Quantum Algorithms for lattice Φ^4 theory

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Motivation

Quantum Field Theories

- **Analytically solvable at weak coupling using perturbative methods**
  - **Some window into strong coupling via dualities (e.g. AdS/CFT)**

- **Lattice field theory**
  - **Non-perturbative formulation**
  - **Simulating dynamics is difficult (exponential size of Hilbert space)**
  - **Finite density is difficult (sign problem)**

Gate complexity estimates

T gate counts

Gauge Group	Transport Coefficients	Heavy-Ion Collisions
$U(1)$	7.35×10^{17}	7.19×10^{23}
$SU(2)$	2.83×10^{34}	3.71×10^{40}
$SU(3)$	3.01×10^{49}	3.22×10^{55}

(Age of the universe ~ $10^{17} s$)

[1] A. Kan and Y. Nam, "Lattice Quantum Chromodynamics and Electrodynamics on a Universal Quantum Computer", arXiv:2107.12769
[2] T. D. Cohen, H. Lamm, S. Lawrence, and Y. Yamauchi (NuQS Collaboration), Phys. Rev. D 104, 094514 (2021)

Scalar Field Theory

ϕ^4 Hamiltonian

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\overbrace{\frac{1}{2}\pi(\mathbf{x})^2 + \frac{1}{2}(\nabla\varphi(\mathbf{x}))^2 + \frac{1}{2}m^2\varphi(\mathbf{x})^2}^{H_0} + \underbrace{\lambda\varphi(\mathbf{x})^4}_{H_\varphi} \right]$$

Simplest field theory:

- Free theory -- relativistic energy-momentum relationship (Klein-Gordon equation)
- ϕ^4 term simplest self-interaction term bounded below
- Scalar fields in nature: Higgs boson, Inflaton(?)

Occupation Basis

Field expansion

$$\varphi(\mathbf{x}) = \sum_{\mathbf{p}\in\Gamma} \frac{1}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{x}} \sqrt{\frac{1}{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a^{\dagger}_{-\mathbf{p}}) \qquad \pi(\mathbf{x}) = \sum_{\mathbf{p}\in\Gamma} \frac{-i}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{x}} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - a^{\dagger}_{-\mathbf{p}})$$

$$[a_p, a_q^{\dagger}] = (2\pi)^d \delta(p-q)$$

Free part

$$: H_0 := \frac{1}{(2\pi)^d} \sum_p \omega_p a_p^{\dagger} a_p$$

Interacting part

$$: H_{\varphi} := \frac{\lambda}{4!} \frac{1}{(2\pi)^d} \sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4\in\Gamma} \delta\left(\sum_{i=1}^{4} \vec{p}_i\right) \prod_{\vec{p}=\vec{p}_1}^{\vec{p}_4} \frac{1}{\sqrt{2\omega_{\vec{p}}}} : \left(a_{\vec{p}} + a^{\dagger}_{-\vec{p}}\right) :$$

Normal ordering

$: a_p a_k^{\dagger} a_q : \equiv a_k^{\dagger} a_p a_q$ etc. (vanishing vacuum expectation value)

Occupation Basis

One-hot encoding

$P^d = V$ distinct momentum modes

M distinct momentum states

$$
\begin{aligned}
|p,n\rangle &= |0_{1,1},\ldots,0_{p-1,M},0_{p,1},\ldots,0_{p,n-1},1_{p,n},0_{p,n+1},\ldots,0_{p,M},0_{p+1,1},\ldots 0_{V,M}\rangle \\
&= \left(\otimes_{j=1}^{p-1}|0_{j,1}\ldots,0_{j,M}\rangle\right)\bigotimes|0_{p,1},\ldots 1_{p,n}\ldots,0_{p,M}\rangle\bigotimes\left(\otimes_{j=p+1}^{V}|0_{j,1}\ldots,0_{j,M}\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
a_p^\dagger &= \sum_n \sqrt{n+1}\left(\sigma_n^-\sigma_{n+1}^+\right)_p \\
a_p &= \sum_n \sqrt{n+1}\left(\sigma_n^+\sigma_{n+1}^-\right)_p,
\end{aligned}
$$

