Discontinuous Galerkin and Flux Reconstruction Methods for Time Stepping (ODE)

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Time Discretization

- Mathematical modeling of physical phenomenon (e.g., for fluid flows, Navier-Stokes equations):

$$u_t + f_x + g_y + h_z = 0$$

$$u(0) = u_0$$

- After the spatial discretization, the above reduces to a time discretization problem (ODE):

$$u_t = f(t, u), \qquad u(0) = u_0$$

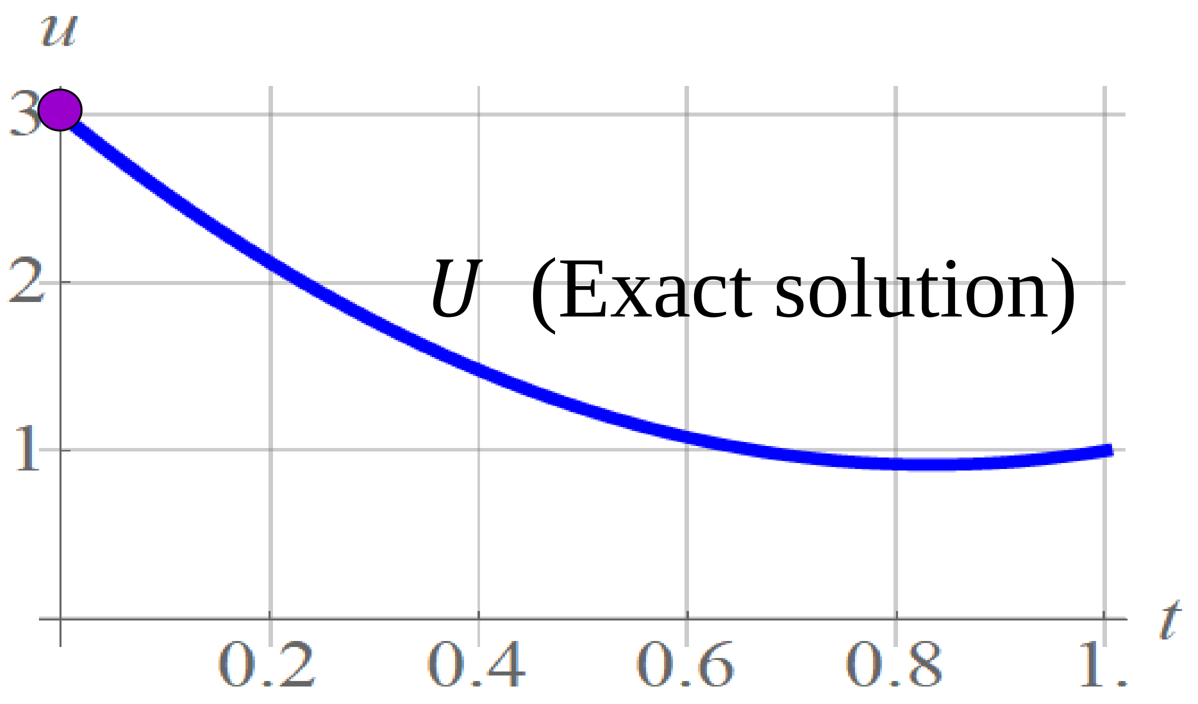
- NASA CFD Vision 2030 Report: Time-stepping remains to be a bottleneck for turbulent flow simulations. (Explicit method leads to too small of a time step size, implicit method leads to too large of a system.)

Outline

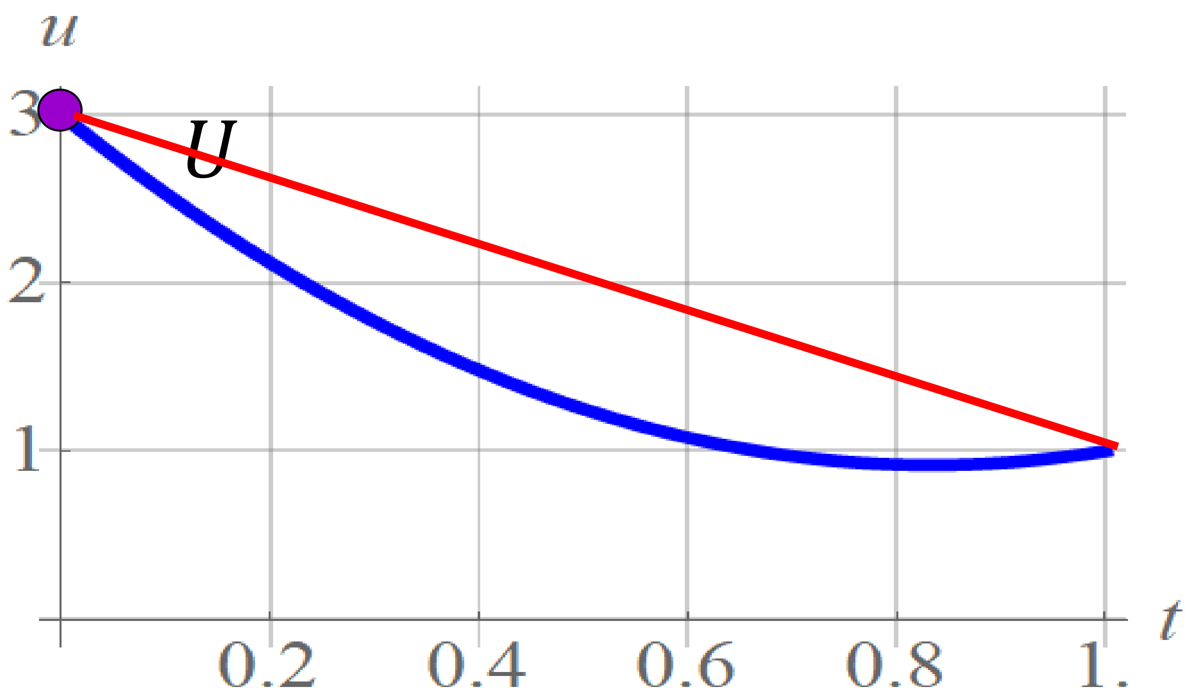
- Discontinuous Galerkin (DG) and flux reconstruction (FR) methods for time discretization
- Resulting implicit Runge-Kutta scheme IRK-DG versus existing IRK methods
- Stability and Accuracy
- Conclusions and discussion

A Simple Example

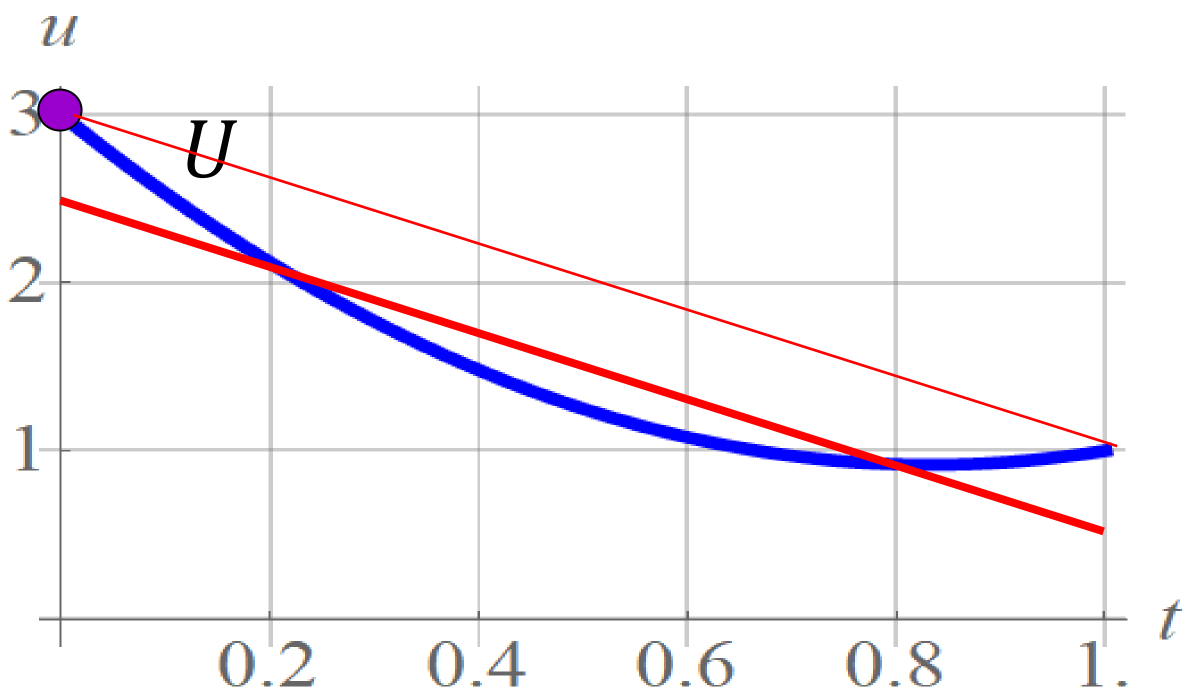
- On $[0,1]$, solve $u'(t) = 6t - 5,\ u(0) = u_n = 3.$
- Exact solution $U(t) = 3t^2 - 5t + 3.$
- Find the **linear** DG solution $u_h = at + b.$



