

Iterated Sigma-Point Kalman Filtering for Trajectory Reconstruction

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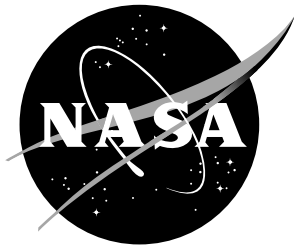
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NASA/TM-20240001235



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March 2024

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Abstract

Two second-order Kalman filtering methods are added to the New Statistical Trajectory Estimation Program (NewSTEP). The two filters are of the class of Sigma-Point Kalman Filters, namely the Unscented Kalman Filter and the Divided Difference Filter. Both filters are implemented with an iterative measurement update step. Results from simulation test case and a flight data test case are provided to show the filter performance. The results indicate that the new second-order filters produce trajectory estimates with reduced uncertainties when compared to the existing first-order filtering method.

Nomenclature

$\tilde{D}_{\Delta z}$	=	First-order divided difference operator
$\tilde{D}_{\Delta z}^2$	=	Second-order divided difference operator
E	=	Expectation operator
$\mathbf{f}$	=	Filter process model
h	=	Divided difference interval factor
$\mathbf{h}$	=	Filter measurement model
$\mathbf{H}$	=	Statistical linearization of $\mathbf{h}$
i	=	Iteration index
$\mathbf{I}$	=	Identity matrix
j	=	Sigma point index
k	=	Integer time index
$\mathbf{K}$	=	Kalman gain
L	=	Length of $\mathbf{x}^a$
n_v	=	Length of $\mathbf{v}$
n_x	=	Length of $\mathbf{x}$
p	=	Matrix column index
$\mathbf{P}$	=	State covariance
$\mathbf{P}^a$	=	Augmented state covariance
$\mathbf{P}_{xy}$	=	State-residual covariance
$\mathbf{P}_{yy}$	=	Residual covariance
$\mathbf{Q}$	=	Process noise covariance
$\mathbf{R}$	=	Measurement noise covariance
$\mathbf{s}$	=	Column of a $\mathbf{S}$ matrix
$\mathbf{S}_v$	=	Cholesky factor of $\mathbf{Q}$
$\mathbf{S}_w$	=	Cholesky factor of $\mathbf{R}$
$\mathbf{S}_x$	=	Cholesky factor of $\mathbf{P}$
$\mathbf{S}_{xv}^{(1)}$	=	First-order divided differences of $\mathbf{f}$ with respect to $\mathbf{v}$
$\mathbf{S}_{xv}^{(2)}$	=	Second-order divided differences of $\mathbf{f}$ with respect to $\mathbf{v}$
$\mathbf{S}_{xx}^{(1)}$	=	First-order divided differences of $\mathbf{f}$ with respect to $\mathbf{x}$
$\mathbf{S}_{xx}^{(2)}$	=	Second-order divided differences of $\mathbf{f}$ with respect to $\mathbf{x}$
$\mathbf{S}_{yw}^{(1)}$	=	First-order divided differences of $\mathbf{h}$ with respect to $\mathbf{w}$

$\mathbf{S}_{yw}^{(2)}$	=	Second-order divided differences of $\mathbf{h}$ with respect to $\mathbf{w}$
$\mathbf{S}_{yx}^{(1)}$	=	First-order divided differences of $\mathbf{h}$ with respect to $\mathbf{x}$
$\mathbf{S}_{yx}^{(2)}$	=	Second-order divided differences of $\mathbf{h}$ with respect to $\mathbf{x}$
$\mathbf{u}$	=	Filter inputs
t	=	Time, s
$\mathbf{v}$	=	Process noise
$w^{(c)}$	=	Covariance weighting
$w^{(m)}$	=	Mean weighting
$\mathbf{w}$	=	Measurement noise
$\mathbf{x}$	=	Filter state
$\mathbf{x}^a$	=	Augmented filter state
$\mathbf{X}^a$	=	Augmented state sigma point
$\mathbf{X}^v$	=	State sigma point corresponding to $\mathbf{v}$
$\mathbf{X}^x$	=	State sigma point corresponding to $\mathbf{x}$
$\mathbf{y}$	=	Measurements
$\mathbf{Y}$	=	Measurement sigma point
$\mathbf{z}$	=	Linear transformation of $\mathbf{x}$
α	=	Unscented Kalman filter tuning parameter
β	=	Unscented Kalman filter tuning parameter
δ	=	Differential operator
κ	=	Unscented Kalman filter tuning parameter
λ	=	Unscented Kalman filter scaling parameter
μ	=	Differential operator
$\hat{}$	=	Estimated quantity
$\bar{}$	=	Predicted quantity