$$
\sigma^+ = \frac{1}{2}(X-iY), \quad \sigma_- = \frac{1}{2}(X+iY)
$$

Occupation Basis

Number operator

$$\hat{a}_p^\dagger \hat{a}_p = \hat{n}_p = \sum_n |1\rangle\langle 1|_{p,n} = \sum_n \left(\frac{I-Z}{2}\right)_{p,n}$$

Free Hamiltonian

$$: H_0 := \frac{1}{(2\pi)^d} \sum_p \omega_p a_p^\dagger a_p = \frac{1}{(2\pi)^d} \sum_{n,p} n\omega_p \left(\frac{I-Z}{2}\right)_{n,p}$$

$$e^{-i:H_0:t} \text{ requires } MV \text{ many } R_Z \text{ gates}$$

Occupation Basis

	$\#(c)R_z$	# T	# CNOT	# H
$e^{-iH_0 t}$	MV	-	-	-
$e^{-iH_{1\varphi} t}$	$\frac{M^4V^2(V-1)}{48}$	$\frac{M^4V^2(V-1)}{4}$	$\frac{11M^4V^2(V-1)}{16}$	$\frac{M^4V^2(V-1)}{24}$
$e^{-iH_{2\varphi} t}$	$\frac{M^3V^2}{3}$	$\frac{8M^3V^2}{3}$	$\frac{20M^3V^2}{3}$	$\frac{2M^3V^2}{3}$
$e^{-iH_{3\varphi} t}$	$2M^2V(V-1)$	$8M^2V(V-1)$	$16M^2V(V-1)$	$3M^2V(V-1)$
$e^{-iH_{4\varphi} t}$	$3MV$	-	$4MV$	$4MV$

$$p_3 = p_1 + k, \quad p_4 = p_2 - k$$

1) $p_1 \neq p_2, \; k \neq 0$

2) $p_1 = p_2, \; k \neq 0$

3) $p_1 \neq p_2, \; k = 0$

4) $p_1 = p_2, \; k = 0$

Linear Combination of Unitaries (LCU)

