Finite-Volume Diffusion Schemes for Svard's Eulerian Governing Equations

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Outline

- Background/motivation

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- Governing equations and discretization

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- Verification

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- Results

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- Conclusions and future work

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- This set of governing equations allows for arguments of well-posedness and existence of solutions
- The result is that the viscous flux has only normal terms and it was believed this would lead to a simpler viscous dissipation matrix and hyperbolic diffusion system.
- In this work, I compare a typical edge-based finite-volume diffusion discretization and a hyperbolic diffusion discretization

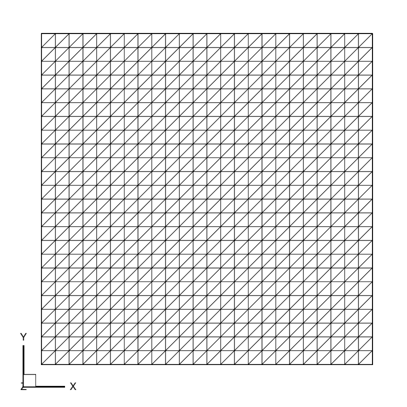
Governing Equations and Discretization

Governing Equations and Discretization

We are considering a 2D unsteady hyperbolic system defined as:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}_x}{\partial x} + \frac{\partial \mathbf{f}_y}{\partial y} = \mathbf{s}(x, y, t), \tag{1}$$

solved on a representative 2D grid.

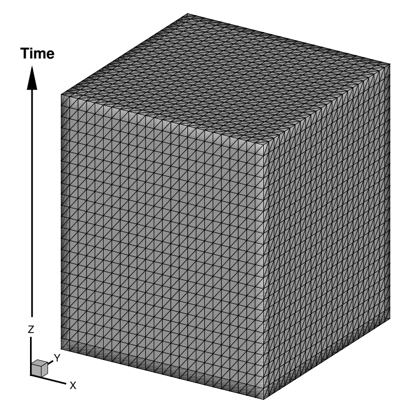


Governing Equations and Discretization

We then transform them to the space-time as follows:

$$\frac{\partial \mathbf{u}}{\partial z} + \frac{\partial \mathbf{f}_x}{\partial x} + \frac{\partial \mathbf{f}_y}{\partial y} = \mathbf{s}(x, y, z), \qquad (2)$$

solved on the extruded space-time grid below.



Governing Equations and Discretization: Navier Stokes

Compressible NS equations:

$$\begin{aligned}
\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) &= 0 \\
\partial_t (\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) &= -\nabla_{\mathbf{x}} p + \nabla_{\mathbf{x}} \cdot \tau \\
\partial_t (E) + \nabla_{\mathbf{x}} \cdot (E\mathbf{v} + p\mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\tau \mathbf{v}) - \nabla_{\mathbf{x}} \cdot \mathbf{q}
\end{aligned} \tag{3}$$

where t is the physical time.

The viscous stress tensor $\boldsymbol{\tau}$ is given by

$$\tau = -\frac{2}{3}\mu(\nabla_{\mathbf{x}} \cdot \mathbf{v})\mathbf{I} + \mu\left(\nabla_{\mathbf{x}}\mathbf{v} + (\nabla_{\mathbf{x}}\mathbf{v})^T\right). \tag{4}$$

Governing Equations and Discretization: Hyperbolic Navier Stokes

Compressible HNS equations:

$$
\begin{aligned}
\partial_\tau \rho + \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\nu_r \mathbf{r}) \\
\partial_\tau (\rho \mathbf{v}) + \partial_t (\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) &= -\nabla_{\mathbf{x}} p + \nabla_{\mathbf{x}} \cdot (\mu_v \widetilde{\tau}) \\
\partial_\tau (\rho E) + \partial_t (\rho E) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} H) &= \nabla_{\mathbf{x}} \cdot (\mu_v \widetilde{\tau} \mathbf{v}) + \nabla_{\mathbf{x}} \cdot \left(\frac{\mu_h}{\gamma(\gamma - 1)} \mathbf{h} \right) \\
T_r \partial_\tau \mathbf{r} &= \nabla_{\mathbf{x}} \rho - \mathbf{r} \\
T_v \partial_\tau \mathbf{g} &= \nabla_{\mathbf{x}} \mathbf{v} - \mathbf{g} \\
T_h \partial_\tau \mathbf{h} &= \nabla_{\mathbf{x}} T - \mathbf{h}
\end{aligned}
\tag{5}
$$

