Hamiltonian Optimal Control of Distributed Lagrangian Systems

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Abstract. This lecture presents a Hamiltonian control method and a distributed optimal control method for distributed Lagrangian systems. The distributed optimal control theory is formulated using a semi-group abstraction resulting in an integro-differential Riccati equation.

1 Introduction

Distributed systems are modeled as systems of partial differential equations (PDEs) which exhibit both temporal and spatial variations in the state variables of the systems. Distributed control can be designed for such systems. Research in this field is extensive and can be found in a few references cited herein [1,2,19,8,9,10,16,17,20]. Systems which can be described by the Hamilton's principle are called Lagrangian systems. In spite of the extensive research, very few distributed control applications have been developed principally due to the lack of distributed sensing capabilities. Many natural and bio-inspired systems on the other hand can sense environmental inputs through sensor systems distributed throughout those systems. For example, avian wings are known to be able to sense air flow over the wings which allows birds to adjust their flight path by actuating their feather systems to change the wing cambers or curvatures [21]. In recent years, advancements in distributed sensor technologies which allow measurements of both temporal and spatial information of physical signals have made steady progress. Distributed fiber optic sensors offer new possibilities in control system designs that could bring distributed control research into a practical reality [24].

In this lecture, we will present a mathematical model and Hamiltonian optimal control of a class of distributed Lagrangian systems [23,25,26,29] coupled to lumped-parameter systems which are modeled by systems of ordinary differential equations (ODEs). A flexible aircraft flight control application illustrates the theoretical development.

2 Distributed Lagrangrian System

Consider a finite-dimensional or lumped-parameter (P) system coupled to an infinite-dimensional or distributed-parameter Lagrangian system (S)

$$\dot{z} = g(z) + A_z z + \int_\Omega r\left(x, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}\right) dx + B_z u_z + B_{zw} u_w + g_z(t) \tag{1}$$

$$m(x)\frac{\partial^2 w}{\partial t^2} - f\left(x, w, \frac{\partial w}{\partial x}, \dots \frac{\partial^m w}{\partial x^m}\right) = q_c\left(x, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}\right) + q_z(x)z + b(x)u_w + g_w(x,t) \tag{2}$$

where $z(t) : [0,\infty) \to \mathbb{R}^k$ is the state vector of (P), $w(x,t) : \Omega \times [0,\infty) \to \mathbb{R}^n$ is the distributed state vector of (S) defined over an open bounded domain $x \in \Omega \subset \mathbb{R}$ with suitable boundary conditions on the boundary $\partial\Omega$ for all $t \in [0,\infty)$, $u_z(t) : [0,\infty) \to \mathbb{R}^q$ is a control vector for (P), and $u_w(t) : [0,\infty) \to \mathbb{R}^p$ is a control vector for (S). The distributed state vector $w(x,t) \in C_c^\infty$ is assumed to be continuous and infinitely differentiable. The initial values are specified by $z(0) = z_0$ and $w(x,0) = w_0(x)$ and $\frac{\partial w(x,0)}{\partial t} = w_{t_0}(x)$.

The function $m(x) > 0 \in \mathbb{R}^{n\times n}$ is a positive-valued function distributed over x. The function $q_c(\dots) \in \mathbb{R}^n$ is a non-homogeneous term that depends on $w(x,t)$ as well as the spatial and temporal partial derivatives of the solution $w(x,t)$. The functions $g_z(t)$ and $g_w(x,t)$ represent bounded exogenous disturbances and are assumed to be known.

The nonlinear state transition function $g(z)$ and control matrix B_z are assumed to be in a second-order controllable form where $z(t) = \left[z_1^\top(t)\ z_2^\top(z)\right]^\top$, $z_i(t) : [0,\infty) \to \mathbb{R}^{\frac{k}{2}}$, $i = 1,2$, $g(z) = \left[z_2^\top(t)\ h^\top(z)\right]^\top$ and $B_z = \left[0\ b_z^\top\right]^\top$.

The control objective is to minimize a performance objective $p(t) : [0,\infty) \to \mathbb{R}^l$, $l \le n$ that exists inside the bounded domain Ω or on the boundary $\partial\Omega$. This performance objective is defined as

$$p(x,t) = \begin{bmatrix} P_z z(t) \\ P_w w(x,t) \end{bmatrix} \tag{3}$$

where $P_z \in \mathbb{R}^{l\times m}$ is a matrix and $P_w \in \mathbb{R}^{l\times n}$ is an differential or integral operator.

Lemma 1: Let $\delta w(x,t)$ be a variation in the solution of (S). If there exists a positive-valued scalar function $\nu\left(x, w, \frac{\partial w}{\partial x}, \dots \frac{\partial^m w}{\partial x^m}\right) > 0 \in \mathbb{R}$ such that

$$\delta w \cdot f = -\delta w \cdot \sum_{i=0}^{m} (-1)^i \frac{\partial^i}{\partial x^i}\left(\frac{\partial \nu}{\partial\left(\frac{\partial^i w}{\partial x^i}\right)}\right) \tag{4}$$

where the $\cdot$ symbol denotes a scalar product operation of two vectors, and

$$\delta\phi_{ij}(x) = \left(\frac{\partial^{j-i-1}\delta w}{\partial x^{j-i-1}}\right) \cdot \frac{\partial^i}{\partial x^i}\left(\frac{\partial \nu}{\partial\left(\frac{\partial^j w}{\partial x^j}\right)}\right) \tag{5}$$

has compact support in Ω with vanishing boundary values on $\partial\Omega$ [27] for all $0 \le i \le m-1$ and $1 \le j \le m$, then $\nu(\dots)$ is called a potential energy density.