Given

$$A = \sum_{i=1}^{L} \alpha_i U_i \qquad \alpha_i > 0$$

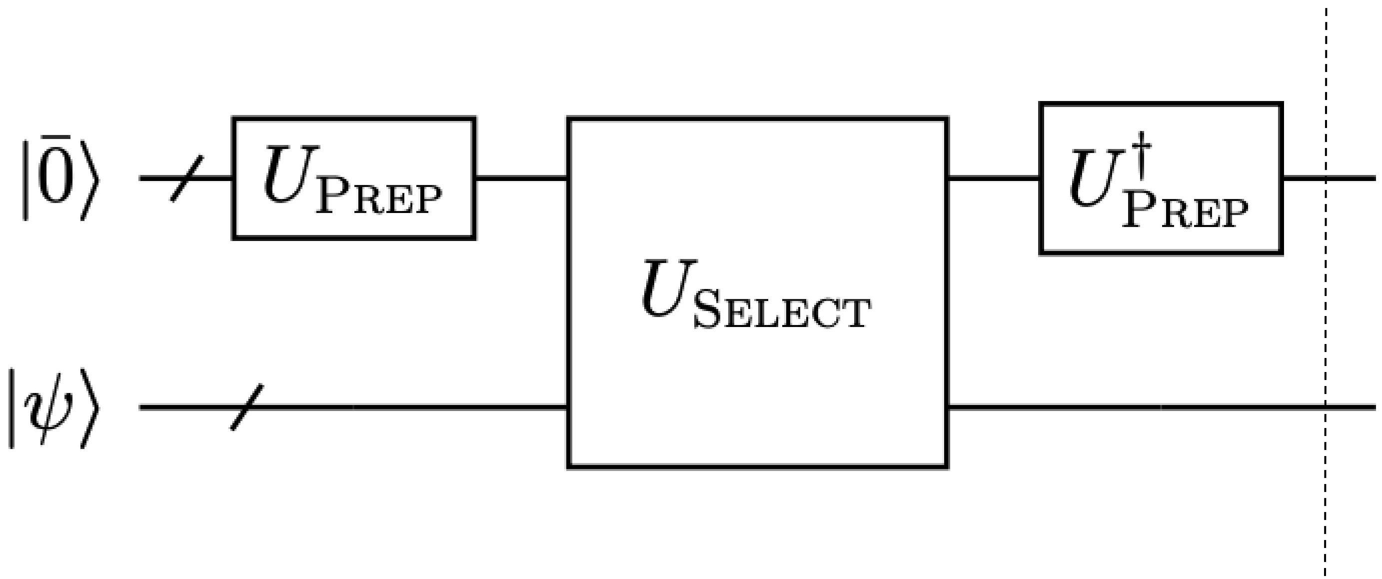
Construct

$$U_{PREP}|\bar{0}\rangle = \frac{1}{\sqrt{|\alpha|_1}} \sum_{i=1}^{L} \sqrt{\alpha_i}|i\rangle \qquad |\alpha|_1 = \sum_{i=1}^{L} \alpha_i$$

$$U_{SELECT} = \sum_{i=1}^{L} |i\rangle\langle i| \otimes U_i$$

"Hamiltonian Simulation Using Linear Combinations of Unitary Operations", A. M. Childs, N. Wiebe, Quantum Information and Computation 12, 901-924 (2012)

Linear Combination of Unitaries (LCU)



Success probability

$$\frac{||A|\psi\rangle||^2}{||\alpha||_1^2}$$

$$\frac{1}{|\alpha|_1}|\bar{0}\rangle \otimes A|\psi\rangle + |\Phi^{\perp}\rangle$$

$$\left(|\bar{0}\rangle\langle\bar{0}| \otimes \mathbb{I}\right)|\Phi^{\perp}\rangle = 0$$

Scalar Field Theory

ϕ^4 Hamiltonian

$$H = \sum_{\vec{x}\in\Omega}\left[\frac{1}{2}\pi^2(x) + (m^2 + D + 1)\phi^2(\vec{x}) + \frac{\lambda}{4!}\phi^4(\vec{x}) - 2\sum_{i=1}^{D}\phi(\vec{x})\phi(\vec{x}+a\hat{x}_i)\right]$$

Rescaled variables

$$\phi = a^{\frac{D}{2}-1}\Phi, \qquad \pi = a^{\frac{D}{2}}\Pi,$$
$$m = aM, \qquad \lambda = a^{4-D}\Lambda$$

Field amplitude basis

$$\hat{\phi}|\phi\rangle = \phi|\phi\rangle$$

$$\hat{\phi}/\Delta\phi = \begin{pmatrix} -k+1 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & k \end{pmatrix}$$

LCU with unequally sized groups of terms

$$\hat{H} = \alpha_0 \left(U_0^{(0)} + \cdots + U_{N_0-1}^{(0)} \right) + \cdots + \alpha_{M-1} \left(U_0^{(M-1)} + \cdots + U_{N_{M-1}-1}^{(M-1)} \right)$$

Let $N_{max} = \max\{N_i\}_{i=0}^{M-1}$,

$$U_{PREP} \left|\overline{0}\right\rangle = \frac{1}{\sqrt{|\alpha|_1 N_{max}}} \left(\sum_{i=0}^{M-1} \sqrt{\alpha_i N_i} \left|i\right\rangle \right) \left(\sum_{m=0}^{N_i-1} \left|m\right\rangle \right) \left(\sum_{k=0}^{\frac{N_{max}}{N_i}-1} \left|k\right\rangle \right)$$

$$U_{SELECT} = \sum_{i=0}^{M-1} \left|i\right\rangle \left\langle i\right| \otimes U_{SUB-SELECT_i}$$

$$U_{PREP}^{\dagger} U_{SELECT} U_{PREP} \left|\overline{0}\right\rangle \left|\psi\right\rangle = \left|\overline{0}\right\rangle \frac{\hat{H}}{|\alpha|_1} \left|\psi\right\rangle$$

