Information-theoretic aspects of scrambling and chaos
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Goals and references

- Goal: to connect "out-of-time-ordered correlators" with as many "information-theoretic" quantities as we can, in hopes to better understand what the OTOC actually measures.

- Today's talk:
    1. Information Scrambling over Bipartitions: Equilibration, Entropy Production, and Typicality
       [Styliaris, Anand, Zanardi; Phys. Rev. Lett. 126, 030601 (2021)]
    2. Information scrambling and chaos in open quantum systems
       [Zanardi, Anand; Phys. Rev. A 103, 062214 (2021)]
    3. BROTOCs and Quantum Information Scrambling at Finite Temperature
       [Anand, Zanardi; Quantum 6, 746 (2022)]
- Related works:
    4. Quantum coherence as a signature of chaos
       [Anand, Styliaris, Kumari, Zanardi; Phys. Rev. Research 3, 023214 (2021)]
    5. Scrambling of Algebras in Open Quantum Systems
       [Andreadakis, Anand, Zanardi; Phys. Rev. A 107, 042217 (2023), Editors' Suggestion]
    6. Scrambling and operator entanglement in local non-Hermitian quantum systems
       [Barch, Anand, Marshall, Rieffel, Zanardi; Phys. Rev. B 108, 134305 (2023), Editors' Suggestion]

Thermalization in closed quantum systems

- Unitary dynamics “preserves” information (about the initial state).
  - E.g., perfect distinguishability: $\langle \psi(0) | \phi(0) \rangle = 0 \implies \langle \psi(t) | \phi(t) \rangle = 0 \quad \forall t \geq 0.$

- Question: so how can closed quantum systems thermalize?
  - Idea: bipartition in the system $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B \implies$ the rest of the system acts as a bath $\implies$ thermalization.
  - **Thermalization**: Equilibration, bath state independence, subsystem state independence, Boltzmann form, etc.
  - **Equilibration**: A system equilibrates if $|\psi(t)\rangle$ evolves towards “some” state and remains “close” to it for almost all times.

- Equilibration of local observables: Let $\rho(t) = \sum \rho_{mn}(0) e^{i[E_n - E_m]t/\hbar} | m \rangle \langle n |$ then if the ‘time-averaged’ variance of an observable A, $\sigma_A^2 := \overline{\left[\mathrm{Tr}\{\rho(t) A\} - \mathrm{Tr}\left\{ \rho_{eq} A \right\} \right]^2}$ is exponentially small then we have ‘apparent equilibration’.

- Formally, given a Hamiltonian H with ‘non-degenerate energy gaps’ + all eigenstates are Haar random $\implies$ every local initial state will equilibrate.
  - $\overline{D\left(\rho_S(t), \omega_S\right)} \leq \frac{d_S}{2}\left(\frac{1}{d^{\mathrm{eff}}(\omega)}\right)$ where $D(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1$ and $1/d^{\mathrm{eff}}(\omega) = \mathrm{Tr}\left[\omega^2\right]$.

Footnotes/references:

1. Peter Reimann Phys. Rev. Lett. 101, 190403 (2008) and Linden et. al, Phys. Rev. E 79:061103 (2009)

Out-of-time-ordered correlators

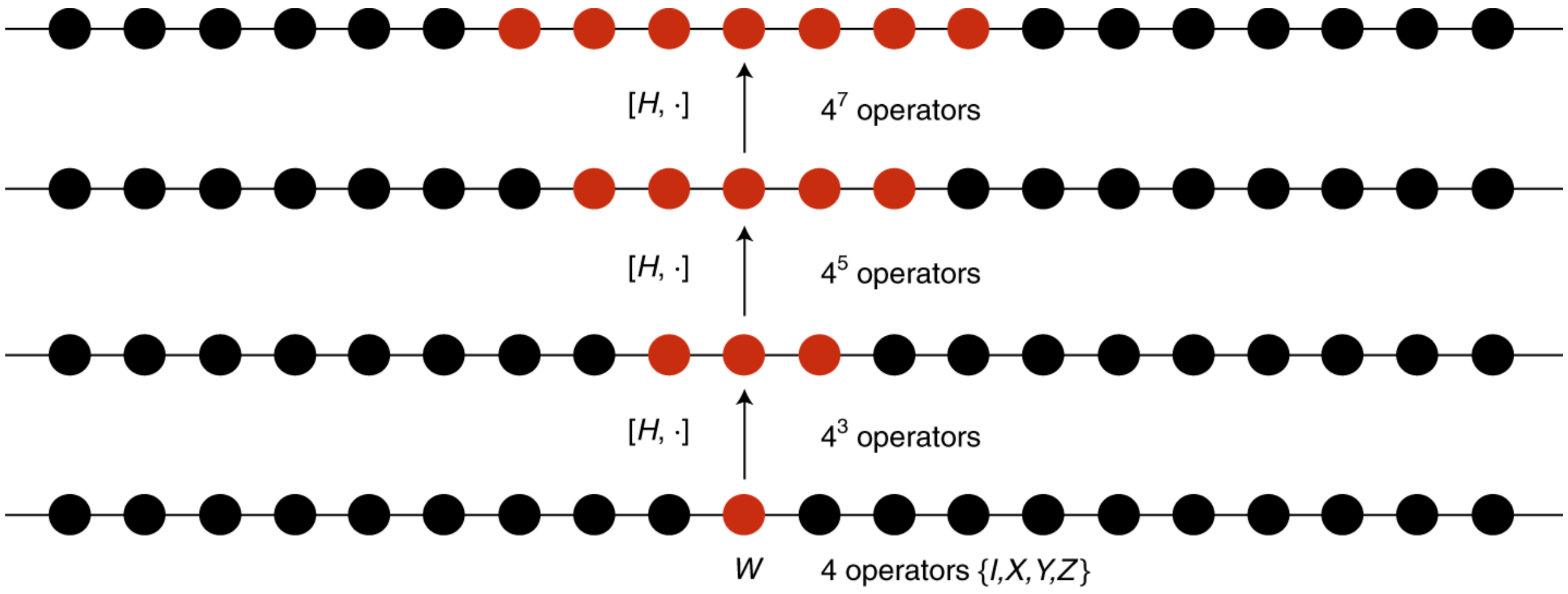
- Given two (local) operators V and W and the time-evolution operator $U_t = e^{-iHt}$ consider the following quantity:
  - $C_{V,W}(\beta, t) := \frac{1}{2}\,\mathrm{Tr}\left([V(t), W]^\dagger [V(t), W]\rho_\beta\right),$
  - Here $\rho_\beta := e^{-\beta H}/Z(\beta)$ is the Gibbs state an inverse temperature β. And $V(t) := U_t^\dagger V U_t$ is the Heisenberg evolved operator.
  - Too complicated…we would like to 'simplify' this quantity.
  - Assume $\beta = 0$ and V, W are unitary. Recall $\|A\|_2^2 := \mathrm{Tr}\left[A^\dagger A\right]$ is the Hilbert-Schmidt norm.
  - $$C_{V,W}(t) = \frac{1}{2d}\|[V(t), W]\|_2^2 = 1 - \underbrace{\frac{1}{d}\mathrm{Re}\,\mathrm{Tr}\left[V^\dagger(t)W^\dagger V(t)W\right]}_{\text{OTOC}} = 1 - \frac{1}{d}\sum_{j=1}^{d}\mathrm{Re}\left\langle j\,|\,V^\dagger(t)W^\dagger V(t)W\,|\,j\right\rangle.$$
  - 'Four point correlation function' that is *not* time-ordered, hence OTOC.
- As a 'generalized' Loschmidt echo (return/survival probability):
  - Recall the Loschmidt echo is defined as, $L(t) := \left|\langle\psi|\,e^{i(H+\epsilon K)t}e^{-iHt}\,|\psi\rangle\right|^2$ where K is a perturbation.
  - $$F_{V,W}(t) = \langle\psi|\,V^\dagger(t)W^\dagger V(t)W\,|\psi\rangle = \underbrace{\langle\psi|\,U_t^\dagger V^\dagger U_t W^\dagger}_{\langle\xi(t)|}\;\vdots\;\underbrace{U_t^\dagger V U_t W\,|\psi\rangle}_{|\chi(t)\rangle}$$