where τ is a pseudotime variable, and $\widetilde{\boldsymbol{\tau}} = -\frac{1}{2} tr(\mathbf{g})\mathbf{I} + \frac{3}{4}\left(\mathbf{g} + \mathbf{g}^T\right)$,

$$
T_r = \frac{L_r^2}{\nu_r},\ T_v = \frac{L_v^2}{\nu_v},\ T_h = \frac{L_h^2}{\nu_h},\ \nu_r = V_{min}^{4/3} M_\infty, \nu_v = \frac{4\mu}{3\rho},\ \nu_h = \frac{\gamma\mu}{\rho Pr}, \tag{6}
$$

Governing Equations and Discretization: Hyperbolic Navier Stokes

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\begin{aligned}
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\partial_\tau (\rho \mathbf{v}) + \partial_t (\rho \mathbf{v}) + \mathrm{div}(\rho \mathbf{v} \otimes \mathbf{v}) &= -\nabla_{\mathbf{x}} p + \nabla_{\mathbf{x}} \cdot (\mu_v \widetilde{\tau}) \\
\partial_\tau (\rho E) + \partial_t (\rho E) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} H) &= \nabla_{\mathbf{x}} \cdot (\mu_v \widetilde{\tau} \mathbf{v}) + \nabla_{\mathbf{x}} \cdot \left(\frac{\mu_h}{\gamma(\gamma - 1)} \mathbf{h} \right) \\
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\end{aligned}
\tag{7}
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where τ is a pseudotime variable, and $\widetilde{\boldsymbol{\tau}} = -\frac{1}{2} tr(\mathbf{g})\mathbf{I} + \frac{3}{4}\left(\mathbf{g} + \mathbf{g}^T\right)$,

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$$

Governing Equations and Discretization: Svard-Eulerian equations

Svard's Eulerian equations:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} \rho), \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} \rho \mathbf{v}), \\
\frac{\partial E}{\partial t} + \nabla_{\mathbf{x}} \cdot (E \mathbf{v} + p \mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} E) + \nabla_{\mathbf{x}} \cdot (\kappa \nabla_{\mathbf{x}} T), \\
p &= \rho R T, \\
\nu &= \alpha \frac{\mu}{\rho} + \beta(\rho, T).
\end{aligned} \tag{9}
$$

Governing Equations and Discretization: Svard-Eulerian equations

Svard's Eulerian equations:

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\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} \rho), \\
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\frac{\partial E}{\partial t} + \nabla_{\mathbf{x}} \cdot (E \mathbf{v} + p \mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} E), \\
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Governing Equations and Discretization: Hyperbolic Eulerian Flow

Compressible HEF equations:

$$
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\partial_\tau (\rho E) + \partial_t (\rho E) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} H) &= \nabla_{\mathbf{x}} \cdot (\nu \mathbf{k}) \\
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T_h \partial_\tau \mathbf{k} &= \nabla_{\mathbf{x}} E - \mathbf{k}
\end{aligned}
\tag{11}
$$

where τ is a pseudotime variable and

$$
T_r = T_v = T_h = \frac{L_h^2}{\nu} \tag{12}
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Governing Equations and Discretization: Hyperbolic Eulerian Flow

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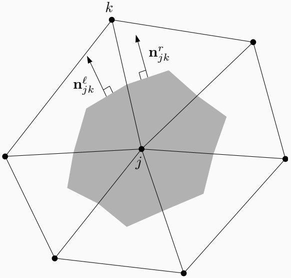
$$

Governing Equations and Discretization: Residual

We discretize the governing equations on a tetrahedral grid where the residual at node j is defined as:

$$\sum_{k \in \{k_j\}} \mathbf{\Phi}_{jk} A_{jk} = s(x_j, y_j, z_j), \tag{15}$$

where Φ_{jk} is the numerical flux and A_{jk} is the directed area vector on the edge that connects nodes j and k.