Abbreviations

ASPIRE	=	Advanced Supersonic Parachute Inflation Research Experiment
CKF	=	Cubature Kalman Filter
DDF	=	Divided Difference Filter
GEOS	=	Goddard Earth Observing System
GPS	=	Global Positioning System
IDDF	=	Iterated Divided Difference Filter
IEKFS	=	Iterated Extended Kalman Filter-Smoother
ISPKF	=	Iterated Sigma Point Kalman Filter
IUKF	=	Iterated Unscented Kalman Filter
IMU	=	Inertial Measurement Unit
INS	=	Inertial Navigation System
NewSTEP	=	New Statistical Trajectory Estimation Program
POST2	=	Program to Optimize Simulated Trajectories II
SPKF	=	Sigma Point Kalman Filter
STEP	=	Statistical Trajectory Estimation Program
UKF	=	Unscented Kalman Filter

1 Introduction

Trajectory reconstruction is typically one of the first steps in evaluating the performance of vehicles from flight missions, including both flight test and operational programs. The reconstructed trajectory can be used to help evaluate pre-flight models, such as aerodynamics and atmosphere, and can be used to determine the as-flown characteristics of the vehicle. The reconstructed data can be used improve modeling capabilities to inform and improve future missions.

The reconstruction process usually involves a statistical estimation technique. NASA has developed a Kalman filter approach for solving trajectory reconstruction problems, with a history dating initially to the mid 1960s with the development of the Statistical Trajectory Estimation Program (STEP) [1,2]. STEP made use of a linearized Kalman filter method for reconstructing the trajectory of flight vehicles, based on Inertial Measurement Unit (IMU) as the process model, and ground-based radar tracking observations, to correct the state predictions based on the Kalman filter method.

An effort was made in the early 2000s to modernize the STEP code, resulting in an Iterative Extended Kalman Filter-Smoother (IEKFS) tool that is known as the New Statistical Trajectory Estimation Program (NewSTEP) [3,4]. The modernization included a new code base implemented in Matlab, an improved filter algorithm based on the IEKFS approach, and support for many other types of measurement data sources such as Global Positioning System (GPS), atmosphere and atmosphere-relative measurements such as those produced by air-data sensors, magnetometers, altimeters, and other sources.

The IEKFS makes use of linearized approximations to solve the sequential optimal filtering problem. These linearizations can form poor approximations of the underlying dynamics when the system is highly nonlinear and/or the filter initial conditions are grossly inaccurate. In these situations, results from the IEKFS can be inaccurate or even divergent [5,6].

Second-order filters have been developed that can more accurately approximate the system dynamics and thus improve on filter performance in cases where the linearizing assumption can be problematic. The second-order Gaussian filter was introduced in [7] and shown to have superior performance compared to the first-order Kalman filter for a vehicle tracking problem. The second-order Gaussian filter performs well, but the performance comes at a cost in implementation as it requires analytical expressions for the Hessian matrices of the dynamics and measurement models.

The class of *Sigma-Point Kalman Filters* (SPKF) was introduced in several papers during the late 1990s [8–11]. These filters use a cluster of points propagated through the nonlinear transformations to approximate the system dynamics rather than relying on linearization. The resulting filters have second-order accuracy and also have the benefit of not requiring any partial derivatives of the dynamics. The two main subclasses of the SPKF are the *Unscented Kalman Filter* (UKF) [8,9] and the *Divided Difference Filter* (DDF) [10,11]. These two filters are derived from different principles: The UKF is based on a sampling approach to approximate probability density functions, and the DDF is based on multi-dimensional interpolation

formulas to approximate the nonlinear process and measurement models. The resulting filters are nearly identical in structure and performance, and both have been shown to be superior to the usual first-order Kalman filter methods in numerous applications. Some relevant examples include time series estimation [12], ballistic missile tracking [13], blended Inertial Navigation Systems (INS) with GPS architectures [14], spacecraft attitude estimation [15,16], spacecraft orbit determination [17], and spacecraft rendezvous relative navigation [18]. Applications to trajectory reconstruction problems have been investigated in [19,20].