Footnotes/references:

1. Unscrambling the physics of out-of-time-order correlators, Brian Swingle, Nature Physics volume 14, 988–990 (2018).

Operator growth of local operators under local Hamiltonian

- 'Nested commutators' in the Heisenberg picture: $W(t) = e^{iHt} W e^{-iHt} = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} [H, \cdots, [H, W], \cdots]$.



- Notice a 'lightcone' in the operator dynamics above: this follows from Lieb-Robinson bounds.
  - for any locally interacting lattice system, there exist positive constants ξ, μ, v_{LR} such that for any two operators a, b, the following bound holds: $\|[a(t), b]\| \leq \xi \min\{\,|\mathrm{supp}(a)|, |\mathrm{supp}(b)|\,\} \|a\| \|b\| e^{-\mu \max\{0,\ \mathrm{d}(\mathrm{supp}(a), \mathrm{supp}(b)) - v_{LR} t\}}$

Footnotes/references:

1. Unscrambling the physics of out-of-time-order correlators, Brian Swingle, Nature Physics volume 14, 988–990 (2018).

Intuitive/‘physics inspired’ remarks

- The faster the ‘operator growth’ the faster information scrambling.
  - Intuition: if the unitary dynamics generates fast ‘local operator entanglement’ then we will have fast scrambling

- Relationship with integrability/chaos:
  - Scrambling rates[1]: Random matrix $\gg$ Chaotic $\gg$ Integrable models.
  - Unfortunately, this is incorrect. The ‘short-time’ growth of the OTOC is not a reliable indicator of integrability vs. chaoticity.

- “Weak quantum chaos”:
  - Recall ‘weak chaos’ in classical systems refers to ergodic systems that only have a **polynomial** (as opposed to exponential) sensitivity to initial conditions.
  - Weak quantum chaos refers to local quantum many-body systems that are quantum chaotic (in the sense of spectral statistics or decay of Loschmidt echoes) but **do not** have exponential growth of the OTOC (commutator).

Footnotes/references:
1. Intuitively one expects this but this is not necessarily true, especially for lattice systems.

Operator entanglement of linear operators

- How do we quantify entanglement of pure bipartite states?
  - Schmidt decomposition: $|\psi\rangle \in \mathscr{H}_{AB}$ then there exist orthonormal bases $|j_A\rangle \in \mathscr{H}_A$ and $|j_B\rangle \in \mathscr{H}_B$ such that $|\psi\rangle = \sum_j \sqrt{\lambda_j}\, |j_A\rangle\, |j_B\rangle$ with $\sum_j \lambda_j = 1$.
  - Von Neumann entropy of reduced state = Shannon entropy of $\{\lambda_j\}$.
- Can we do the same for operators?
  - Yes! Operator space is *also* a Hilbert space (equipped with Hilbert-Schmidt inner product).
  - $X = \sum_j \sqrt{\lambda_j} V_j \otimes W_j$ where $\langle V_j, V_k \rangle = d_A \delta_{jk}$ and $\langle W_j, W_k \rangle = d_B \delta_{j,k}$.
  - Normalization: $\sum_j \lambda_j = \frac{1}{d} \|X\|_2^2 = \frac{1}{d} \mathrm{Tr}\left[X^\dagger X\right]$. For unitaries $= 1$.
  - Linear entropy $S_{\mathrm{lin}}(\rho) = 1 - \mathrm{Tr}\left[\rho^2\right]$ of the 'probability distribution' obtained from $\{\lambda_j\}$ gives us the operator entanglement. I.e., operator entanglement of a unitary U is obtained as $1 - \|\vec{\lambda}\|^2$.
- Analytical formula with 'local swap' operators:
  - $E_{\mathrm{op}}(U) = 1 - \dfrac{1}{d^2} \mathrm{Tr}\left[\mathbb{S}_{AA'} U^{\otimes 2} \mathbb{S}_{AA'} U^{\dagger \otimes 2}\right]$ where $\mathscr{H} \cong \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_{A'} \otimes \mathscr{H}_{B'}$ and $\mathbb{S}_{AA'}$ swaps $A \leftrightarrow A'$.

Footnotes/references:

1. Paolo Zanardi Phys. Rev. A 63, 040304(R) (2001)

Operator entanglement and entangling power

- Operator entanglement and Choi-Jamiolkowski isomorphism:
  - Choi state: $|U\rangle := U_{AB} \otimes \mathbb{I}_{A'B'} |\Phi^+\rangle$ where $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |j\rangle |j\rangle$ is a maximally entangled state between $\mathscr{H} \otimes \mathscr{H}'$.
  - This is equivalent to 'vectorization' of a matrix (stacking columns to generate a vector from a matrix).
  - Notice, this Choi state is maximally entangled across $AB|A'B'$.
  - Consider the following partition: $AA'|BB'$. Let us compute entanglement across this:
  - $\rho_{AA'}(U) = \mathrm{Tr}_{BB'}\left[|U\rangle\langle U|\right]$ is the reduced state
  - Linear entropy: $S_{\mathrm{lin}}(\rho_{AA'}(U)) = 1 - \mathrm{Tr}\left[\rho_{AA'}(U)^2\right]$ gives us the operator entanglement of U across $A|B$.
  - Recall, this is equivalent to computing $E_{\mathrm{op}}(U) = 1 - \frac{1}{d^2} \mathrm{Tr}\left[\mathbb{S}_{AA'} U^{\otimes 2} \mathbb{S}_{AA'} U^{\dagger \otimes 2}\right]$.
- Entangling power of a unitary = average entanglement generated by U when acting on random product states.
  - $e_p(U) := \mathbb{E}_{\psi_A, \phi_B}\left[S_{\mathrm{lin}} \mathrm{Tr}_B\left(U\left(|\psi_A\rangle \otimes |\phi_B\rangle\right)\right)\right]$
  - $e_P(U) = \alpha\left(E_{\mathrm{op}}(U) + E_{\mathrm{op}}(US_{AB}) - E_{\mathrm{op}}(S_{AB})\right)$, where $\alpha = d/\left(\sqrt{d}+1\right)^2$.

Footnotes/references:

1. P. Zanardi Phys. Rev. A 63, 040304(R) (2001)
2. G. Styliaris, N. Anand, and P. Zanardi Phys. Rev. Lett. 126, 030601 (2021).