Proof: We define $\delta\nu(\dots)$ due to $\delta w(x,t)$ such that

$$\delta \int_\Omega \delta w \cdot f dx = -\int_\Omega \delta\nu dx \tag{6}$$

Upon successive integration by parts, we obtain

$$\int_\Omega \delta w \cdot f dx = -\int_\Omega \delta w \cdot \sum_{i=0}^{m} (-1)^i \frac{\partial^i}{\partial x^i}\left(\frac{\partial \nu}{\partial\left(\frac{\partial^i w}{\partial x^i}\right)}\right) dx - \sum_{j=1}^{m}\sum_{i=0}^{j-1} (-1)^i \delta\phi_{ij}(x)\Big|_{\partial\Omega} \tag{7}$$

Since $\delta\phi_{ij}(x)$ has compact support, $\delta\phi_{ij}(x) = 0$ on $\partial\Omega$. This immediately leads to Eq. (4). The boundary conditions for (S) thus satisfy

$$\left.\frac{\partial^{j-i-1}w}{\partial x^{j-i-1}}\cdot\frac{\partial^i}{\partial x^i}\left(\frac{\partial \nu}{\partial\left(\frac{\partial^j w}{\partial x^j}\right)}\right)\right|_{\partial\Omega} = 0 \tag{8}$$

for all $0 \le i \le j-1$ and $1 \le j \le m$. It follows that

$$f = -\sum_{i=0}^{m}(-1)^i\frac{\partial^i}{\partial x^i}\left(\frac{\partial \nu}{\partial\left(\frac{\partial^i w}{\partial x^i}\right)}\right) \tag{9}$$

Consider the following generalized quadratic potential energy density of arbitrary order $2m$:

$$\nu = \sum_{i=0}^{m}\frac{1}{2}\frac{\partial^i w}{\partial x^i}\cdot k_i(x)\frac{\partial^i w}{\partial x^i} \tag{10}$$

where $k_i(x) \ge 0 \in \mathbb{R}^{n\times n}$, $i = 0,1,\ldots,m$, can be positive semi-definite but $\nu(\ldots)$ is a positive-definite function. Thus, the distributed system (S) is $2m^{\text{th}}$-order in space and second-order in time. If $m = 0$, then the system reverts to a standard spring-mass system. If $m > 0$ and the generalized potential energy density includes the first term on the right-hand side of Eq. (10), then (S) is a coupled PDE-ODE system.

Let $\mathscr{H}$ be a Hilbert space with the an inner product definition

$$\langle \xi, \upsilon\rangle = \int_\Omega \upsilon\cdot m(x)\,\xi\,dx \tag{11}$$

on a Lebesgue square integrable inner product space $\mathscr{L}^2(\Omega)$where $\xi(x) \in \mathscr{H}(\Omega)$ and $\upsilon(x) \in \mathscr{H}(\Omega)$. Let $L : \mathscr{H} \to \mathscr{H}$ be a linear operator on $\mathscr{H}(\Omega)$ defined by

$$L = m^{-1}(x)\sum_{i=0}^{m}(-1)^i\frac{\partial^i}{\partial x^i}\cdot\left(k_i(x)\frac{\partial^i}{\partial x^i}\right) \tag{12}$$

Theorem 1: Let $L^* : \mathscr{H} \to \mathscr{H}$ be an linear operator on $\mathscr{H}(\Omega)$. Then, L is a formally self-adjoint operator such that

$$\langle \upsilon, L\xi\rangle = \langle \xi, L^*\upsilon\rangle \tag{13}$$

Moreover, L is a self-adjoint operator if the boundary conditions for L and L^* are equivalent [28].

Proof: $\langle \upsilon(x), L\xi(x)\rangle$ is expressed as

$$\langle \upsilon, L\xi\rangle = \int_\Omega \upsilon\cdot\sum_{i=0}^{m}(-1)^i\frac{\partial^i}{\partial x^i}\left(k_i(x)\frac{\partial^i \xi}{\partial x^i}\right)dx \tag{14}$$

Upon successive integration by parts, we get

$$\langle \upsilon, L\xi \rangle = \int_\Omega \xi \cdot \sum_{i=0}^{m} (-1)^i \frac{\partial^i}{\partial x^i} \left(k_i(x) \frac{\partial^i \upsilon}{\partial x^i} \right) dx$$
$$- \sum_{j=1}^{m} \sum_{i=0}^{j-1} (-1)^i \left[\frac{\partial^{j-i-1} \upsilon}{\partial x^{j-i-1}} \cdot \frac{\partial^i}{\partial x^i} \left(k_j(x) \frac{\partial^j \xi}{\partial x^j} \right) - \frac{\partial^{j-i-1} \xi}{\partial x^{j-i-1}} \cdot \frac{\partial^i}{\partial x^i} \left(k_j(x) \frac{\partial^j \upsilon}{\partial x^j} \right) \right]_{\partial\Omega} \tag{15}$$

The vanishing boundary term in Eq. (15) results in the following boundary conditions associated with the adjoint operator L^*. Then, $\langle \upsilon, L\xi \rangle = \langle \xi, L^* \upsilon \rangle$. Therefore, L is a self-adjoint operator since $L^* = L$. Moreover, L is also a positive-definite operator since

$$\langle w, Lw \rangle = 2 \int_\Omega \nu dx > 0 \tag{16}$$

Thus, (S) in Eq. (2) can be expressed in a linear operator form as

$$m(x) \frac{\partial^2 w}{\partial t^2} + m(x) Lw = q\left(x, t, z, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, u_w \right) \tag{17}$$

Example 1: Consider the motion of a rectangular space structure where $(x, y) \in \Omega \subset \mathbb{R}^2$. The operator L is given by

$$m(x) Lw = (-1)^2 \left[\frac{\partial^2}{\partial x^2} \; \frac{\partial^2}{\partial y^2} \right] \left(\begin{bmatrix} k_{11}(x,y) & k_{12}(x,y) \\ k_{12}(x.y) & k_{22}(x,y) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \end{bmatrix} \right)$$
$$= \frac{\partial^2}{\partial x^2} \left(k_{11}(x,y) \frac{\partial^2 w}{\partial x^2} + k_{12}(x,y) \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial^2}{\partial y^2} \left(k_{12}(x) \frac{\partial^2 w}{\partial x^2} + k_{22}(x) \frac{\partial^2 w}{\partial y^2} \right) \tag{18}$$

The general homogeneous solution of Eq. (17) with $q(\dots) = 0$ can be expressed in the form

$$w(x,t) = \sum_{i=1}^{\infty} \phi_i(x) \theta_i(t) = \phi(x) \theta(t) \tag{19}$$

where $\phi_i(x) : \Omega \to \mathbb{R}^n$ is a solution of the eigenvalue problem $(L - \lambda)\phi = 0$ and $\theta_i(t) : [0, \infty) \to \mathbb{R}$. It follows that

$$\langle \phi, L\phi \rangle = \lambda \langle \phi, \phi \rangle \tag{20}$$

where $\{\phi_i(x)\}_{i=1}^{\infty}$ form a set of orthogonal basis functions. Due to $q(\dots)$ as a function of $w(x,t)$ and $\frac{\partial w}{\partial x}$, $\phi(x)$ represents the solution of Eq. (17) in a weak sense.

