Fundamentals of Electromagnetics

Maxwell's Equations and Wave Propagation

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Topics

- **Review of electric and magnetic fields**
- **Maxwell’s/Heaviside’s Equations**
- **Wave propagation**
- **Speed of propagation, frequency, and wavelength**
- **Characteristic impedance**
- **Poynting Vector**

- ***Material taken from “Fundamentals of Electromagnetics” video series***
  - *Publicly available on YouTube; search for above title*
  - *Direct playlist link:*
    - https://youtube.com/playlist?list=PLtrpQ-gPvnJn2r9Mw49jij7Ky0mb6RJYF&si=UxEKqVRgsR9w6nZ7

What We've Learned So Far

Gauss's Law for Electric Field:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \qquad \oiint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$$

Charge produces electric field

Ampère's Law:

$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{d\vec{E}}{dt} \qquad \oint \vec{H} \cdot d\vec{l} = \oiint \vec{J} \cdot d\vec{s} + \frac{d}{dt} \oiint \varepsilon \vec{E} \cdot d\vec{s}$$

Current (moving charge) produces magnetic field

Faraday's Law:

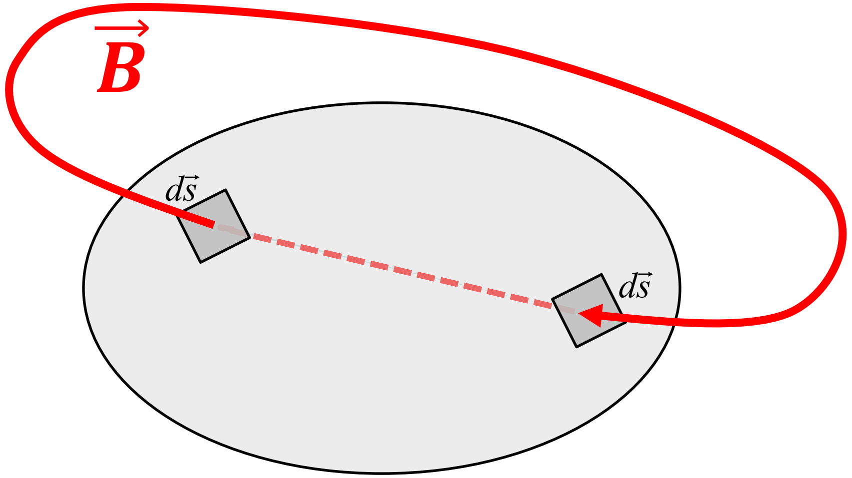
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \qquad \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oiint \vec{B} \cdot d\vec{s}$$

Magnetic flux induces potential

Gauss's Law for Magnetic Field

$$\oiint \vec{\boldsymbol{B}} \cdot \boldsymbol{d\vec{s}} = 0$$

$$\nabla \cdot \vec{\boldsymbol{B}} = 0$$

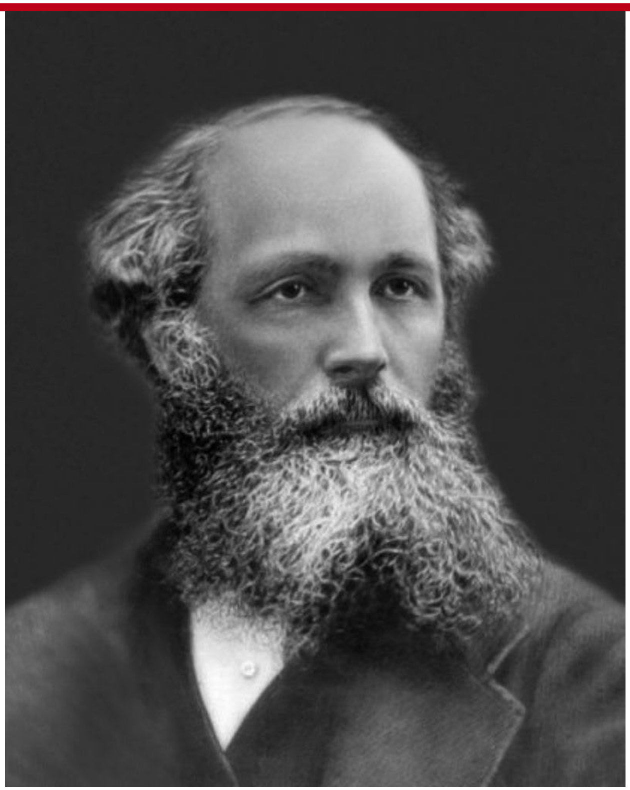


No magnetic flow sources
No net magnetic flux through any closed surface
All magnetic field lines form closed loops

“Maxwell’s” Equations

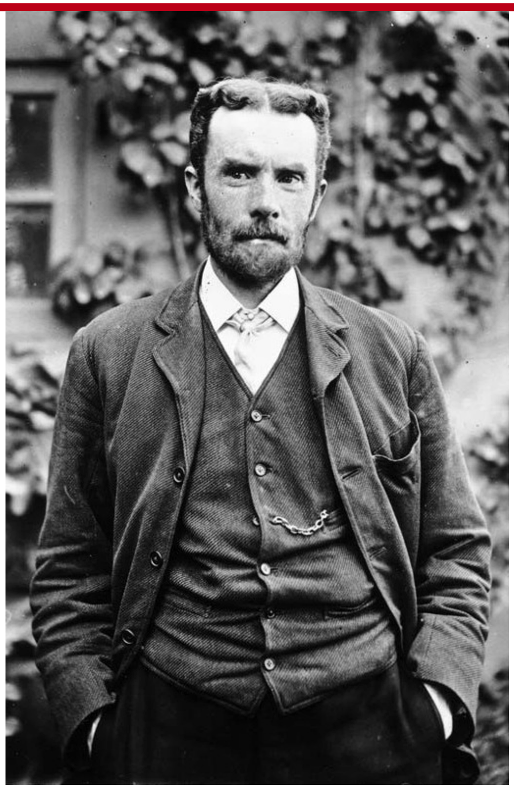
	Differential forms	Integral forms
Gauss’s Law for Electric Field:	$\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}}=\frac{\rho}{\varepsilon}$	$\oiint\vec{\boldsymbol{E}}\cdot\boldsymbol{d}\vec{\boldsymbol{s}}=\frac{Q}{\varepsilon}$
Ampère’s Law:	$\boldsymbol{\nabla}\times\vec{\boldsymbol{H}}=\vec{\boldsymbol{J}}+\varepsilon\frac{\boldsymbol{d}\vec{\boldsymbol{E}}}{\boldsymbol{dt}}$	$\oint\vec{\boldsymbol{H}}\cdot\boldsymbol{d}\vec{\boldsymbol{l}}=\oiint\vec{\boldsymbol{J}}\cdot\boldsymbol{d}\vec{\boldsymbol{s}}+\frac{d}{dt}\oiint\varepsilon\,\vec{\boldsymbol{E}}\cdot\boldsymbol{d}\vec{\boldsymbol{s}}$
Faraday’s Law:	$\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}=-\frac{\boldsymbol{d}\vec{\boldsymbol{B}}}{\boldsymbol{dt}}$	$\oint\vec{\boldsymbol{E}}\cdot\boldsymbol{d}\vec{\boldsymbol{l}}=-\frac{d}{dt}\oiint\vec{\boldsymbol{B}}\cdot\boldsymbol{d}\vec{\boldsymbol{s}}$
Gauss’s Law for Magnetic Field:	$\boldsymbol{\nabla}\cdot\vec{\boldsymbol{B}}=0$	$\oiint\vec{\boldsymbol{B}}\cdot\boldsymbol{d}\vec{\boldsymbol{s}}=0$

Maxwell’s/Heaviside’s Equations



James Clerk Maxwell
13 June 1831
5 November 1879

“A Treatise on Electricity and Magnetism” (1873) included > 20 equations



Oliver Heaviside
18 May 1850
3 February 1925

Synthesized them into the set of 4 equations we know, love, and use to this day

Differential Forms

Differential forms

Gauss's Law for Electric Field:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

Ampère's Law:

$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{d\vec{E}}{dt}$$

Faraday's Law:

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Gauss's Law for Magnetic Field:

$$\nabla \cdot \vec{B} = 0$$

(Yes, I know I've been saying I prefer the integral forms, but the differential forms facilitate the following discussion...)

Differential Forms

$$\nabla \cdot \vec{E} = \boxed{\frac{\rho}{\varepsilon}}$$

$$\nabla \times \vec{H} = \boxed{\vec{J}} + \varepsilon \frac{d\vec{E}}{dt}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \cdot \vec{B} = 0$$

Far from charge & current sources:

$$\rho = 0$$

$$\vec{J} = 0$$

Differential Forms

$$\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}}=0$$

$$\boldsymbol{\nabla}\times\vec{\boldsymbol{H}}=\varepsilon\frac{\boldsymbol{d}\vec{\boldsymbol{E}}}{\boldsymbol{dt}}$$

$$\left.\begin{aligned}\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}&=-\frac{\boldsymbol{d}\vec{\boldsymbol{B}}}{\boldsymbol{dt}}\\[2em]\boldsymbol{\nabla}\cdot\vec{\boldsymbol{B}}&=0\end{aligned}\right\}\ \vec{\boldsymbol{B}}=\mu\vec{\boldsymbol{H}}$$

Differential Forms - Source Free Medium

“Source free” medium

$$\nabla \cdot \vec{\boldsymbol{E}} = 0$$

$$\left.\begin{aligned} \nabla \times \vec{\boldsymbol{H}} &= \varepsilon \frac{\boldsymbol{d}\vec{\boldsymbol{E}}}{dt} \\ \nabla \times \vec{\boldsymbol{E}} &= -\mu \frac{\boldsymbol{d}\vec{\boldsymbol{H}}}{dt} \end{aligned}\right\}$$

Symmetric relationship

$$\nabla \cdot \vec{\boldsymbol{H}} = 0$$

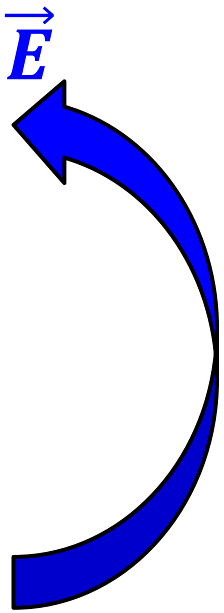
Maxwell's Equations & Wave Propagation

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{H} = \boxed{\varepsilon \frac{d\vec{E}}{dt}}$$

$$\nabla \times \vec{E} = -\mu \frac{d\vec{H}}{dt}$$

$$\nabla \cdot \vec{H} = 0$$



A time-varying electric field...