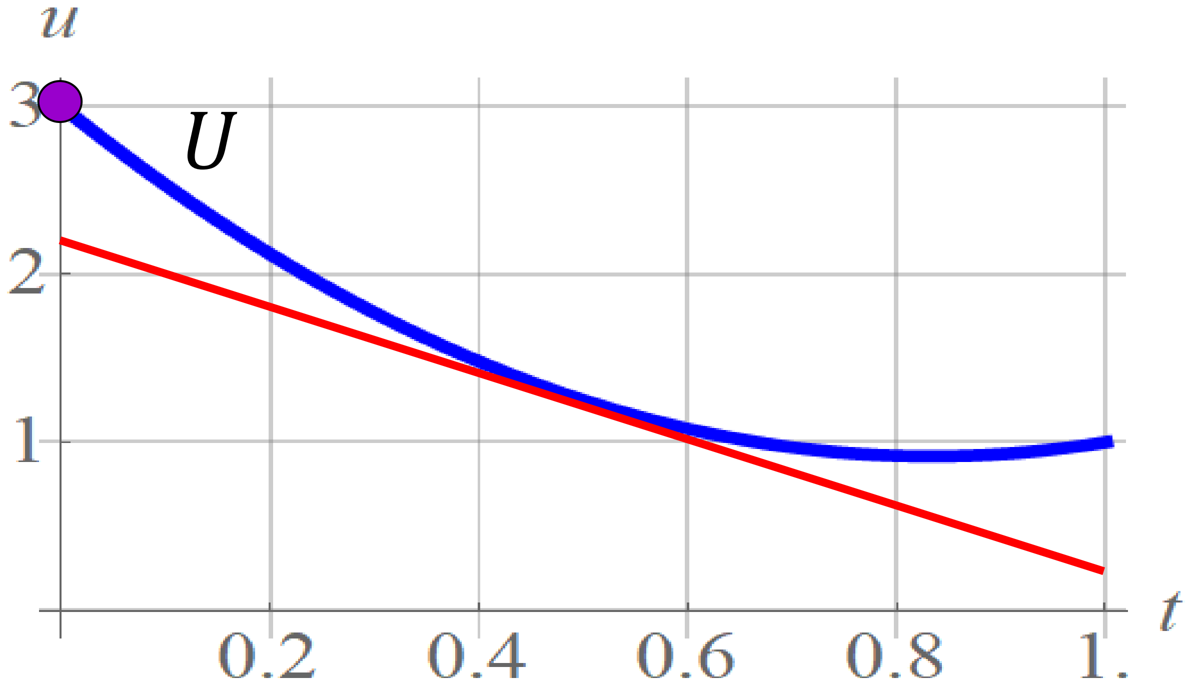
Linear DG Solution



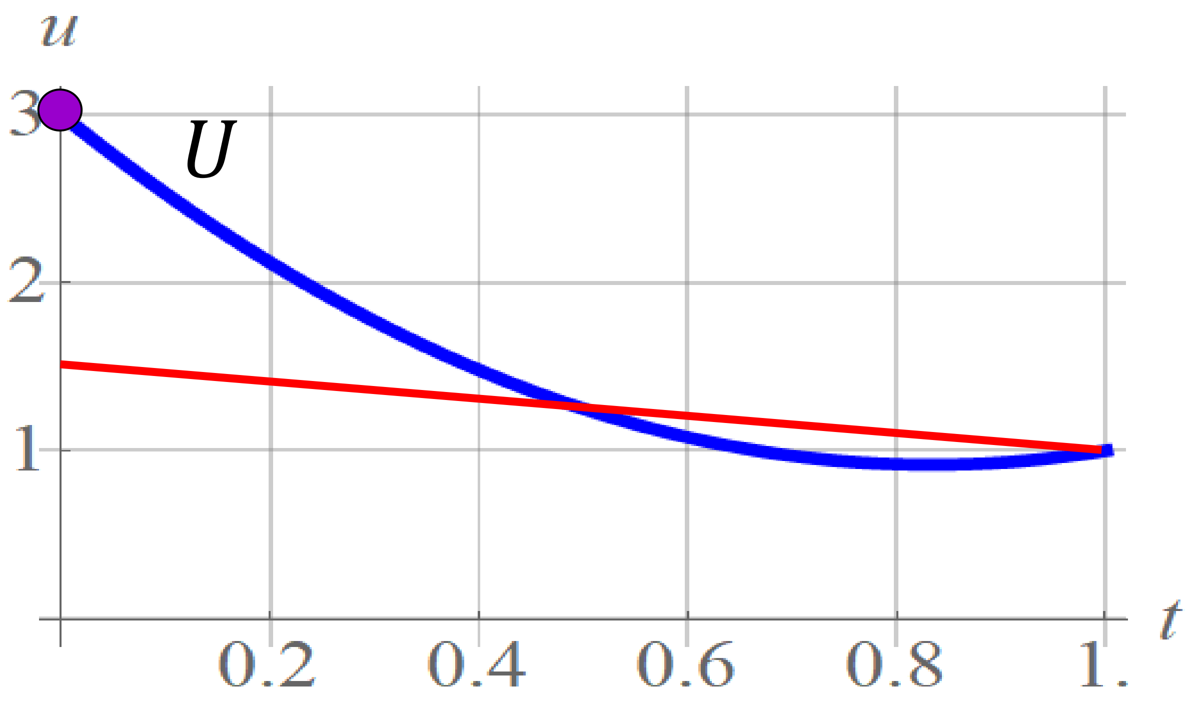
Linear DG Solution



Linear DG Solution



Linear DG Solution



DG Formulation

- On $[0, 1]$, $u'(t) = f(t) = 6t - 5$; $u_n = 3$; DG: $u_h(0) \neq u_n$.
- DG solves $u'(t) = f(t)$ in an average sense: $v = 1$ and $v = t$,

$$\int_0^1 u_h'(t)v(t)dt \approx \int_0^1 f(t)v(t)dt$$

- To involve u_n, integrate by parts $\int_0^1 u_h' v dt = [u_h v]_0^1 - \int_0^1 u_h v' dt$

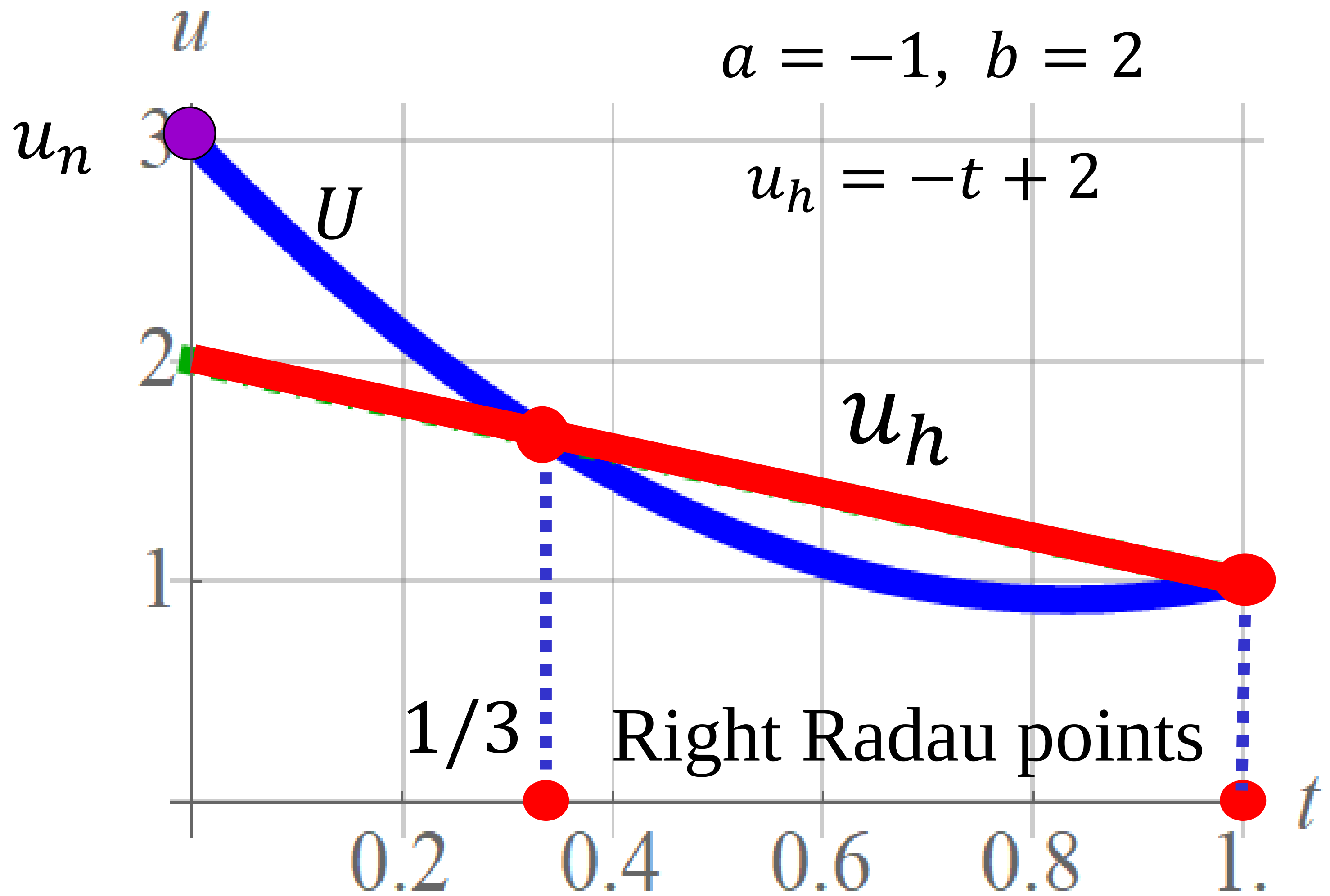
$$\int_0^1 u_h'(t)v(t)dt = u_h(1)v(1) - u_h(0)v(0) - \int_0^1 u_h(t)v'(t)dt$$