Stencil for the edge-based (EB) discretization, showing the directed area vector on edge jk

We extend this to be 2^{nd}-order accurate using the U-MUSCL scheme

$$\mathbf{w}_L = \kappa\frac{\mathbf{w}_j + \mathbf{w}_k}{2} + (1-\kappa)\left[\mathbf{w}_j + \nabla\mathbf{w}_j^{LSQ}\cdot(\mathbf{x}_k - \mathbf{x}_j)\right], \quad (16)$$

$$\mathbf{w}_R = \kappa\frac{\mathbf{w}_j + \mathbf{w}_k}{2} + (1-\kappa)\left[\mathbf{w}_k - \nabla\mathbf{w}_k^{LSQ}\cdot(\mathbf{x}_k - \mathbf{x}_j)\right], \quad (17)$$

with $\kappa = \frac{1}{2}$.

Governing Equations and Discretization: Fluxes

The numerical flux is defined as:

$$\mathbf{\Phi}_{jk} \quad = \quad \mathbf{\Phi}_{jk}^{inv}|(\hat{\hat{n}}_x, \hat{\hat{n}}_y)| + \mathbf{\Phi}_{jk}^{vis}|(\hat{\hat{n}}_x, \hat{\hat{n}}_y)| + \mathbf{\Phi}_{jk}^{time}|\hat{\hat{n}}_t|, \tag{18}$$

where the norm $n_{jk} = (n_x, n_y, n_t)$ is normalized in space-time as $\hat{\mathbf{n}}_{jk} = \mathbf{n}_{jk}/|\mathbf{n}_{jk}| = (\hat{n}_x, \hat{n}_y, \hat{n}_t)$.

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We use a $\hat{\hat{}}$ notation to indicate normalization in space or time alone. For space this is:

$$\hat{\hat{\mathbf{n}}}_{xy} = \frac{\hat{\mathbf{n}}_{xy}}{|\hat{\mathbf{n}}_{xy}|} = \frac{(\hat{n}_x, \hat{n}_y)}{|(\hat{n}_x, \hat{n}_y)|} = \frac{(\hat{n}_x, \hat{n}_y)}{\sqrt{\hat{n}_x^2 + \hat{n}_y^2}}, \tag{19}$$

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and for time it is:

$$\hat{\hat{n}}_t = \frac{\hat{n}_t}{|\hat{n}_t|}. \tag{20}$$

Governing Equations and Discretization: Fluxes

We use Roe's flux for the invisicid flux

$$\boldsymbol{\Phi}_{jk}^{inv} = \frac{1}{2}\left[\mathbf{f}^{inv}(\mathbf{w}_L) + \mathbf{f}^{inv}(\mathbf{w}_R)\right] - \frac{1}{2}\left|\mathbf{A}^{inv}(\mathbf{w}_L, \mathbf{w}_R)\right|(\mathbf{u}_R - \mathbf{u}_L)\,, \qquad (21)$$

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an upwind temporal flux

$$\boldsymbol{\Phi}_{jk}^{time} = \frac{1}{2}\left[\mathbf{u}_L + \mathbf{u}_R\right]\hat{\hat{n}}_t - \frac{|\hat{\hat{n}}_t|}{2}(\mathbf{u}_R - \mathbf{u}_L), \qquad (22)$$

Governing Equations and Discretization: Fluxes

We use Roe's flux for the invisicid flux

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and use an upwind flux for the viscous flux making it hyperbolic

$$\mathbf{\Phi}_{jk}^{vis} = \frac{1}{2}\left[\mathbf{f}^{vis}(\mathbf{w}_L) + \mathbf{f}^{vis}(\mathbf{w}_R)\right] - \frac{1}{2}\left|\mathbf{A}^{vis}(\mathbf{w}_L, \mathbf{w}_R)\right|(\mathbf{u}_R - \mathbf{u}_L) \tag{23}$$

