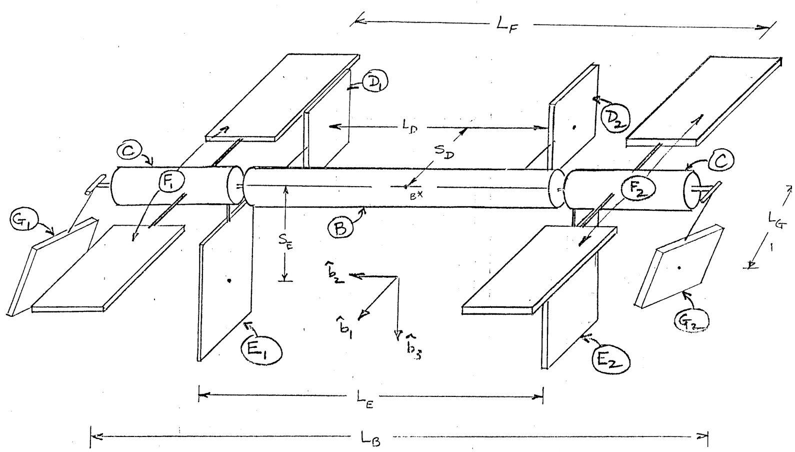
# Application of Kane's Method to Various Aerospace Mechanical Systems

Dr. Carlos M. Roithmayr

Vehicle Analysis Branch
NASA Langley Research Center
Hampton, VA

Aerospace Control and Guidance Systems Committee
Meeting 131
October 27, 2023

SPACE STATION MODEL FOR SD/EXACT TEST CASE



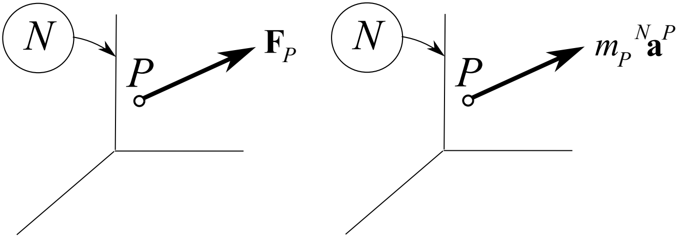

Derive dynamical equations of motion for a system of rigid bodies attached to one another

# Possible Approaches

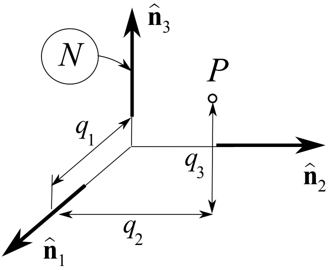
- momentum principles
- Newton-Euler method
- D'Alembert's principle
- Lagrange's equations
- Hamilton's canonical equations
- Boltzmann-Hamel equations
- Gibbs' equations
- **Kane's method**
  - Kane's equations have the simplest form and are derived with the least amount of labor[1]

[1] Kane, T. R., and Levinson, D. A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control*, Vol. 3, No. 2, 1980, pp. 99–112.

# Kane's Method, Single Particle



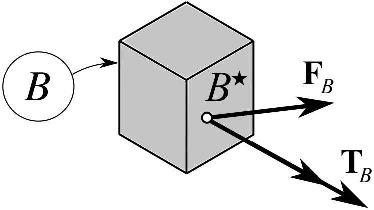

$$ {}^N\mathbf{v}_r^P \cdot \left(\mathbf{F}_P - m_P\,{}^N\mathbf{a}^P\right) = 0 \qquad (r = 1, 2, 3) \qquad (1) $$




The velocity of $P$ in $N$

$$
\begin{aligned}
{}^N\mathbf{v}^P &= \dot{q}_1\hat{\mathbf{n}}_1 + \dot{q}_2\hat{\mathbf{n}}_2 + \dot{q}_3\hat{\mathbf{n}}_3 \\
&\triangleq u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3
\end{aligned} \tag{2}
$$

