) \frac{\partial}{\partial y_1}$$

y_1 -- Streamwise direction

$(\boldsymbol{y_T}) = (y_2, y_3)$ - Cross Stream vector

$U(\boldsymbol{y_T})$ = Mean Streamwise velocity

$c^2(\boldsymbol{y_T})$ = Mean speed of sound (squared)



Solution for Inhomogeneous Partial Differential Equation
Green's Function

- The Green's function is the solution due to a point source
  - Know Green's function → Obtain solution for any source

$$\left(\frac{D^3G^a}{D\tau^3} - \frac{\partial}{\partial y_i}c^2(y_T)\frac{\partial}{\partial y_i}\frac{DG^a}{D\tau}\right) - 2\frac{\partial}{\partial y_i}c^2(y_T)\frac{dU(y_T)}{dy_i}\frac{\partial G^a}{\partial y_1}$$
$$= -\delta(x - y)\,\delta(t - \tau)$$

- In this equation, coefficients do not depend on:
  1. Time, τ (by construction)
  2. Streamwise coordinate, y_1 (Approximation, parallel mean flow)

➢ Modal (Fourier Transform) solution in these independent variables

  ❖ Trade independent variables (τ, y_1) for parameters (ω, k_1)

Governing Equation for ‘reduced’ Green’s Function

$$\left[\frac{\partial}{\partial y_i}\frac{c^2(\boldsymbol{y}_T)}{\left(\omega - kU(\boldsymbol{y}_T)\right)^2}\frac{\partial}{\partial y_i} + 1 - \frac{k^2c^2(\boldsymbol{y}_T)}{\left(\omega - kU(\boldsymbol{y}_T)\right)^2}\right]\hat{G}_0(\boldsymbol{y}_T|\boldsymbol{x}_T; k, \omega) = \frac{\delta(\boldsymbol{y}_T - \boldsymbol{x}_T)}{(2\pi)^2}$$

Boundary Conditions:

Bounded at $y_T = 0$

Outgoing as $y_T \to \infty$

Two-dimensional problem (y_2, y_3)

To be solved for given frequency ω and streamwise wavenumber k $(= {}^{\omega cos\theta}/_{c_\infty}$, far-field)

Mean streamwise velocity, $U(\boldsymbol{y}_T)$, and sound speed, $c^2(\boldsymbol{y}_T)$, given from Reynolds-averaged Navier-Stokes solution

Numerical methods for two classes of problems

Polar coordinates (r, φ)

$$y_T = r = \sqrt{y_2^2 + y_3^2} \qquad \varphi = tan^{-1}(y_2/y_3)$$

1. Axisymmetric Jets: $U = U(y_T), c^2 = c^2(y_T)$
- Depends only on r, independent of φ
- $U = U(r), c^2 = c^2(r)$

2. Non-axisymmetric jets $U = U(\boldsymbol{y_T}), c^2 = c^2(\boldsymbol{y_T})$
- Depends on both r and φ
- $U = U(r, \varphi), c^2 = c^2(r, \varphi)$

Governing Equation for ‘Reduced’ Green’s Function
Axisymmetric Jet

- Polar coordinates (r, φ)
- Mean flow (equation coefficients) depends only on r
- Fourier series in φ - Trade φ for n

Second-order **ordinary** differential equation with **variable coefficients**

$$\left[\frac{1}{r}\frac{d}{dr}\frac{r\,c^2(r)}{(\omega - kU(r))^2}\frac{d}{dr} + 1 - \frac{k^2 c^2(r)}{(\omega - kU(r))^2}\left(k^2 + \frac{n^2}{r^2}\right)\right] G^n(r|r'; k, \omega, n) = \frac{\delta(r - r')}{(2\pi)^3 r}$$

Solve in terms of two independent homogeneous solutions

Numerical Solution of 'Reduced' Green's Function

Axisymmetric Jet

Numerical problem reduces to solving a second-order ordinary differential equation

Regular singular point at $r = 0$

General Form: $$w_n''(r) + p(r)\, w_n''(r) + q(r)\, w_n(r) = 0 \quad ; \quad n = 0,1,\ldots,N$$

Express as system of two first-order equations

Put: $$v_n = w_n'$$

Then: $$w_n' = v_n \quad ; \quad v_n' = -(pv_n + qw_n) = f(r, v_n, w_n)$$

Solve first-order system $\begin{Bmatrix} v_n \\ w_n \end{Bmatrix}$ by 'marching' from 'initial' condition at $r = r_0$

Numerical Solution of 'Reduced' Green's Function

Axisymmetric Jet

'March' solution from 'initial' condition at $r = r_0$, with $\begin{Bmatrix} v_n(r_0) \\ w_n(r_0) \end{Bmatrix}$ known, to r_M

Method: 'Classical' Fourth-order Runge-Kutta for system of first-order equations

First-order system: $v_n' = f(r, v_n, w_n) \quad ; \quad w_n' = v_n$

For $i = 0, M-1$

$$\begin{Bmatrix} v_n(r_{i+1}) \\ w_n(r_{i+1}) \end{Bmatrix} = \begin{Bmatrix} v_n(r_i) \\ w_n(r_i) \end{Bmatrix} + \begin{Bmatrix} \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\ \Delta v_n(r_i) + \frac{\Delta}{6}[k_1 + k_2 + k_3] \end{Bmatrix}$$

k_1, k_2, k_3, k_4 are known at r_i

But need 'initial' condition $\begin{Bmatrix} v_n(r_0) \\ w_n(r_0) \end{Bmatrix}$ → How to get this ?

Numerical Solution of ‘Reduced’ Green’s Function

Axisymmetric Jet

‘Initial’ condition for marching solution

Second-order ordinary differential equation with Regular Singular point at $r = 0$

$$w_n''(r) + p(r)\, w_n''(r) + q(r)\, w_n\ (r) = 0$$

Series solution around $r = 0$

Method of Frobenius: $w_n = r^n \sum_{m=0}^{\infty} a_m r^m$

Expand coefficients in Taylor series around $r = 0$

$$rp = \sum_{m=0}^{\infty} p_m\, r^m \qquad r^2 q = \sum_{m=0}^{\infty} q_m\, r^m$$

$$rp \approx p_0 + p_1 r + \ p_2 r^2 + p_3 r^3 + p_4 r^4$$
$$r^2 q \approx q_0 + q_1 r + \ q_2 r^2 + q_3 r^3 + q_4 r^4$$