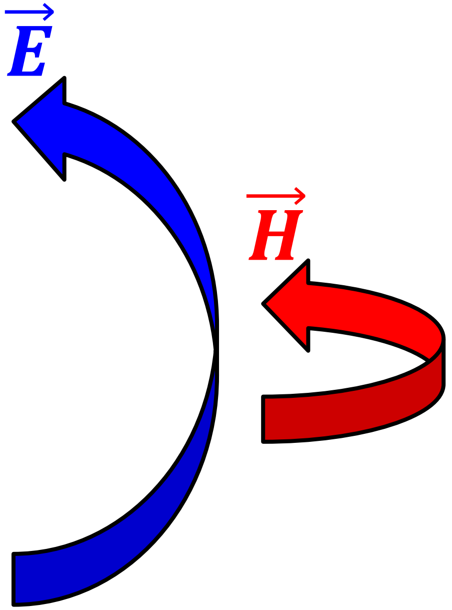
Maxwell's Equations & Wave Propagation (cont.)

$$\nabla \cdot \vec{E} = 0$$

$$\boxed{\nabla \times \vec{H} = \varepsilon \frac{d\vec{E}}{dt}}$$

$$\nabla \times \vec{E} = -\mu \frac{d\vec{H}}{dt}$$

$$\nabla \cdot \vec{H} = 0$$



…is accompanied by a time-varying magnetic field that curls around it…

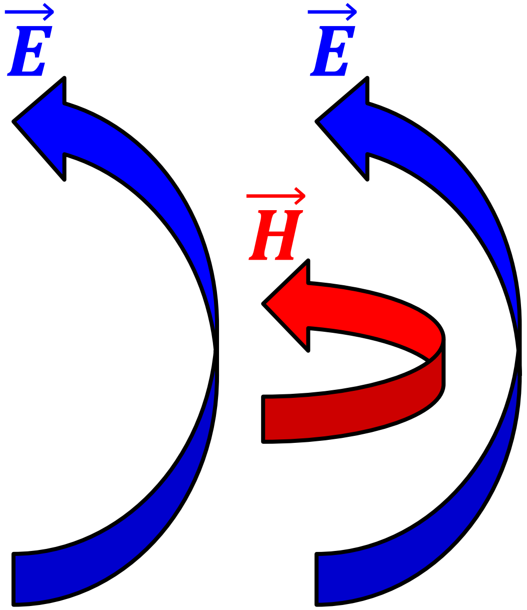
Maxwell's Equations & Wave Propagation (cont.)

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{H} = \varepsilon \frac{d\vec{E}}{dt}$$

$$\boxed{\nabla \times \vec{E} = -\mu \frac{d\vec{H}}{dt}}$$

$$\nabla \cdot \vec{H} = 0$$



...which is accompanied by a time-varying electric field that curls around it...

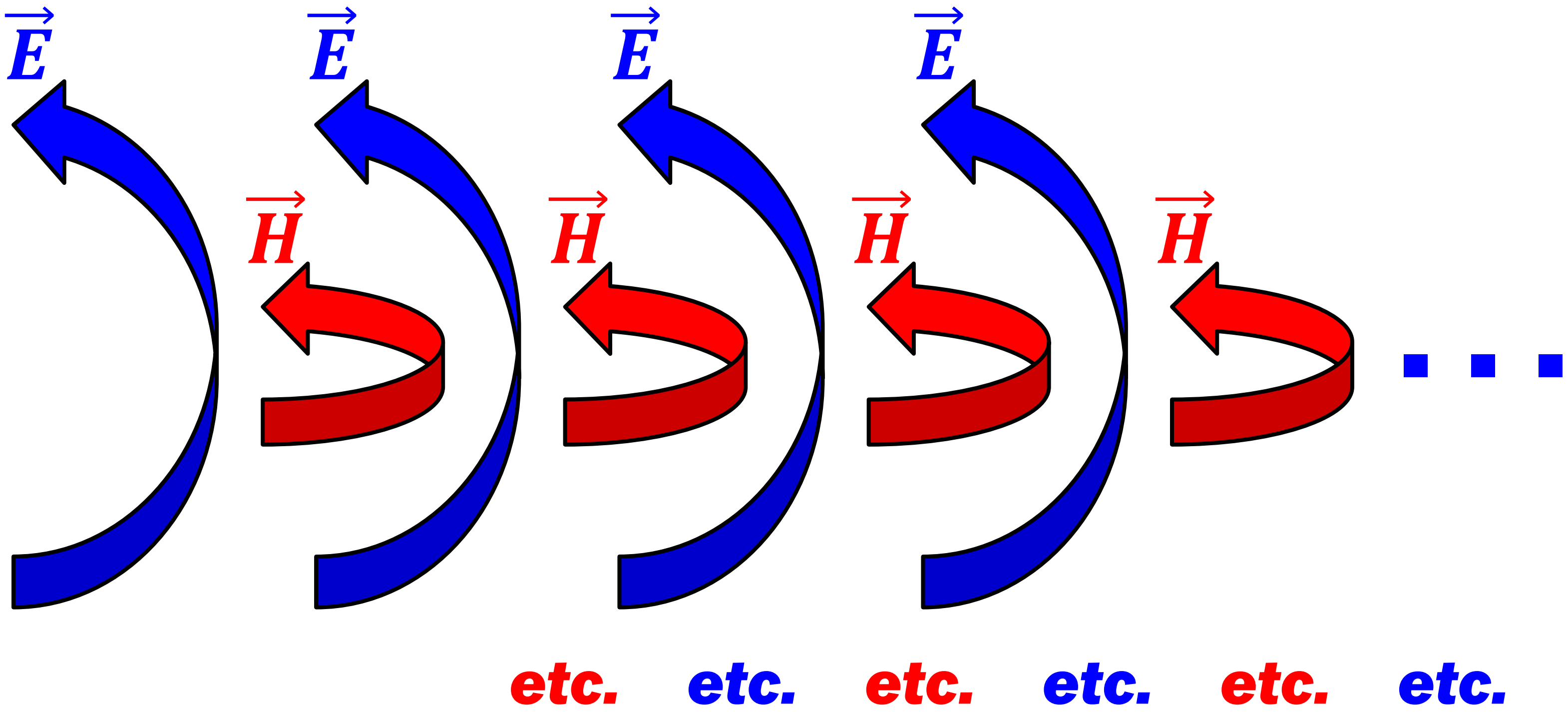
Maxwell's Equations & Wave Propagation

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{H} = \varepsilon \frac{d\vec{E}}{dt}$$

$$\nabla \times \vec{E} = -\mu \frac{d\vec{H}}{dt}$$

$$\nabla \cdot \vec{H} = 0$$



Maxwell's Equations & Wave Propagation – the Math

$$\underline{\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}} = 0}$$

$$\underline{\boldsymbol{\nabla}\times\vec{\boldsymbol{H}} = \varepsilon\frac{\boldsymbol{d}\vec{\boldsymbol{E}}}{dt}}$$

$$\boxed{\boldsymbol{\nabla}\times\vec{\boldsymbol{E}} = -\mu\frac{\boldsymbol{d}\vec{\boldsymbol{H}}}{dt}}$$

$$\boldsymbol{\nabla}\cdot\vec{\boldsymbol{H}} = 0$$

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}\right) = \boldsymbol{\nabla}\times\left(-\mu\frac{\partial\vec{\boldsymbol{H}}}{\partial t}\right)$$

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}\right) = -\mu\frac{\partial}{\partial t}\underline{\left(\boldsymbol{\nabla}\times\vec{\boldsymbol{H}}\right)}$$

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}\right) = -\mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2}$$

Vector identity (proof in backup):

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}\right) = \boldsymbol{\nabla}\underbrace{\left(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}}\right)}_{=\,0} - \boldsymbol{\nabla}^2\vec{\boldsymbol{E}}$$

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}\right) = -\boldsymbol{\nabla}^2\vec{\boldsymbol{E}}$$

$$\boldsymbol{\nabla}^2\vec{\boldsymbol{E}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0$$

Wave equation for electric field

Maxwell's Equations & Wave Propagation – the Math (cont.)

$$\nabla \cdot \vec{E} = 0$$

$$\boxed{\nabla \times \vec{H} = \varepsilon \frac{d\vec{E}}{dt}}$$

$$\underline{\nabla \times \vec{E} = -\mu \frac{d\vec{H}}{dt}}$$

$$\underline{\nabla \cdot \vec{H} = 0}$$

Parallel approach for H-field

$$\nabla \times \left(\nabla \times \vec{H}\right) = \nabla \times \left(\varepsilon \frac{d\vec{E}}{dt}\right)$$

$$\nabla \times \left(\nabla \times \vec{H}\right) = \varepsilon \frac{\partial}{\partial t} \underline{\left(\nabla \times \vec{E}\right)}$$

$$\nabla \times \left(\nabla \times \vec{H}\right) = -\mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