- *Motion variables*, $u_r$, can be time derivatives of generalized coordinates, $q_r$
- *Partial velocities* are simply the vector coefficients of the motion variables in the expression for ${}^N\mathbf{v}^P$; that is, ${}^N\mathbf{v}_r^P = \hat{\mathbf{n}}_r \ (r = 1, 2, 3)$



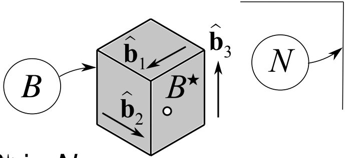

$$ {}^N\mathbf{v}_r^{B^\star} \cdot \left(\mathbf{F}_B - m_B\, {}^N\mathbf{a}^{B^\star}\right) + {}^N\omega_r^B \cdot \left(\mathbf{T}_B - \frac{{}^N d\, {}^N\mathbf{H}^{B/B^\star}}{dt}\right) = 0 \quad (r = 1, \ldots, 6) \quad (3) $$

- Contribution of $B$ to $r$th *generalized active force*:

$$ (F_r)_B = {}^N\mathbf{v}_r^{B^\star} \cdot \mathbf{F}_B + {}^N\omega_r^B \cdot \mathbf{T}_B $$

- Contribution of $B$ to $r$th *generalized inertia force*:

$$ (F_r^\star)_B = -\,{}^N\mathbf{v}_r^{B^\star} \cdot m_B\, {}^N\mathbf{a}^{B^\star} - {}^N\omega_r^B \cdot \frac{{}^N d\, {}^N\mathbf{H}^{B/B^\star}}{dt} $$



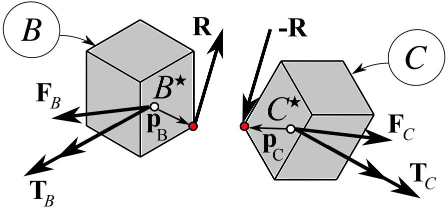

The velocity of $B^\star$ in $N$

$$^N\mathbf{v}^{B^\star} \triangleq u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{4}$$

The angular velocity of $B$ in $N$

$$^N\boldsymbol{\omega}^B = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \triangleq u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3 \tag{5}$$

- *Motion variables*, $u_4$, $u_5$, $u_6$, can be *linear combinations* of the time derivatives of generalized coordinates
- *Partial angular velocities* are simply the vector coefficients of the motion variables in the expression for $^N\boldsymbol{\omega}^B$; that is, $^N\boldsymbol{\omega}_r^B = \mathbf{0}$ $(r = 1, 2, 3)$, $^N\boldsymbol{\omega}_r^B = \hat{\mathbf{b}}_{r-3}$ $(r = 4, 5, 6)$




$$
\begin{aligned}
&{}^N\mathbf{v}_r^{B^\star} \cdot \left(\mathbf{F}_B + \mathbf{R} - m_B\, {}^N\mathbf{a}^{B^\star}\right) \\
&+ {}^N\omega_r^B \cdot \left(\mathbf{T}_B + \mathbf{p}_B \times \mathbf{R} - \frac{{}^N d\, {}^N\mathbf{H}^{B/B^\star}}{dt}\right) \\
&+ {}^N\mathbf{v}_r^{C^\star} \cdot \left(\mathbf{F}_C - \mathbf{R} - m_C\, {}^N\mathbf{a}^{C^\star}\right) \\
&+ {}^N\omega_r^C \cdot \left(\mathbf{T}_C - \mathbf{p}_C \times \mathbf{R} - \frac{{}^N d\, {}^N\mathbf{H}^{C/C^\star}}{dt}\right) = 0 \quad (r = 1, \ldots, 9)
\end{aligned}
\tag{6}
$$

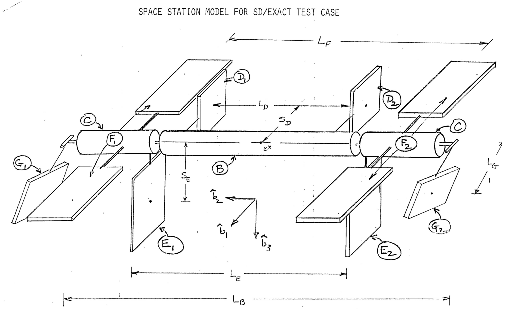
It can be shown that the constraint force **R** does not contribute to the equations of motion