Consider a weak formulation of (S) with $\upsilon(x) \triangleq D^m \phi(x) \in \mathscr{W}^{m,2} = \mathscr{H}^m(\Omega)$, where $\mathscr{W}^{m,2}(\Omega)$ is a Sobolev space with the $\mathscr{L}^2$ Lebesque measure, defined as the weak derivative of $\phi(x) \in H^m(\Omega)$ of order m [27] such that

$$\int_\Omega \phi \frac{\partial^m \varphi}{\partial x^m} dx = (-1)^m \int_\Omega \upsilon \varphi dx \tag{21}$$

for all test functions $\varphi(x) \in C_c^{\infty}(\Omega)$ with compact support in Ω.

Lemma 1: Let $\phi(x)$ be a steady-state solution of (S) that solves the steady-state equation

$$m(x) L\phi = q\left(x, \phi, \frac{\partial \phi}{\partial x}\right) = q_0(x) + \sum_{i=0}^{1} q_{i+1}(x) \frac{\partial^i \phi}{\partial x^i} \tag{22}$$

where $q(\ldots)$ only depends on $\phi(x)$ and x.

Let $\mathscr{B}(\phi, \varphi) : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ be a bilinear mapping defined by

$$\mathscr{B}(\phi, \varphi) = \left\langle \varphi, m^{-1}(x) q_0(x) \right\rangle = \langle \varphi, L\phi \rangle - \left\langle \varphi, \sum_{i=0}^{l<m} m^{-1}(x) q_{i+1}(x) \frac{\partial^i \phi}{\partial x^i} \right\rangle \tag{23}$$

Then, (S) admits a unique weak solution $\phi(x)$ if

$$\mathscr{B}(\phi, \phi) \geq \kappa \|\phi\|^2 \tag{24}$$

for some $\kappa > 0$, which implies

$$\left\langle \phi, \sum_{i=0}^{l} m^{-1}(x) q_{i+1}(x) \frac{\partial^i \phi}{\partial x^i} \right\rangle < 2 \int_\Omega \nu(\phi)\, dx \tag{25}$$

Proof: Since $\phi(x)$ and $\varphi(x)$ are infinitely differentiable, the bilinear mapping $\mathscr{B}(\phi, \varphi)$ is a continuous bounded functional. This implies $|\mathscr{B}(\phi, \varphi)| \leq \eta \|\phi\| \|\varphi\|$. If $\mathscr{B}(\xi, \varphi)$ satisfies Ineq. (24), then $\mathscr{B}(\xi, \varphi)$ is coercive [27]. Thus, (S) possesses finite energy. According to the Lax-Milgram theorem [27], Eq. (22) admits a weak solution if

$$\mathscr{B}(\phi, \phi) = \langle \phi, L\phi \rangle - \left\langle \phi, \sum_{i=0}^{l} m^{-1}(x) q_{i+1}(x) \frac{\partial^i \phi}{\partial x^i} \right\rangle \geq \kappa \|\phi\|^2 > 0 \tag{26}$$

Since $\langle \phi, L\phi \rangle = 2 \int_\Omega \nu(\phi)\, dx$, it follows that

$$\left\langle \phi, \sum_{i=0}^{l} m^{-1}(x) q_{i+1}(x) \frac{\partial^i \phi}{\partial x^i} \right\rangle < 2 \int_\Omega \nu(\xi)\, dx \tag{27}$$

Example 2: Consider the steady-state equation

$$m(x) Lw = q_0(x) + q_1(x) w \tag{28}$$

with $q_1(x) > 0$. Let $w(x) = \phi(x)\theta$. Then,

$$\left(\langle \phi, L\phi \rangle - \left\langle \phi, m^{-1}(x) q_1(x) \phi \right\rangle\right) \theta = \left\langle \phi, m^{-1}(x) q_0(x) \right\rangle \tag{29}$$

Clearly, if $\left\langle \phi, m^{-1}(x) q_1(x) \phi \right\rangle = 2 \int_\Omega \nu(\xi)\, dx = \langle \phi, L\phi \rangle$, the solution θ is unbounded. If $\left\langle \phi, m^{-1}(x) q_1(x) \phi \right\rangle > 2 \int_\Omega \nu(\xi)\, dx$, then fundamentally the notion of positive-valued potential energy ceases to exist.

3 Hamiltonian Control

The Hamiltonian function which represents the storage energy of distributed system (S) is defined as [23]

$$H = \frac{1}{2}\left\langle \frac{\partial w}{\partial t}, \frac{\partial w}{\partial t} \right\rangle + \frac{1}{2}\langle w, Lw \rangle \tag{30}$$

A stabilizing control is a control policy that renders that the system dissipative for which $\dot{H} \leq 0$. The time derivative of the Hamiltonian function for zero disturbance is evaluated as

$$\dot{H} = \left\langle \frac{\partial w}{\partial t}, \frac{\partial w}{\partial t} + Lw \right\rangle = \left\langle \frac{\partial w}{\partial t}, m^{-1}(x)\left(q_c + q_z z + b u_w\right) \right\rangle \tag{31}$$

The right hand side of Eq. (31) represents the rate of excess energy which must be decreased for the system to be dissipative.