Substitute series’ into differential equation, equate powers of r to determine a_m

$$w_n(r_0) = r_0^n [a_0 + a_1 r_0 + a_2 r_0^2 + a_3 r_0^3] \qquad v_n(r_0) = n r_0^{n-1} \left[a_1 + 2a_2 r_0 \ + 3a_3 r_0^2\right]$$

Numerical Solution of ‘Reduced’ Green’s Function

Axisymmetric Jet

Solution Procedure Summary:

Numerical solution for Green’s function:

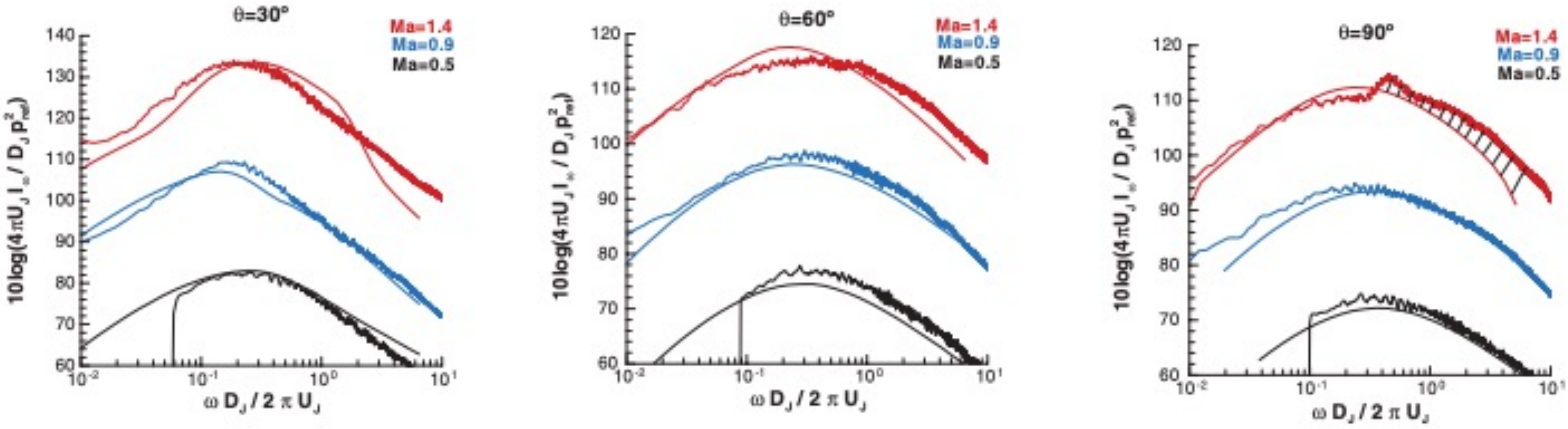
- Fourier-decompose in the streamwise and azimuthal directions and in time
- Solve numerically for the independent azimuthal modes at given frequency and observer angle
- Sum azimuthal modes to obtain ‘reduced’ Green’s function

Prediction of far-field acoustic spectrum:

- Insert ‘reduced’ Green’s function into formula for acoustic spectrum
- Integrate over source volume

Example Results: Axisymmetric (Round) Jet

Predictions of Noise from Round, unheated turbulent compared with experimental data (NASA GRC SHJAR)



Governing Equation for 'Reduced' Green's Function

Non-Axisymmetric Jet

$$\left[\frac{\partial}{\partial y_i}\frac{c^2(\boldsymbol{y}_T)}{\left(\omega - kU(\boldsymbol{y}_T)\right)^2}\frac{\partial}{\partial y_i} + 1 - \frac{k^2 c^2(\boldsymbol{y}_T)}{\left(\omega - kU(\boldsymbol{y}_T)\right)^2}\right]\hat{G}_0(\boldsymbol{y}_T|\boldsymbol{x}_T;k,\omega) = \frac{\delta(\boldsymbol{y}_T - \boldsymbol{x}_T)}{(2\pi)^2}$$

- Coefficients (mean flow) depends on (y_2, y_3) (ie. both (r, φ))

- Azimuthal Fourier modes of Green's function are coupled with those of the mean flow
  - System of simultaneous linear algebraic equations for azimuthal Fourier modes
  - Feasible for certain classes of mean flows

- More general method based on finite volume technique

Governing Equation for 'Reduced' Green's Function

Non-Axisymmetric Jet

Far-field solution, specified incoming wave

$$\frac{\partial}{\partial y_i}\frac{c^2(\boldsymbol{y_T})}{\left[1-M\left(\boldsymbol{y}_\perp\right)\cos\theta\right]^2}\frac{\partial g\left(\boldsymbol{y}_\perp;\varphi,\theta:\omega\right)}{\partial y_i}+\omega^2\left\{1-\frac{\left(c^2(\boldsymbol{y_T})/c_\infty^2\right)\cos^2\theta}{\left[1-M\left(\boldsymbol{y}_\perp\right)\cos\theta\right]^2}\right\}g\left(\boldsymbol{y}_\perp;\varphi,\theta:\omega\right)=0 \quad ; \quad i=2,3$$

$$\boldsymbol{y}_\perp=\left\{y_2,y_3\right\}$$

Boundary Conditions:

$$g\left(\boldsymbol{y}_\perp;\varphi,\theta:\omega\right)\rightarrow\frac{\left(\omega/c_\infty\right)^2 e^{-i(\omega/c_\infty)y_\perp\sin\theta\cos(\varphi-\varphi_0)}e^{i\pi/4}}{2\left(2\pi\right)^2\sqrt{2\pi\sin\theta\,\omega/c_\infty}}+\text{outgoing waves}$$

as $y_\perp\rightarrow\infty$

$$g\left(\boldsymbol{y}_\perp;\varphi,\theta:\omega\right)\quad\text{Bounded at } y_\perp=0$$

Numerical Solution of Green's Function: Finite Volume Method

$$\iint_R \frac{\partial}{\partial y_i} \frac{c^2(\boldsymbol{y_T})}{\left[1 - M\left(\boldsymbol{y}_\perp\right)\cos\theta\right]^2} \frac{\partial g\left(\boldsymbol{y}_\perp ; \varphi, \theta : \omega\right)}{\partial y_i} dR + \omega^2 \iint_R \left\{1 - \frac{\left(c^2(\boldsymbol{y_T})/c_\infty^2\right)\cos^2\theta}{\left[1 - M\left(\boldsymbol{y}_\perp\right)\cos\theta\right]^2}\right\} g\left(\boldsymbol{y}_\perp ; \varphi, \theta : \omega\right) dR = 0 \quad ; \quad i = 2,3$$