We use the HNS20G formulation for both the NS and EF equation viscous fluxes, for NS this is:

$$\phi_{jk}^{vis} = \begin{bmatrix} 0 \\ -\mu_v \left[\widetilde{\tau}_{xx}\hat{\hat{n}}_x + \widetilde{\tau}_{yx}\hat{\hat{n}}_y\right] \\ -\mu_v \left[\widetilde{\tau}_{xy}\hat{\hat{n}}_x + \widetilde{\tau}_{yy}\hat{\hat{n}}_y\right] \\ -\mu_v \left[\left(\vec{\widetilde{\tau}}_x \vec{u} + \frac{\mu_h h_x}{\gamma(\gamma-1)}\right)\hat{\hat{n}}_x + \left(\vec{\widetilde{\tau}}_y \vec{u} + \frac{\mu_h h_y}{\gamma(\gamma-1)}\right)\hat{\hat{n}}_y\right] \\ -\rho\hat{\hat{n}}_x \\ -\rho\hat{\hat{n}}_y \\ -u\hat{\hat{n}}_x \\ -u\hat{\hat{n}}_y \\ -v\hat{\hat{n}}_x \\ -v\hat{\hat{n}}_y \\ -T\hat{\hat{n}}_x \\ -T\hat{\hat{n}}_y \end{bmatrix} \tag{24}$$

Governing Equations and Discretization: Fluxes

In contrast, the HEF flux is:

$$\phi_{jk}^{vis} = \begin{bmatrix} -\nu(r_x n) \\ -\nu\left[\rho(u_x n) + (r_x n)u\right] \\ -\nu\left[\rho(v_x n) + (r_x n)v\right] \\ -\nu(E_x n) \\ -\rho\hat{\hat{n}}_x \\ -\rho\hat{\hat{n}}_y \\ -u\hat{\hat{n}}_x \\ -u\hat{\hat{n}}_y \\ -v\hat{\hat{n}}_x \\ -v\hat{\hat{n}}_y \\ -E\hat{\hat{n}}_x \\ -E\hat{\hat{n}}_y \end{bmatrix} \tag{25}$$

Where we define the following variables $u_x n = u_x\hat{\hat{n}}_x + u_y\hat{\hat{n}}_y$, $v_x n = v_x\hat{\hat{n}}_x + v_y\hat{\hat{n}}_y$, $r_x n = r_x\hat{\hat{n}}_x + r_y\hat{\hat{n}}_y$, $E_x n = E_x\hat{\hat{n}}_x + E_y\hat{\hat{n}}_y$.

Governing Equations and Discretization: Dissipation

By diagonalizing the matrix and using the local preconditioning approach we get a dissipation matrix of the form:

$$|PA| = \begin{bmatrix} a\nu & 0 & 0 \\ 0 & \frac{\nu nx^2}{a\nu} & \frac{\nu nxny}{a\nu} \\ 0 & \frac{\nu nxnxy}{a\nu} & \frac{\nu ny^2}{a\nu} \\ \dots & \dots & \dots \end{bmatrix} \tag{26}$$

where $a\nu = \sqrt{\frac{\nu}{T_h}}$. Note that this (reduced) matrix is block diagonal. Additionally, it lacks the additional dissipation vectors HNS20G requires to maintain strong coupling for design-order accurate computation of the velocity gradients.