# Advantages of Generalized Forces

- Generalized active forces
  - Constraint forces do not appear in Kane's equations of motion
    - Forces exerted on particles across smooth surfaces
    - Contact forces exerted by two bodies rolling on each other
  - Constraint forces do appear when using Newton-Euler or D'Alembert's method; extra work to eliminate them
  - If constraint forces are of interest, Kane shows how to bring them into evidence
- Generalized inertia forces
  - Forming Kane's generalized inertia forces is much easier than
    - Forming the system kinetic energy and then differentiating it (Lagrange's Eqs.)
    - Forming the Gibbs function and then differentiating it (Gibbs' method)

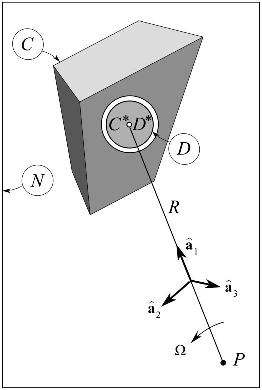
# Applications

Space Station Multi Rigid Body Simulation (SSMRBS) at JSC. Simulations for analysis of control moment gyroscope momentum management, and reaction control system propellant usage. Every configuration in the ISS assembly sequence.



# Spacecraft with Magnetic Damper[2]

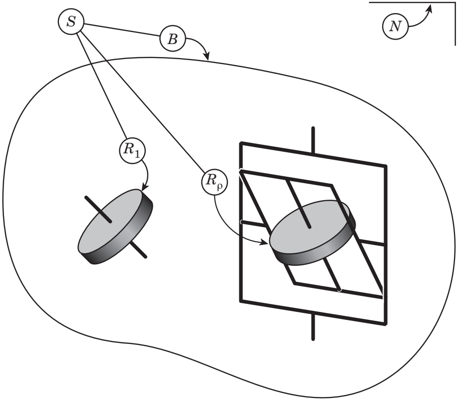
Simulations of attitude motion of gravity-gradient stabilized spacecraft containing a passive magnetic damper (ball with magnet inside spherical cavity, surrounded by viscous fluid). Long Duration Exposure Facility, Space Station Freedom.




[2] Roithmayr, C. M., Hu, A., and Chipman, R., "Motion of a Spacecraft with Magnetic Damper," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 4, 1996, pp. 980–982.

# Combined Attitude Control and Energy Storage

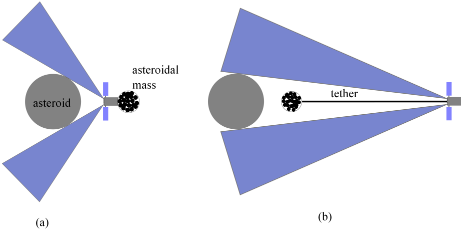
Simulations of combined control of attitude motion, momentum management, and energy storage on spacecraft carrying control moment gyroscopes and flywheels for energy storage.

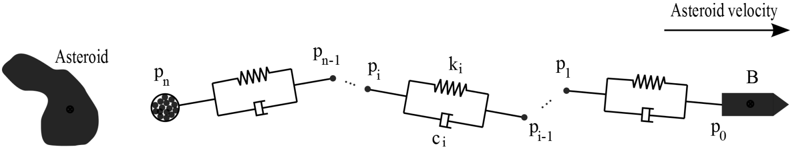



[3] Roithmayr, C. M., Karlgaard, C. D., Kumar, R. R., and Bose, D. M., "Integrated Power and Attitude Control with Spacecraft Flywheels and Control Moment Gyroscopes," *Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 5, 2004, pp. 859–873.