Let $u_v(x,t) = b(x) u_w(t)$ be a virtual control signal. A stabilizing control for (S) is designed as

$$u_v^* = -q_c - q_z z - m(x) c \frac{\partial w}{\partial t} + \mu m(x) Lw \tag{32}$$

where $c > 0$ and $\mu < 1$ which can be established by Lax-Milgram theorem [27]. Applying the Lax-Milgram theorem, the solution exists if

$$\mathscr{B}(\phi,\phi) = \langle \phi, (1-\mu) L\phi \rangle = (1-\mu)\langle \phi, L\phi \rangle \geq \kappa \|\phi\|^2 > 0 \tag{33}$$

which implies $\mu < 1$.

Using superposition principle, we write $u_w^*(t) = \sum_{i=1}^{\infty} u_{m_i}(t) = e u_m(t)$ where $u_{m_i}(t)$ is a distributed control associated with the eigenfunction $\phi_i(x)$. Let $B_m = \left\langle \phi, m^{-1}(x) be \right\rangle$. Then, the distributed control $u_m(t)$ is obtained as

$$u_m = B_m^{-1}\left\langle \phi, m^{-1}(x) u_v^* \right\rangle \tag{34}$$

The stabilizing control for (P) is designed as

$$u_z^* = -b_z^{-1} h(z) - K_z z - \mathscr{K}_w w \tag{35}$$

The distributed feedback gain operator $\mathscr{K}_w$ is obtained by minimizing a weighted least-squares cost function

$$\min_{\mathscr{K}_w} J = \left(F - B_z \mathscr{K}_w w\right)^\top W_z \left(F - B_z \mathscr{K}_w w\right) \tag{36}$$

where $W_z \geq 0$ is a positive semi-definite weighting matrix and

$$F = \int_\Omega \left[r dx + B_{zw} e B_m^{-1} \phi \cdot \left(u_v^* + q_z z\right)\right] dx \tag{37}$$

The weighted least-squares solution yields

$$\mathscr{K}_w w = \left(B_z^\top W_z B_z\right)^{-1} B_z^\top W_z \int_\Omega \left[r dx + B_{zw} e B_m^{-1} \phi \cdot \left(u_v^* + q_z z\right)\right] dx \tag{38}$$

Without loss of generality, let $r = r_1(x)w + r_2(x)\frac{\partial w}{\partial x} + r_3(x)\frac{\partial w}{\partial t}$ and $q_c = q_1(x)w + q_2(x)\frac{\partial w}{\partial x} + q_3(x)\frac{\partial w}{\partial t}$. Then, the distributed feedback gain operator K_w is obtained as

$$\mathscr{K}_w = \left(B_z^\top W_z B_z\right)^{-1} B_z^\top W_z \int_\Omega \mathscr{K}\mathscr{D} dx \tag{39}$$

where $\mathscr{D}() = \left[\, ()^\top \ \frac{\partial()}{\partial x}^\top \ \frac{\partial()}{\partial t}^\top \,\right]^\top$ is a differential operator and $\mathscr{K} = \left[\, \mathscr{K}_1 \ \mathscr{K}_2 \ \mathscr{K}_3 \,\right]$ with

$$\mathscr{K}_1 = r_1 - B_{zw} e B_m^{-1} \phi \cdot [q_1 - \mu m(x) L] \tag{40}$$

$$\mathscr{K}_2 = r_2 - B_{zw} e B_m^{-1} \phi \cdot q_2 \tag{41}$$

$$\mathscr{K}_3 = r_3 - B_{zw} e B_m^{-1} \phi \cdot [q_3 + m(x) c] \tag{42}$$

The feedback gain K_z is a stabilizing gain if there exists positive definite matrices $P > 0$ such that $A_c^\top P + P A_c = -Q < 0$ where

$$A_c = A_z - B_{zw} e B_m^{-1} \int_\Omega \phi \cdot q_z dx - B_z K_z \tag{43}$$

Let $u_w(t) = u_w^*(t) + \tilde{u}_w(t)$ and $u_z(t) = u_z^*(t) + \tilde{u}_z(t)$. The closed-loop system (P+S) becomes

$$\dot{z} = A_c z + \mathscr{A}_{zw} w + B_z \tilde{u}_z + B_{zw} \tilde{u}_w + g_z \tag{44}$$

$$m(x)\frac{\partial^2 w}{\partial t^2} + m(x) c \frac{\partial w}{\partial t} + (1-\mu) m(x) L w = b(x)\tilde{u}_w + g_w \tag{45}$$

where $\mathscr{A}_{zw}$ is an operator given by

$$\mathscr{A}_{zw} = \left[I - B_z \left(B_z^\top W_z B_z\right)^{-1} B_z^\top W_z \right] \int_\Omega \mathscr{K}\mathscr{D} dx \tag{46}$$

It can be shown that the closed-loop system (P+S) is stable [29].

4 Distributed Optimal Control

The closed-loop system (P+S) can be expressed abstractly in a semi-group form as

$$\frac{\partial \mathscr{X}}{\partial t} = \mathscr{A}\mathscr{X} + \mathscr{B} u + \mathscr{V} \tag{47}$$

where $\mathscr{X}(x,t) = \left[z^\top(t) \ w^\top(x,t) \ \frac{\partial w^\top(x,t)}{\partial t} \right]^\top : (\Omega \oplus \Delta) \times [0,\infty) \to \mathbb{R}^{m+2n}$ is an abstract state vector with Δ a domain defined by the inner product definition

$$\langle z, z \rangle = \int_\Delta z^\top z dx \triangleq z^\top z \tag{48}$$

$u(t) = \left[\, \tilde{u}_z^\top(t) \ \tilde{u}_w^\top(t) \,\right]^\top : [0,\infty) \to \mathbb{R}^{p+q}$ is the incremental control vector, $\mathscr{A} \in C_c^\infty(\Omega)$ is a semi-group operator with compact support and continuous derivatives, $\mathscr{B} \in \mathscr{C}^0(\Omega)$

is a bounded semi-group operator, and $\mathscr{V}(x,t) \in \mathscr{C}^0(\Omega) \times [0,\infty)$ represents a bounded disturbance.