Verification

Verification: Exponential Solution

We used the method of manufactured solutions (MMS) with an exponential solution to calculate the truncation and discretization errors. The solution is:

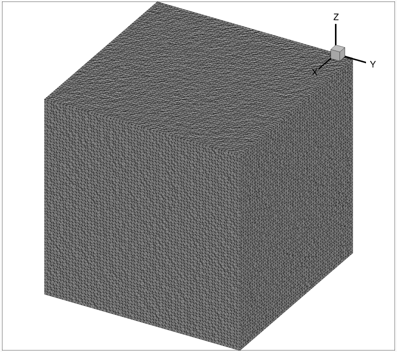
$$a = a_0 + a_{scale} exp(a_x x + a_y y + a_t t), \tag{27}$$

and the parameters to define it are below.

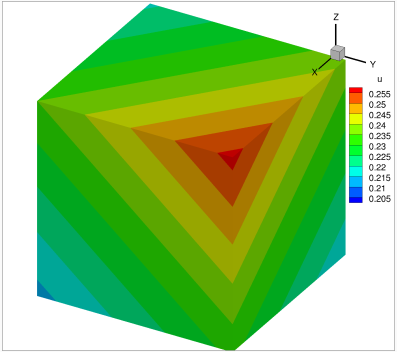
variable	a_0	a_{scale}	a_x	a_y	a_t
ρ	.5	1.0	0.525	0.550	0.575
u	.1	0.1	0.125	0.150	0.175
v	.2	0.2	0.225	0.250	0.275
p	.4	0.714	0.425	0.450	0.475

Parameters used to define the exact solution.

Verification: Exponential Solution



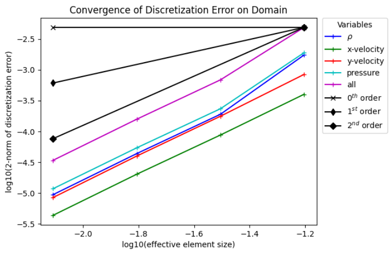
(a) Coarsest perturbed mesh



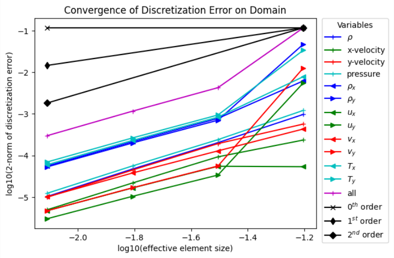
(b) Solution on coarsest mesh

Example mesh and solution.

Verification: Exponential Solution Discretization Error



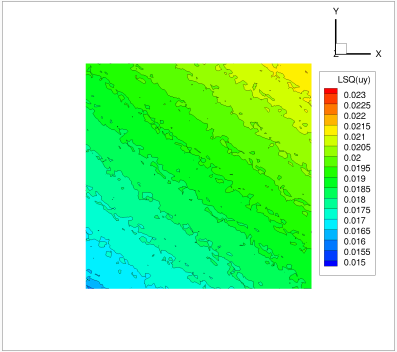
(a) Alpha damping discretization error convergence



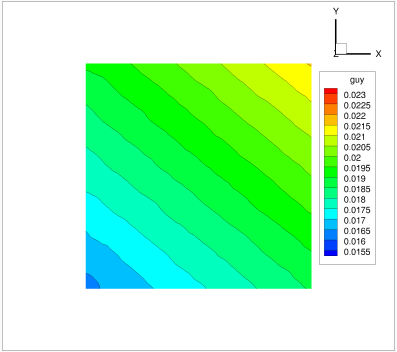
(b) HEF discretization error convergence

Discretization error convergence.

Verification: Exponential Solution



(a) $LSQ(u_x)$ on a 32^3-node grid.



(b) $g_{ux}(= u_x)$ on a 32^3-node grid.

Gradient calculation on arbitrary perturbed mesh

Results

Results: Boundary Layer Solution

- We used the FUN3D sketch to solution framework (combining EGADS, *refine*, and ruby scripts).