$u_h(0) \to u_n$

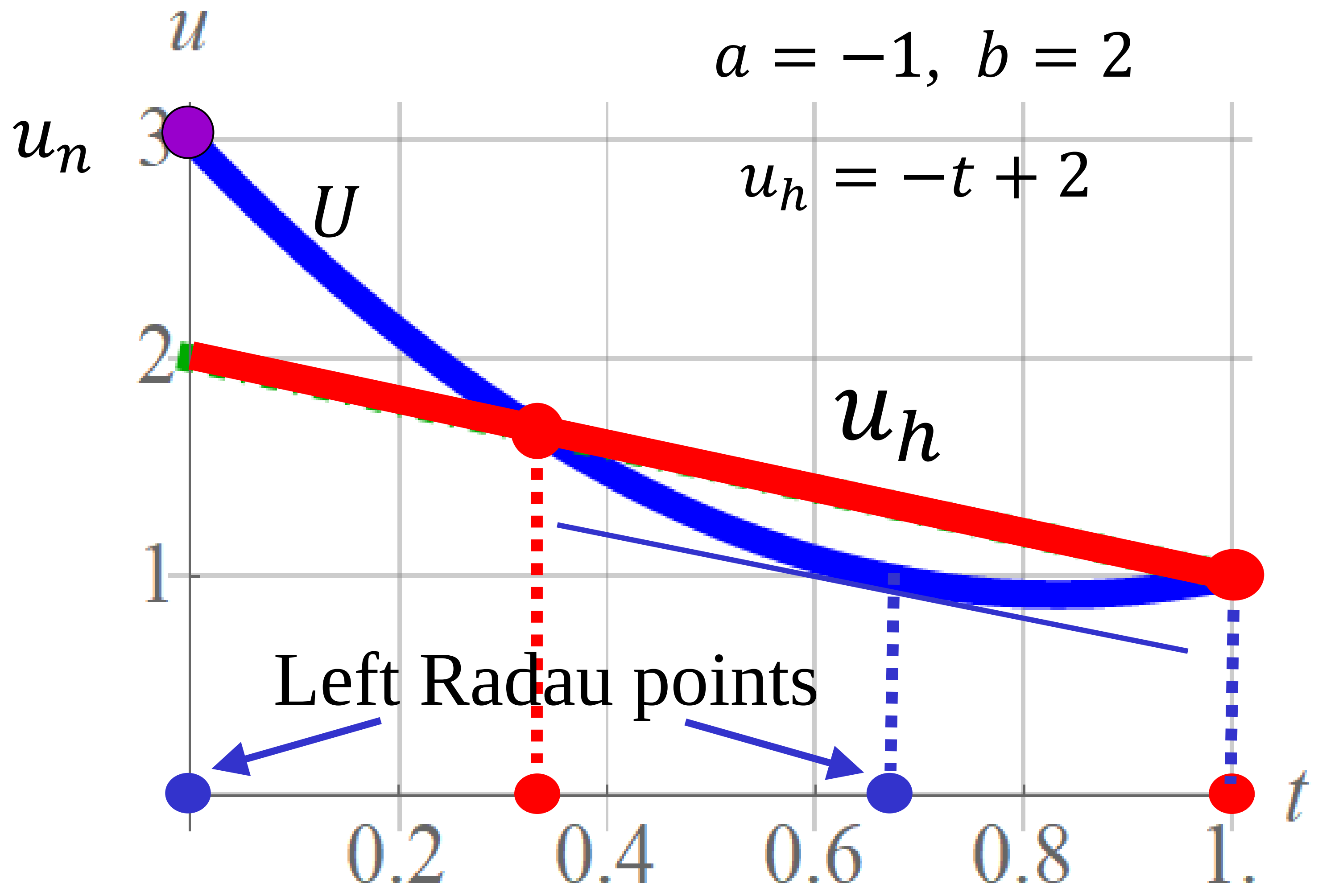
- DG method: Solution $u_h = u_h(t) = at + b$ is required to satisfy