Divergence Theorem:

$$\int_S \frac{c^2(\boldsymbol{y_T})}{\left[1 - M\left(\boldsymbol{y}_\perp\right)\cos\theta\right]^2} \widehat{n_i} \frac{\partial g\left(\boldsymbol{y}_\perp ; \varphi, \theta : \omega\right)}{\partial y_i} dS + \omega^2 \iint_R \left\{1 - \frac{\left(c^2(\boldsymbol{y_T})/c_\infty^2\right)\cos^2\theta}{\left[1 - M\left(\boldsymbol{y}_\perp\right)\cos\theta\right]^2}\right\} g\left(\boldsymbol{y}_\perp ; \varphi, \theta : \omega\right) dR = 0 \quad ; \quad i = 2,3$$

No Mean Flow Derivatives !

$$\int_S F\left(\boldsymbol{y}_\perp\right) \widehat{n_i} \frac{\partial g\left(\boldsymbol{y}_\perp ; \varphi, \theta : \omega\right)}{\partial y_i} dS + \omega^2 \iint_R H\left(\boldsymbol{y}_\perp\right) g\left(\boldsymbol{y}_\perp ; \varphi, \theta : \omega\right) dR = 0 \quad ; \quad i = 2,3$$

Numerical Solution of Green's Function: Finite Volume Method -- General Control Volume

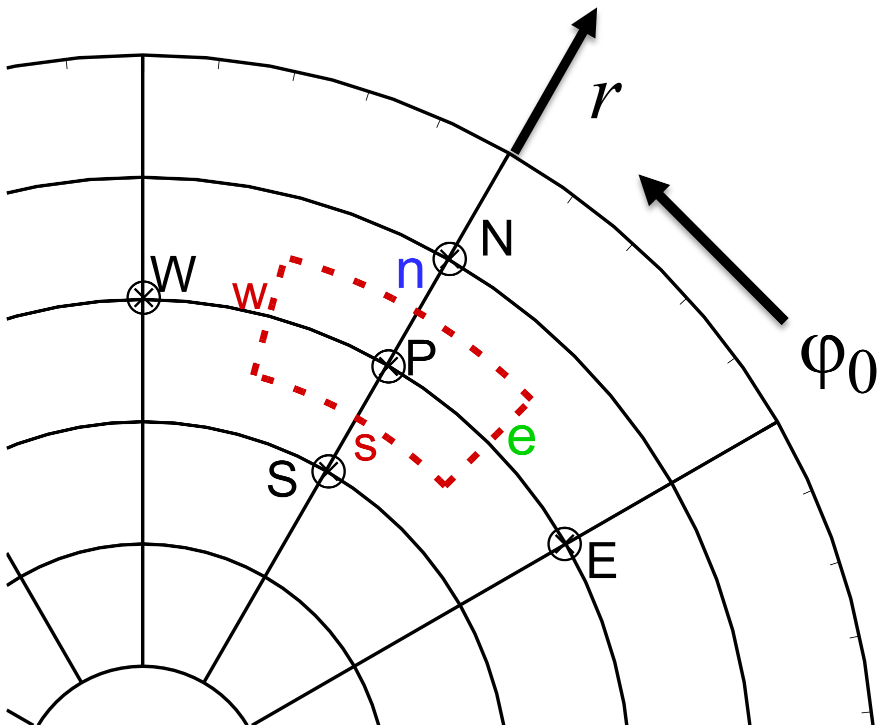
Approximation of Surface (line) Integrals:

'e' Face: $\int_{S_e} F(\boldsymbol{y}_\perp)\hat{n}_i \frac{\partial g(\boldsymbol{y}_\perp;\varphi,\theta:\omega)}{\partial y_i} dS \approx -F_e \frac{1}{r_e}\frac{\partial g}{\partial \varphi_0}\bigg|_e (r_{ne}-r_{se})$

$$\approx -\frac{1}{2}[F_E+F_P]\frac{1}{r_e}\frac{g_P-g_E}{\varphi_0^P-\varphi_0^E}(r_{ne}-r_{se})$$

'n' Face: $\int_{S_n} F(\boldsymbol{y}_\perp)\hat{n}_i \frac{\partial g(\boldsymbol{y}_\perp;\varphi,\theta:\omega)}{\partial y_i} dS \approx F_n \frac{\partial g}{\partial r}\bigg|_n r_n(\varphi_0^{nw}-\varphi_0^{ne})$

$$\approx \frac{1}{2}[F_N+F_P]\frac{g_N-g_P}{r_N-r_P} r_n(\varphi_0^{nw}-\varphi_0^{ne})$$



Approximation of Volume (area) integrals:

$$\iint_R H(\boldsymbol{y}_\perp) g(\boldsymbol{y}_\perp;\varphi,\theta:\omega) dR \approx H_P g_P R_P \qquad R_P = \int_{\varphi_e}^{\varphi_w}\int_{r_s}^{r_n} r\,dr\,d\varphi_0$$

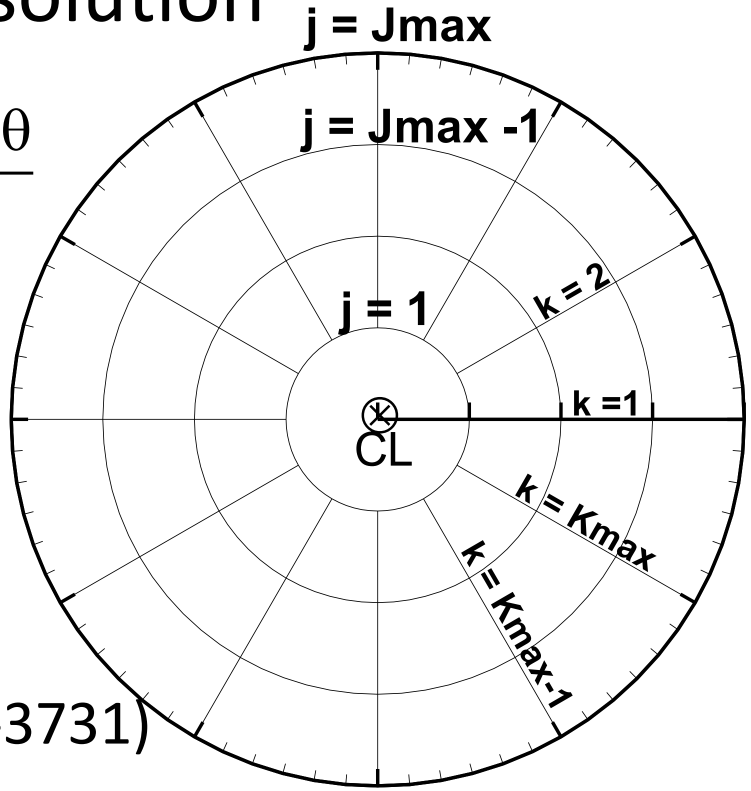
Numerical Solution of Green's Function: Finite Volume Method -- Boundary Conditions

