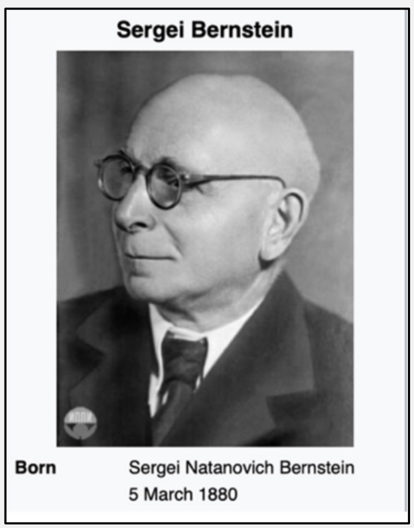
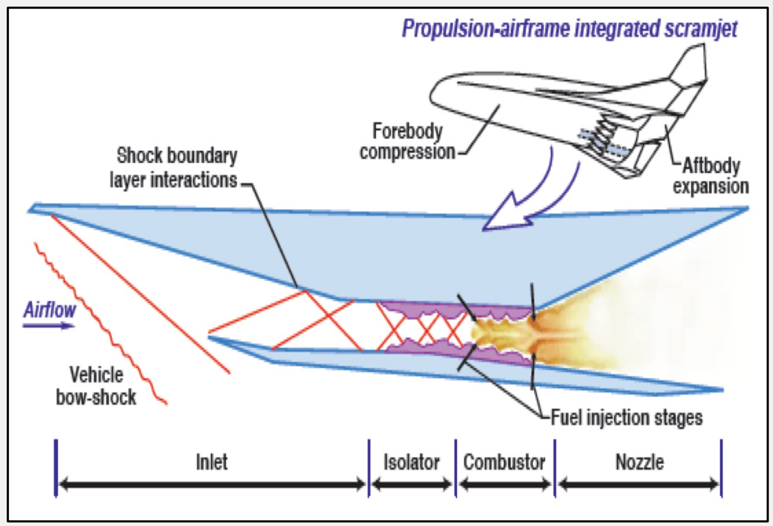
Constrained Field Reconstruction Using Bernstein Polynomial Derived Operators

Aaron H. Auslender, Robin L. Hunt and Louis M. Edelman
NASA Langley Research Center
Hampton, Virginia 23681-2199