Scrambling and operator entanglement

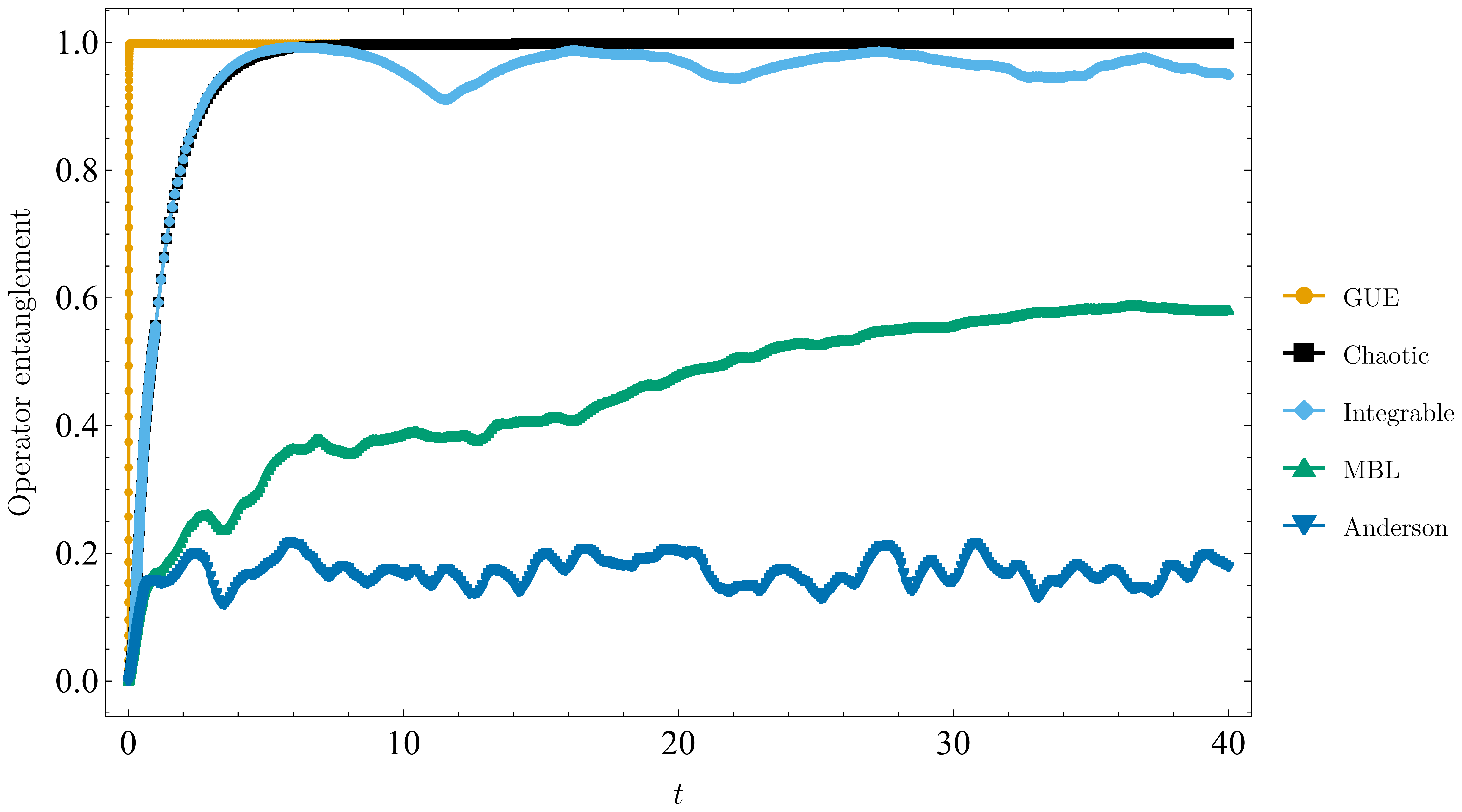
- Setup: bipartite Hilbert space, $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B \cong \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, with local operators $V_A \equiv V \otimes \mathbb{I}_B, W_B \equiv \mathbb{I}_A \otimes W$.
  - 'Bipartite OTOC': $G(t) := \mathbb{E}_{V_A, W_B}\left[C_{V_A, W_B}(t)\right]$, where $\mathbb{E}_{V_A, W_B}$ denotes Haar averaging V_A, W_B over the corresponding Haar measures on $\mathscr{H}_A, \mathscr{H}_B$.
  - Main result: the bipartite OTOC quantifies *exactly* the operator entanglement of $U_t = e^{-iHt}$. That is, $$G(t) = 1 - \frac{1}{d^2}\,\mathrm{Tr}\left(S_{AA'}U_t^{\otimes 2}S_{AA'}U_t^{\dagger\otimes 2}\right).$$
  - Measure concentration + Levy's lemma => deviations from the average are exponentially suppressed.
  - Notice that this quantity can be estimated by averaging over a unitary 1-design such as Pauli operators on each subsystem.
  - [Yan, Cincio, Zurek; Phys. Rev. Lett. 124, 160603 (2020)]: (Assumes weak coupling + Markovianity to show that) bipartite OTOC = thermal average of Loschmidt Echo signals.

- Operational distinction: notice that this OTOC measures operator entanglement and *not* entangling power => it is maximal for a SWAP unitary, which actually generates *no* dynamical entanglement on states!

Footnotes/references:

1. P. Zanardi Phys. Rev. A 63, 040304(R) (2001)
2. G. Styliaris, N. Anand, and P. Zanardi Phys. Rev. Lett. 126, 030601 (2021).

Diagnosing quantum chaos with bipartite OTOCs



- $H_{\text{TFIM}} = -\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z - g \sum_{j=1}^{L} \sigma_j^x - h \sum_{j=1}^{L} \sigma_j^z$, chaotic if $g \neq 0 \neq h$. Integrable if $h = 0, g \neq 0$.
- $H_{\text{MBL}} = -\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z - \sum_{j=1}^{L} g_j \sigma_j^x - h \sum_{j=1}^{L} \sigma_j^z$, where we draw each from the uniform distribution each $g_j \in [-W, W]$. $h = 0$ is Anderson localized.

Long-time average of operator entanglement

- Short-time growth of operator entanglement: $G(U_t) = f(H)t^2 + O(t^3)$, where $f(H) = \frac{2}{d}\left\| H - \mathrm{Tr}_B[H] \otimes \frac{\mathbb{I}_B}{d_B} - \frac{\mathbb{I}_A}{d_A} \otimes \mathrm{Tr}_A[H] \right\|_2^2$.
  - The above expression tells us that growth rate is controlled by interaction across the 'boundary' $A|B$ in H.
  - Unfortunately, short-time growth cannot distinguish chaotic and integrable models in lattice systems…

- What about long-time behavior?
  - Finite system + unitary evolution => no equilibration in the strict sense.
  - Infinite-time average of an observable, $\overline{f(t)} := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt$. If limit exists then $\overline{f(t)} = \lim_{t\to\infty} f(t)$.
  - No-resonance condition (NRC): both the energy levels and the energy gaps are non-degenerate.
  - Spectral decomposition: $H = \sum_k E_k \left|\phi_k\right\rangle\left\langle\phi_k\right|$ and $\rho_k^{(\chi)} := \mathrm{Tr}_{\overline{\chi}}\left(\left|\phi_k\right\rangle\left\langle\phi_k\right|\right)$ where $\chi = \{A, B\}$. Then,

$$\overline{G(t)}^{\mathrm{NRC}} = 1 - \frac{1}{d^2}\sum_{\chi\in\{A,B\}}\left(\left\| R^{(\chi)} \right\|_2^2 - \frac{1}{2}\left\| R_D^{(\chi)} \right\|_2^2 \right)$$ with $R_{kl}^{(\chi)} := \left\langle \rho_k^{(\chi)}, \rho_l^{(\chi)} \right\rangle$.