- Far-Field: Sommerfeld radiation for scattered solution

$$\frac{\partial g\left(\boldsymbol{y}_{\perp};\varphi,\theta:\omega\right)}{\partial r}+\kappa g\left(\boldsymbol{y}_{\perp};\varphi,\theta:\omega\right)\rightarrow\Lambda\left(\boldsymbol{y}_{\perp};\varphi,\theta:\omega\right)\quad;\quad\kappa=\frac{i\omega\sin\theta}{c_{\infty}}$$

  – Second-order central difference at j = Jmax – 1/2

$$\left[-1+\frac{1}{2}\left(r_{k,J}-r_{k,J-1}\right)\kappa\right]g_{k,J-1}+\left[1+\frac{1}{2}\left(r_{k,J}-r_{k,J-1}\right)\kappa\right]g_{k,J}\approx \left(r_{k,J}-r_{k,J-1}\right)\Lambda_{k,J-1/2}\left(\boldsymbol{y}_{\perp};\varphi,\theta:\omega\right)$$



- Centerline: (Giuliani, Chen, Beach and Bakhle; AIAA 2014-3731)

$$g_{CL}=\frac{1}{\text{Kmax}}\sum_{k=1}^{\text{Kmax}}g_{k,1}$$

Numerical Solution of Green's Function: Finite Volume Method -- Algebraic Equations

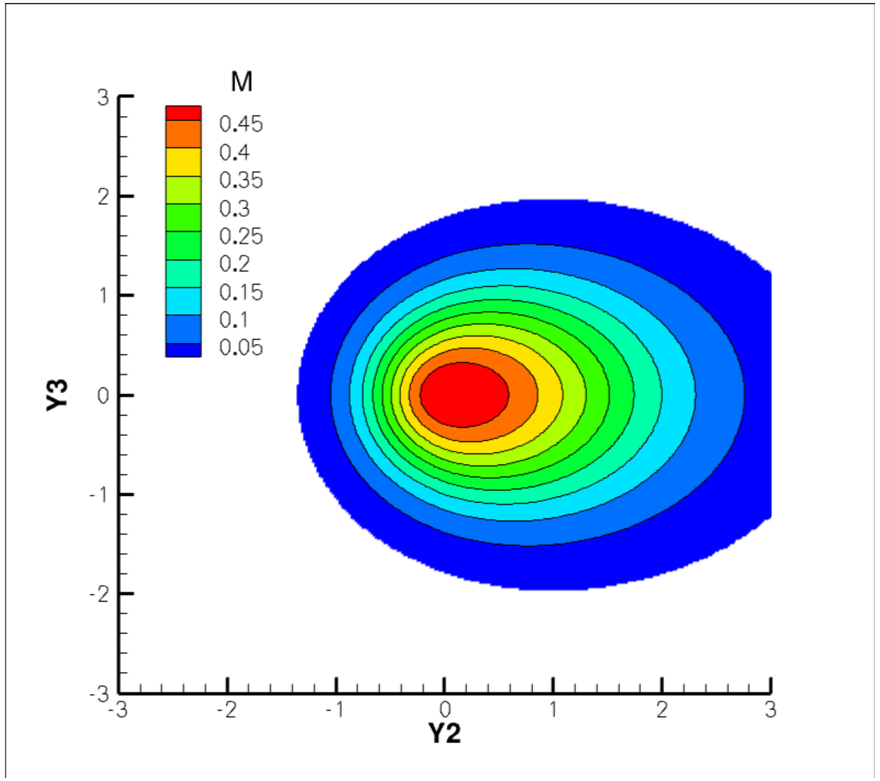
- Sum contributions from all Control Volumes
- Special CVs for periodicity, centerline, outer boundary
- Banded system of equations -- solve directly using a sparse system algorithm

$$A^{(k,j)} g_{k,j-1} + B^{(k,j)} g_{k-1,j} + C^{(k,j)} g_{k,j} + D^{(k,j)} g_{k+1,j} + E^{(k,j)} g_{k,j+1} = 0 \quad ; \; k = 1, \mathrm{Kmax}, \; j = 1, \mathrm{Jmax}$$

Sample Solutions

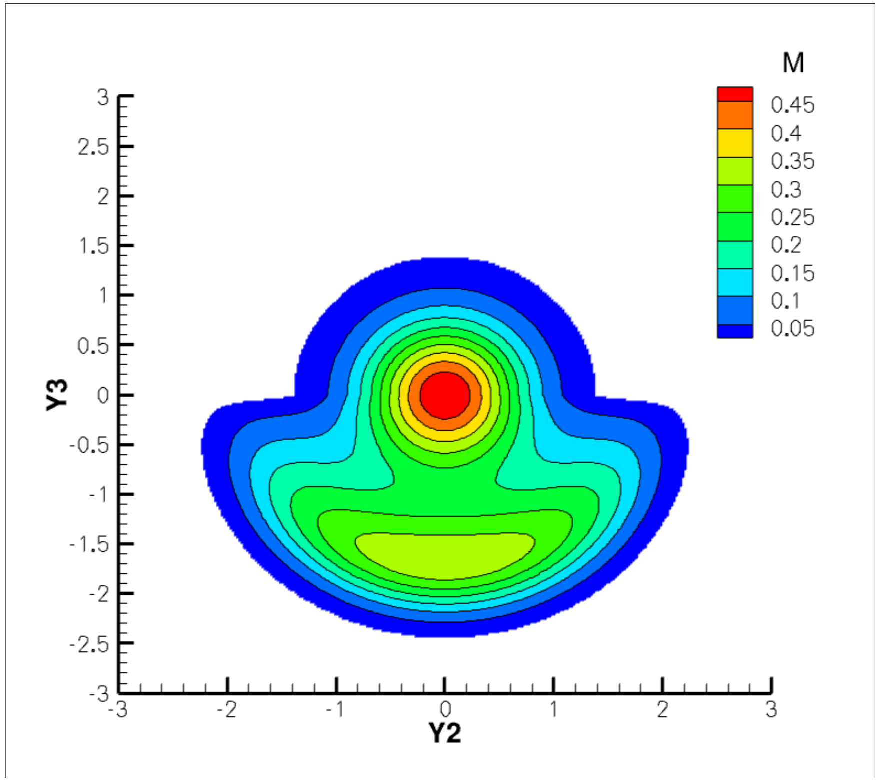
- Two analytical test cases
- Compare with hybrid spectral/finite-difference method, Leib(2013)