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- We used the FUN3D sketch to solution framework (combining EGADS, *refine*, and ruby scripts).
- This entailed 10 adaptation cycles with mesh size doubling every other iteration.
- The Mach Hessian metric computed by *refine* off the alpha-damping viscous discretization
- The final HEF results shown are from solving the hyperbolic equations on the final adapted mesh

Results: Boundary Layer Solution

The exact solution is defined by:

$$\mathbf{w}_{exact} = \begin{bmatrix} 1.0 \\ 1.0 \\ 0.1 \\ 1.0 \end{bmatrix} - \begin{bmatrix} 0.1 \\ 1.0 \\ 0.1 \\ 0.1 \end{bmatrix} \exp(-\eta), \quad \eta = y\sqrt{\frac{R(z)}{x - 0.2}}, \tag{28}$$

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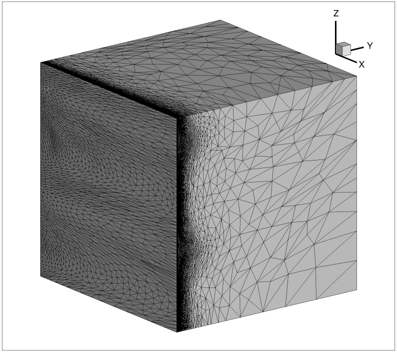
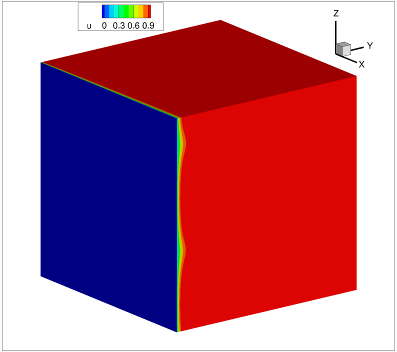
where $R(z)$ is a time-dependent parameter defined by

$$R(z) = 10^5 \left[1 + 0.75 \sin(4\pi z)\right]. \tag{29}$$

Results: Boundary Layer Solution - Mesh and Solution



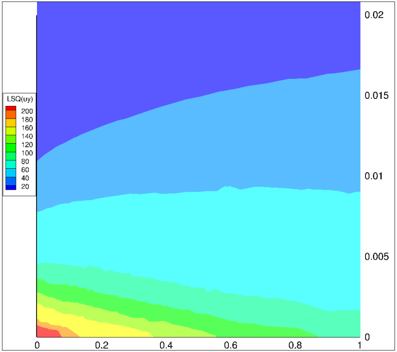
(a) Grid.



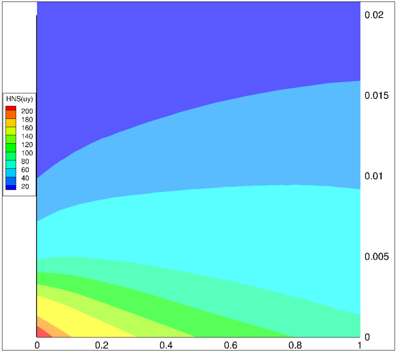
(b) X-velocity contours.

Grid and solution for MMS boundary layer test case.

Results: Boundary Layer Solution - Gradient Comparison



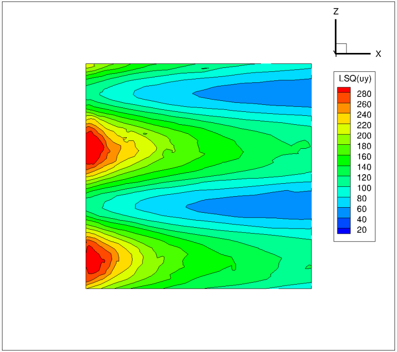
(a) Alpha: $LSQ(u_y)$ at $z = 0.619$.



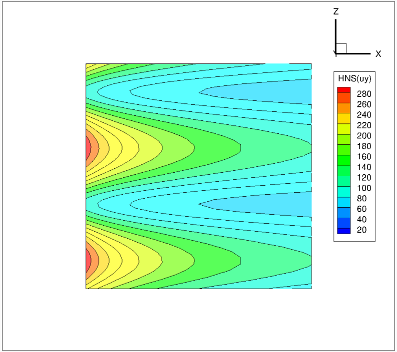
(b) HNS: g_{uy} at $z = 0.619$.