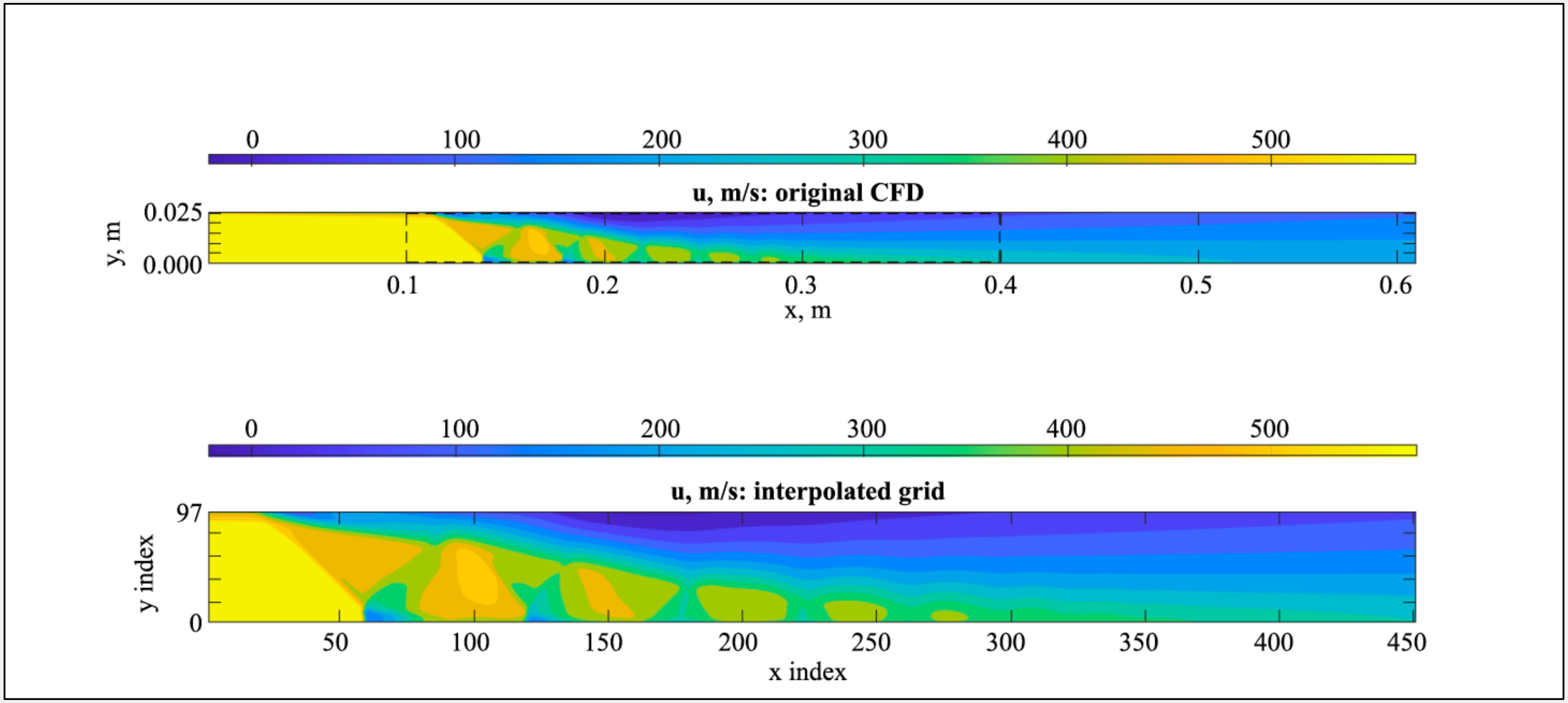
Outline

- Introduction
- Equation Formulation
- Solution Methodology
- Results for the Mach-2.0 Normal Shock Canonical Cases
- Results for the Mach-2.5 CFD Reconstruction Case
- Conclusions

Introduction

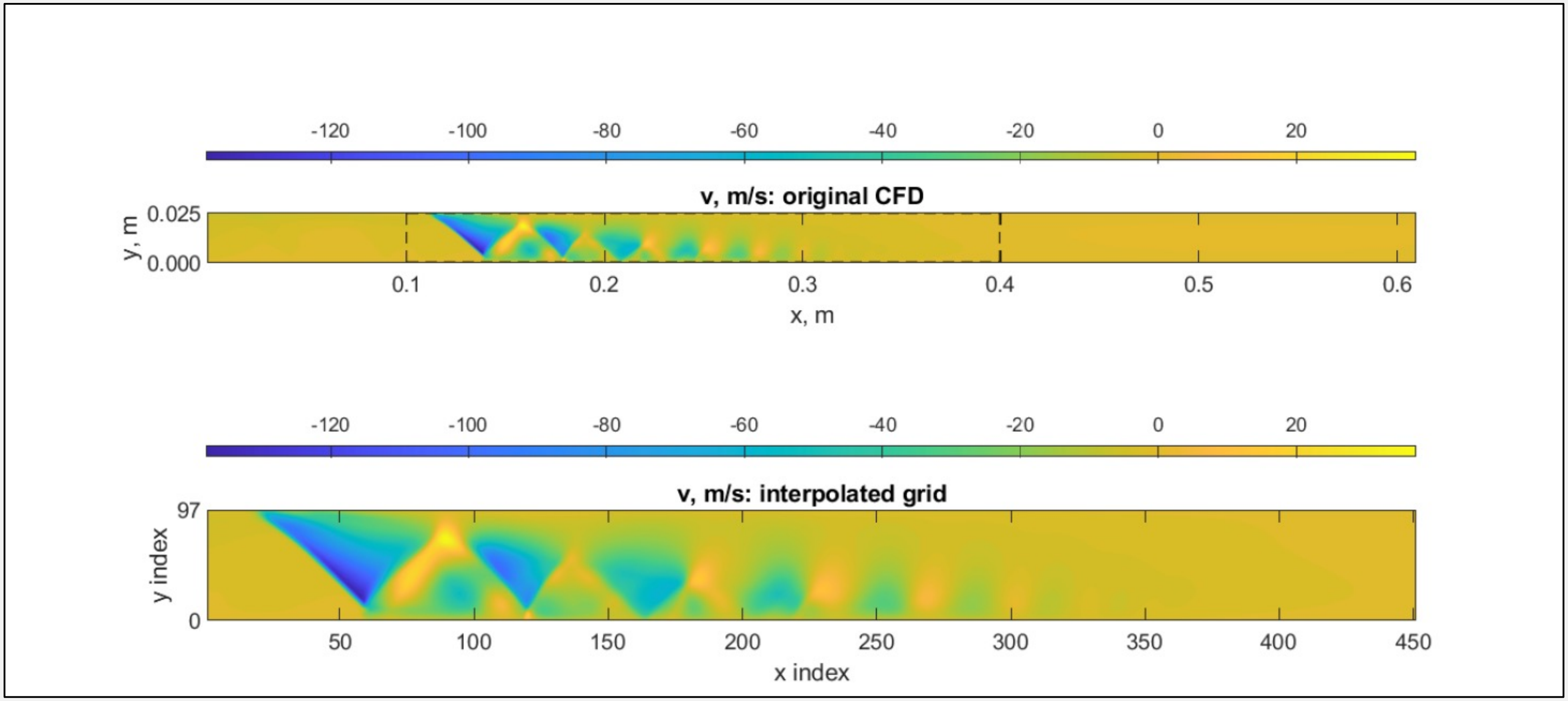
- An algorithm for reconstructing a density field from a prescribed velocity field is presented. This technique seeks to supplement existing diagnostic measurement methods, so that complex fluid-dynamic structure(s) within the isolator component of a hypersonic engine can be more thoroughly characterized and studied.
- This method recasts the mass-conservation equation as a modified Sylvester equation employing high-order operators derived from modified Bernstein polynomial expansions.
- To demonstrate its practical utility, this analytic methodology is applied to two canonical cases and a CFD-derived dataset obtained from a Mach-2.5, back-pressured isolator simulation.