Offset Stream



$$M(r,\varphi)=M_C e^{-(1+\alpha-2\alpha\cos\varphi)r^2}$$

Fluid Shield



$$M(r,\varphi)=M_C\left[e^{-ar^2}+br^2e^{-c(r-1)^4}h(\varphi)\right]$$

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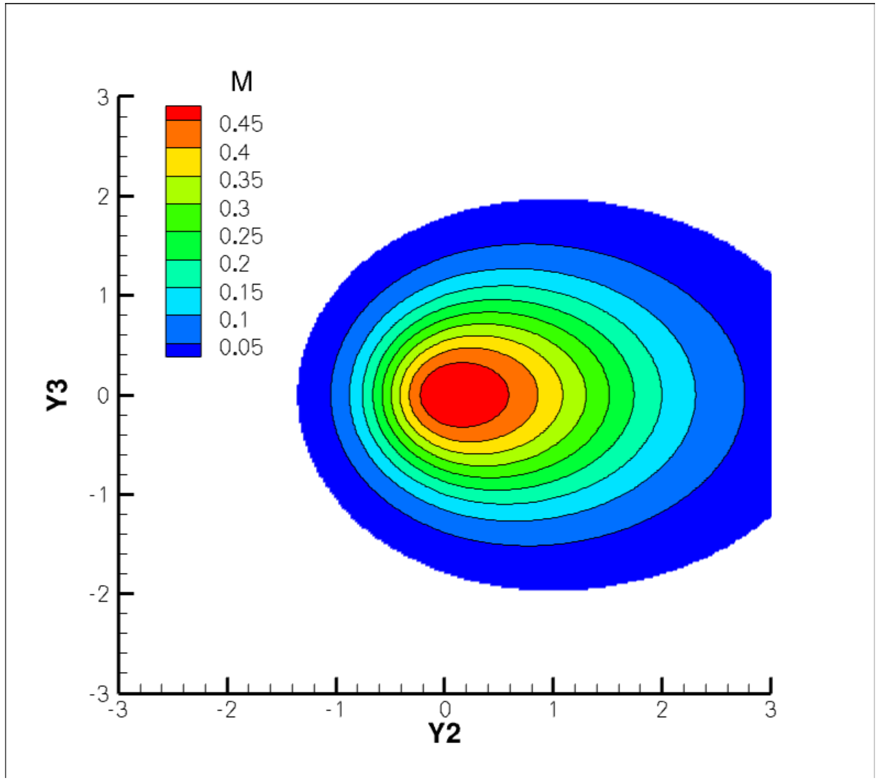
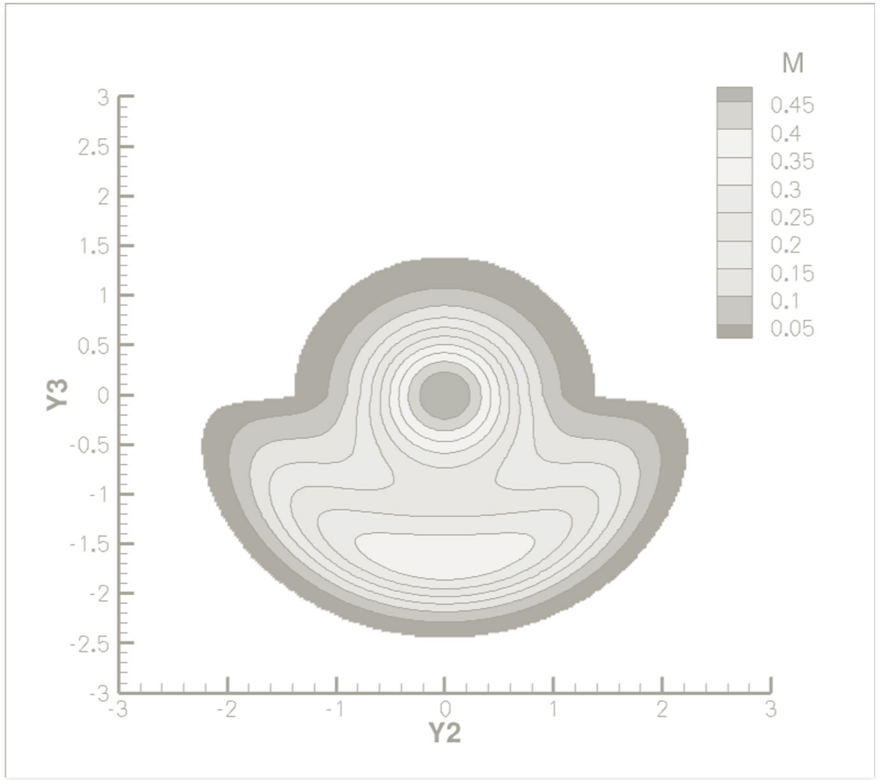


$$M\left(r,\varphi\right)=M_{C}e^{-\left(1+\alpha-2\alpha\cos\varphi\right)r^{2}}$$