Vector identity (proof in backup):

$$\nabla \times \left(\nabla \times \vec{H}\right) = \nabla\underbrace{\left(\nabla \cdot \vec{H}\right)}_{=\,0} - \nabla^2 \vec{H}$$

$$\nabla \times \left(\nabla \times \vec{H}\right) = -\nabla^2 \vec{H}$$

Wave equation for magnetic field

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Maxwell’s Equations & Wave Propagation – the Math (cont.)

$$\boldsymbol{\nabla}^2\vec{\boldsymbol{E}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0 \qquad \boldsymbol{\nabla}^2\vec{\boldsymbol{H}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{H}}}{\partial t^2} = 0$$

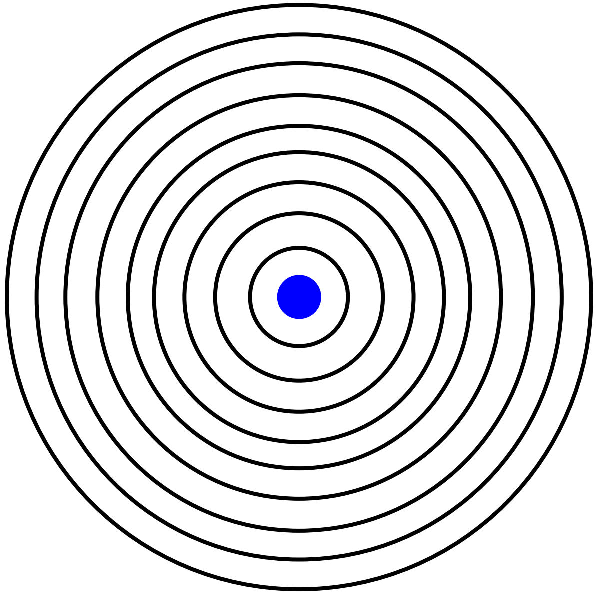
3-dimensional 2nd order differential equations

$$\left.\begin{aligned}
&\vec{\boldsymbol{a}}_x\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\right)\\
&+\vec{\boldsymbol{a}}_y\left(\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2}\right)\\
&+\vec{\boldsymbol{a}}_z\left(\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial z^2}\right)\\
&-\mu\varepsilon\left[\vec{\boldsymbol{a}}_x\frac{\partial^2 E_x}{\partial t^2}+\vec{\boldsymbol{a}}_y\frac{\partial^2 E_y}{\partial t^2}+\vec{\boldsymbol{a}}_z\frac{\partial^2 E_z}{\partial t^2}\right]\\
&=0
\end{aligned}\right\}$$

Similar equation for H

Anyone for simplification…?

Point Source Approximation



At sufficient distance from source, it can be approximated as a point source

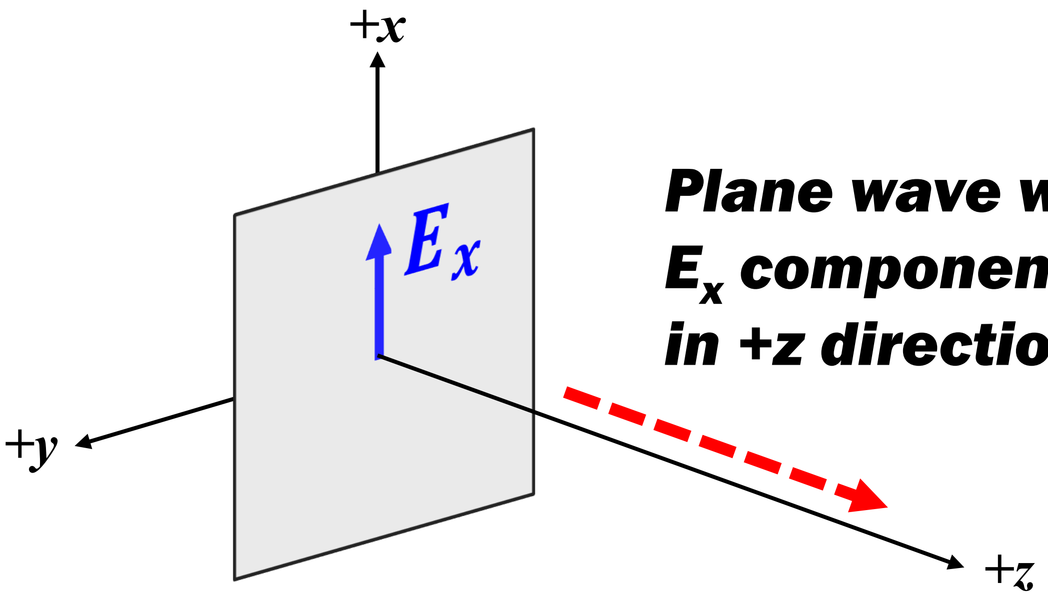
Energy propagates outward as expanding spherical wave

Plane Wave Approximation

Farther still from source, small portion of spherical wave front can be approximated as uniform plane wave

MUCH simpler to analyze…

Maxwell's Equations & Wave Propagation – the Math (cont.)



Plane wave with only E_x component travelling in +z direction

Maxwell’s Equations & Wave Propagation – the Math (cont.)

$$\nabla^2 \vec{\boldsymbol{E}} - \mu\varepsilon \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} = 0$$

$$\begin{aligned}
&\vec{\boldsymbol{a}}_x\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\right)\\
&\boxed{\begin{aligned}&+\vec{\boldsymbol{a}}_y\left(\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2}\right)\\
&+\vec{\boldsymbol{a}}_z\left(\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial z^2}\right)\end{aligned}}\\
&-\mu\varepsilon\left[\vec{\boldsymbol{a}}_x\frac{\partial^2 E_x}{\partial t^2}+\boxed{\vec{\boldsymbol{a}}_y\frac{\partial^2 E_y}{\partial t^2}+\vec{\boldsymbol{a}}_z\frac{\partial^2 E_z}{\partial t^2}}\right]\\
&=0
\end{aligned}$$

$$E_y, E_z = 0$$

Maxwell’s Equations & Wave Propagation – the Math (cont.)

$$\nabla^2\vec{\boldsymbol{E}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0$$

$$\vec{\boldsymbol{a}}_x\left(\underbrace{\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}}_{\substack{dx,\ dy\\ \text{terms} = 0}}+\frac{\partial^2 E_x}{\partial z^2}\right)$$

$$-\vec{\boldsymbol{a}}_x\mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}$$

$$= 0$$

Maxwell’s Equations & Wave Propagation – the Math (cont.)

$$\boldsymbol{\nabla}^2\vec{\boldsymbol{E}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0$$

$$\vec{\boldsymbol{a}}_x\frac{\partial^2 E_x}{\partial z^2}$$

$$-\vec{\boldsymbol{a}}_x\mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}$$

$$= 0$$

Maxwell's Equations & Wave Propagation – the Math (cont.)

$$\boldsymbol{\nabla}^2\vec{\boldsymbol{E}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0$$

$$\vec{\boldsymbol{a}}_x\frac{\partial^2 E_x}{\partial z^2} - \vec{\boldsymbol{a}}_x\mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} = 0$$

$$\frac{\partial^2 E_x}{\partial z^2} - \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} = 0$$

1-dimensional 2nd order differential equation

$$E(z,t) = E_0 e^{j(\omega t - \beta z)}$$

General solution

Angular temporal frequency $\omega = 2\pi f = \frac{2\pi}{T}$ $T = period\ (temporal\ )$

Wavenumber $\beta = \frac{2\pi}{\lambda}$ $\lambda = wavelength\ (spatial)$

Parallel relationship between time-variance and spatial propagation

Maxwell’s Equations & Wave Propagation – the Math (cont.)

$$\frac{\partial^2 E_x}{\partial z^2} - \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} = 0 \qquad E(z,t) = E_0 e^{j(\omega t - \beta z)}$$

$$[-\beta^2 + \omega^2\mu\varepsilon]\underbrace{\left[E_0 e^{j(\omega t - \beta z)}\right]}_{\text{This term = 0 is not useful solution}} = 0$$

This term = 0 is not useful solution

$$-\beta^2 + \omega^2\mu\varepsilon = 0$$

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$\frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Back to this presently…

Maxwell's Equations & Wave Propagation – the Math (cont.)