Let $\mathscr{M}$ be a semi-group operator in $\mathscr{C}^0(\Omega)$. We define the following inner product with the norm definition:

$$\langle \mathscr{X}, \mathscr{X} \rangle = \|\mathscr{X}\|^2 = \int_{\Omega\oplus\Delta} \mathscr{X} \cdot \mathscr{M}\mathscr{X}\, dx = \int_{\Delta} z^\top z dx + \int_{\Omega} w^\top w dx + \int_{\Omega} \frac{\partial w^\top}{\partial t} m(x) \frac{\partial w}{\partial t} dx \tag{49}$$

The objective is to minimize the performance objective signal $p(t)$ with the linear-quadratic regulator (LQR) cost functional

$$J = \frac{1}{2}\int_0^\infty \left(\langle \mathscr{M}^{-1}\mathscr{Q}\chi, \chi \rangle + u^\top R u \right) dt \tag{50}$$

where $\mathscr{Q} \in \mathscr{C}_c^\infty(\Omega)$ is a semi-group operator with $Q_z = Q_z^\top > 0 \in \mathbb{R}^{l\times l}$, $Q_w = Q_w^\top > 0 \in \mathbb{R}^{l\times l}$, $R = \operatorname{diag}(R_z, R_w) \in \mathbb{R}^{(p+q)\times(p+q)}$, $R_z = R_z^\top > 0 \in \mathbb{R}^{q\times q}$, and $R_w = R_w^\top > 0 \in \mathbb{R}^{p\times p}$ are the weighting matrices.

Theorem 2: The optimal control that minimizes the LQR cost functional in Eq. (50) is given by

$$u = -R^{-1} \langle \mathscr{B}, \Psi W \langle \Psi, \chi \rangle + \Psi S \langle \Psi, \mathscr{V} \rangle \rangle \tag{51}$$

where W and S are steady-state solutions of an integro-differential Riccati equation and its auxiliary equation and $\Psi(x) = \begin{bmatrix} I & 0 & 0 \\ 0 & \phi(x) & 0 \\ 0 & 0 & \phi(x) \end{bmatrix} : \Omega \oplus \Delta \in \mathbb{R}^{(m+2n)\times(m+2nN)}$, $N \to \infty$.

Proof: Using the inner product definition in Eq. (49), the cost functional is expressed as

$$J = \frac{1}{2}\int_0^\infty \left(\langle \mathscr{M}^{-1}\mathscr{Q}\chi, \chi \rangle + u^\top R u \right) dt + \int_0^\infty \left\langle \lambda, -\frac{\partial \mathscr{X}}{\partial t} + \mathscr{A}\mathscr{X} + \mathscr{B}u + \mathscr{V} \right\rangle dt \tag{52}$$

where $\lambda(x,t) : (\Omega \oplus \Delta) \times [0,\infty) \to \mathbb{R}^{m+2n}$ is an abstract adjoint vector.

The variation of the cost functional δJ due to the state variation $\delta\chi(x,t)$ and the control variation $\delta u(t)$ is computed as

$$\begin{aligned} \delta J = &\int_0^\infty \left(\langle \mathscr{M}^{-1}\mathscr{Q}\chi, \delta\chi \rangle + u^\top R \delta u \right) dt \\ &+ \int_0^\infty \left\langle \frac{\partial \lambda}{\partial t} + \mathscr{A}^*\lambda, \delta\chi \right\rangle dt + \int_0^\infty \langle \mathscr{B}^*\lambda, \delta u \rangle dt - \langle \lambda, \delta\chi \rangle|_0^\infty \end{aligned} \tag{53}$$

where $\mathscr{A}^* = \mathscr{M}^{-1}\mathscr{A}^\top\mathscr{M}$ and $\mathscr{B}^* = \mathscr{M}^{-1}\mathscr{B}^\top\mathscr{M}$ are the adjoint semi-group operators corresponding to the semi-group operators $\mathscr{A}$ and $\mathscr{B}$, respectively. The necessary conditions of optimality are obtained as

$$\left\langle \mathscr{M}^{-1}\mathscr{Q}\chi + \frac{\partial \lambda}{\partial t} + \mathscr{A}^*\lambda, \delta\chi \right\rangle = 0 \tag{54}$$

$$u^\top R \delta u + \langle \mathscr{B}^*\lambda, \delta u \rangle = 0 \tag{55}$$

The transversality condition is also obtained as

$$\langle \lambda, \delta\chi \rangle|_0^\infty = 0 \tag{56}$$

which yields $\lambda(x,t) = 0$ as $t \to \infty$.

The optimal control solution are obtained as

$$u = -R^{-1} \langle \mathscr{B}, \lambda \rangle \tag{57}$$

We define the solution of $\chi(x,t)$ as

$$\chi(x,t) = \Psi(x)\,\eta(t) \tag{58}$$

and assume the adjoint solution of the form

$$\lambda = \Psi W \langle \Psi, \chi \rangle + \Psi S \langle \Psi, \mathscr{V} \rangle \tag{59}$$

where $W(t) : [0,\infty) \to \mathbb{R}^{(m+2nN)\times(m+2nN)}$ and $S(t) : [0,\infty) \to \mathbb{R}^{(m+2nN)\times(m+2nN)}$.