Fluid Shield

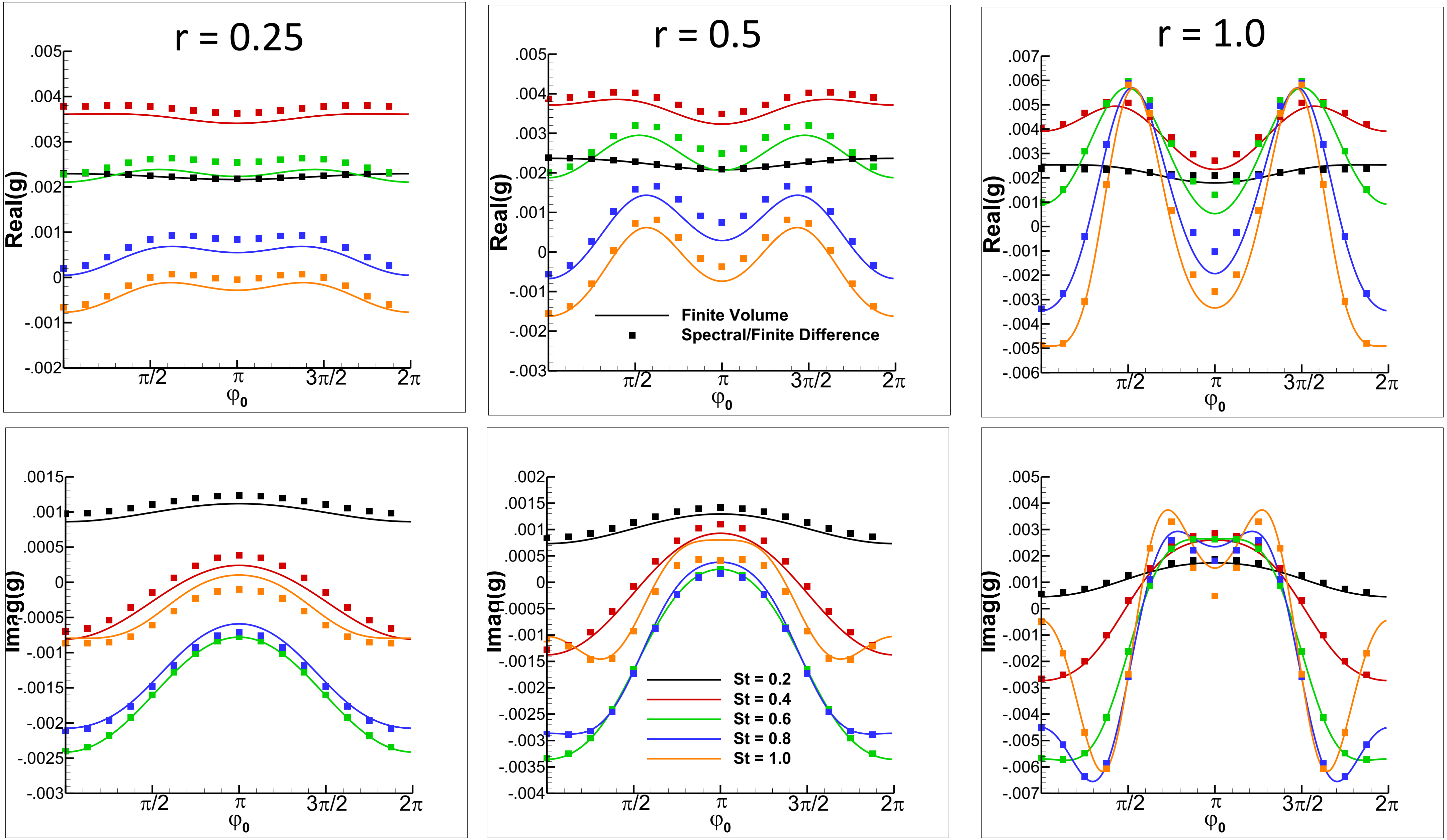


$$M\left(r,\varphi\right)=M_{C}\left[e^{-ar^{2}}+br^{2}e^{-c\left(r-1\right)^{4}}h\left(\varphi\right)\right]$$

Results: Offset Jet $\theta = 30^\circ$; $\varphi = 0^\circ$

$\mathrm{Real}(g)$

$\mathrm{Imag}(g)$



Sample Solutions

- Two analytical test cases
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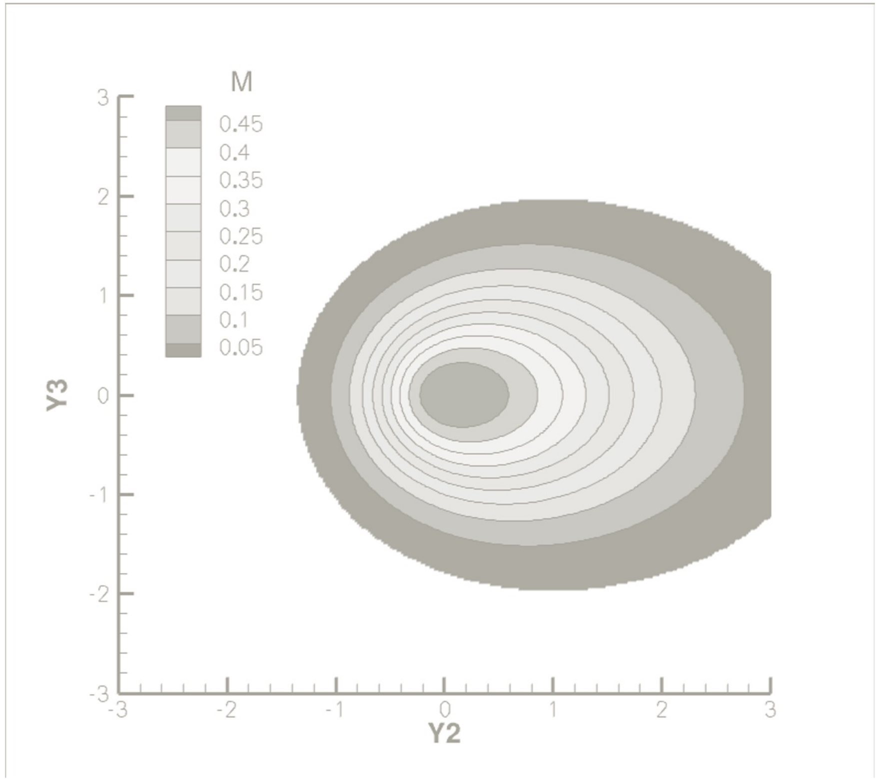
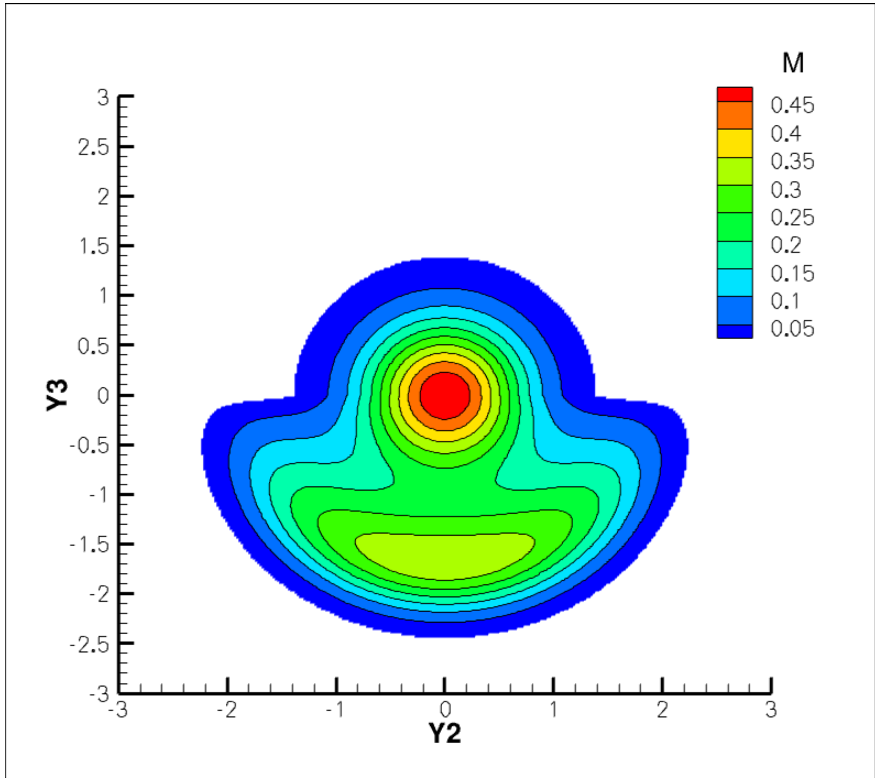