(VULCAN-CFD Axial-Velocity Generated Dataset)



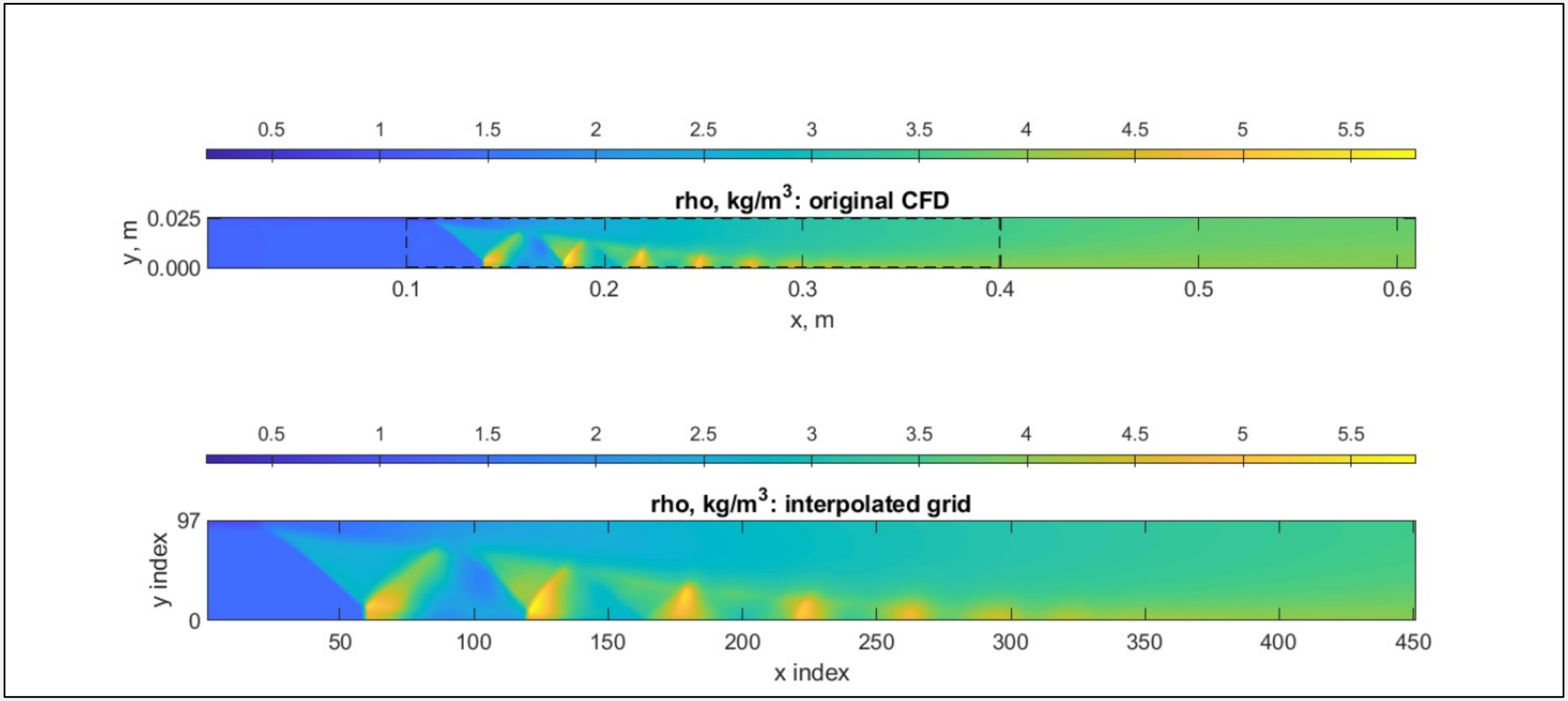
Isolator Dynamics Research Laboratory Computationally Derived U-Velocity Field