Substituting into Eqs. (54) and (58) yields

$$\Big\langle \mathscr{M}^{-1}\mathscr{Q}\Psi + \Psi\dot{W}\langle \Psi,\Psi\rangle + \Psi W \Big\langle \Psi, \mathscr{A}\Psi - \mathscr{B}R^{-1}\langle \mathscr{B}, \Psi W\langle\Psi,\Psi\rangle + \Psi S\langle \Psi,\mathscr{V}\rangle\rangle + \mathscr{V}\Big\rangle \\ + \Psi\dot{S}\langle\Psi,\mathscr{V}\rangle + \Psi S\left\langle \Psi, \frac{\partial \mathscr{V}}{\partial t}\right\rangle + \mathscr{A}^*\left[\Psi W\langle\Psi,\Psi\rangle + \Psi S\langle\Psi,\mathscr{V}\rangle\right], \Psi \Big\rangle = 0 \tag{60}$$

Separating the equation into two parts gives

$$\Big\langle \mathscr{M}^{-1}\mathscr{Q}\Psi + \Psi\dot{W}\langle\Psi,\Psi\rangle + \Psi W\langle\Psi,\mathscr{A}\Psi\rangle - \Psi W\Big\langle \Psi, \mathscr{B}R^{-1}\langle\mathscr{B},\Psi W\langle\Psi,\Psi\rangle\rangle\Big\rangle \\ + \mathscr{A}^*\Psi W\langle\Psi,\Psi\rangle, \Psi\Big\rangle = 0 \tag{61}$$

$$\left\langle \Psi W\Big\langle \Psi, -\mathscr{B}R^{-1}\langle\mathscr{B},\Psi S\langle\Psi,\mathscr{V}\rangle\rangle + \mathscr{V}\Big\rangle + \Psi\dot{S}\langle\Psi,\mathscr{V}\rangle + \Psi S\left\langle\Psi,\frac{\partial\mathscr{V}}{\partial t}\right\rangle, \Psi \right. \\ \left. + \mathscr{A}^*\Psi S\langle\Psi,\mathscr{V}\rangle, \Psi\right\rangle = 0 \tag{62}$$

Equation (61) is an integro-differential Riccati equation subject to the transversality condition $\dot{W}(\infty) = 0$ and $\dot{S}(\infty) = 0$

The optimal control $u(t)$ is obtained as

$$u = K_\chi \int_\Omega \Psi^\top \mathscr{M}\chi\,dx + K_\mathscr{V}\int_\Omega \Psi^\top\mathscr{M}\mathscr{V}\,dx \tag{63}$$

where

$$K_\chi = -R^{-1}\left(\int_\Omega \mathscr{B}^\top\mathscr{M}\Psi\,dx\right)W \tag{64}$$

$$K_\mathscr{V} = -R^{-1}\left(\int_\Omega \mathscr{B}^\top\mathscr{M}\Psi\,dx\right)S \tag{65}$$

5 Aircraft Flight Control Application

Consider the lateral-directional dynamics of the aircraft coupled to the structural dynamics of the flexible wing. The aircraft motion is described by

$$mV\dot{\beta} = Y_\beta \beta + Y_p p + (Y_r - mV) r + mg\phi + Y_{\delta_r} \delta_r + Y_g \tag{66}$$

$$\begin{aligned} I_{xx}\dot{p} + I_{xz}\dot{r} =& L_\beta + L_p + L_r + \int_{-L}^{L} \left[l_w(x) w + l_{w_x}(x) \frac{\partial w}{\partial x} + l_{w_t}(x) \frac{\partial w}{\partial t} \right] x \cos\Lambda \, dx \\ &+ L_{\delta_r}\delta_r + L_\delta \delta + L_g \end{aligned} \tag{67}$$

$$I_{xz}\dot{p} + I_{zz}\dot{r} = N_\beta + N_p + N_r + N_{\delta_r}\delta_r + N_\delta \delta + N_g \tag{68}$$

$$\dot{\phi} = p \tag{69}$$

where $\alpha(t)$ is the aircraft angle of attack, $q(t)$ is the aircraft pitch rate, $w(x,t) = \begin{bmatrix} v(t) & \theta(t) \end{bmatrix}^\top$ is a wing elastic state vector due to the wing vertical bending displacement $v(x,t)$ and wing torsional displacement $\theta(t)$ in anti-symmetric motion, $\delta(t)$ is a vector of flight control surface deflections on the aircraft wing, $\delta_r(t)$ is the rudder control surface deflection, $Y_g(t)$ is aerodynamic side force due to atmospheric gust, $L_g(t)$ is the aerodynamic rolling moment due to gust, and $N_g(t)$ is the aerodynamic yawing moment due to gust.

The bending-torsion motion of an aircraft wing is described by the aeroelastic equations of motion

$$\begin{aligned} \rho A \frac{\partial^2 v}{\partial t^2} + \rho A e_{cg} \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v}{\partial x^2} \right) =& l_p p + l_{v_x} \frac{\partial v}{\partial x} + l_{v_t} \frac{\partial v}{\partial t} + l_{\theta_t} \frac{\partial \theta}{\partial t} \\ &+ l_\delta \delta + l_g(x,t) + v_n(x,t) \end{aligned} \tag{70}$$

$$\rho I_{xx} \frac{\partial^2 \theta}{\partial t^2} + \rho A e_{cg} \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(GJ \frac{\partial \theta}{\partial x} \right) = m_p p + m_{v_x} \frac{\partial v}{\partial x} + m_{v_t} \frac{\partial v}{\partial t} + m_{\theta_t} \frac{\partial \theta}{\partial t} \tag{71}$$

$$+ m_\delta \delta + m_g(x,t) + \theta_n(x,t) \tag{72}$$

where x denotes the coordinate along the wing elastic axis, $EI(x)$ is the bending stiffness, $GJ(x)$ is the torsional stiffness, $l_g(x,t)$ is aerodynamic lift due to gust, $m_g(x,t)$ is the aerodynamic pitching moment due to gust, and $v_n(x,t)$ and $\theta_n(x,t)$ are the process noise due to atmospheric turbulence.

The gust is modeled as a two-component sinusoidal vertical and lateral gust with a 1° equivalent angle of attack and a 1° equivalent angle of sideslip at a gust frequency of 30 rad/sec. The vertical gust generates a symmetric gust load on the aircraft wing and the lateral gust generates an asymmetric gust load on the vertical tail. The gust load is defined as a differential operator

$$p(x,t) = \begin{bmatrix} I & 0 & 0 \\ 0 & -EI \frac{\partial^2}{\partial x^2} \delta(x - x_v) & 0 \\ 0 & 0 & GJ \frac{\partial}{\partial x} \delta(x - x_\theta) \end{bmatrix} \begin{bmatrix} z(t) \\ v(x,t) \\ \theta(x,t) \end{bmatrix} \tag{73}$$

where $z(t) = \left[\beta(t)\ p(t)\ r(t)\ \phi(t)\right]^\top$ is the aircraft state vector and $\delta(x - x_0)$ is the Dirac delta function.