  - This contains information about (state) entanglement across the full system of Hamiltonian eigenstates => 'infinite temperature' quantity.

Footnotes/references:

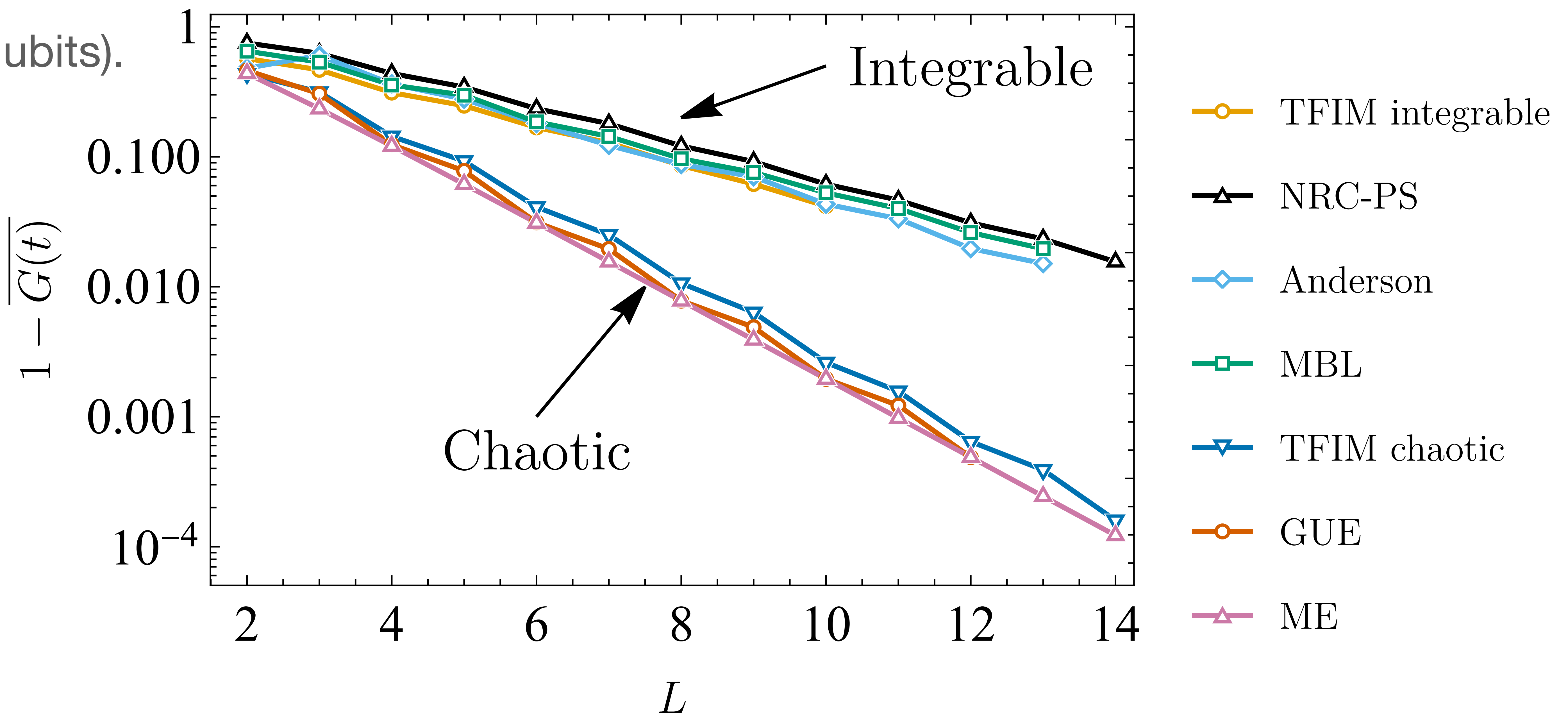
1. G. Styliaris, N. Anand, and P. Zanardi Phys. Rev. Lett. 126, 030601 (2021).

Two analytical results + numerical simulations

- 'Maximally entangled' Hamiltonian: $d_A = d_B = \sqrt{d} = 2^{L/2}$ and $|\phi_k\rangle$ are maximally entangled across $A|B$ then
  - $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}} = \left(1 - \frac{1}{d}\right)^2 \implies \overline{S_2(t)} = \overline{-\log\left(1 - G(t)\right)} \approx L$ is the time average operator 2-Renyi entropy
  - Exponentially close to $G^{\max} = 1 - \dfrac{1}{d}$. Implies small temporal fluctuations (e.g., Markov inequality).
- If all the eigenstates are product states (and NRC holds):

$$H_{\text{NRC-PS}} := \sum_{j,k=1}^{d_A,d_B} E_{j,k} |\phi_j^{(A)}\rangle\langle\phi_j^{(A)}| \otimes |\phi_k^{(B)}\rangle\langle\phi_k^{(B)}|$$
, where the spectrum is from the Gaussian Unitary Ensemble (GUE). Then,

$$\overline{G_{\mathrm{PS}}(t)}^{\mathrm{NRC}} = \left(1 - \frac{1}{\sqrt{d}}\right)^2 \implies \overline{S_2(t)} \approx \frac{1}{2}L$$
(half of the qubits).



Footnotes/references:
1. BROTOCs and Quantum Information Scrambling at Finite Temperature. Anand, Zanardi; Quantum 6, 746 (2022).