# Gravity Tractor[4]

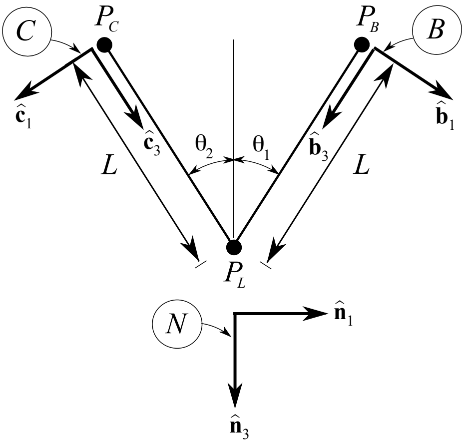
Dynamics and control of a tethered enhanced gravity tractor performing asteroid deflection. Equations of motion and simulation code for tether, modeled as particles connected by springs and dampers.




[4] Shen, H., Roithmayr, C. M., and Li, Y., "Dynamics and Control of a Tethered Enhanced Gravity Tractor Performing Asteroid Deflection," AAS Guidance and Control Conference, Breckenridge, CO, February 1–7, 2018.

Analysis of scissor-mode motion of a two-parachute system




[5] Pei, J., Roithmayr, C. M., Barton, R. L., and Matz, D. A., "Modal Analysis of a Two-Parachute System," 25th Aerodynamic Decelerator Conference, AIAA, 2019.

Equations of motion for a generic multibody tilt-rotor aircraft. (Many current simulations are based on single rigid body model and ignore multibody dynamics.)

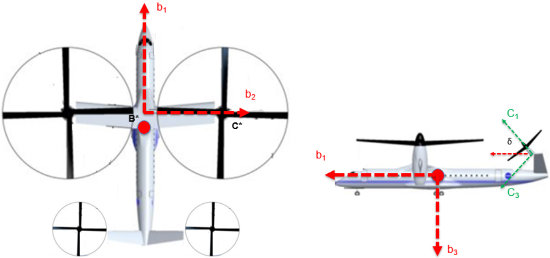
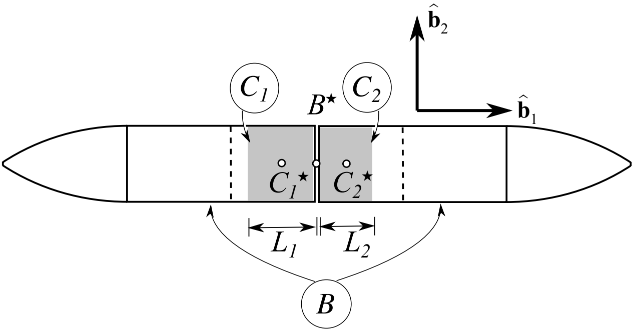



Image credit: https://rotorcraft.arc.nasa.gov/Research/Programs/LCTR.html

[6] Pei, J., and Roithmayr, C. M., "Equations of Motion for a Generic Multibody Tilt-rotor Aircraft," AIAA Aviation Forum, June 27, 2022.

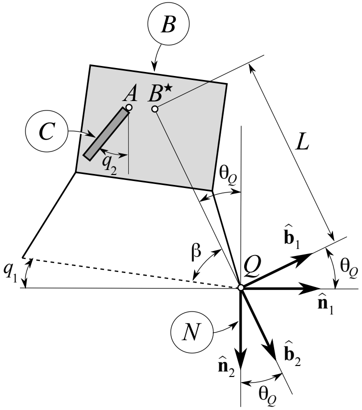
# On-Orbit Refueling[7]

Attitude dynamics of on-orbit refueling configurations. Correct application of the angular momentum principle accounts for the change over time in the stack's mass distribution, position of the center of mass, and terms associated with moving mass, in the attitude motion of two docked spacecraft.



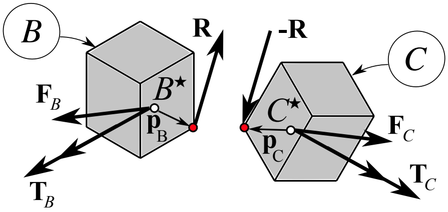

[7] Pei, J., and Roithmayr, C. M., "Attitude Dynamics of On-orbit Refueling Configurations," AAS 22-096, Astrodynamics Specialist Conference, Charlotte, NC, August 7–11, 2022.