(VULCAN-CFD Transverse-Velocity Generated Dataset)



Isolator Dynamics Research Laboratory Computationally Derived V-Velocity Field

(VULCAN-CFD Density Generated Dataset)



Isolator Dynamics Research Laboratory Computationally Derived Density Field

Equation Formulation

- Bernstein polynomials have attributes of positivity, continuity, and form a partition of unity over a closed interval.
- Bernstein polynomials are highly compatible with the construction of derivative operators, a feature that is exploited by this numerical reconstruction technique.

$$(1) \quad \mathfrak{J}_B(x) \equiv \sum_{i=0}^{n} F\left(a + \left(\frac{i}{n}\right)(b-a)\right)\binom{n}{i} \left(\frac{x-a}{b-a}\right)^{(i)} \left(\frac{b-x}{b-a}\right)^{(n-i)} \equiv \sum_{i=0}^{n} F_{(i,n)}\, B_i^n(x)$$

$$(2) \quad a \leq x \leq b$$

$$(3) \quad n = \frac{(b-a)}{\Delta x}$$

- Various relationships involving the derivative operator are available.

- In this study, exact-derivative operator stencils are constructed employing the corresponding Bernstein-derivative operators.

$$(4)\quad \mathfrak{I}_B^{(q)}(x) = \frac{d^q}{dx^q}\sum_{i=0}^{n} F_{(i,n)}\, B_i^n(x) = \sum_{i=0}^{n} F_{(i,n)} \frac{d^q}{dx^q} B_i^n(x) = \sum_{i=0}^{n} a_{(i,n)}^{(q)}\, B_i^n(x)$$

$$(5)\quad \mathfrak{I}_B^{(q)}(x) = \sum_{i=0}^{n} a_{(i,n)}^{(q)}\, B_i^n(x) = \sum_{i=0}^{n} F_{(i,n)}^{(q)}\, B_i^n(x) \approx \sum_{i=0}^{n} \mathfrak{I}_{B}{}_{(i,n)}^{(q)}\, B_i^n(x)$$

$$(6)\quad \mathfrak{I}_{B}{}_{(i,n)}^{(q)} \approx a_{(i,n)}^{(q)} = \sum_{k=-q}^{q}\sum_{m=0}^{q} \frac{q!\,(-1)^{(m+q)}}{(n\,\Delta x)^q} \binom{q}{m}\binom{i}{m+k}\binom{n-i}{q-(m+k)} F_{(i-k,n)}$$

$$(7)\quad \mathfrak{I}_{B}{}_{(i,n)}^{(q)} \approx \sum_{k=-q}^{q}\sum_{m=0}^{q} \frac{q!\ (-1)^{(m+q)}}{(n\,\Delta x)^q} \binom{q}{m}\binom{i}{m+k}\binom{n-i}{q-(m+k)} \sum_{s=0}^{N\to\infty} \frac{(-k)^{(s)}\ (\Delta x)^{(s)}}{s!} F_{(i,n)}^{(s)}$$