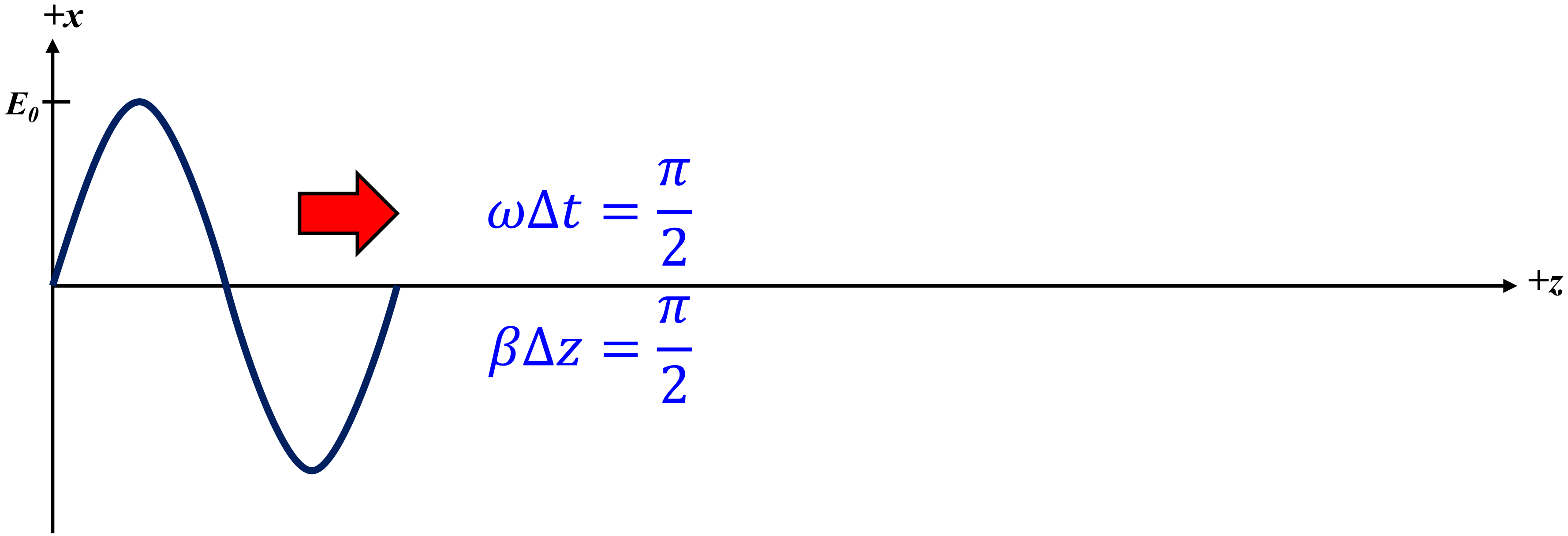
$$E(z,t) = E_0 e^{j(\omega t - \beta z)}$$

$$\Re e[E(z,t)] = E_0 \cos(\omega t - \beta z)$$

Maxwell's Equations & Wave Propagation – the Math (cont.)

$$E(z,t) = E_0 e^{j(\omega t - \beta z)}$$

$$\Re e[E(z,t)] = E_0 \cos(\omega t - \beta z)$$



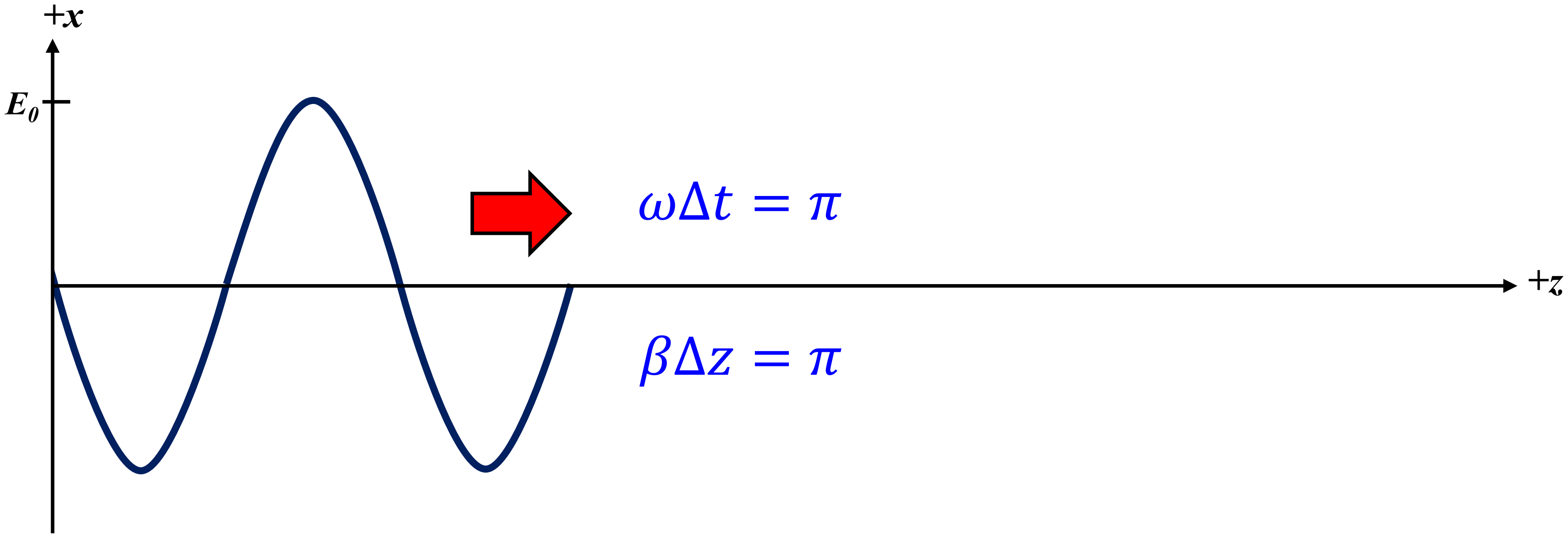
$$\omega \Delta t = \frac{\pi}{2}$$

$$\beta \Delta z = \frac{\pi}{2}$$

Maxwell's Equations & Wave Propagation – the Math (cont.)

$$E(z,t) = E_0 e^{j(\omega t - \beta z)}$$

$$\Re e[E(z,t)] = E_0 \cos(\omega t - \beta z)$$



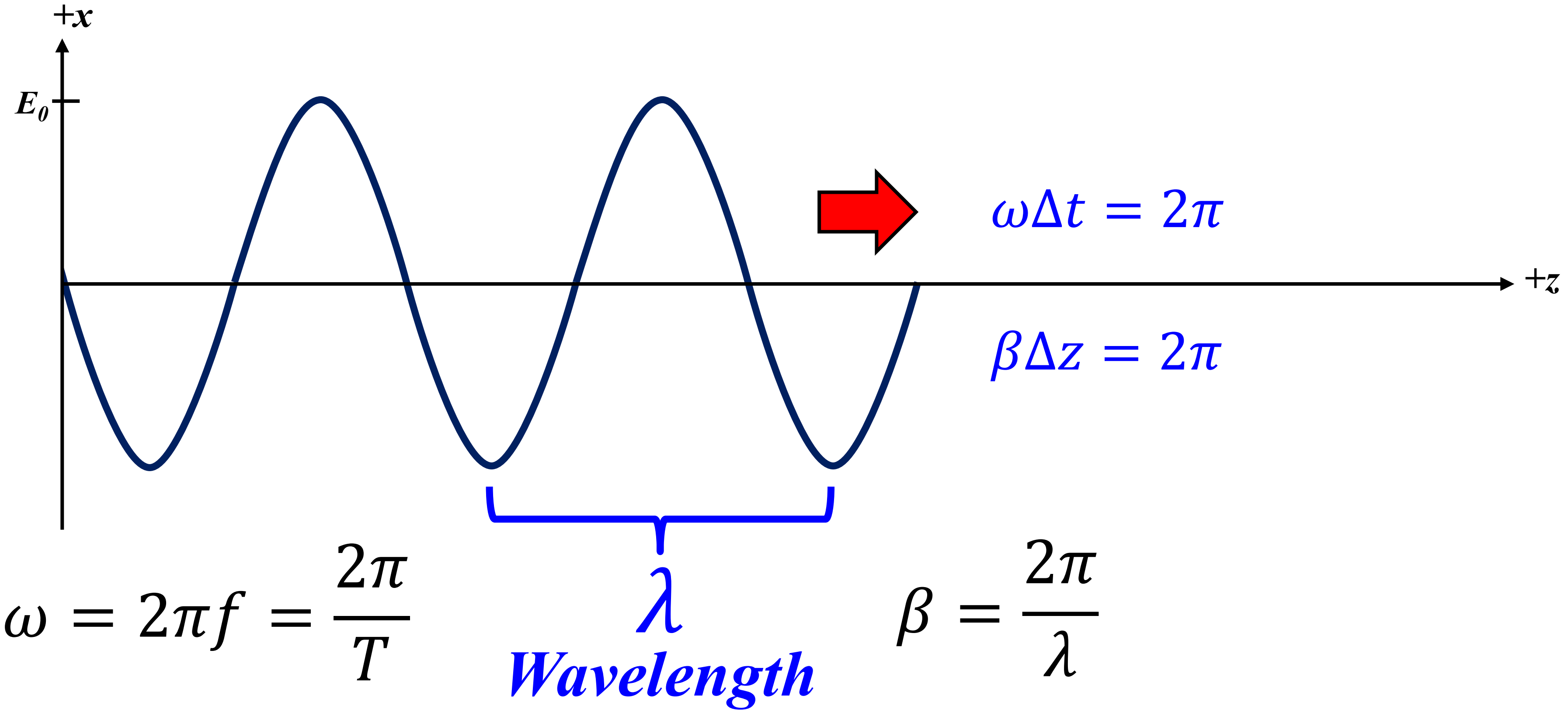
$\omega \Delta t = \pi$

$\beta \Delta z = \pi$

Maxwell's Equations & Wave Propagation – the Math (cont.)