Bipartite OTOC and local entropy production

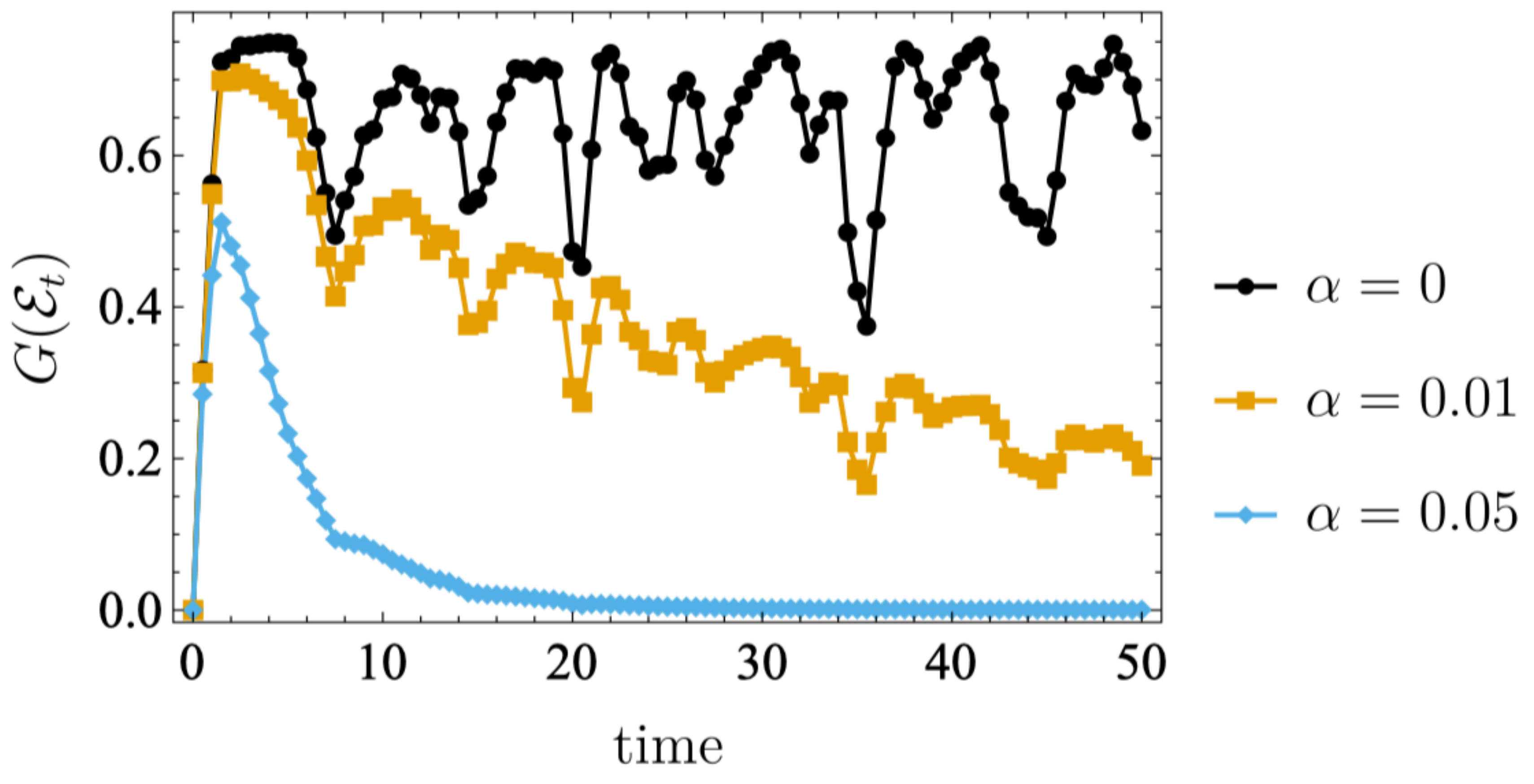
- Local entropy production under 'reduced' dynamics:
  - Let $\Lambda_t^{(A)}\left(\psi_A\right) := \mathrm{Tr}_B\left[U_t\left(\psi_A \otimes I_B/d_B\right)U_t^{\dagger}\right]$ and $S_{\mathrm{lin}}(\rho) := 1 - \mathrm{Tr}(\rho^2)$, then
  - $G(t) = \frac{d_A + 1}{d_A}\mathbb{E}_U\left(S_{\mathrm{lin}}\left[\Lambda_t^{(A)}\left(\left|\psi_U\right\rangle\left\langle\psi_U\right|\right)\right]\right)$ where $\left|\psi_U\right\rangle := U\left|\psi_0\right\rangle$ corresponds to Haar random pure states in $\mathscr{H}_A$.
  - Notice that, $1 - S_{\mathrm{lin}} = \mathrm{Tr}\left(\mathbb{S}\rho^{\otimes 2}\right)$ where $\mathbb{S}$ is the swap operator between the two Hilbert spaces. This formulation allows for the experimental estimation of $G(t)$ from measuring linear entropy of random initial states.

Footnotes/references:

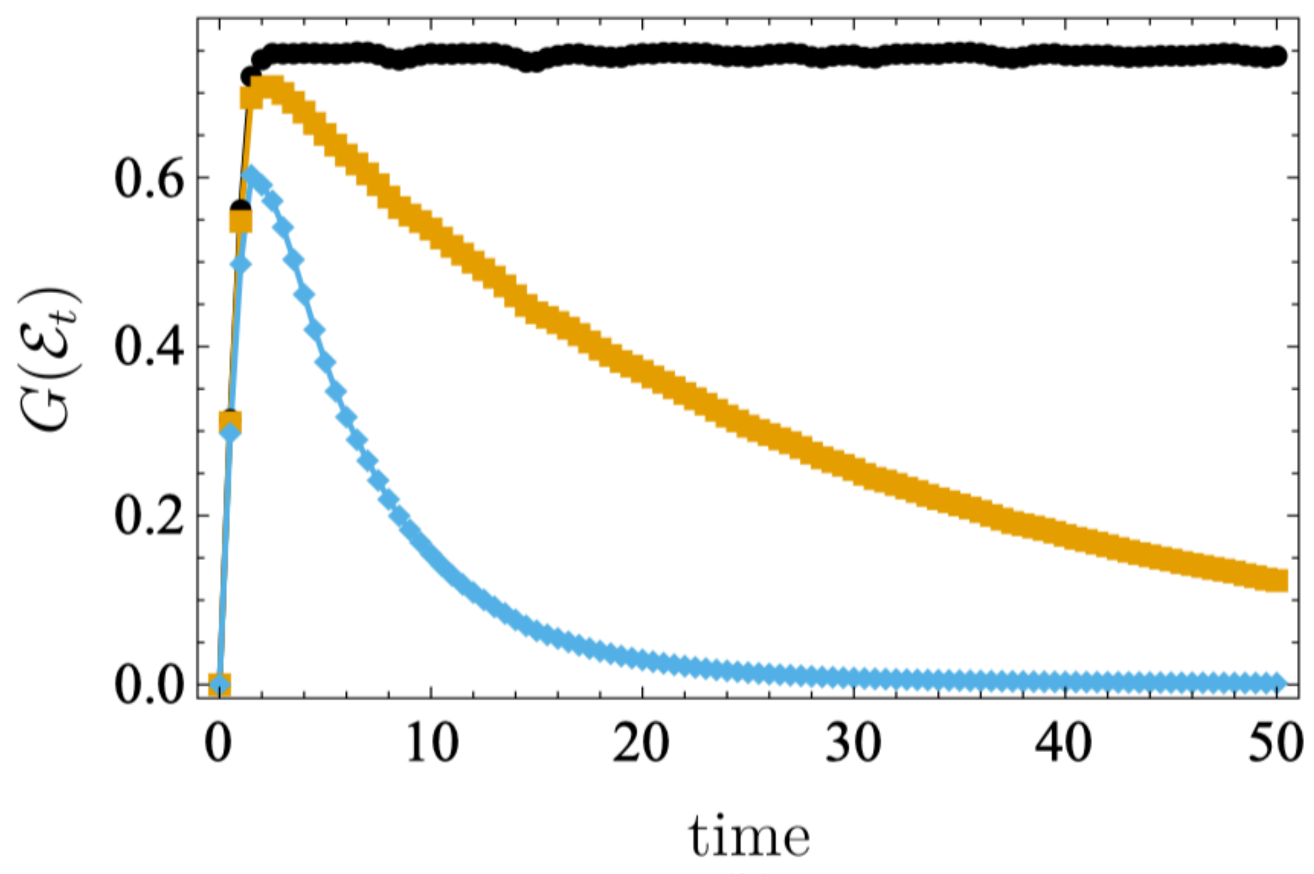
1. G. Styliaris, N. Anand, and P. Zanardi Phys. Rev. Lett. 126, 030601 (2021).