The control input vector $u_w(t) = \left[\delta_1(t) \cdots \delta_p(t)\right]^\top \in \mathbb{R}^p$ comprises p discrete control surface deflections that span the wing. Figure 1 illustrates several distributed control surfaces of a flexible wing aircraft. The wing elastic motion is measured by fiber-optic strain sensors placed on the wing. The bending displacement $v(x,t)$ and torsional rotation $\theta(x,t)$ can be directly computed from these measurements.

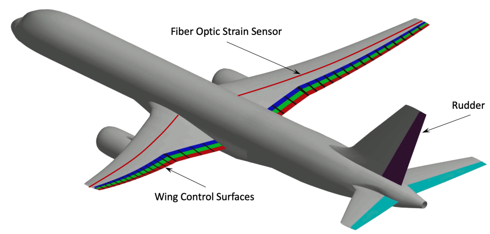
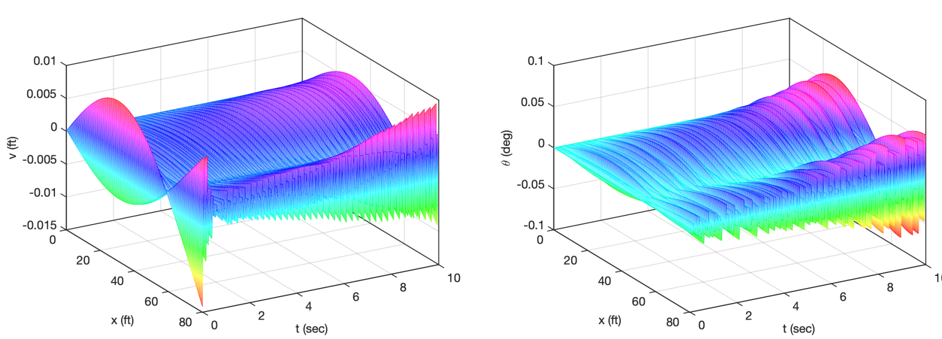


Fig. 1. Flexible Wing Aircraft with Distributed Fiber-Optic Strain Sensors

The first unstable mode of the elastic wing occurs above an airspeed of 453 ft/sec. Figures 2(a) and (b) show the unstable open-loop response of the wing elastic motion at 500 ft/sec. Figure 3 shows the unstable aircraft response.

A Hamiltonian stabilizing control is designed with $\mu = 0$ and $c = 0.01\omega_f$, respectively, where ω_f is the flutter frequency. In addition, the rudder feedback control is designed by the standard LQR method to improve lateral-directional handling qualities of the aircraft. Figures 4(a) and (b) show the stable closed-loop response of the wing elastic motion at 500 ft/sec with the Hamiltonian control. The stable closed-loop response of the aircraft is shown in Fig. 5(a) along with the control surface deflections in Fig. 5(b).



(a) (b)

Fig. 2. Wing Bending and Torsional Displacement without Stabilizing Control

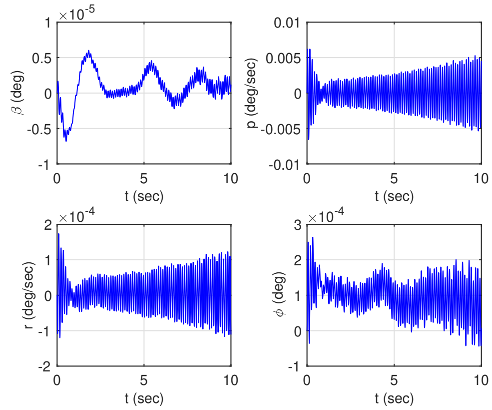
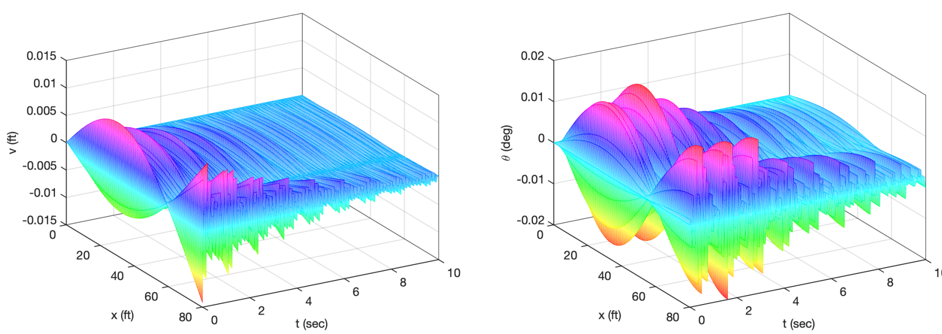


Fig. 3. Aircraft States without Stabilizing Control



(a) (b)

Fig. 4. Wing Bending and Torsional Displacement with Hamiltonian Control

A sinusoidal gust disturbance with a frequency of 30 rad/sec is introduced. Figures 6(a) and (b) show the wing bending and torsional moments, respectively. The maximum wing bending moment is 293,748 ft-lb and the maximum wing torsional moment is 74,779 ft-lb. A gust load alleviation (GLA) distributed optimal control is designed and implemented to reduce the wing bending and torsional moments. Figures 7(a) and (b) show the wing bending and torsional moments, respectively, with the distributed optimal control. Both the maximum bending and torsional moments are substantially reduced to 63,130 ft-lb and 48,568 ft-lb, respectively. The gust load reduction is substantial; 4.65 times for the maximum bending moment and 1.54 times for the maximum torsional moment.