$$E(z,t) = E_0 e^{j(\omega t - \beta z)}$$

$$\Re e[E(z,t)] = E_0 \cos(\omega t - \beta z)$$



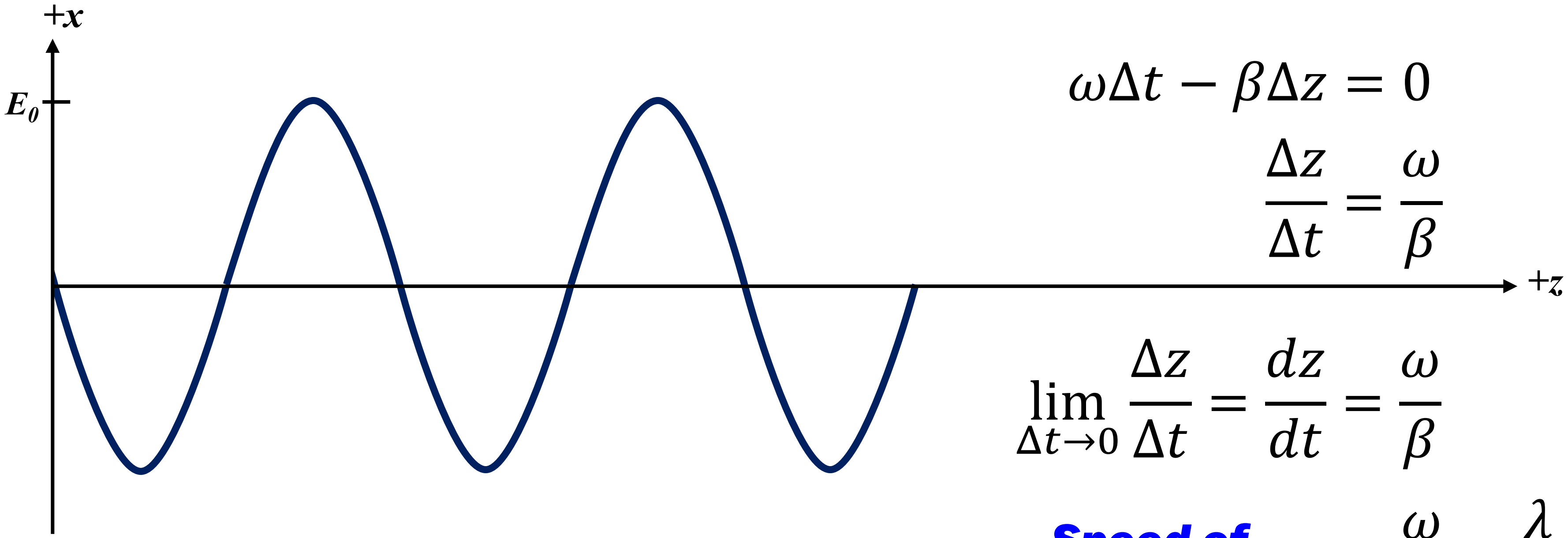
$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$\beta = \frac{2\pi}{\lambda}$$

Maxwell's Equations & Wave Propagation – the Math (cont.)

$$E(z,t) = E_0 e^{j(\omega t - \beta z)}$$

$$\Re e[E(z,t)] = E_0 \cos(\omega t - \beta z)$$



$$\omega \Delta t - \beta \Delta z = 0$$

$$\frac{\Delta z}{\Delta t} = \frac{\omega}{\beta}$$

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt} = \frac{\omega}{\beta}$$

Speed of propagation $u = \frac{\omega}{\beta} = \frac{\lambda}{T}$

Speed of Propagation

Speed of propagation $u = \frac{\omega}{\beta}$ **Recall:** $\frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$ ➡ $u = \frac{1}{\sqrt{\mu\varepsilon}}$

Free space $u = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ $\mu_0 = 4\pi \times 10^{-7}\,\frac{H}{m}$ $\varepsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}\,\frac{F}{m}$

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\frac{1}{9} \times 10^{-16}\,\frac{H \cdot F}{m^2}}} \qquad H \cdot F = sec^2$$

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\ \times 10^{8}\,m/sec\ \approx 300{,}000\,km/sec$$

$$\approx 186{,}000\,miles/sec$$

Speed of light in free space

Speed of Light in Free Space

$$\left.\begin{aligned} c &\approx 3\ \times 10^{8}\ m/sec \\ c &= 299{,}792{,}458\ m/sec \end{aligned}\right\}$$

Agrees to within < 0.1%

$$\mu_0 = 1.25663706212(19) \times 10^{-6}\ H/m$$

$$\mu_0 \approx 4\pi \times 10^{-7}\ H/m$$

$$\mu_0 \approx \underbrace{1.25663706}14359\ldots \times 10^{-6}\ H/m$$

Agrees to 9 significant digits

< 1 part in 10^9

$$\varepsilon_0 = 8.8541878128(13) \times 10^{-12}\ F/m$$

$$\varepsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}\ F/m$$

$$\varepsilon_0 \approx 8.841941282883\ \ldots\ \times 10^{-12}\ F/m$$

Agrees to within < 0.2%

Frequency and Wavelength

$$c \approx 3 \times 10^8 \; m/sec$$

Approximate value facilitates quick estimates of wavelength

$$c = \frac{\omega}{\beta} = \frac{\lambda}{T} = \lambda f$$

$$\lambda = \frac{c}{f} \approx \frac{3 \times 10^8 \; m/sec}{f} \approx \frac{300 \; Mm/sec}{f}$$

When circuit element is significant fraction of wavelength, it will behave as transmission line and/or antenna

$$\lambda \approx \frac{300}{f_{MHz}} \; (m)$$

$$\lambda \approx \frac{0.3}{f_{GHz}} \; (m)$$

f = 30 MHz:

λ = 10 m

λ/10 = 1 m

≈ 3 feet

Cables…?

What About H?

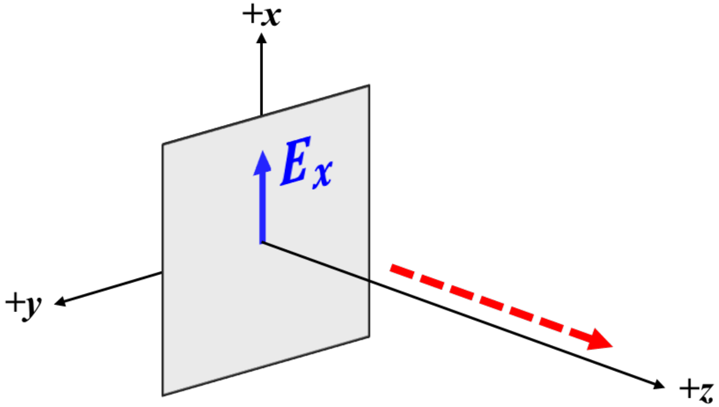
$$\nabla^2\vec{\boldsymbol{E}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0$$

$$E_x(z,t) = E_0 e^{j(\omega t-\beta z)}$$

$$\nabla^2\vec{\boldsymbol{H}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{H}}}{\partial t^2} = 0$$

$$H(z,t) = H_0 e^{j(\omega t-\beta z)}$$

Direction and relative magnitude?



$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{a}}_x E_x$$

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{a}}_x E_0 e^{j(\omega t-\beta z)}$$

$$\nabla\times\vec{\boldsymbol{E}} = -\mu\frac{\partial\vec{\boldsymbol{H}}}{\partial t}$$

$$\nabla\times\vec{\boldsymbol{E}} = \begin{vmatrix} \vec{\boldsymbol{a}}_x & \vec{\boldsymbol{a}}_y & \vec{\boldsymbol{a}}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \vec{\boldsymbol{a}}_y\frac{\partial E_x}{\partial z}$$

$$= -\vec{\boldsymbol{a}}_y j\beta E_0 e^{j(\omega t-\beta z)}$$

H vector is orthogonal to E vector

What About H? (cont.)

$$\nabla^2\vec{\boldsymbol{E}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0 \qquad \nabla^2\vec{\boldsymbol{H}} - \mu\varepsilon\frac{\partial^2\vec{\boldsymbol{H}}}{\partial t^2} = 0$$

$$E_x(z,t) = E_0 e^{j(\omega t - \beta z)} \qquad H(z,t) = H_0 e^{j(\omega t - \beta z)}$$

$$\nabla \times \vec{\boldsymbol{E}} = -\mu\frac{\partial\vec{\boldsymbol{H}}}{\partial t}$$

$$-\vec{\boldsymbol{a}}_{\boldsymbol{y}} j\beta E_0 e^{j(\omega t-\beta z)} = -\mu\left(j\omega\vec{\boldsymbol{H}}\right) = -j\omega\mu\vec{\boldsymbol{H}}$$

$$-\vec{\boldsymbol{a}}_{\boldsymbol{y}} j\beta E_0 e^{j(\omega t-\beta z)} = -\vec{\boldsymbol{a}}_{\boldsymbol{y}} j\omega\mu H_0 e^{j(\omega t-\beta z)}$$

$$\beta E_0 = \omega\mu H_0$$

$$\frac{E_0}{H_0} = \frac{\omega\mu}{\beta} \qquad \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\frac{E_0}{H_0} = \frac{\mu}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

Characteristic Impedance

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu}{\varepsilon}} \qquad \frac{V/m}{A/m}$$

Characteristic impedance of medium

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad (\Omega)$$

Free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx \sqrt{\frac{4\pi \times 10^{-7}\ H/m}{\left(\frac{1}{36\pi} \times 10^{-9}\ F/m\right)}}$$