Scrambling in open quantum systems

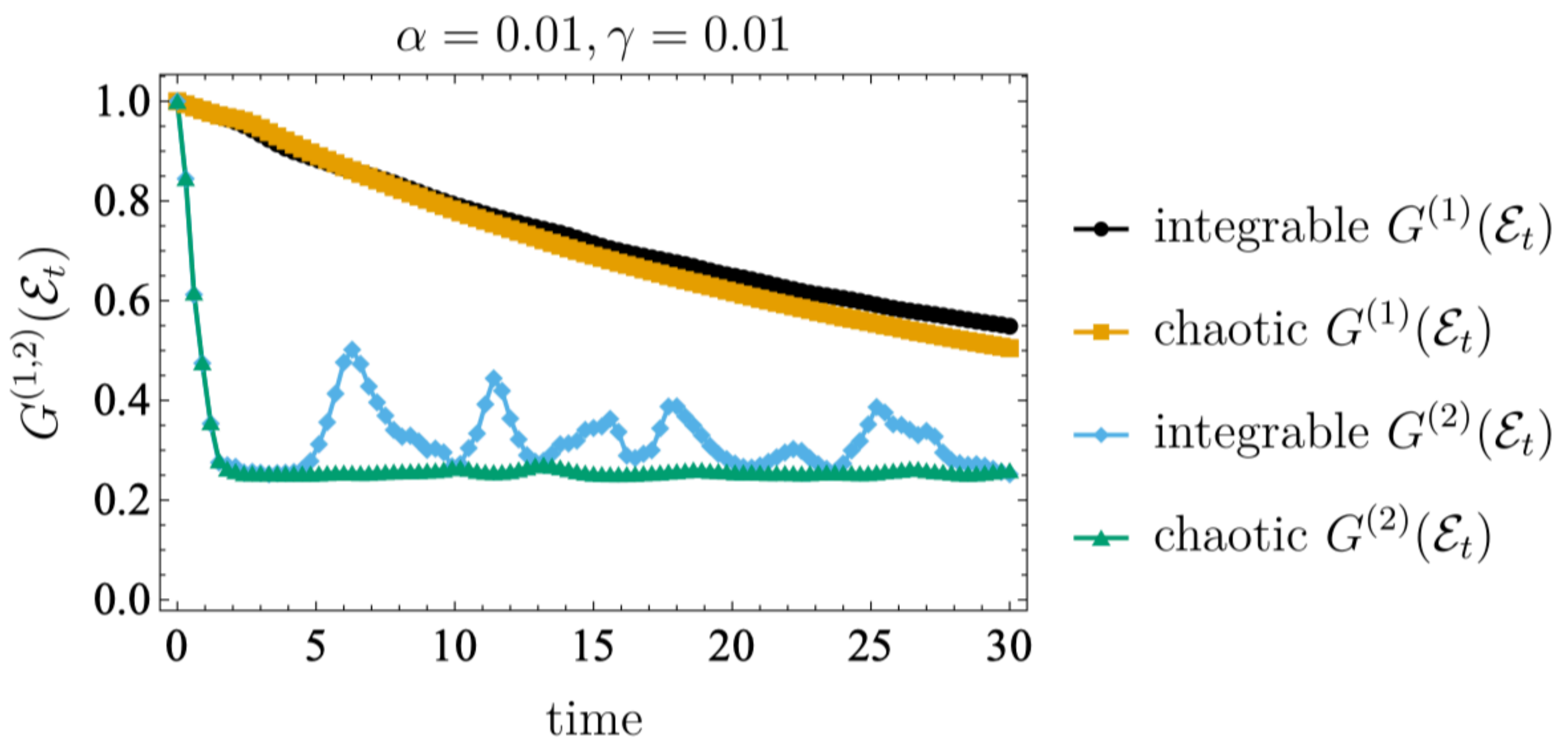
- Given a unital quantum channel $\mathscr{E}$, the OTOC in open quantum systems is $G(\mathscr{E}) := \frac{1}{2d}\mathbb{E}_{V_A,W_B} \left\| \left[\mathscr{E}\left(V_A\right), W_B\right] \right\|_2^2$.
  - The open 'bipartite OTOC' displays a 'competition' between scrambling and decoherence:
$$G(\mathscr{E}) = \frac{1}{d^2}(\underbrace{d_B \operatorname{Tr}\left[S\mathscr{E}^{\otimes 2}\left(S_{AA'}\right)\right]}_{\text{decoherence}} - \underbrace{\operatorname{Tr}\left[S_{AA'}\mathscr{E}^{\otimes 2}\left(S_{AA'}\right)\right]}_{\text{scrambling}}).$$
  - At the level of quantum states we see a competition between local and global entropy production: Let $\Lambda\left(\psi_A\right) := \mathscr{E}\left(\psi_A \otimes \frac{\mathbb{I}_B}{d_B}\right)$, then, $G(\mathscr{E}) \propto \mathbb{E}_{\psi_A}\left[S_{\text{lin}}\left(\operatorname{Tr}_B\left[\Lambda(\psi_A)\right]\right)\right] - d_B\mathbb{E}_{\psi_A}\left[S_{\text{lin}}\left(\Lambda(\psi_A)\right)\right]$.
- Numerical simulations: $H_{\text{TFIM}} = -\sum_j \sigma_j^z\sigma_{j+1}^z - g\sum_j \sigma_j^x - h\sum_j \sigma_j^z$ with dephasing at the boundaries: $\sqrt{\alpha}\sigma_1^z, \sqrt{\alpha}\sigma_L^z$.



(a)



(b)



Footnotes/references:

1. Information scrambling and chaos in open quantum systems. Zanardi, Anand; Phys. Rev. A 103, 062214 (2021).

Crash course in resource theory of quantum coherence

- Fact: quantum systems exist in a (linear) superposition of quantum states.
- Formally, let $\mathscr{H} \cong \mathbb{C}^d$ and $\mathbb{B} = \{\,|j\rangle\}_{j=1}^{d}$ a "preferred" orthonormal basis. $|\psi\rangle = \sum_j c_j |j\rangle$ is "basis-dependent".
- $$\implies \rho = |\psi\rangle\langle\psi| = \underbrace{\sum_{j=1}^{d} \left|c_j\right|^2 |j\rangle\langle j|}_{\text{diagonal/incoherent}} + \underbrace{\sum_{j\neq k} c_j c_k^* |j\rangle\langle k|}_{\text{off}-\text{diagonal/coherent}}$$
- E.g., Qubit, $\mathscr{H} \cong \mathbb{C}^2$. Consider two different bases, $\mathbb{B} = \{\,|0\rangle, |1\rangle\}$and $\mathbb{B}' = \{\,|\pm\rangle\}$. Then, $|+\rangle$ is coherent w.r.t. $\mathbb{B}$ while it is incoherent w.r.t. $\mathbb{B}'$.
- Incoherent states: all diagonal density matrices w.r.t. $\mathbb{B}$. $\mathscr{I}_{\mathbb{B}} = \{\delta \mid \delta = \sum_{j=1}^{d} p_j |j\rangle\langle j|, p_j \geq 0, \sum_{j=1}^{d} p_j = 1.\}$
- Incoherent operations: CP-maps $\mathscr{E}$ such that $\mathscr{E}(\mathscr{I}_{\mathbb{B}}) \subseteq \mathscr{I}_{\mathbb{B}}$. E.g., permutations, diagonal unitaries, dephasing superoperator.
- Coherence measure: functionals $c_{\mathbb{B}} : \mathscr{S}(\mathscr{H}) \to \mathbb{R}_0^+$ such that (i) vanishes on incoherent states, $\rho \in \mathscr{I}_{\mathbb{B}} \implies c_{\mathbb{B}}(\rho) = 0$ and (ii) non-increasing under incoherent operations, $c_{\mathbb{B}}\left(\mathscr{E}(\rho)\right) \leq c_{\mathbb{B}}(\rho)$.