$$M(r,\varphi) = M_C e^{-(1+\alpha-2\alpha\cos\varphi)r^2}$$

Fluid Shield



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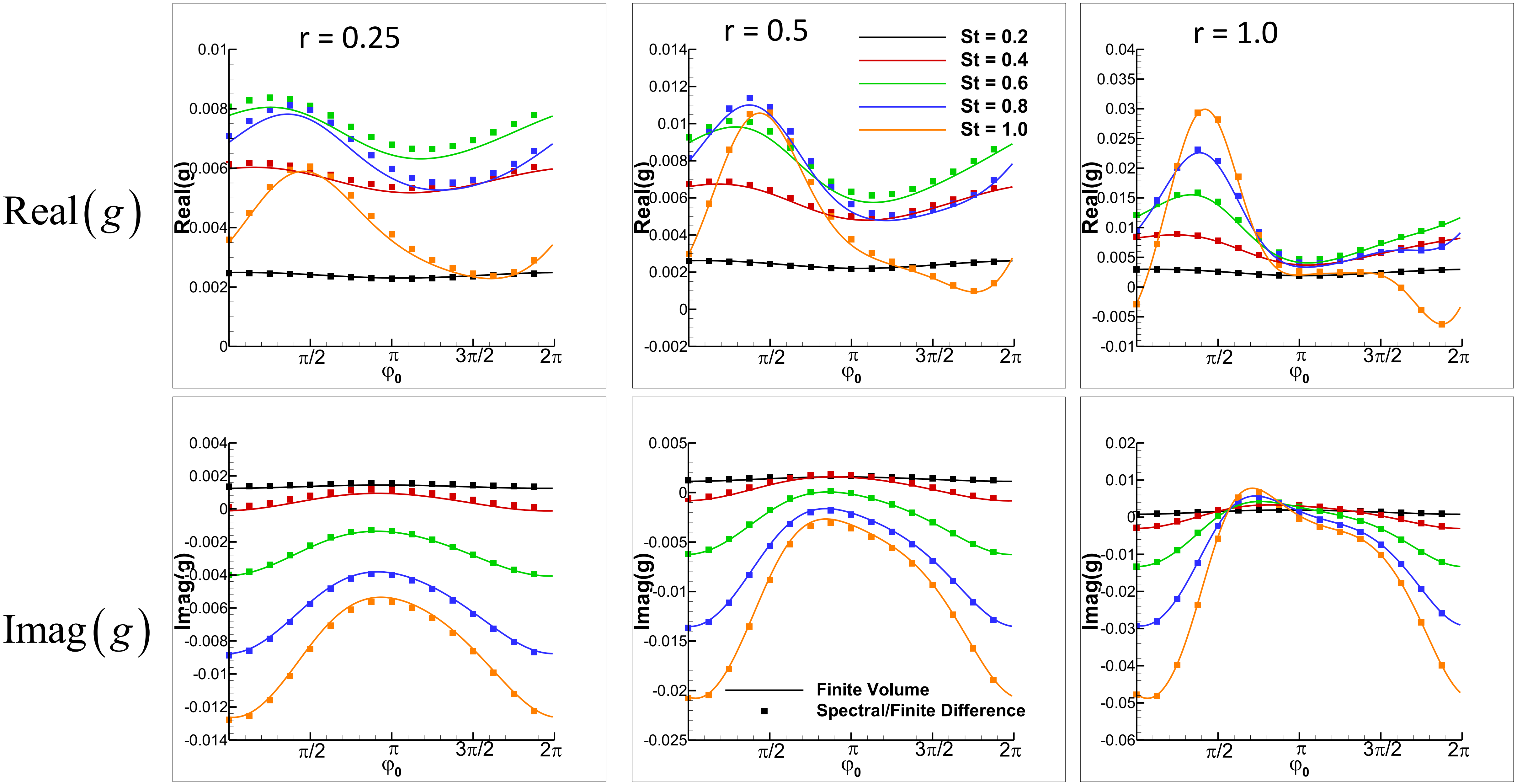
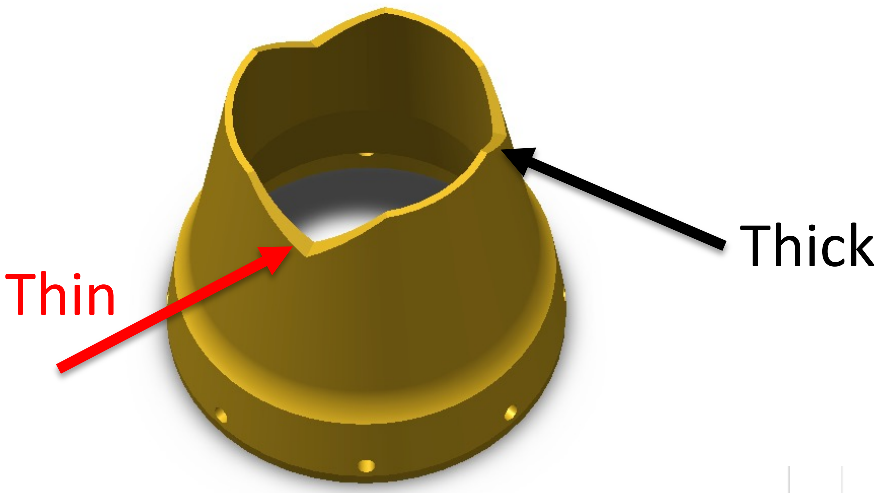
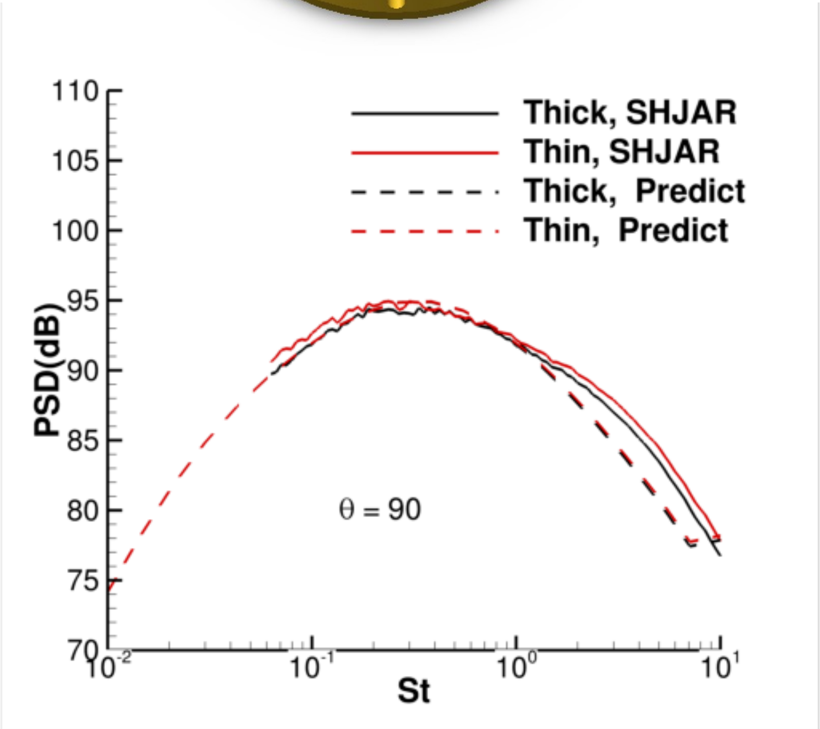
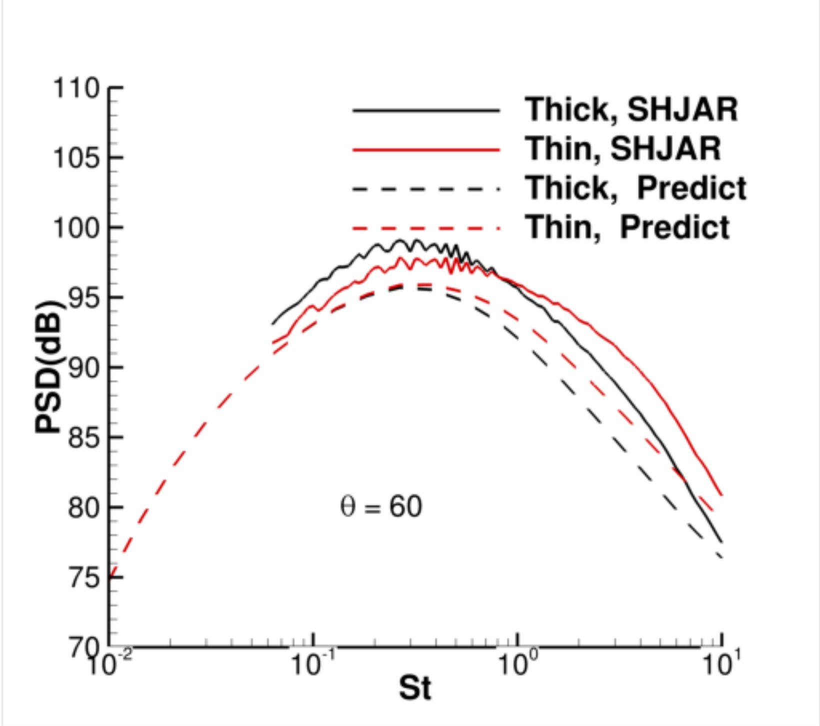