LCU decompositions

Equal weight LCU

$$\frac{\hat{\phi}}{\Delta\phi} = \frac{1}{2}\sum_{m=0}^{2k-1} U^{(m)}$$

$$U^{(m)} = -\sum_{i=0}^{m-1} |i\rangle\langle i| + \sum_{i=m}^{2k-1} |i\rangle\langle i| = \sum_{i=0}^{2k-1} \left[2\Theta(i-m) - 1\right] |i\rangle\langle i|$$

$$\hat{\phi}/\Delta\phi = \begin{pmatrix} -k+1 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & k \end{pmatrix}$$

Unequal weight LCU[1]

$$\frac{\hat{\phi}}{\Delta\phi} = \frac{1}{2}\mathbb{I} - \frac{1}{2}\sum_{j=0}^{\log_2 2k-1} 2^j Z_j$$

$O\left(|\Omega| \log_2^4 k\right)$ T gates

$$\#\text{qubits} \sim O(|\Omega| \log_2 k)$$

(either way)

$$n = \frac{1}{2}\left(\underbrace{+1\cdots+1}_{k+n} \quad \underbrace{-1\cdots-1}_{k-n}\right)$$

[1] "Quantum Algorithms for Simulating the Lattice Schwinger Model", A. F. Shaw, P. Lougovski, J. R. Stryker, N. Wiebe, Quantum 4, 306 (2020)

Equal weight LCU

Use comparator (adapted from Gidney adder[1])

$$\texttt{CMP}_{A,B,C}\,|i\rangle\,|j\rangle\,|0\rangle \;=\; |i\rangle\,|j\rangle\,|j<i\rangle \;=\; |i\rangle\,|j\rangle\,|\Theta(i-j-1)\rangle$$

$O(|\Omega|\log_2 k)$ T gates

to simplify

$$U_{PREP_\phi^\dagger} U_{SELECT_\phi} U_{PREP_\phi} |\bar{0}\rangle|\psi\rangle|0\rangle_{anc}$$

$$= \left(H^{\otimes m}\texttt{CMP}^\dagger Z_{anc}\texttt{CMP}\,H^{\otimes m}\right)|\bar{0}\rangle|\psi\rangle|0\rangle_{anc}$$

$$= \left(|\bar{0}\rangle\frac{\hat{\phi}}{\phi_{max}}|\psi\rangle + |\Phi^\perp\rangle\right)|0\rangle_{anc}$$

[1] "Halving the cost of quantum addition", Craig Gidney, Quantum 2, 74 (2018)

Equal weight LCU

Other terms in the Hamiltonian

$$\left(\frac{\hat{\phi}}{\Delta\phi}\right)^2 = \frac{1}{2}\sum_{i=0}^{2k^2-1} U^{(i)}, \quad \text{where}$$