Footnotes/references:

1. Colloquium: Quantum coherence as a resource. A. Streltsov, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. 89, 041003 (2017)

Quantifying coherence for mixed states

- "Distance based measures": minimize the distance from the set of "free states".
- Define the "dephasing superoperator," $\mathscr{D}_{\mathbb{B}}(X) := \sum_{j=1}^{d} \Pi_j X \Pi_j$.
- We will focus on two measures for quantum coherence:

  $\mathbf{c}_{\mathbb{B}}^{(\mathrm{rel})}(\rho) := \min_{\sigma \in \mathscr{I}_{\mathbb{B}}} S(\rho \| \sigma)$, where $S(\rho \| \sigma) = \mathrm{Tr}\left(\rho \log(\rho)\right) - \mathrm{Tr}\left(\rho \log(\sigma)\right)$ is the quantum relative entropy.
- $\mathrm{c}_{\mathrm{B}}^{(2)}(\rho) := \min_{\sigma \in \mathscr{I}_{\mathrm{B}}} \|\rho - \sigma\|_2^2$, where $\| X \|_2^2 := \mathrm{Tr}\left(X^\dagger X\right)$.
- $\mathbf{c}_{\mathbb{B}}^{(\mathrm{rel})}(\rho) = S\left(\mathscr{D}_{\mathbb{B}}(\rho)\right) - S(\rho)$ and $\mathrm{c}_{\mathrm{B}}^{(2)}(\rho) = \| \rho - \mathscr{D}_{\mathbb{B}}(\rho) \|_2^2 = \sum_{j \neq k} \left| \rho_{j,k} \right|^2$.
- Coherence of unitary dynamics: given a set of incoherent states, how much coherence does it generate on average under the action of U => "coherence-generating power" (CGP).
- Formally: Pure incoherent states and averaging over the d states: $\mathscr{C}_{\mathbb{B}}(\mathscr{U}) = \frac{1}{d} \sum_{j=1}^{d} \mathbf{c}_{\mathbb{B}}\left(\mathscr{U}\left(\Pi_j\right)\right)$.

OTOCs and coherence-generating power

- Result 1): Assume, V, W are nondegenerate unitaries with spectral decomposition, $V = \sum_{j=1}^{d} \exp[i\theta_j]\Pi_j$ and $W = \sum_{j=1}^{d} \exp[i\phi_j]\Pi_j$.
  Then, $C_{V,W}(t) = \mathscr{C}_{\mathbb{B}}(\mathscr{U}_t) - \frac{1}{d}\mathrm{Re}\left(\sum_{j\neq l, k\neq m} \exp[i(\theta_l - \theta_j + \phi_m - \phi_k)]\mathrm{Tr}\left[\Pi_k(t)\Pi_j\Pi_m(t)\Pi_l\right]\right)$.

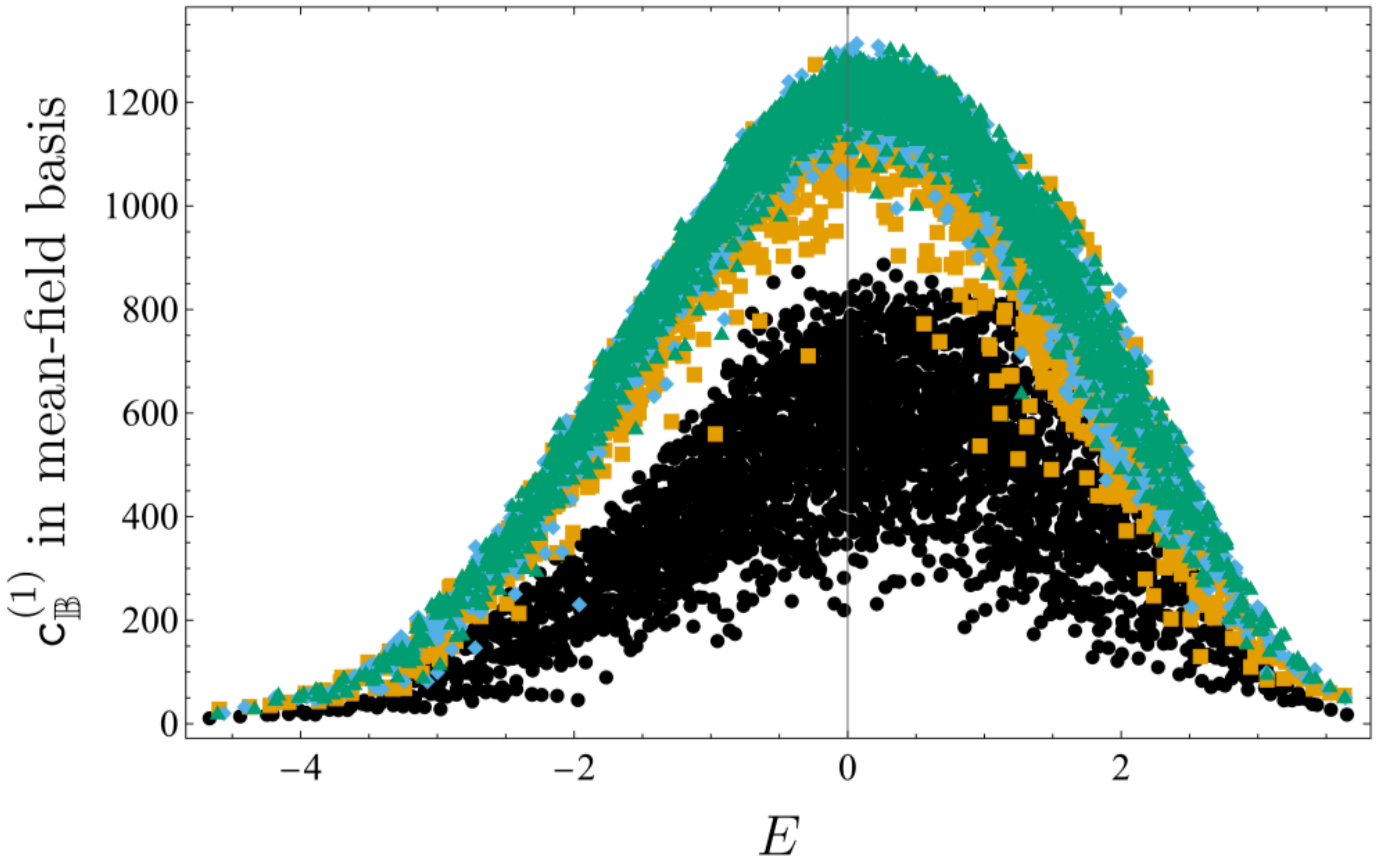
- Result 2): further assume $\{\theta_j, \phi_j\}_j$ independently and identically distributed on the interval $[0,2\pi)$, then,
  $\mathbb{E}_{\theta,\phi}\left[C_{V,W}(t)\right] = \mathscr{C}_{\mathbb{B}}(\mathscr{U}_t)$.