Solution Methodology

- Approximations to the exact spatial derivative operators are explicitly available via a matrix inversion. This information is the crux of the algorithm.

$$(8) \quad \begin{pmatrix} \mathfrak{J}B_{(i,n)}^{(0)} \\ \mathfrak{J}B_{(i,n)}^{(1)} \\ \vdots \\ \mathfrak{J}B_{(i,n)}^{(N)} \end{pmatrix} \approx [\Pi_{(i,n)}^{(N,\Delta x)}] \begin{pmatrix} F_{(i,n)}^{(0)} \\ F_{(i,n)}^{(1)} \\ \vdots \\ F_{(i,n)}^{(N)} \end{pmatrix}$$

- Applying the proposed form of the spatial derivative operator to the mass-conservation equation results in a modified Sylvester equation.

$$(11) \quad \left[[X_{(\rho_L,\rho_R)}] \circ [U_{CFD}]\right]\left[D_{(x)}^{(B)}\right] + [D_{(y)}^{(B)}]^T \left[[X_{(\rho_L,\rho_R)}] \circ [V_{CFD}]\right] = 0$$

$$(10) \quad \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

Solution Methodology

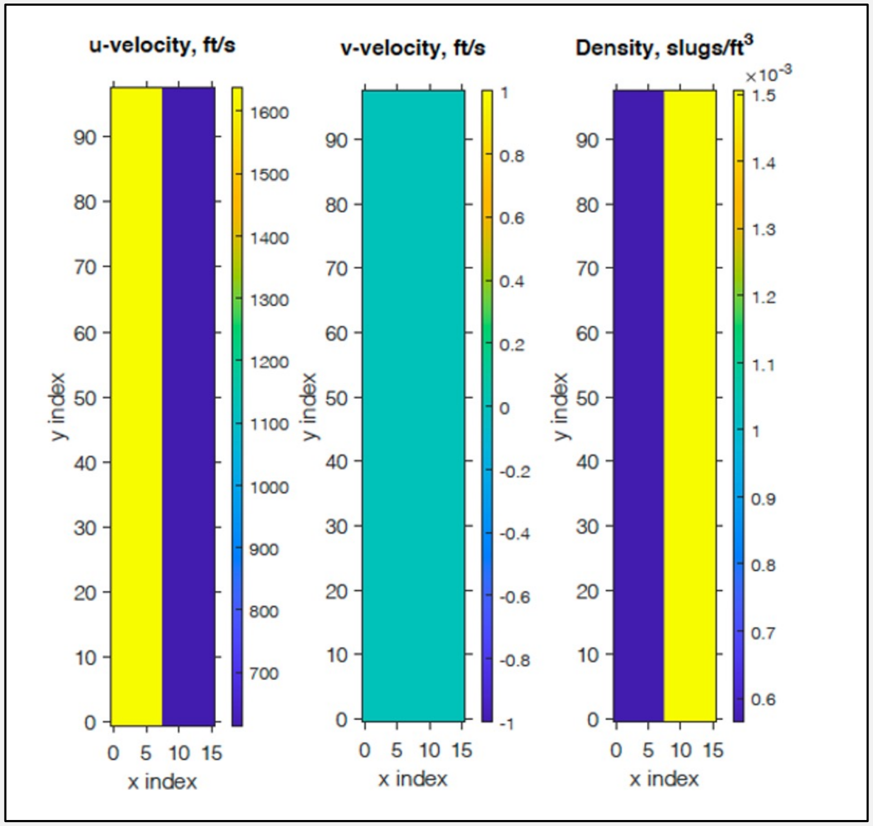
- The density field is obtained by solving for the least squares best-fit, two-dimensional field corresponding to the prespecified two-dimensional velocity field.

$$(13)\quad \left\| \begin{pmatrix} [M_R] \\ \mu_1 \left[[D^{(B)}_{(y)}]^T \, [D^{(B)}_{(y)}]^T \, [D^{(B)}_{(y)}]^T \right] \\ \mu_2 \, [I] \end{pmatrix} \rho_R - \begin{pmatrix} [-M_L] \, \rho_L \\ 0 \\ 0 \end{pmatrix} \right\|_2^2 \equiv \| \, [A] \, \rho_R - b \, \|_2^2 \;\; \text{st} \;\; [C] \, \rho_R = \text{Max}(d, d_{\min})$$

$$(14)\quad [A^T A]^{-1} = \left[[Q] \;\; [\Sigma^+]^{-1} \, [Q]^T \right]_{SVD}$$