Simulations of the effects of propellant slosh on the touchdown stability of landing vehicles. Collision dynamics, and equations of motion in between collisions.




[8] Roithmayr, C. M., and Pei, J., "Effects of Propellant Slosh on Touchdown Stability for Landing Vehicles," accepted for publication, *Journal of Spacecraft and Rockets*, 2023.

# Backup Charts




$$ {}^N\mathbf{v}^{C\star} = {}^N\mathbf{v}^{B\star} + {}^N\omega^{B} \times \mathbf{p}_B + {}^N\omega^{C} \times (-\mathbf{p}_C) $$

$$ {}^N\mathbf{v}_r^{C\star} = {}^N\mathbf{v}_r^{B\star} + {}^N\omega_r^{B} \times \mathbf{p}_B - {}^N\omega_r^{C} \times \mathbf{p}_C $$

$$ \begin{aligned} {}^N\mathbf{v}_r^{C\star} \cdot (-\mathbf{R}) &= -{}^N\mathbf{v}_r^{B\star} \cdot \mathbf{R} - {}^N\omega_r^{B} \times \mathbf{p}_B \cdot \mathbf{R} + {}^N\omega_r^{C} \times \mathbf{p}_C \cdot \mathbf{R} \\ &= -{}^N\mathbf{v}_r^{B\star} \cdot \mathbf{R} - {}^N\omega_r^{B} \cdot \mathbf{p}_B \times \mathbf{R} + {}^N\omega_r^{C} \cdot \mathbf{p}_C \times \mathbf{R} \end{aligned} $$

cf. Eqs. (6)

System $S$ is made of $\nu$ particles $P_i$, each of mass $m_i$ $(i = 1, \ldots, \nu)$, moving in a Newtonian reference frame $N$.

$$\mathbf{F}_1 = m_1 \, {}^N\mathbf{a}^{P_1} \tag{7}$$

$$\mathbf{F}_2 = m_2 \, {}^N\mathbf{a}^{P_2} \tag{8}$$

$$\ldots$$

$$\mathbf{F}_\nu = m_\nu \, {}^N\mathbf{a}^{P_\nu} \tag{9}$$

or, a single vector equation

$$\sum_{i=1}^{\nu} \left( \mathbf{F}_i - m_i \, {}^N\mathbf{a}^{P_i} \right) = \mathbf{0} \tag{10}$$

from which one can obtain a scalar equation

$$\sum_{i=1}^{\nu} \left( \mathbf{F}_i - m_i \, {}^N\mathbf{a}^{P_i} \right) \cdot \mathbf{v} = \mathbf{0} \cdot \mathbf{v} = 0 \tag{11}$$

where **v** is *any* vector

# Basic Statement of Kane's Method

For a holonomic system possessing $n$ degrees of freedom in frame $N$

$$\sum_{i=1}^{\nu} \left( \mathbf{F}_i - m_i \, {}^{N}\mathbf{a}^{P_i} \right) \cdot {}^{N}\mathbf{v}_r^{P_i} = 0 \qquad (r = 1, \ldots, n) \tag{12}$$

where ${}^{N}\mathbf{v}_r^{P_i}$ is called the $r$th *holonomic partial velocity* of particle $P_i$ in $N$. (More about how to find partial velocities later.)

Kane calls $F_r$ the $r$th *generalized active force* for $S$ in $N$, and defines it as

$$F_r \triangleq \sum_{i=1}^{\nu} \mathbf{F}_i \cdot {}^N\mathbf{v}_r^{P_i} \qquad (r = 1, \ldots, n) \tag{13}$$

$F_r^\star$ is the $r$th *generalized inertia force* for $S$ in $N$, defined as

$$F_r^\star \triangleq \sum_{i=1}^{\nu} -m_i\, {}^N\mathbf{a}^{P_i} \cdot {}^N\mathbf{v}_r^{P_i} \qquad (r = 1, \ldots, n) \tag{14}$$

Kane's Equations:

$$F_r + F_r^\star = 0 \qquad (r = 1, \ldots, n) \tag{15}$$

# Contribution of a Rigid Body to Generalized Active Forces

Let the set of contact forces and distance forces acting on a rigid body $B$ be equivalent to a single force $\mathbf{F}_B$ applied at the mass center, $B^\star$, together with a couple whose torque is $\mathbf{T}_B$.