This paper describes the implementation of both the UKF and the DDF to the NewSTEP trajectory reconstruction code. Both filters are implemented with an iterative measurement update step to further improve performance. Chapter 2 describes the mathematical formulation of the proposed filter methods. Chapter 3 provides the results of sample trajectory reconstruction problems. The first of these problems is a simulated test case in which the “truth” states are known for comparison. The second case involves real flight data as inputs to the newly implemented filters. Truth data is not known in the second test case, but performance of the filters are explored with measurement residuals in comparison to the reconstruction performed by the IEKF.

2 Iterative Sigma-Point Kalman Filters

This chapter describes the two classes of sigma point Kalman filters implemented for the trajectory reconstruction problem: the Iterative Unscented Kalman Filter and the Iterative Divided Difference Filter. As with the IEKF implementation in NewSTEP, both SPKFs are implemented as a fixed-interval smoothing problem, with a forward and backward pass through the data combined with a Fraser-Potter smoother as described in [3,4].

The filter dynamics and measurement models in NewSTEP are of the form

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k, t_k) \quad (1)$$

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k, t_k) + \mathbf{w}_k \quad (2)$$

where $\mathbf{x}$ is the filter state, $\mathbf{y}$ are the measurements, $\mathbf{u}$ are known inputs, $\mathbf{v}$ are zero-mean process noise inputs with covariance $\mathbf{Q}$, and $\mathbf{w}$ are zero-mean measurement noise inputs with covariance $\mathbf{R}$. The index k is an integer time counter. The function $\mathbf{f}$ represents numerical integration of the continuous-time equations of motion from time t_k to t_{k+1} , and the function $\mathbf{h}$ relates the states to the measurements at time t_k .

2.1 Iterative Unscented Kalman Filter (IUKF)

Let the quantities $\hat{\mathbf{x}}_{k-1}$ and $\hat{\mathbf{P}}_{k-1}$ be the state estimate and state estimate error covariance matrix at time t_{k-1} , respectively. Additionally, let $\mathbf{v}_{k-1}$ and $\mathbf{Q}_{k-1}$ be the process noise and process noise covariance matrix at time t_{k-1} . The state and process noise are augmented as $\hat{\mathbf{x}}_{k-1}^a$. Thus, at time t_{k-1} , the augmented a priori

state and covariance estimate are the following

$$\begin{aligned}\hat{\mathbf{x}}_{k-1}^a &= \begin{bmatrix} \hat{\mathbf{x}}_{k-1} \\ \mathbf{v}_{k-1} \end{bmatrix} \\ \hat{\mathbf{P}}_{k-1}^a &= \begin{bmatrix} \hat{\mathbf{P}}_{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{k-1} \end{bmatrix}\end{aligned}\quad (3)$$

Defining the length of the state vector $\hat{\mathbf{x}}_{k-1}$ to be n_x and length of the process noise vector to be n_v , the new length of the augmented vector is $L = n_x + n_v$.

To perform the Unscented Transform, using $\hat{\mathbf{x}}_{k-1}^a$ and $\hat{\mathbf{P}}_{k-1}^a$, a fixed number of sigma points are chosen to be propagated through the nonlinear system model [8, 9]. The matrix of $L \times (2L + 1)$ sigma points is formed by

$$\hat{\mathbf{X}}_{k-1}^a = \begin{bmatrix} \hat{\mathbf{x}}_{k-1}^a & \hat{\mathbf{x}}_{k-1}^a \pm \left(\sqrt{(L + \lambda) \hat{\mathbf{P}}_{k-1}^a} \right)_{j=1, \dots, L} \end{bmatrix} \quad (4)$$

where λ is a user defined scaling parameter and $(*)_j$ denotes the j th column of the square root of the augmented covariance matrix. The matrix square root is typically computed as a Cholesky factor [21].

Each sigma point is then propagated to time step t_k using the given nonlinear process model

$$\bar{\mathbf{X}}_{j,k}^x = \mathbf{f} \left(\hat{\mathbf{X}}_{j,k-1}^x, \mathbf{u}_{k-1}, \hat{\mathbf{X}}_{j,k-1}^v \right) \quad j=0, \dots, 2L \quad (5)$$

where the subscript j denotes the j th sigma point, $j = 0 \dots 2L$, and superscripts 'x' and 'v' indicate the state and process noise portions of the sigma point vector, respectively.