$$u_h(1)v(1) - u_n v(0) - \int_0^1 u_h(t)v'(t)dt = \int_0^1 f(t)v(t)dt$$

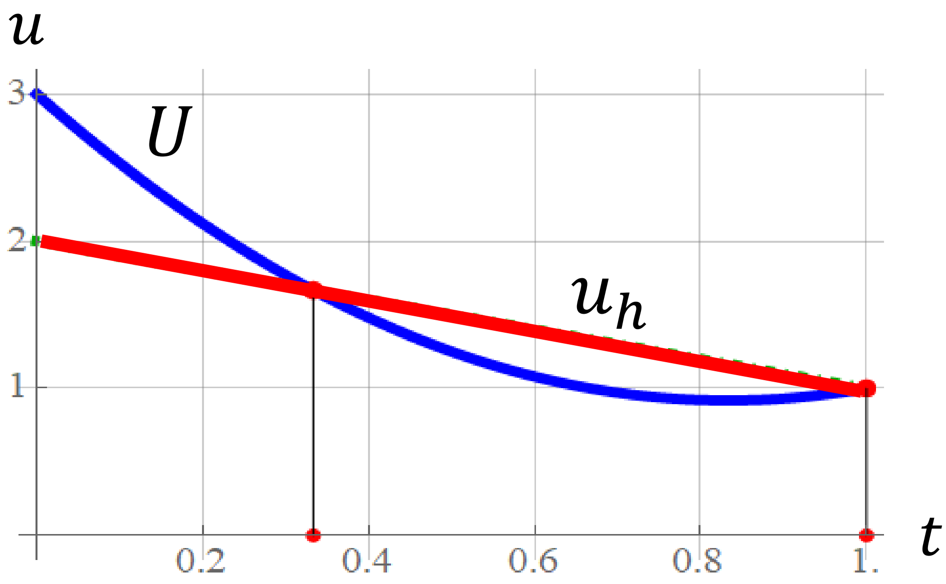
DG Solution

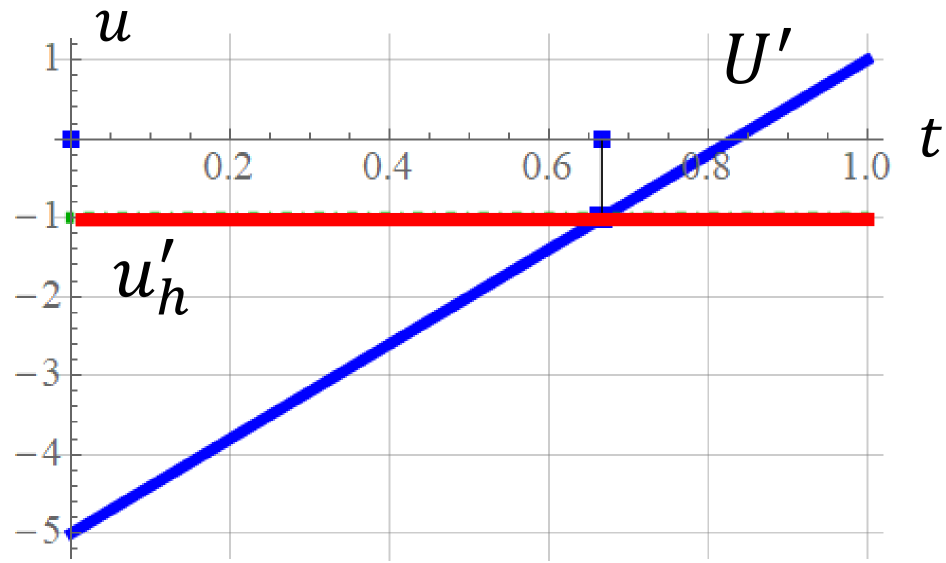


DG Solution

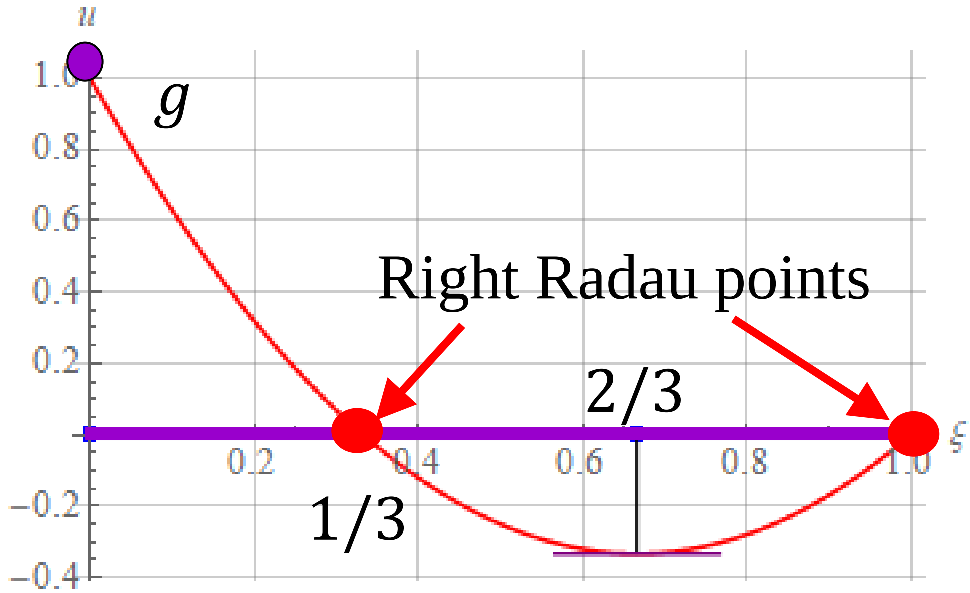


DG Solution





Connection between DG and FR: Approximating a Jump by a Polynomial



Approximating the jump from 1 at $\xi = 0$ to 0 for $0 < \xi \leq 1$ by a polynomial of degree $k + 1$ defined by $k + 2$ conditions:

$g(0) = 1$ and g vanishes at the $k + 1$ right Radau points

Then g is the right Radau polynomial $R_{R,k+1}$, and

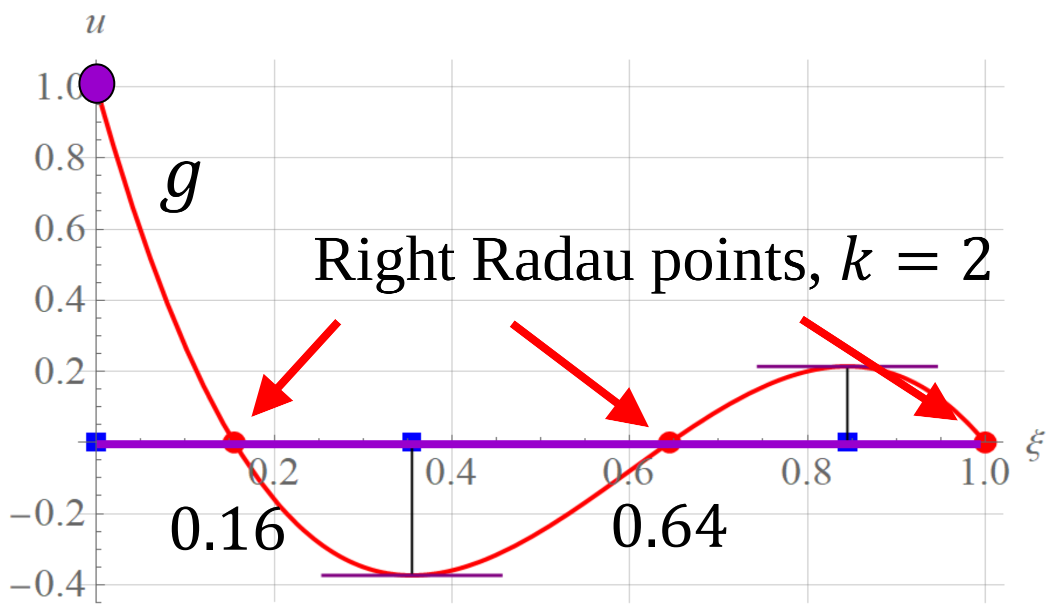
$$U = u_h + [u_n - u_h(0)]g$$

Correction Function g for arbitrary k

g: Right Radau polynomial of degree $k+1$ defined by $g(0)=1$ and, for $m=0,\ldots,k$,

$$\int_0^1 g(t)\, t^m dt = 0.$$

Or equivalently, g vanishes at the $k+1$ right Radau points.



DG and Collocation

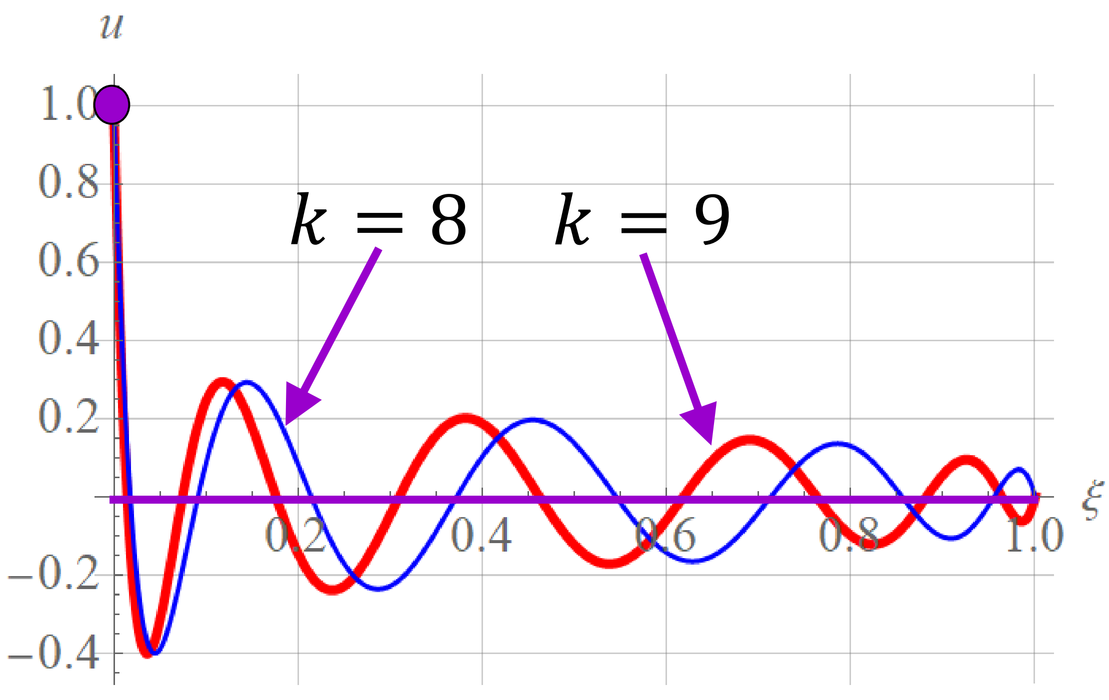
Concerning g, loosely put,

- **approximating zero in an average sense via k + 1 conditions (DG)**

is equivalent to

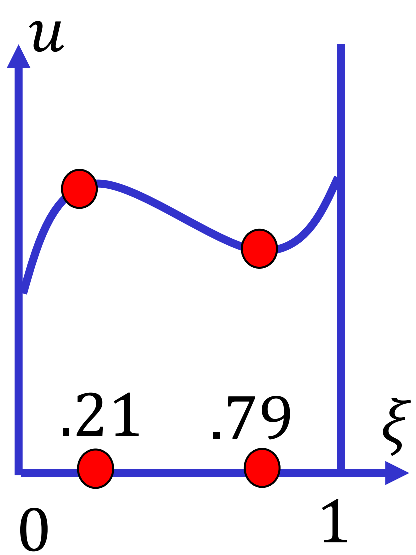
- **vanishing at the k + 1 right Radau points (collocation)**

Correction Functions (Radau Polynomial) of degree $k + 1$ for $k = 8$ and $k = 9$



$(k+1)$-Point Quadratures for DG

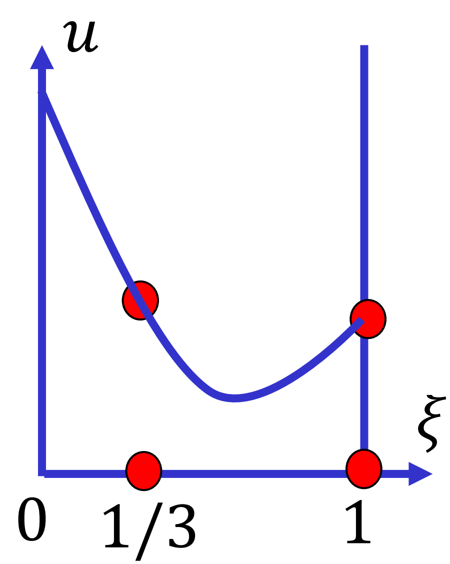
2-Point Quadrature $$\int_0^1 f(\xi)d\xi \approx b_1 f(\xi_1) + b_2 f(\xi_2)$$