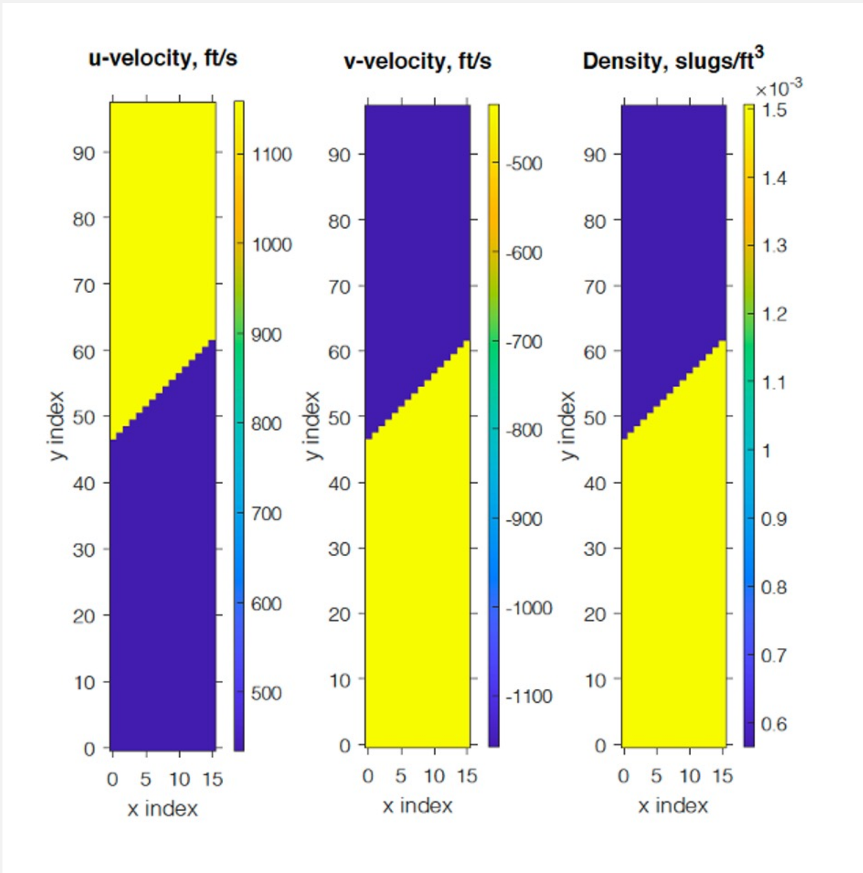
$$(15)\quad (\hat{\rho}_R)_{(\mu_1, \mu_2)} = [A^T A]^{-1} \, (A^T b) + [[A^T A]^{-1} \, [C]^T] \;\; [[C] \;\; [A^T A]^{-1} \, [C]^T]^{-1} \, \big(d - [C] \, [A^T A]^{-1} \, (A^T b) \big)$$

Results for the Mach-2.0 Normal Shock Canonical Cases (Grid-Aligned Mach-2.0 Normal Shock Case)