- Coherence of Hamiltonian eigenstates to distinguish integrable and chaotic phases.

- $H = \frac{1}{4}\sum_{j=1}^{L-1}\left(J_{xy}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y\right) + J_z\sigma_j^z\sigma_{j+1}^z\right)$

  $+\frac{1}{2}\left(\sum_{j=1}^{L}\omega\sigma_j^z + \epsilon_\delta\sigma_\delta^z\right)$ where $\delta \in \{1,2,\ldots,L\}$ is the defect site.

- Defect in boundary vs. bulk => integrable vs. chaos.



Footnotes/references:

1. Quantum coherence as a signature of chaos. Anand, Styliaris, Kumari, Zanardi; Phys. Rev. Research 3, 023214 (2021).

Regularized OTOCs and finite-temperature scrambling

- Regularized OTOC, $F_\beta^{(r)}(t) := \mathrm{Tr}\left[W_t^\dagger y V^\dagger y W_t y V y\right]$ with $y^4 = \rho_\beta$.
    - E.g., Maldacena, Shenker, Stanford's "bound on chaos": $\frac{\partial}{\partial t}\log\left(F_\beta^{(d)}(t) - F_\beta^{(r)}(t)\right) \le \frac{2\pi}{\beta}$.
- "Thermofield double state": a (canonical) purification of a Gibbs state, $|\psi(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_{j=1}^{d}\exp\left[-\beta E_j/2\right]|j\rangle|j\rangle$.
- 'Bipartite regularized OTOC' (BROTOC) is equal to the purity of the time-evolved TDS:

$$\mathbb{E}_{V_A,W_B}[F_\beta^{(r)}(t)] = \frac{Z(\beta/2)^2}{dZ(\beta)}P_{AA'}\left(|\psi(\beta/2,t)\rangle_{ABA'B'}\right)$$

    - Infinite temperature => operator entanglement. Zero temperature => GS purity (if Hamiltonian is non-degenerate).
- Averaging over global Haar random unitaries = analytically-continued spectral form factor.

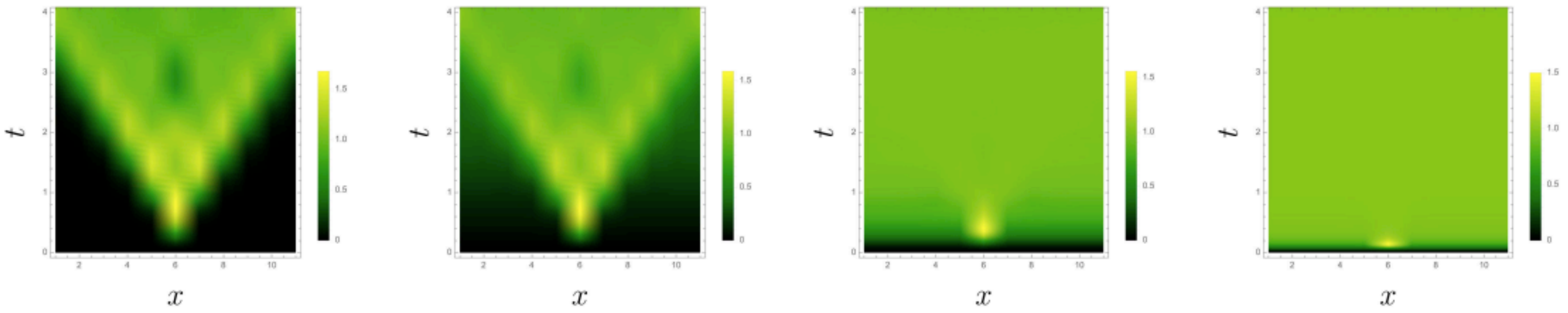
$$\mathbb{E}_{A_1,B_1,A_2\in\mathcal{U}(\mathcal{H})}\left[F_\beta^{(A_1,B_1,A_2,B_2)}(t)\right] = \frac{\mathcal{R}_4^{(H)}(\beta/4,t)}{(d^3\mathcal{Z}(\beta))}.$$ Here $\mathcal{R}_4^{(H)}(\beta,t) := \left|\mathrm{Tr}\left[e^{(-\beta+it)H}\right]\right|^4$.

Footnotes/references:

1. BROTOCs and Quantum Information Scrambling at Finite Temperature. Anand, Zanardi; Quantum 6, 746 (2022).

Weak quantum chaos and non-Hermitian scrambling

- Lieb-Robinson bounds:
  - for any locally interacting lattice system, there exist positive constants ξ, μ, v_{LR} such that for any two operators a, b, the following bound holds: $\| [a(t), b] \| \leq \xi \min\{ |\mathrm{supp}(a)|, |\mathrm{supp}(b)| \} \|a\| \|b\| e^{-\mu \max\{0,\ \mathrm{d}(\mathrm{supp}(a), \mathrm{supp}(b)) - v_{LR} t\}}$
  - Even for local non-Hermitian Hamiltonians, LR bounds are violated!
  - Consider 'effective' non-Hermitian Hamiltonians for a continuously monitored system.
  - A 'normalized OTOC': $C_{V,W}(t) = 1 - \frac{1}{d} \sum_{j=1}^{d} \frac{\mathrm{Re}\langle j | W_t^\dagger V^\dagger W_t V | j \rangle}{\| W_t | j \rangle \| \ \| W_t V | j \rangle \|}$.
  - A local non-Hermitian Hamiltonian: $H_{\mathrm{I}} = J \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + h \sum_{j=1}^{L} \sigma_j^z + g \sum_{j=1}^{L} \left(e^{\beta} \sigma_j^+ + e^{-\beta} \sigma_j^- \right)$



(a) $\beta = 0$ (b) $\beta = 0.25$ (c) $\beta = 1$ (d) $\beta = 2$

Footnotes/references:

1. Scrambling and operator entanglement in local non-Hermitian quantum systems, [Barch, Anand, Marshall, Rieffel, Zanardi; Phys. Rev. B 108, 134305 (2023), Editors' Suggestion]

Summary of results

- Averaging the OTOC over local, Haar-random unitaries => operator entanglement of the dynamical unitary; and is closely related to the entangling power of the unitary.
- Averaging the OTOC over diagonal unitaries => coherence-generating power of dynamics
- In open quantum systems, there is a competition between environmental decoherence and information scrambling (i.e., scrambling can be masked because of decoherence).
- Regularized OTOC when averaged over local Haar-random unitaries => purity of the thermofield double state.
- Our formalism can be generalized to an algebraic framework that can detect scrambling between an algebra and its commutant.
- Good for detecting integrability-vs-chaos, localization transition, decoherence-vs-scrambling, measurement-induced phase transitions, etc.

Thank you!