Gauss

$$\frac{1}{2}f(.21) + \frac{1}{2}f(.79)$$

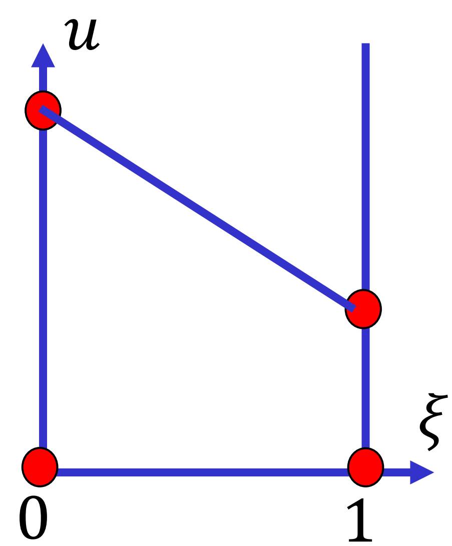
Exact for a cubic f



Right Radau

$$\frac{3}{4}f\left(\frac{1}{3}\right) + \frac{1}{4}f(1)$$

Exact for a parabolla f



Equidistance

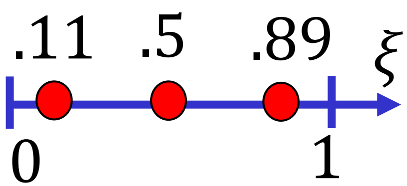
$$\frac{1}{2}f(0) + \frac{1}{2}f(1)$$

Exact for a linear f

$(k+1)$-Point Quadratures

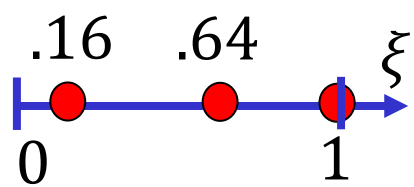
$$\int_0^1 f(\xi)d\xi \approx \sum_{i=1}^{k+1} b_i f(\xi_i)$$

$\boldsymbol{k = 2}$



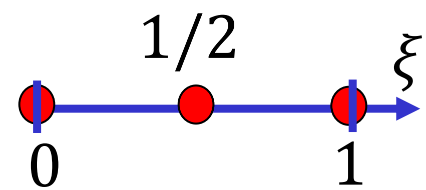
Gauss

Exact for polynomials of degree $2k+1$



Right Radau

Exact for polynomials of degree $2k$



Equidistance

Exact for polynomials of degree $k+1$

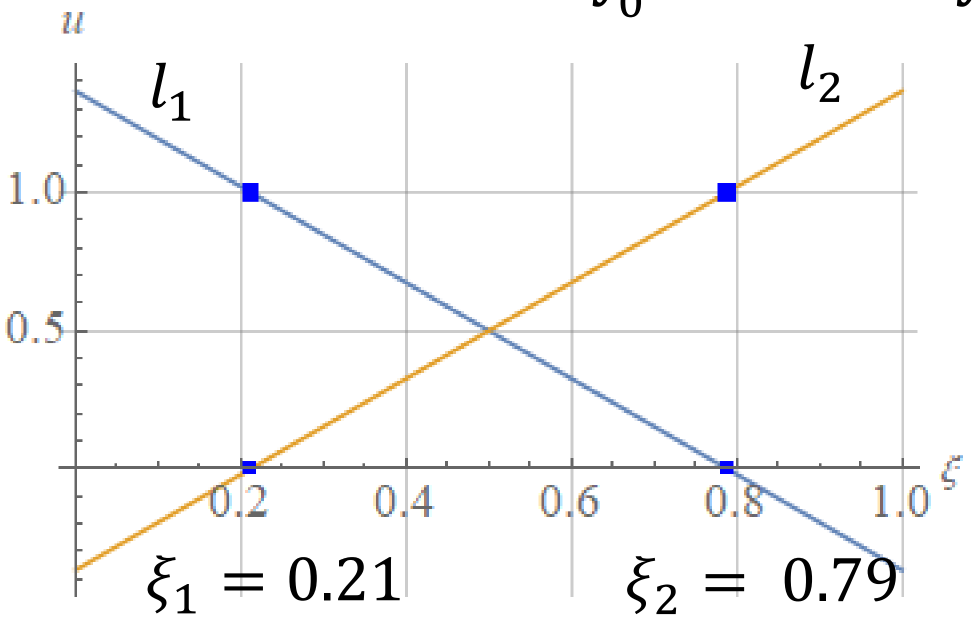
DG Solution Under Gauss Quadrature

$U' = f_h, \quad U(0) = u_n;$

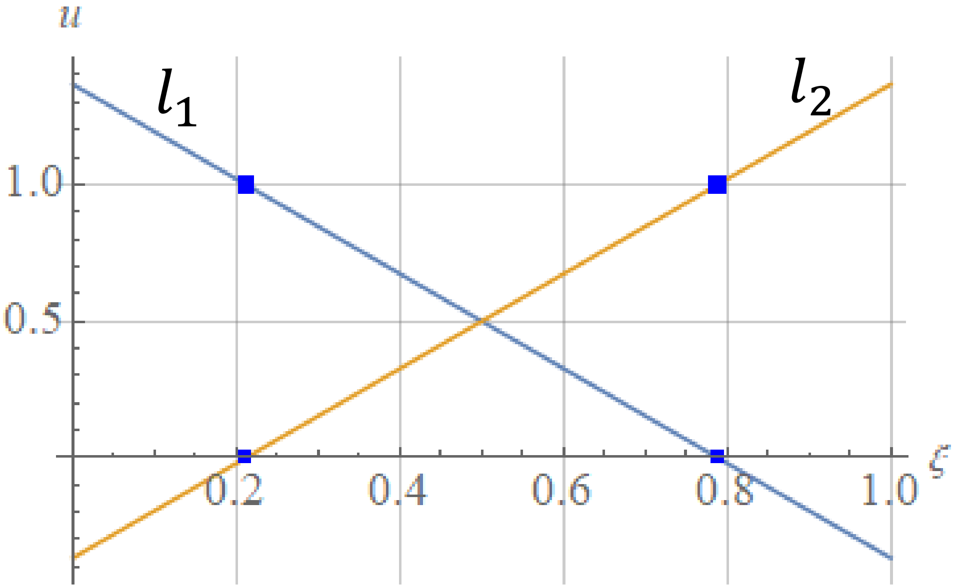
2 Gauss points; linear $f_h = f_1 l_1 + f_2 l_2$

$U(\xi) = u_n + \int_0^\xi f_h d\eta;$

$$\int_0^\xi f_h d\eta = f_1 \int_0^\xi l_1(\eta) d\eta + f_2 \int_0^\xi l_2(\eta) d\eta$$