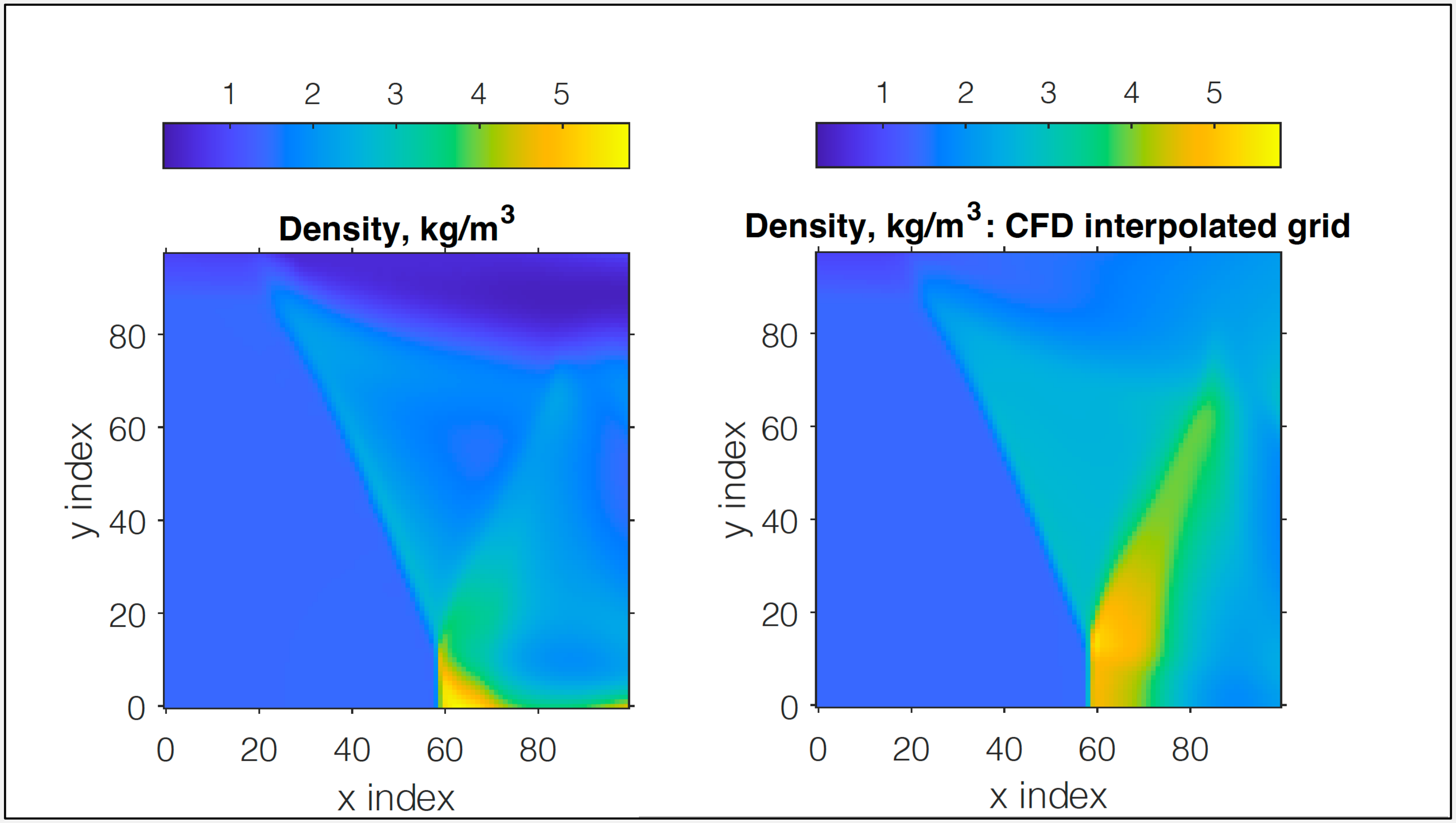
- A Mach-2 normal shock with a constant specific-heat-ratio of 1.40 generates a density ratio of 2.667, as do the reconstructed solutions.

Results for the Mach-2.0 Normal Shock Canonical Cases (Non-Grid-Aligned Mach-2.0 Normal Shock Case)



- The non-grid-aligned case has a prescribed 45-degree rotation of the planar normal shock front, as well as specified y-axis intercept.

Results for the Mach-2.5 CFD versus Analytic Reconstructed Density Fields



Reconstructed Density Field vs Computational Data

Conclusions

- The derived density field is engineeringly acceptable but lacks a rigorous "match" to the CFD density field . Note that the two solutions are not derived using the same spatial resolution, with the CFD being significantly more resolved prior to the mapping procedure (to the coarser grid). An additional consequence of the mapping procedure is the loss of information associated with the reduction in boundary condition points (along the centerline and the solid surfaces). This effectively alters the span to the column space of the $[A]$ matrix, which ultimately dictates which solutions are obtainable.

- One potentially germane option is to investigate the "splitting" of the $[A]$ matrix into a low-rank matrix and a sparse matrix to access a more suitable inverse problem. Another alternative is to perform the minimization using alternative matrix norms. Clearly, numerous options need to be explored if higher fidelity reconstructions are to be obtained in a reliable manner.