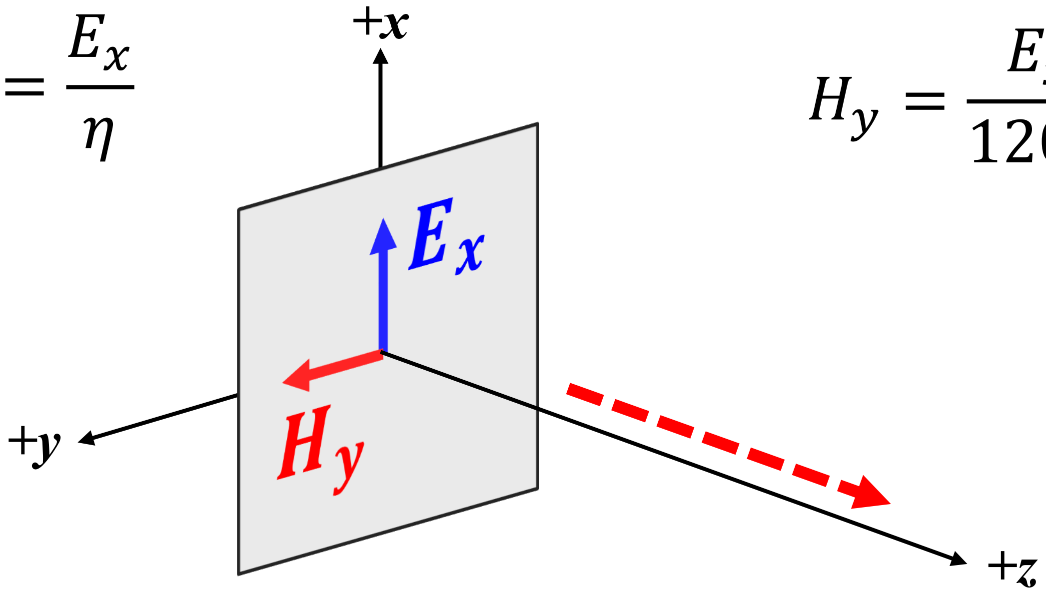
$$\eta_0 \approx \sqrt{144\pi^2 \times 10^2\ H/F} \qquad H/F = \Omega^2$$

$$\left.\begin{aligned} \eta_0 &\approx 120\pi\ \Omega \\ \eta_0 &\approx 377\ \Omega \end{aligned}\right\}$$

You'll see these again…

Characteristic Impedance (cont.)

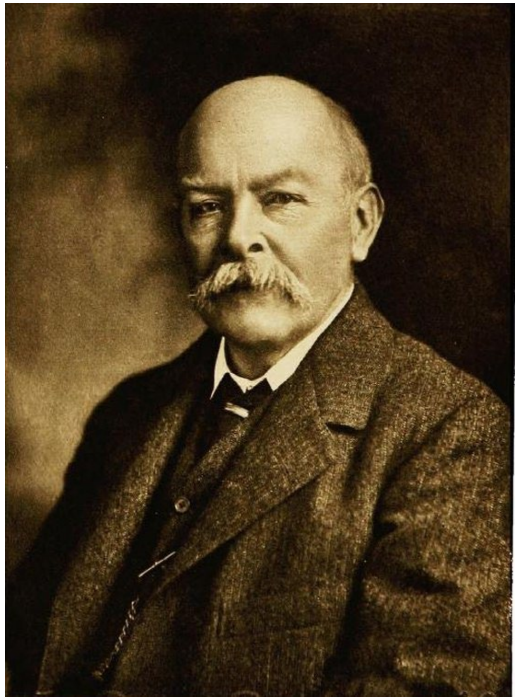
$$H_y = \frac{E_x}{\eta}$$



Free space:

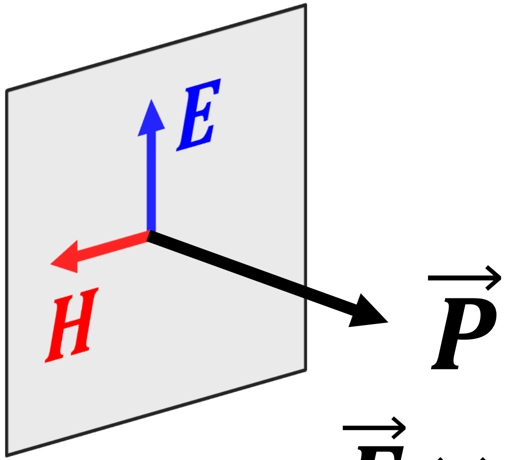
$$H_y = \frac{E_x}{120\pi} = \frac{E_x}{377\ \Omega}$$

Poynting Vector



John Henry Poynting
9 September 1852
30 March 1914

Discovered vector in 1884



$$\vec{P} = \vec{E} \times \vec{H}$$

Poynting vector

Points in direction of propagation

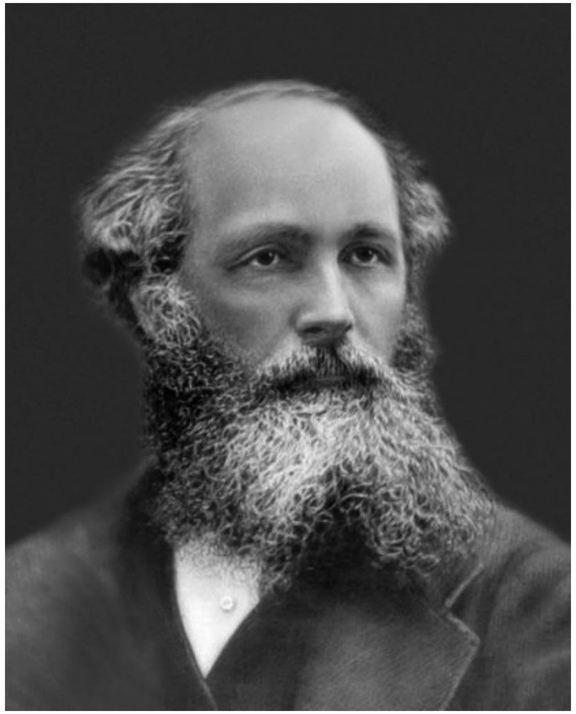
Orthogonal to both E and H vectors

$$|\vec{P}| = |E||H|$$

Power density

$$\frac{V}{m} \cdot \frac{A}{m} = \frac{W}{m^2}$$

Theory and Experiment: Two Sides of the Same Coin



Maxwell proposed the existence of electromagnetic waves because the math said they should exist

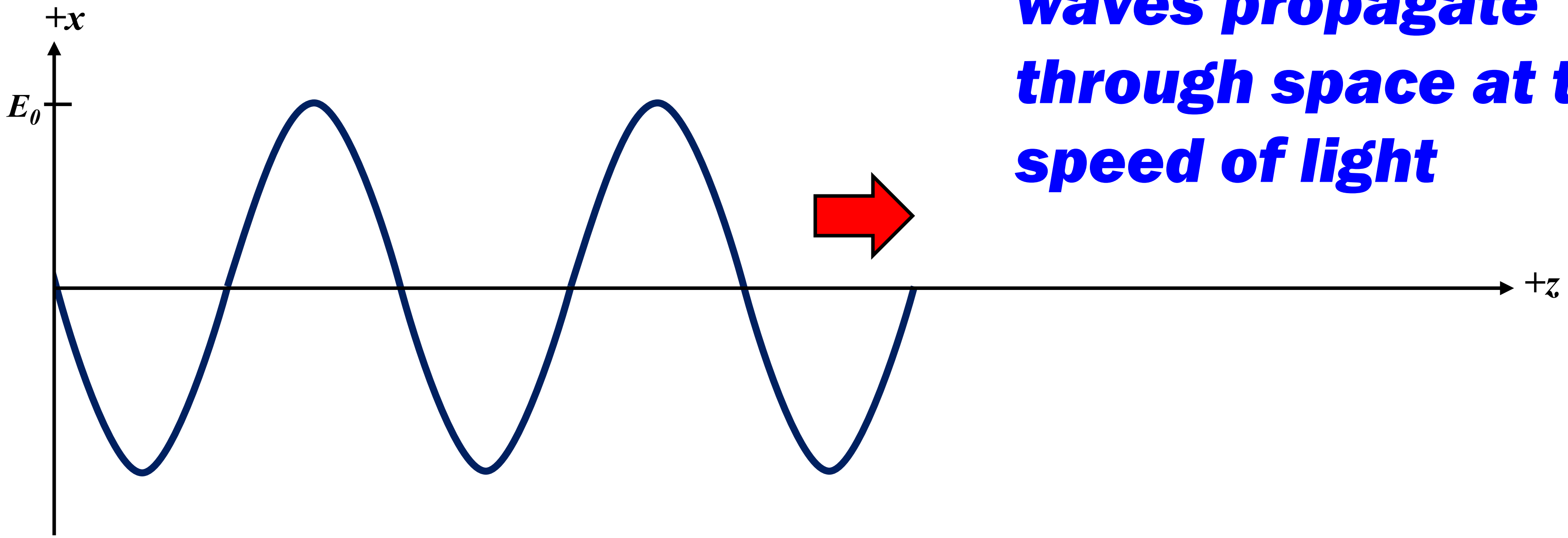
Correctly concluded that light itself is an electromagnetic wave