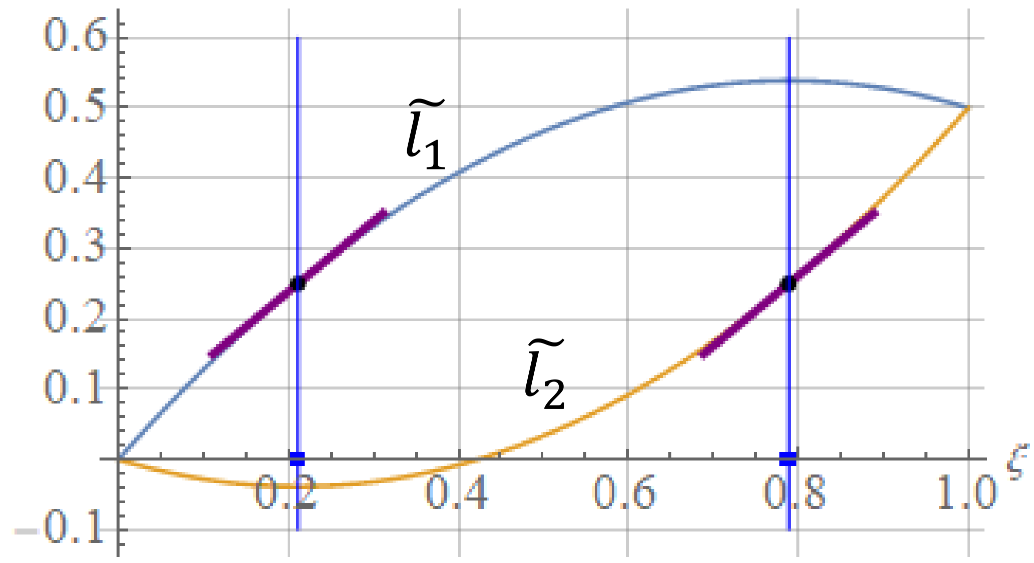


DG with Gauss Quadrature

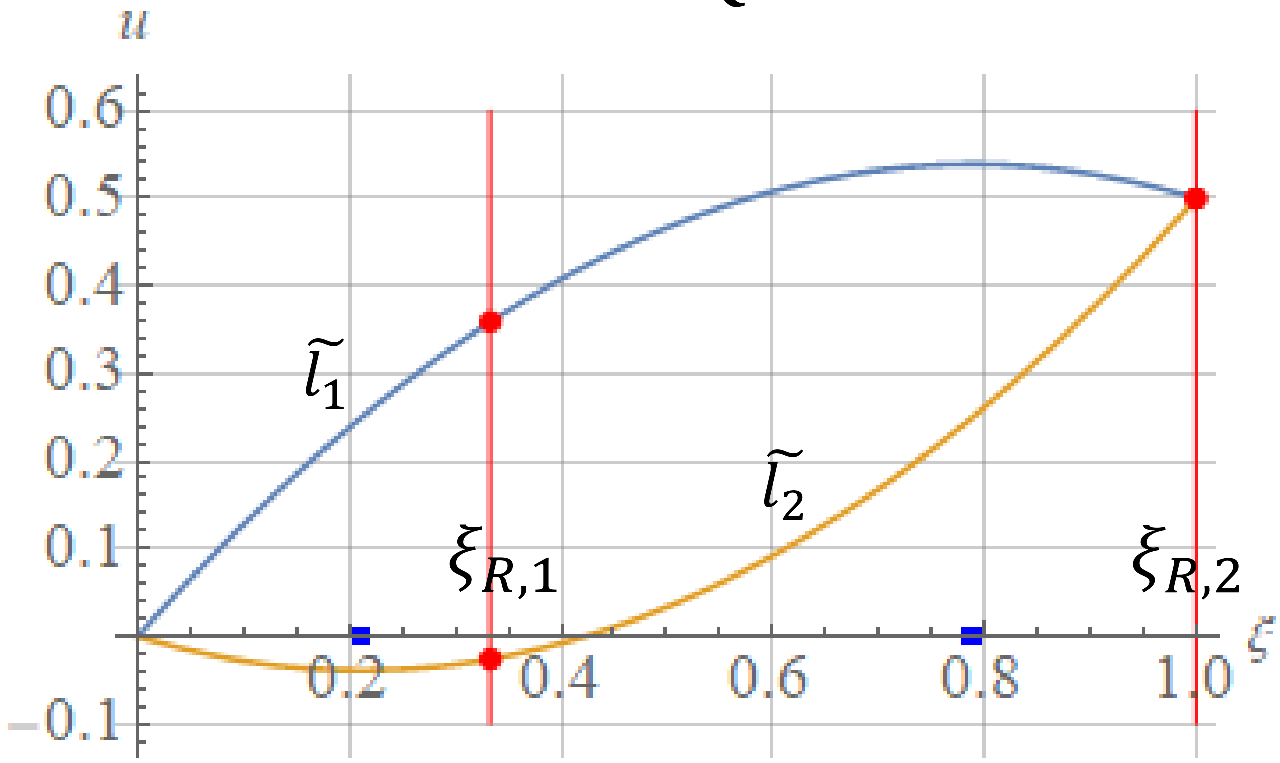


$$\widetilde{l_1}(\xi) = \int_0^{\xi} l_1(\eta)d\eta$$

$$\widetilde{l_2}(\xi) = \int_0^{\xi} l_2(\eta)d\eta$$



Under Gauss Quadrature

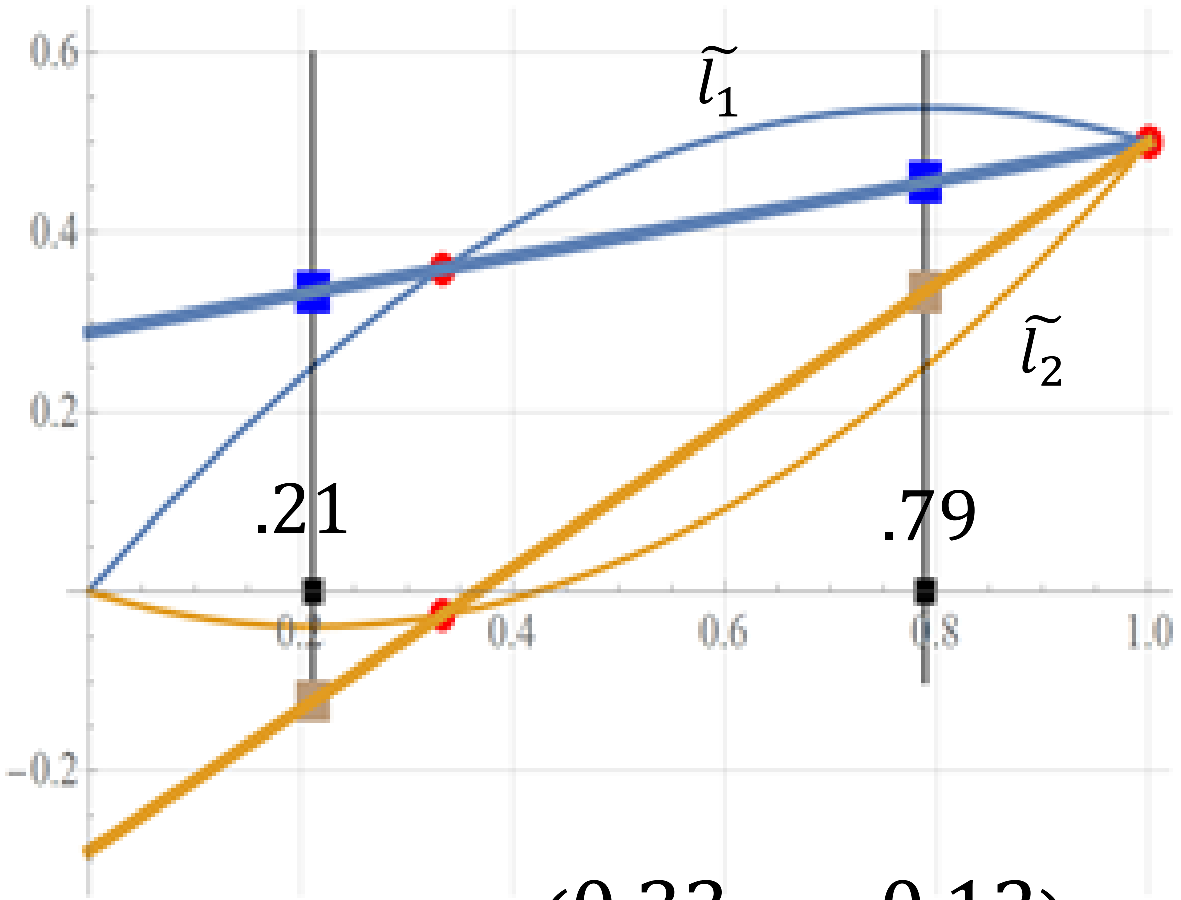


$U(\xi) = u_n + f_1\tilde{l_1}(\xi) + f_2\tilde{l_2}(\xi)$; $U = u_h + [u_n - u_h(0)]g$

At the right Radau points, $i = 1, 2$

$$u_h(\xi_{R,i}) = U(\xi_{R,i}) = u_n + f_1\tilde{l_1}(\xi_{R,i}) + f_2\tilde{l_2}(\xi_{R,i})$$

DG with Gauss Quadrature



$$\begin{pmatrix} 0.33 & -0.12 \\ 0.46 & 0.33 \end{pmatrix}$$

IRK Method: DG-Gauss

$$\begin{pmatrix} .33 & -.12 \\ .46 & .33 \end{pmatrix}$$

Butcher Tableau

$$\begin{array}{c|cc} .21 & .33 & -.12 \\ .79 & .46 & .33 \\ \hline * & .5 & .5 \end{array}$$

$$u_{n,1} = u_n + h\,[\,.33f\big(t_n + .21h, u_{n,1}\big) - .12f\big(t_n + .79h, u_{n,2}\big)]$$

$$u_{n,2} = u_n + h\,[\,.46f\big(t_n + .21h, u_{n,1}\big) + .33f\big(t_n + .79h, u_{n,2}\big)]$$

$$u_{n+1} = u_n + h\,[\,.5f\big(t_n + .21h, u_{n,1}\big) + .5\,f\big(t_n + .79h, u_{n,2}\big)]$$