Wall normal gradient contours on XY plane for $z = 0.619$

Results: Boundary Layer Solution - Gradient Comparison



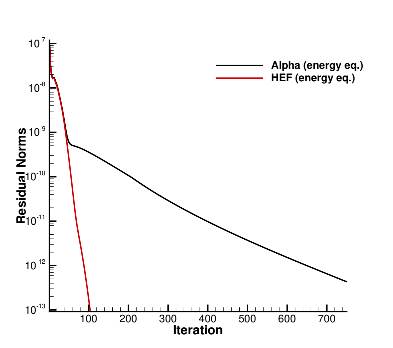
(a) Alpha: $LSQ(u_y)$ at $y = 0$.



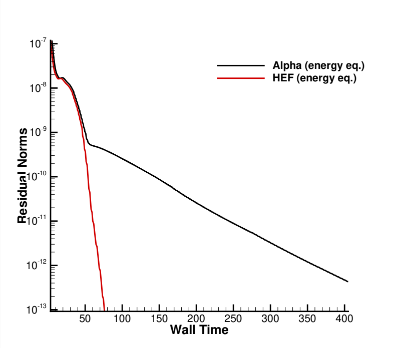
(b) HNS: g_{uy} at $y = 0$.

Wall normal gradient contours on XZ plane for $y = 0.0$

Results: Boundary Layer Solution - Time to Solution Comparison



(a) Iterations to solution



(b) CPU time to soluion

Comparison of CPU time and iterations to convergence for boundary layer MMS.

Results: Infinite Cylinder

- Infinite cylinder in $M = 0.2$, $Re = 200$ crossflow.
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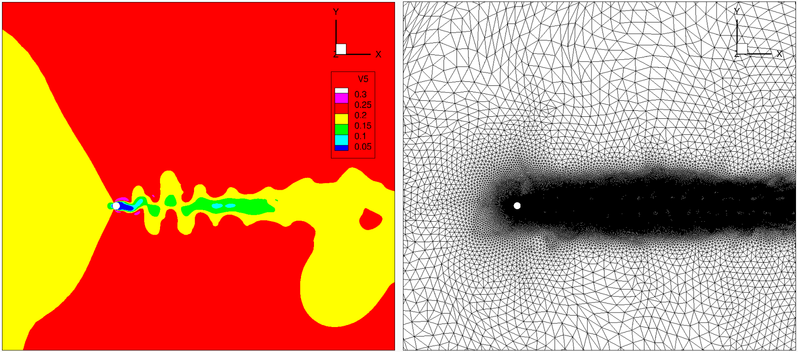
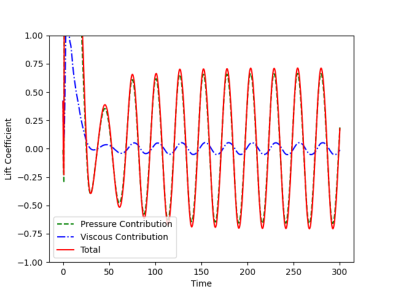
- We look at the 13^{th} mesh ($26.5M$ nodes) and compare the results of HEF discretization to those of the alpha-damping one.

Results: Infinite Cylinder - Solution Evolution

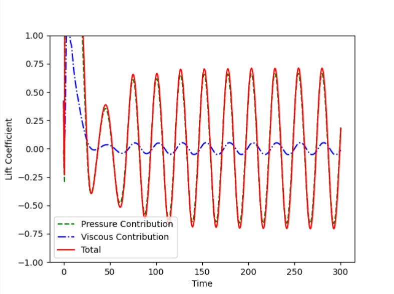


Solution for 13^{th} spatiotemporal triangulation

Results: Infinite Cylinder - Comparison of Lift Coefficient



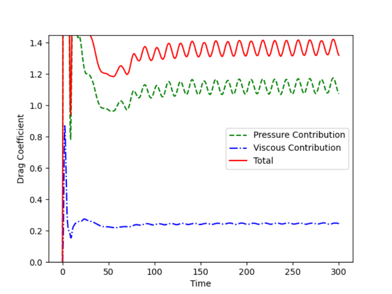
(a) Alpha damping.