The contribution of $B$ to $F_r$ is given by

$$(F_r)_B = {}^N\mathbf{v}_r^{B^\star} \cdot \mathbf{F}_B + {}^N\boldsymbol{\omega}_r^B \cdot \mathbf{T}_B \qquad (r = 1, \ldots, n) \tag{16}$$

where ${}^N\mathbf{v}_r^{B^\star}$ is the $r$th partial velocity of $B^\star$ in $N$, and ${}^N\boldsymbol{\omega}_r^B$ is the $r$th *partial angular velocity* of $B$ in $N$.

# Contribution of a Rigid Body to Generalized Inertia Forces

The contribution of $B$ to $F_r^\star$ is

$$(F_r^\star)_B = {}^N\mathbf{v}_r^{B\star} \cdot \mathbf{R}^\star + {}^N\omega_r^B \cdot \mathbf{T}^\star \qquad (r = 1, \ldots, n) \tag{17}$$

*Inertia force* for $B$ in $N$:

$$\mathbf{R}^\star \triangleq -m_B\, {}^N\mathbf{a}^{B\star} \tag{18}$$

where $m_B$ is the mass of $B$, and ${}^N\mathbf{a}^{B\star}$ is the acceleration in frame $N$ of the mass center of $B$.
*Inertia torque* for $B$ in $N$:

$$\mathbf{T}^\star \triangleq -\left(\underline{\mathbf{I}} \cdot {}^N\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^B \times \underline{\mathbf{I}} \cdot {}^N\boldsymbol{\omega}^B\right) \tag{19}$$

where $\underline{\mathbf{I}}$ is the inertia dyadic of $B$ for $B^\star$, ${}^N\boldsymbol{\omega}^B$ is the angular velocity of $B$ in $N$, and ${}^N\boldsymbol{\alpha}^B$ is the angular acceleration of $B$ in $N$.

When the configuration in $N$ of a system $S$ can be described with $n$ *generalized coordinates* $q_r$, one can define $n$ *motion variables* $u_r$ as linear combinations of the time derivatives of $q_r$,

$$u_r \triangleq \sum_{s=1}^{n} Y_{rs}\dot{q}_s + Z_r \qquad (r = 1, \ldots, n) \tag{20}$$

where $Y_{rs}$ and $Z_r$ $(r, s = 1, \ldots, n)$ are functions of $q_1, \ldots, q_n$ and the time $t$. Must be able to solve Eqs. (20) uniquely for $\dot{q}_1, \ldots, \dot{q}_n$.

One of the chief disadvantages of using Lagrange's equations is that state variables cannot be $u$'s and must be $\dot{q}$'s.

The velocity in any reference frame $A$ of a particle $P$ belonging to $S$ can be expressed uniquely in terms of motion variables and partial velocities ${}^A\mathbf{v}_r^P$,

$$ {}^A\mathbf{v}^P = \sum_{r=1}^{n} {}^A\mathbf{v}_r^P u_r + {}^A\mathbf{v}_t^P \tag{21} $$

The angular velocity in any reference frame $A$ of a rigid body $B$ belonging to $S$ can be expressed uniquely in terms of motion variables and partial angular velocities ${}^A\boldsymbol{\omega}_r^B$,

$$ {}^A\boldsymbol{\omega}^B = \sum_{r=1}^{n} {}^A\boldsymbol{\omega}_r^B u_r + {}^A\boldsymbol{\omega}_t^B \tag{22} $$

where ${}^A\mathbf{v}_r^P$, ${}^A\boldsymbol{\omega}_r^B$ $(r = 1, \ldots, n)$, ${}^A\mathbf{v}_t^P$, and ${}^A\boldsymbol{\omega}_t^B$ are functions of $q_1, \ldots, q_n$ and the time $t$.