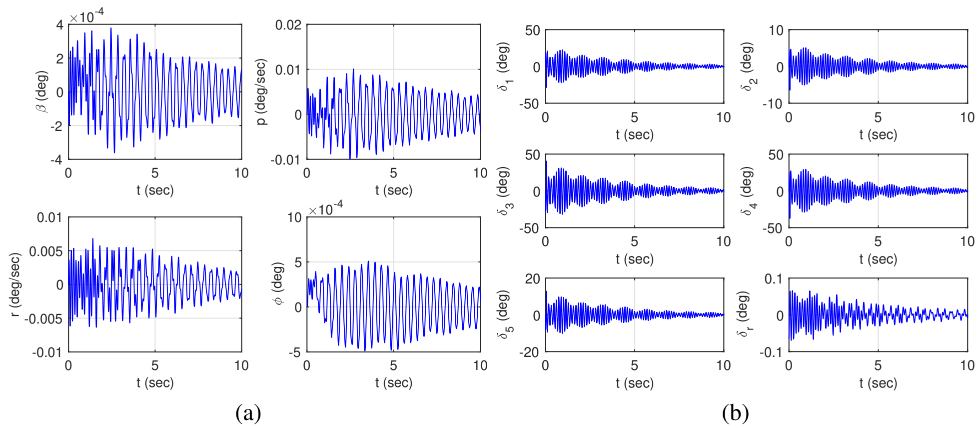


Fig. 5. Aircraft States and Control Surface Deflections with Hamiltonian Control

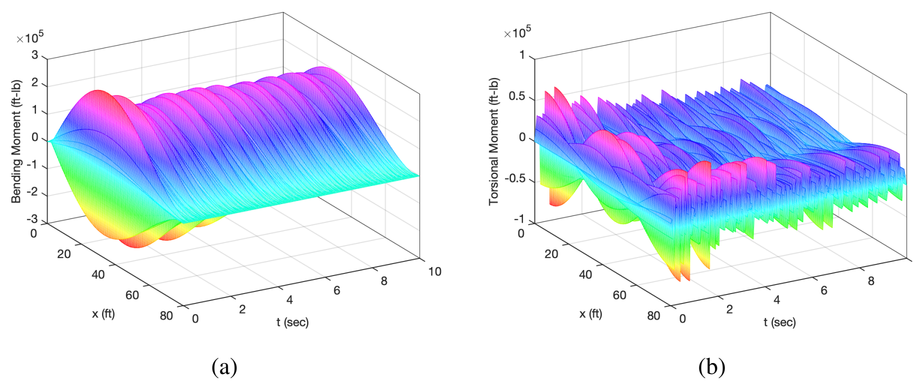
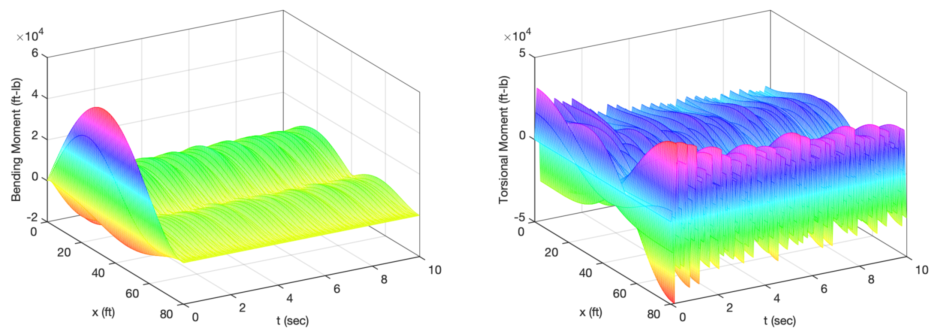


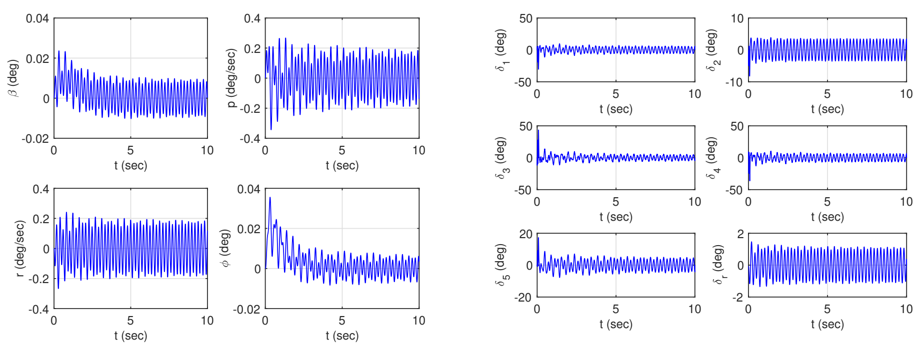
Fig. 6. Wing Bending and Torsional Moments with Hamiltonian Control During Gust Encounter

Figure 8(a) shows the aircraft response with the distributed optimal control along with the control surface deflections in Fig. 8(b). The effect of the distributed optimal control on the aircraft response is small due to the dominant response to the atmospheric turbulence modeled as process noise for the wing. The control surface deflections on the wing are much smaller with than without the distributed optimal control. There are small large initial transients up to 43.5° but overall the deflections are less than 10°. The maximum rudder deflection is 1.5° which is about the same as that without the GLA control. A redesign could be implemented to increase the rudder deflection. However, typically the rudder deflection is restricted to a small operating limit well less than 10° at high speed flight.



(a) (b)

Fig. 7. Wing Bending and Torsional Moments with Hamiltonian and Distributed Optimal Control



(a) (b)

Fig. 8. Aircraft States and Control Surface Deflections with Hamiltonian and Distributed Optimal Control

6 Conclusion

This lecture presents a Hamiltonian control method and distributed optimal control method for a class of distributed Lagrangian systems coupled to lumped-parameter systems. The Hamiltonian method provides a stabilizing control for the distributed Lagrangian system. A distributed optimal control method is developed for the coupled systems using a semi-group abstraction. The optimal control solution leads to a Riccati equation in terms of the inner product of the semi-group operator. A flexible aircraft flight control application illustrates the effectiveness of the theory.

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