Heinrich Rudolf Hertz
22 February 1857
1 January 1894

Later confirmed Maxwell’s predictions through his experiments with radio waves

Maxwell's Equations and Wave Propagation: Summary

$$\Re e[E(z,t)] = E_0 \cos(\omega t - \beta z)$$



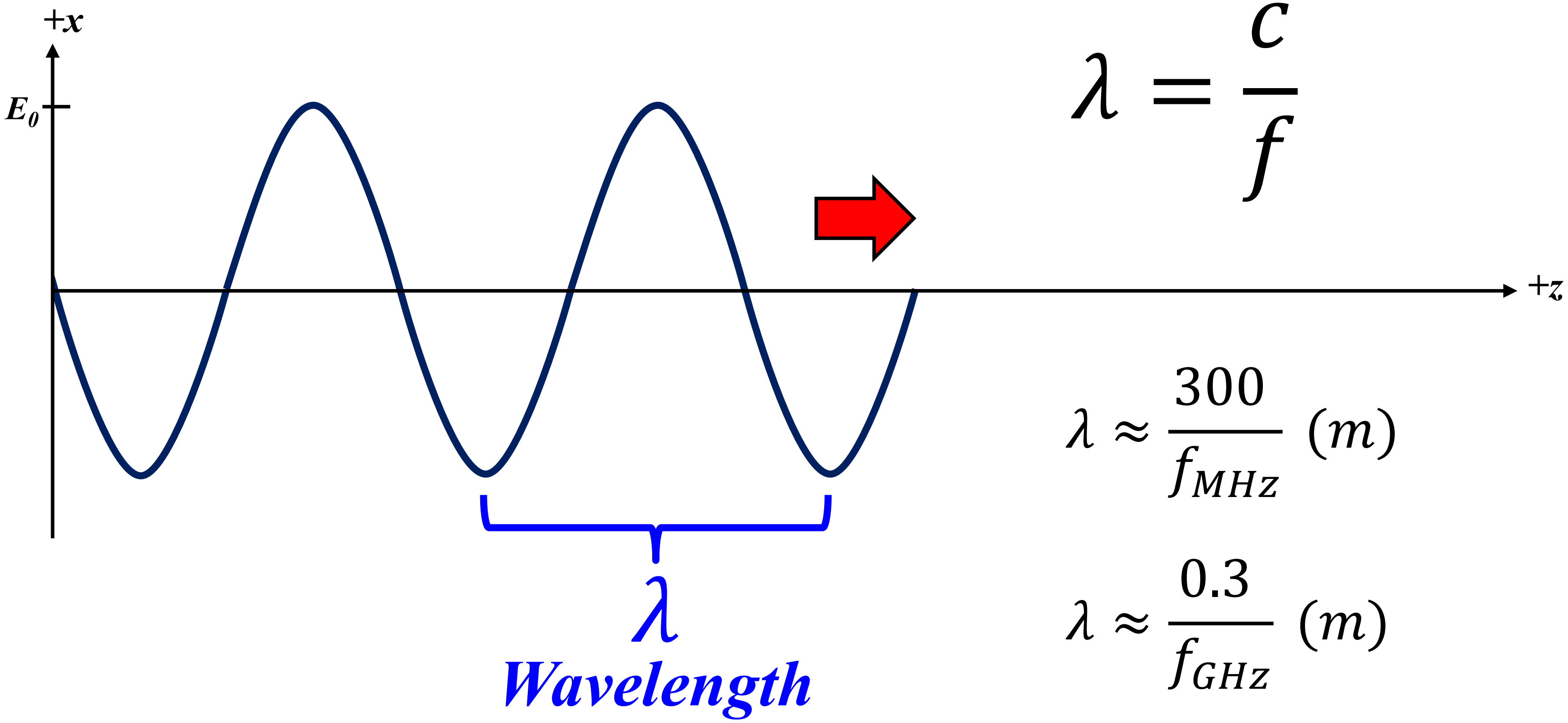
Electromagnetic waves propagate through space at the speed of light

Speed of propagation $u = \frac{1}{\sqrt{\mu\varepsilon}}$

Free space $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 3 \times 10^8 \, m/sec$

Maxwell’s Equations and Wave Propagation: Summary (cont.)

$$\Re e[E(z,t)] = E_0 \cos(\omega t - \beta z)$$



$$\lambda = \frac{c}{f}$$

$$\lambda \approx \frac{300}{f_{MHz}}\ (m)$$

$$\lambda \approx \frac{0.3}{f_{GHz}}\ (m)$$

Maxwell's Equations and Wave Propagation: Summary (cont.)

Characteristic impedance of medium

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Free space

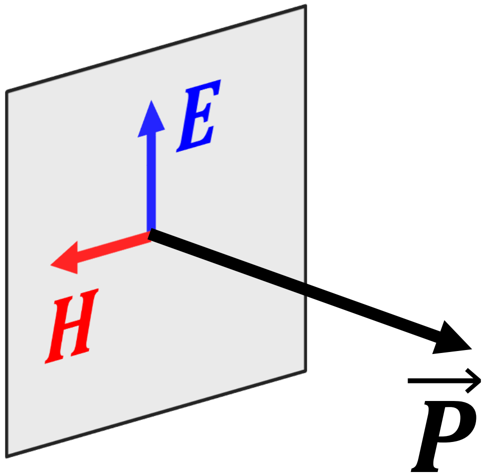
$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\eta_0 \approx 120\pi\ \Omega$$

$$\eta_0 \approx 377\ \Omega$$

Maxwell's Equations and Wave Propagation: Summary (cont.)

$$\frac{E}{H} = \eta$$



$$\vec{P} = \vec{E} \times \vec{H}$$

Poynting vector

Uniform plane wave far from sources:

- ***E and H vectors are orthogonal***
- ***E/H is given by characteristic impedance of medium η***
- ***Wave propagates in direction normal to plane given by Poynting Vector***
- ***Power density given by magnitude of Poynting vector, which is given by the product of E and H magnitudes***

$$|\vec{P}| = |E||H|$$

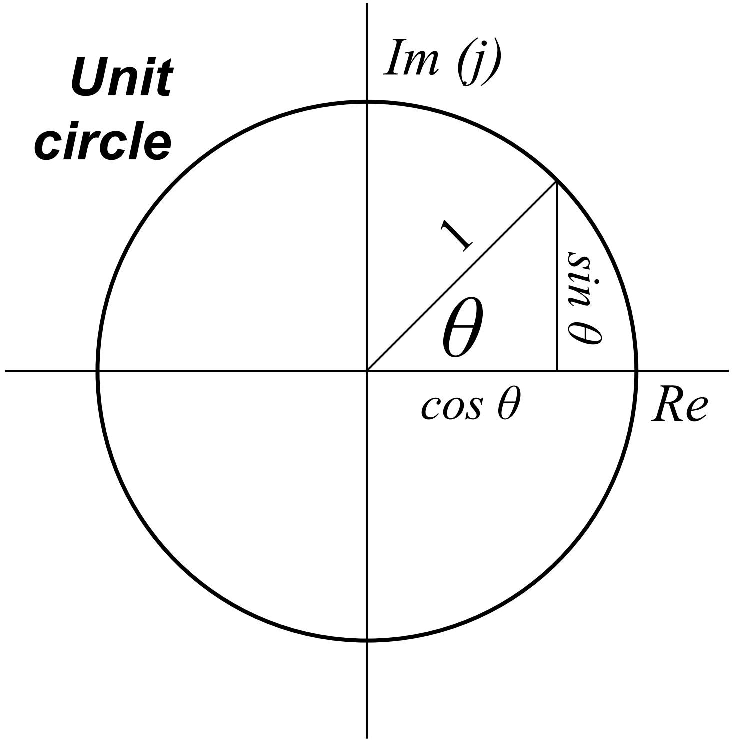
Power density (W/m^2)

Maxwell’s Equations and Wave Propagation: Summary (cont.)

- All very useful analytical tools that are used throughout any work in electromagnetics...
- ...provided by the math...
- ...and verified by experiment.
- You will see them again...

Maxwell’s Equations and Wave Propagation BACKUP

Trigonometric / Exponential Identities



$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$j\sin\theta = \tfrac{1}{2}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\frac{d}{dx}e^{ax} = ae^{ax} \qquad \int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

Hyperbolic trigonometric functions (you may encounter them):

$$\cosh x = \tfrac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\sinh x = \tfrac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Vector Calculus Basics

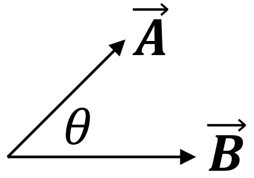
Vector components (Cartesian coordinates)

$$\vec{\boldsymbol{A}} = \vec{\boldsymbol{a}}_{\boldsymbol{x}} A_x + \vec{\boldsymbol{a}}_{\boldsymbol{y}} A_y + \vec{\boldsymbol{a}}_{\boldsymbol{z}} A_z \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{a}}_{\boldsymbol{x}} B_x + \vec{\boldsymbol{a}}_{\boldsymbol{y}} B_y + \vec{\boldsymbol{a}}_{\boldsymbol{z}} B_z$$

Dot (scalar) product

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = \left|\vec{\boldsymbol{A}}\right|\left|\vec{\boldsymbol{B}}\right| \cos\theta$$



Cross product

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = \begin{bmatrix} \vec{\boldsymbol{a}}_x & \vec{\boldsymbol{a}}_y & \vec{\boldsymbol{a}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$= \vec{\boldsymbol{a}}_{\boldsymbol{x}}\left(A_y B_z - A_z B_y\right) + \vec{\boldsymbol{a}}_{\boldsymbol{y}}(A_z B_x - A_x B_z) + \vec{\boldsymbol{a}}_{\boldsymbol{z}}\left(A_x B_y - A_y B_x\right)$$

$$\left|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}\right| = \left|\vec{\boldsymbol{A}}\right|\left|\vec{\boldsymbol{B}}\right| \sin\theta$$

Vector Calculus Basics (cont.)