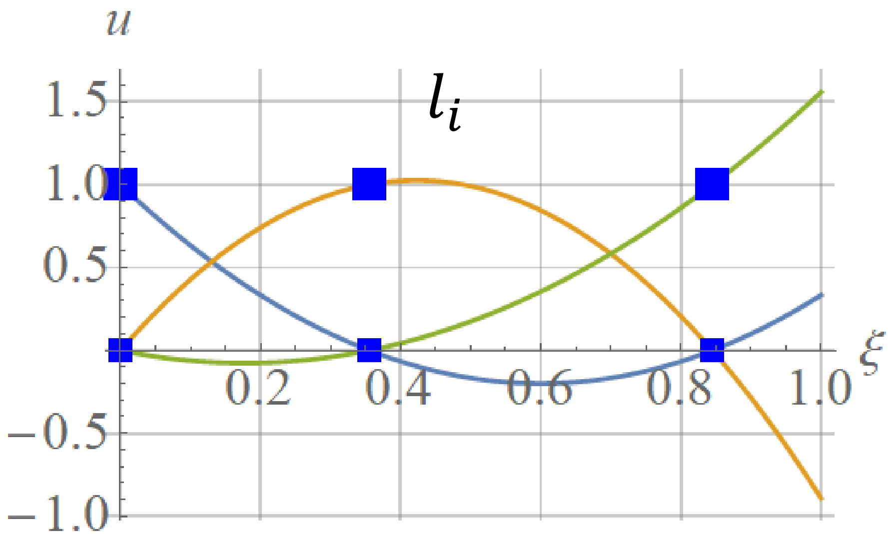
DG-Gauss and Gauss Collocation Methods

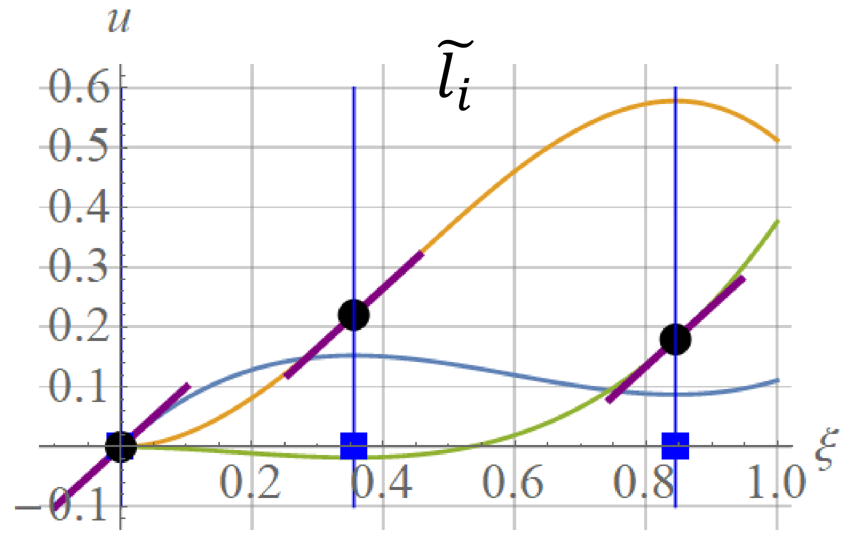
$$\begin{array}{c|cc} .21 & .33 & -.12 \\ .79 & .46 & .33 \\ \hline * & .5 & .5 \end{array} \qquad \begin{array}{c|cc} .21 & .25 & -.04 \\ .79 & .54 & .25 \\ \hline * & .5 & .5 \end{array}$$

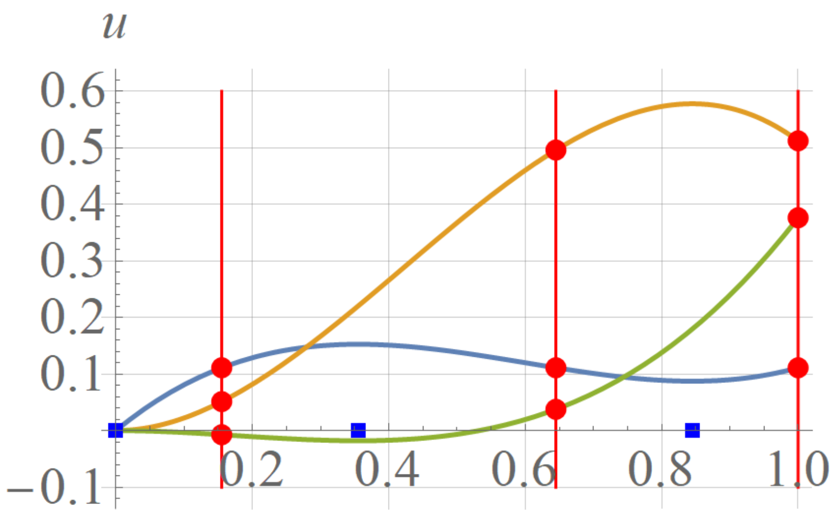
DG-Gauss	Gauss-Collocation
3rd-order accurate	4rd-order accurate
L-stable	Not L-stable

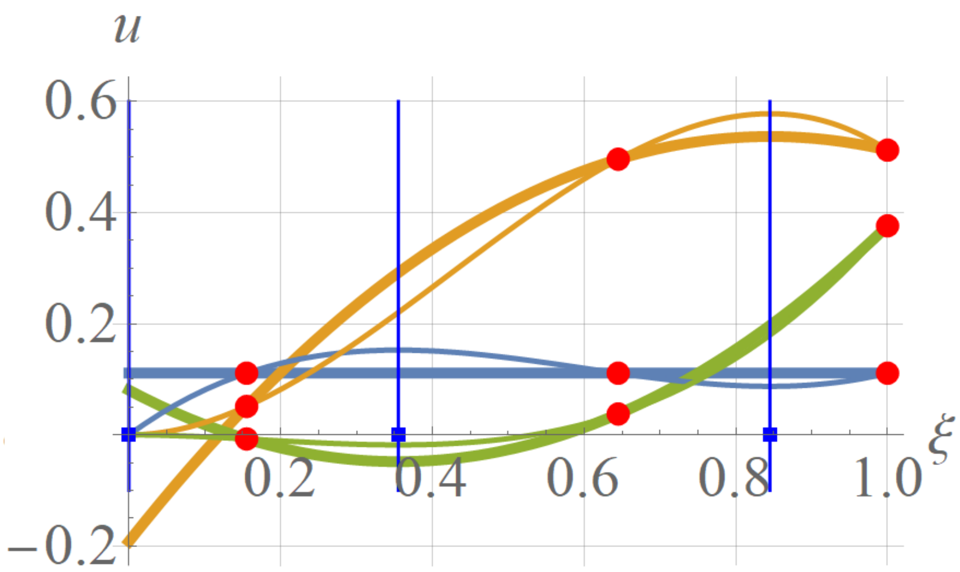
We can adjust numerical dissipation by blending these methods

IRK-DG by Left Radau Quadrature

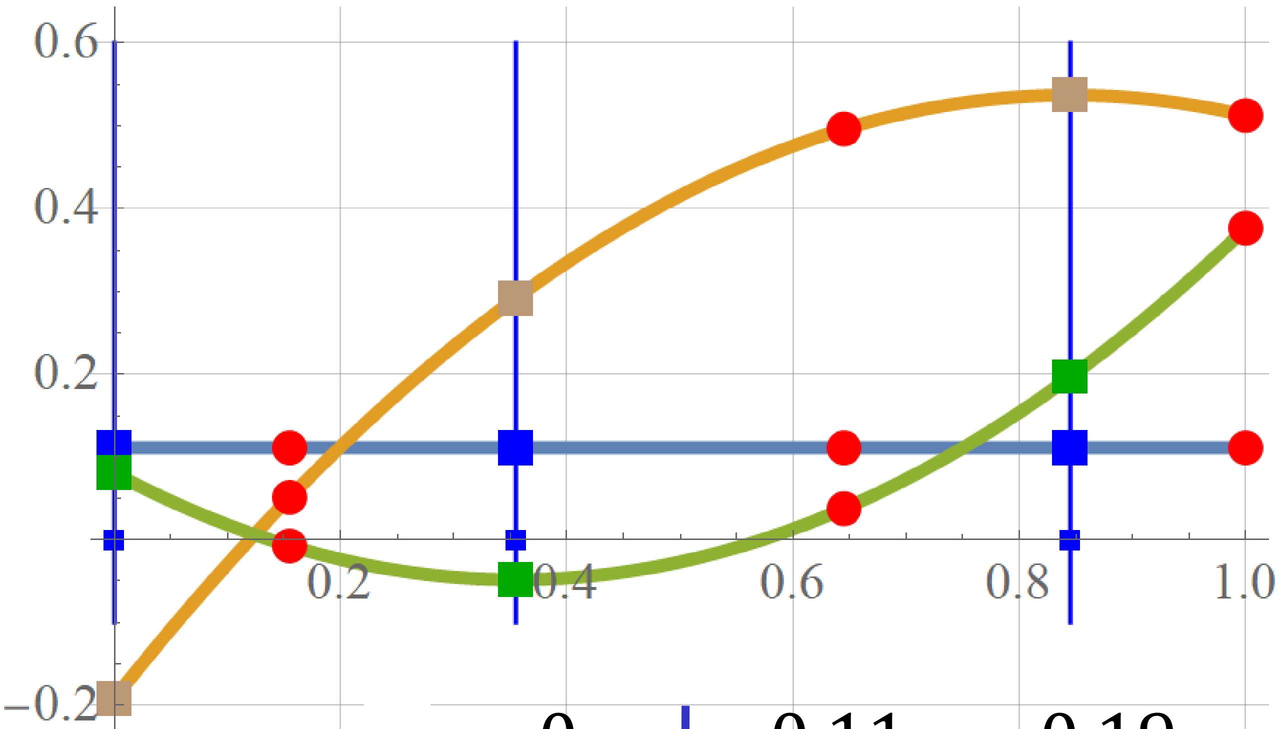








IRK-DG with Left Radau Quadrature



0	0.11	−0.19	0.081
0.35	0.11	0.29	−0.048
0.85	0.11	0.54	0.19
	0.11	0.51	0.37

IRK-DG

DG	**IRK Counterpart**
Left Radau Quadrature	Radau IA
Right Radau Quadrature	Radau IIA
Gauss Quadrature	DG-Gauss

Right Radau Quadrature

$$U = u_h + [u_n - u_h(0)]g$$

Under the $k+1$ point right Radau quadrature,

- The DG, CG (continuous Galerkin), FR, and collocation methods are all equivalent: they yield the same solution U.