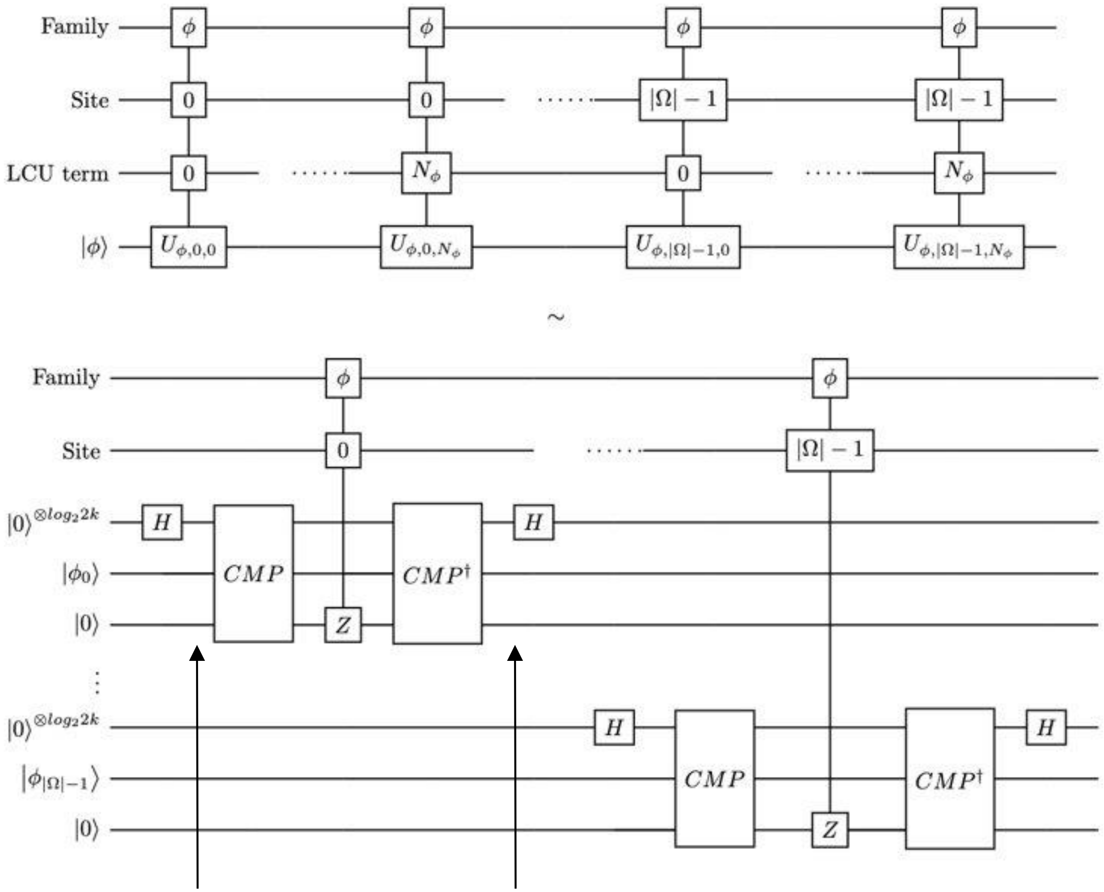
$$\pi^2 = \mathcal{F}^\dagger \phi^2 \mathcal{F}$$

$$U^{(i)} = \sum_{j=0}^{2k-1} \left[2\Theta(k^2 + (k-j-1)^2 - i - 1) - 1\right] |j\rangle\langle j|$$

$$\left(\frac{\hat{\phi}}{\Delta\phi}\right)^4 = \frac{1}{2}\sum_{i=0}^{2k^4-1} U^{(i)}, \qquad \text{where}$$

$$U^{(i)} = \sum_{j=0}^{2k-1} \left[2\Theta(k^4 + (k-j-1)^4 - i - 1) - 1\right] |j\rangle\langle j|$$

Equal weight LCU



$O(|\Omega| \log_2^2 k)$ T gates

Insert $U_{initial}$ . . . and its inverse for ϕ^2, π^2, ϕ^4
(involves arithmetic and Fourier transform)

T gate complexity

$$\mathrm{Count}(\mathrm{T})_{\phi\phi} \sim O\left(|\Omega| D \log_2\left(|\Omega| D k\right)\right)$$

$$\mathrm{Count}(\mathrm{T})_{\phi^2,\pi^2,\phi^4} \sim O\left(|\Omega|(\log_2^2 k + \log_2 |\Omega|)\right)$$

$$k = \frac{\phi_{max}}{\Delta\phi} \qquad |\Omega| : \text{ Lattice size} \qquad D : \text{ spatial dimensionality}$$

Thank you!

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