Del operator $$\nabla = \vec{\boldsymbol{a}}_x \frac{\partial}{\partial x} + \vec{\boldsymbol{a}}_y \frac{\partial}{\partial y} + \vec{\boldsymbol{a}}_z \frac{\partial}{\partial z}$$

Gradient $$\nabla V = \vec{\boldsymbol{a}}_x \frac{\partial V}{\partial x} + \vec{\boldsymbol{a}}_y \frac{\partial V}{\partial y} + \vec{\boldsymbol{a}}_z \frac{\partial V}{\partial z}$$

(spatial variance of scalar field)

Divergence $$\nabla \cdot \vec{\boldsymbol{A}} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint_S \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{s}}$$

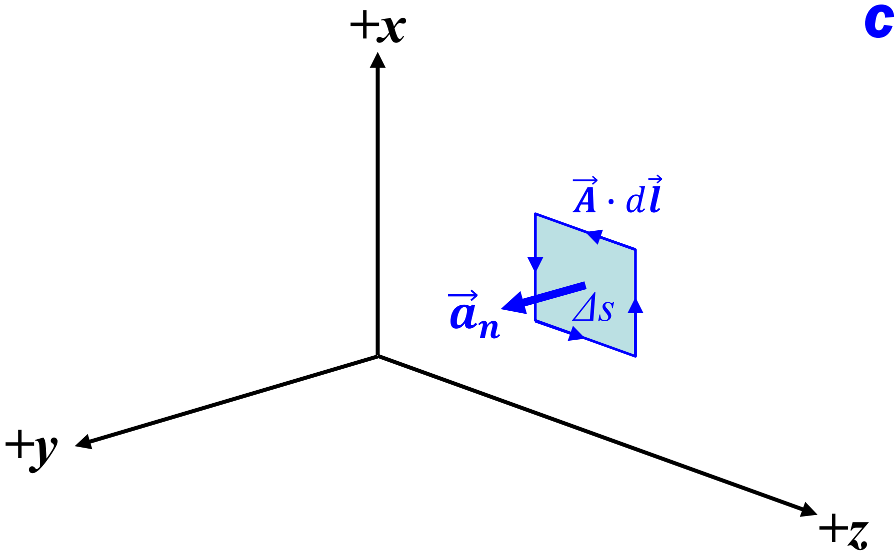
(Net outward flux of vector field per unit volume around a point as volume tends to zero)

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence Theorem $$\int_V \nabla \cdot \vec{\boldsymbol{A}}\, dv = \oint_S \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{s}}$$

Volume integral of the divergence of a vector field equals the total outward flux of the vector field through the surface bounding the volume.

Vector Calculus Basics (cont.)



Curl

$$\nabla \times \vec{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\vec{a}_n \oint_C \vec{A} \cdot d\vec{l} \right]$$

(tendency of vector field to curl or rotate about a point in space)

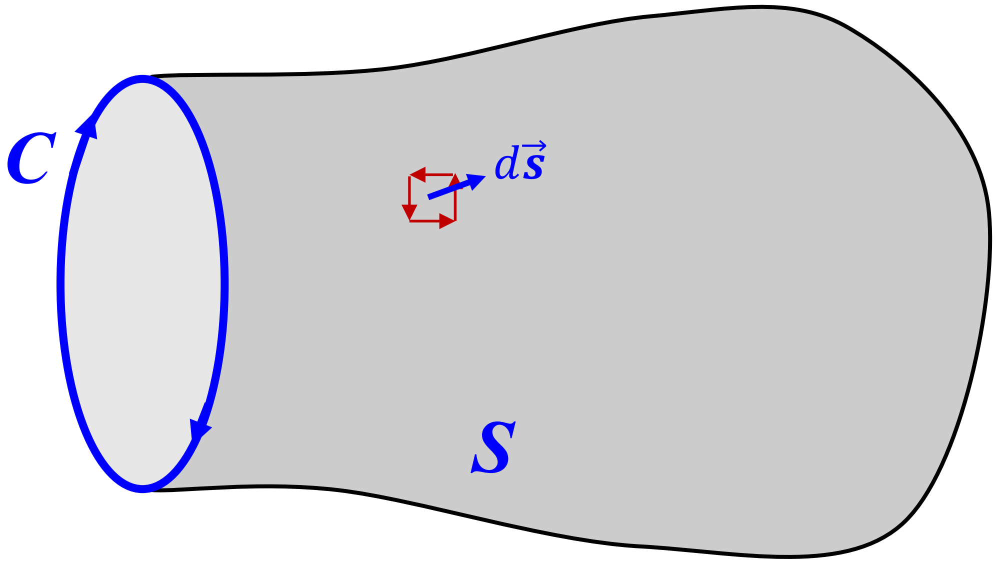
$$\nabla \times \vec{A} = \begin{bmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$= \vec{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \vec{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \vec{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Vector Calculus Basics (cont.)

Stokes' Theorem: $$\iint_S (\nabla \times \vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{s}} = \oint_C \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{l}}$$

The surface integral of the curl of a vector field over an open surface equals the closed line integral of the vector along the contour bounding the surface.



Vector Identity Proof

Proof of:

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

Gradient of divergence

$$\nabla\left(\nabla \cdot \vec{A}\right) = \vec{a}_x\left(\frac{\partial^2 A_x}{\partial^2 x} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}\right) + \vec{a}_y\left(\frac{\partial^2 A_y}{\partial^2 y} + \frac{\partial^2 A_z}{\partial y \partial z} + \frac{\partial^2 A_x}{\partial x \partial y}\right) + \vec{a}_z\left(\frac{\partial^2 A_z}{\partial^2 z} + \frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z}\right)$$

Laplacian

$$\nabla^2 \vec{A} = \vec{a}_x\left(\frac{\partial^2 A_x}{\partial^2 x} + \frac{\partial^2 A_x}{\partial^2 y} + \frac{\partial^2 A_x}{\partial^2 z}\right) + \vec{a}_y\left(\frac{\partial^2 A_y}{\partial^2 y} + \frac{\partial^2 A_y}{\partial^2 x} + \frac{\partial^2 A_y}{\partial^2 z}\right) + \vec{a}_z\left(\frac{\partial^2 A_z}{\partial^2 z} + \frac{\partial^2 A_z}{\partial^2 x} + \frac{\partial^2 A_z}{\partial^2 y}\right)$$

Vector Identity Proof (cont.)

Proof of:

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Curl of curl:

$$\nabla \times (\nabla \times \vec{A}) = \vec{a}_x\left[\left(\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial^2 y}\right) - \left(\frac{\partial^2 A_x}{\partial^2 z} - \frac{\partial^2 A_z}{\partial x \partial z}\right)\right] + \vec{a}_y\left[\left(\frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_y}{\partial^2 z}\right) - \left(\frac{\partial^2 A_y}{\partial^2 x} - \frac{\partial^2 A_x}{\partial x \partial y}\right)\right] + \vec{a}_z\left[\left(\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial^2 x}\right) - \left(\frac{\partial^2 A_z}{\partial^2 y} - \frac{\partial^2 A_y}{\partial y \partial z}\right)\right]$$

$$= \vec{a}_x\left[\left(\frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}\right) - \left(\frac{\partial^2 A_x}{\partial^2 y} + \frac{\partial^2 A_x}{\partial^2 z}\right)\right] + \vec{a}_y\left[\left(\frac{\partial^2 A_z}{\partial y \partial z} + \frac{\partial^2 A_x}{\partial x \partial y}\right) - \left(\frac{\partial^2 A_y}{\partial^2 x} + \frac{\partial^2 A_y}{\partial^2 z}\right)\right] + \vec{a}_z\left[\left(\frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z}\right) - \left(\frac{\partial^2 A_z}{\partial^2 x} + \frac{\partial^2 A_z}{\partial^2 y}\right)\right]$$

$$\begin{aligned}
\nabla \times (\nabla \times \vec{A}) = {} & \vec{a}_x\left[\left(\boxed{\frac{\partial^2 A_x}{\partial^2 x}} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}\right) - \left(\boxed{\frac{\partial^2 A_x}{\partial^2 x}} + \frac{\partial^2 A_x}{\partial^2 y} + \frac{\partial^2 A_x}{\partial^2 z}\right)\right] \\
& + \vec{a}_y\left[\left(\boxed{\frac{\partial^2 A_y}{\partial^2 y}} + \frac{\partial^2 A_z}{\partial y \partial z} + \frac{\partial^2 A_x}{\partial x \partial y}\right) - \left(\boxed{\frac{\partial^2 A_y}{\partial^2 y}} + \frac{\partial^2 A_y}{\partial^2 x} + \frac{\partial^2 A_y}{\partial^2 z}\right)\right] \\
& + \vec{a}_z\left[\left(\boxed{\frac{\partial^2 A_z}{\partial^2 z}} + \frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z}\right) - \left(\boxed{\frac{\partial^2 A_z}{\partial^2 z}} + \frac{\partial^2 A_z}{\partial^2 x} + \frac{\partial^2 A_z}{\partial^2 y}\right)\right]
\end{aligned}$$

Vector Identity Proof (cont.)

Proof of:

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \Bigg[\underbrace{\vec{a}_x\left(\frac{\partial^2 A_x}{\partial^2 x} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}\right) + \vec{a}_y\left(\frac{\partial^2 A_y}{\partial^2 y} + \frac{\partial^2 A_z}{\partial y \partial z} + \frac{\partial^2 A_x}{\partial x \partial y}\right) + \vec{a}_z\left(\frac{\partial^2 A_z}{\partial^2 z} + \frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z}\right)}_{\nabla\left(\nabla \cdot \vec{A}\right)}\Bigg]$$

$$-\Bigg[\underbrace{\vec{a}_x\left(\frac{\partial^2 A_x}{\partial^2 x} + \frac{\partial^2 A_x}{\partial^2 y} + \frac{\partial^2 A_x}{\partial^2 z}\right) + \vec{a}_y\left(\frac{\partial^2 A_y}{\partial^2 y} + \frac{\partial^2 A_y}{\partial^2 x} + \frac{\partial^2 A_y}{\partial^2 z}\right) + \vec{a}_z\left(\frac{\partial^2 A_z}{\partial^2 z} + \frac{\partial^2 A_z}{\partial^2 x} + \frac{\partial^2 A_z}{\partial^2 y}\right)}_{\nabla^2 \vec{A}}\Bigg]$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$ **Q.E.D.**