- They all reduce to the Radau IIA scheme.

Stability and Accuracy

On [0, 1], solve

$$u' = z\,u, \qquad u(0) = 1$$

Exact solution:

$$u_{\text{Exact}}(t) = e^{z\,t}$$

After one step of size $h = t = 1$, the exact

solution is

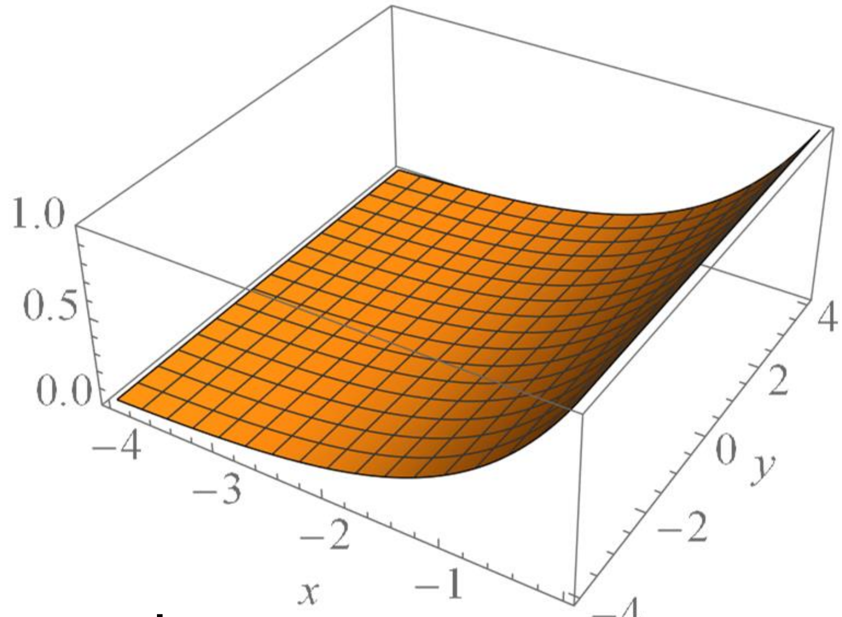
$$u_{\text{Exact}}(1) = e^{z} = e^{x+\mathbb{i}y}$$

Stability Function $R(z)$

$z = x + iy$

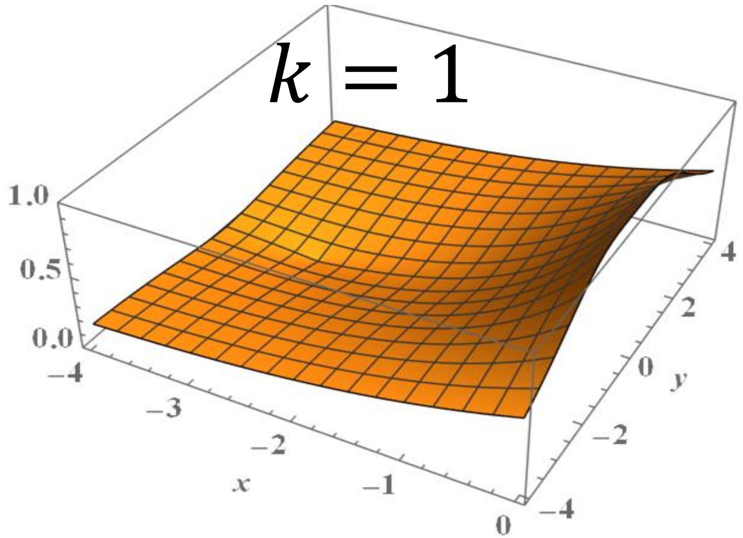
Exact Solution

$$\left|e^{x+iy}\right|$$

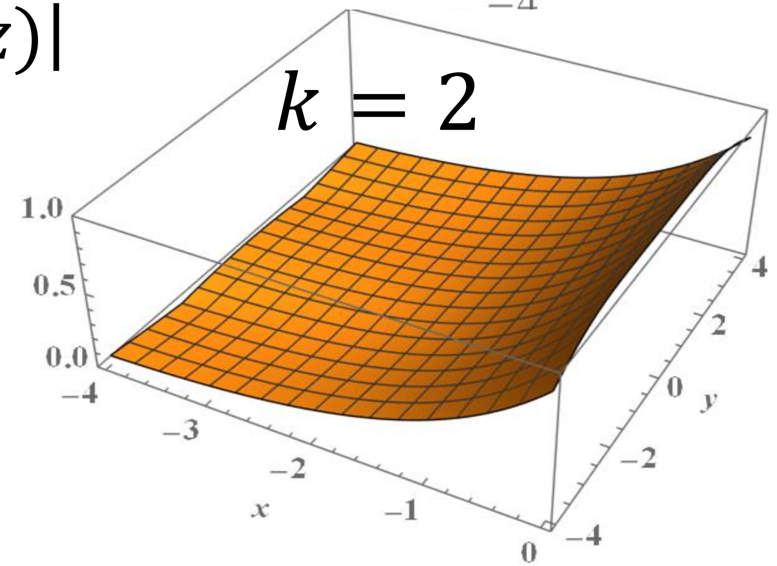


$|R(z)|$

$k = 1$



$k = 2$



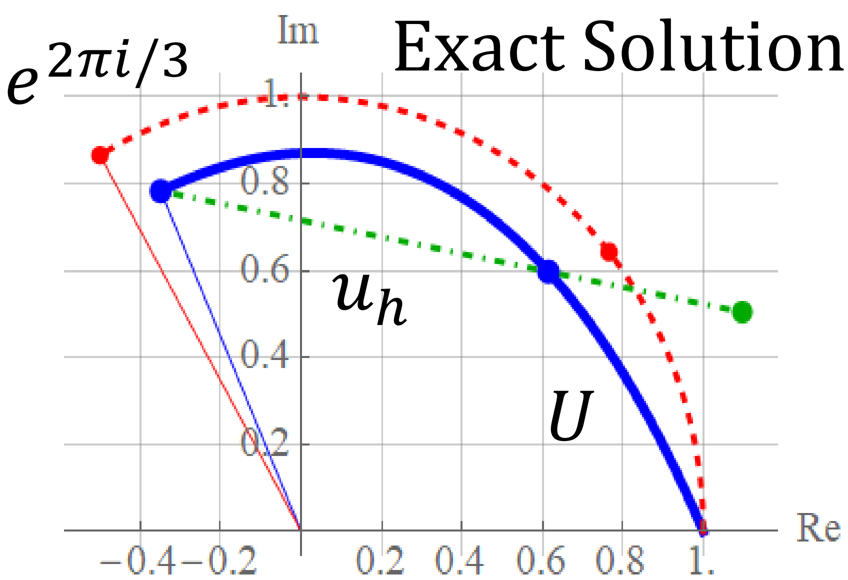
Linear DG Solution

$$u_{\text{Exact}}(1) = e^z$$

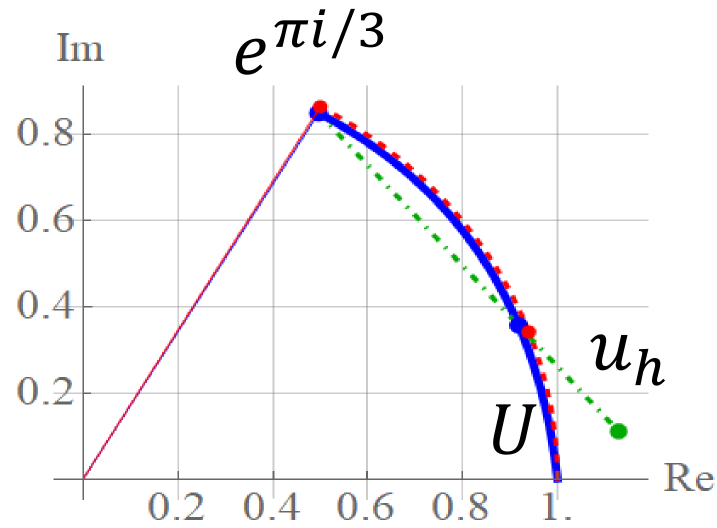
$$R_1(z) = \frac{2z + 6}{z^2 - 4z + 6}$$

$$E_1 = e^z - R_1(z) = \frac{z^4}{72} + \frac{19z^5}{1080} + \cdots$$

Example of DG Solutions



$z = 2\pi i/3$



$z = \pi i/3$

$\mathrm{Er} = |u_{\mathrm{Exact}}(1) - u_h(1)|$

$\mathrm{Er}_1 \approx 0.17$; $\mathrm{Er}_2 \approx 0.015$;

$\mathrm{Er}_1/\mathrm{Er}_2 \approx 11.2$ then 15 and 15.7; third-order accuracy

Conclusions and Discussion

- DG method for ODE was formulated from FR perspective (constructive and geometric).
- IRK-DG methods: (left Radau) Radau IA, (right Radau) Radau IIA, and DG-Gauss.
- The approach provides intuition and clarifies relations among DG, CG, FR, and collocation, methods.
- **An effective iteration procedure for these IRK methods remains to be found.**

Thank you.