(b) HEF.

Lift coefficient vs. time

Results: Infinite Cylinder - Comparison of Drag Coefficient



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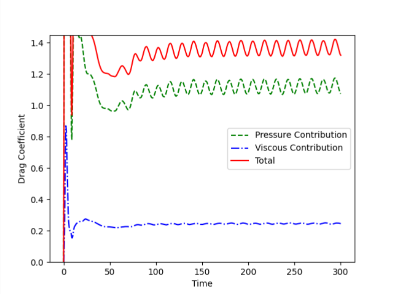


(b) HEF.

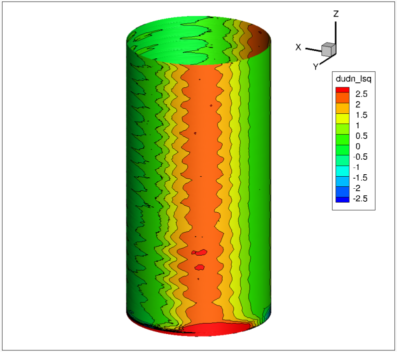
Drag coefficient vs. time

Results: Infinite Cylinder - Comparison of Engineering Coefficients

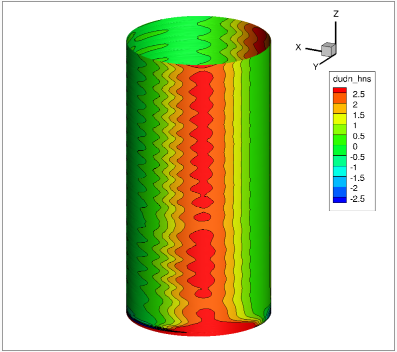
	St	$c_{L_{rms}}$	$c_{D_{avg}}$
reference	0.1957	0.4244	1.3365
NS values	0.1961	0.4941	1.3376
EF values	0.1964	0.4991	1.3699
HEF values	0.1969	0.4986	1.3697

Comparison of engineering quantities to reference values.

Results: Infinite Cylinder - Comparison of Normal Gradient of x-velocity



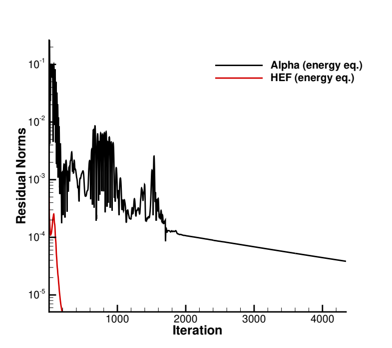
(a) LSQ gradient.



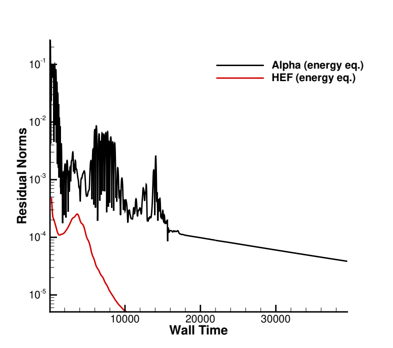
(b) HEF gradient.

Normal gradient on cylinder surface.

Results: Infinite Cylinder - Comparison of Time to Solution



(a) Iterations to solution



(b) CPU time to soluion

Comparison of CPU time and iterations to convergence for boundary layer MMS.

Conclusion and Future Work

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- Verified these results with MMS and compared with NS discretizations
- Demonstrated that the HEF solver is faster in time to solution than the alpha-damping solver and has better accuracy in the gradients
- The HEF discretization has a simpler viscous dissipation matrix

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- Implement the equations in 3 physical dimensions and benchmark on realistic configurations.
- High-order finite-volume/difference schemes with one flux per edge due to the lack of tangent terms in the viscous fluxes.

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Thank you!