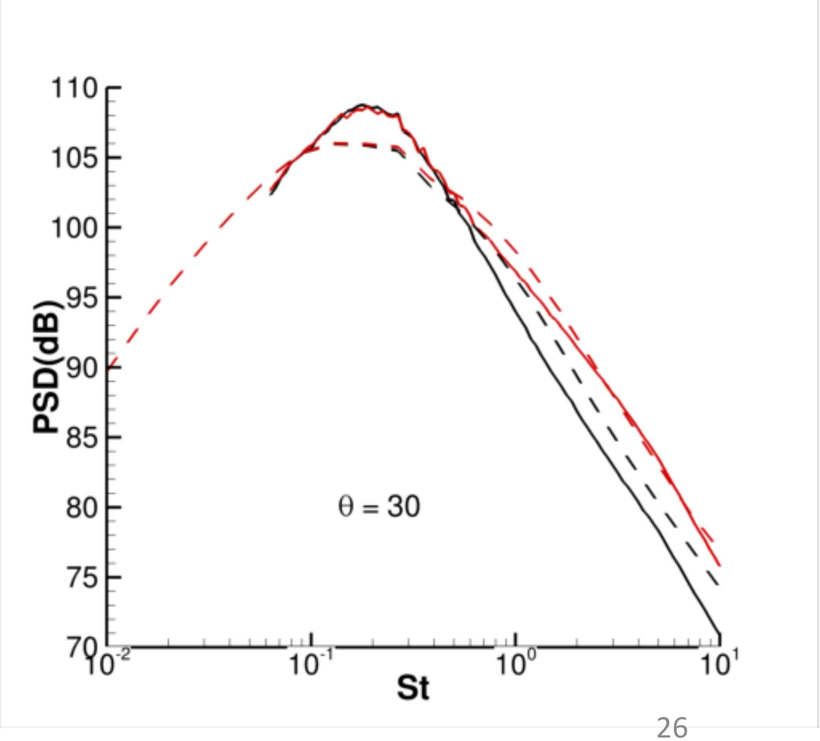
Example Results: Chevon (Non-Axisymmetric) Jet



Predictions of Noise from unheated turbulent jets from a chevon nozzle compared with experimental data (NASA GRC SHJAR)







Summary

- Combined analytical analysis and approximations to allow use of simple numerical methods to obtain noise prediction results for turbulent jets
- Reduced-order model requiring ‘moderate’ computational resources while retaining the most significant physical effects
- Numerical implementations resulted in several ‘production’ codes currently in use at NASA

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