After the Unscented Transform has been applied, the prior state estimate and covariance are calculated using a weighted average

$$\begin{aligned}\bar{\mathbf{x}}_k &= \sum_{j=0}^{2L} w_j^{(m)} \bar{\mathbf{X}}_{j,k}^x \\ \bar{\mathbf{P}}_k &= \sum_{j=0}^{2L} w_j^{(c)} [\bar{\mathbf{X}}_{j,k}^x - \bar{\mathbf{x}}_k] [\bar{\mathbf{X}}_{j,k}^x - \bar{\mathbf{x}}_k]^T\end{aligned}\quad (6)$$

where the weights $w_j^{(m)}$ and $w_j^{(c)}$ are given by

$$\begin{aligned}w_0^{(m)} &= \lambda / (L + \lambda) \\ w_0^{(c)} &= \lambda / (L + \lambda) + (1 - \alpha^2 + \beta) \\ w_j^{(m)} &= w_j^{(c)} = \frac{1}{2(L + \lambda)} \quad j = 1, \dots, 2L\end{aligned}\quad (7)$$

The variables α and β are tuning parameters. The scaling parameter λ is defined as $\lambda = \alpha^2 (L + \kappa) - L$, where κ is another tuning parameter. Some common parameter settings and derived quantities are shown in Table 1 [22, 23], where CKF

is the Cubature Kalman Filter [24]. Note that negative values of λ can lead to numerical errors that cause the covariance estimate to become non-positive definite. This error can be rectified by forming the covariance estimate around the nominal sigma point, $\bar{\mathbf{X}}_{0,k}^x$, rather than the sample mean, $\bar{\mathbf{x}}_k$. This approach is known as the *modified* UKF [8, 25]. A nearest-positive-definite matrix solution method [26] can be used in the event of other numerical errors that arise to yield a non-positive-definite covariance.

Table 1: Common UKF Tuning Parameter Settings

Parameter	UKF1 [8]	UKF2 [12]	CKF [24]
α	$\sqrt{3/L}$	10^{-3}	1
β	$3/L - 1$	2	0
κ	0	0	0
λ	$3 - L$	$L(10^{-6} - 1)$	0
$\sqrt{L + \lambda}$	$\sqrt{3}$	$10^{-3}\sqrt{L}$	$\sqrt{L}$
$w_0^{(m)}$	$1 - L/3$	$1 - 10^6$	0

Using the prior state estimate, a new set of sigma points are generated before going onto the iterative measurement update. With the new sigma points, each are propagated through the output estimate model as the following

$$\bar{\mathbf{Y}}_{j,k}^i = \mathbf{h}(\bar{\mathbf{X}}_{j,k}^{x,i}, \mathbf{u}_k, \bar{\mathbf{X}}_{j,k}^v) \quad j = 0, \dots, 2L \quad (8)$$

The superscript i denotes the iteration index. Using the output estimate sigma points and variable weights defined in (7), the predicted measurement and associated covariances are calculated as

$$\begin{aligned} \bar{\mathbf{y}}_k^i &= \sum_{j=0}^{2L} w_j^{(m)} \bar{\mathbf{Y}}_{j,k}^i \\ \mathbf{P}_{yy,k}^i &= \sum_{j=0}^{2L} w_j^{(c)} [\bar{\mathbf{Y}}_{j,k}^i - \bar{\mathbf{y}}_k^i][\bar{\mathbf{Y}}_{j,k}^i - \bar{\mathbf{y}}_k^i]^T + \mathbf{R}_k \\ \mathbf{P}_{xy,k}^i &= \sum_{j=0}^{2L} w_j^{(c)} [\bar{\mathbf{X}}_{j,k}^{x,i} - \bar{\mathbf{x}}_k^i][\bar{\mathbf{Y}}_{j,k}^i - \bar{\mathbf{y}}_k^i]^T \end{aligned} \quad (9)$$

The measurement is then applied as a corrective update to the prior time estimate as the following [22]

$$\begin{aligned} \mathbf{K}_k^i &= \mathbf{P}_{xy}^i (\mathbf{P}_{yy}^i)^{-1} \\ \mathbf{H}_k^i &= (\mathbf{P}_{xy}^i)^T (\bar{\mathbf{P}}_k)^{-1} \\ \hat{\mathbf{x}}_k^{i+1} &= \bar{\mathbf{x}}_k + \mathbf{K}_k^i (\mathbf{y}_k - \bar{\mathbf{y}}_k^i - \mathbf{H}_k^i (\bar{\mathbf{x}}_k - \hat{\mathbf{x}}_k^i)) \end{aligned} \quad (10)$$

where the initial value for $\hat{\mathbf{x}}_k^i$ is initial set to the prior time estimate of the state. If the difference between $\hat{\mathbf{x}}_k^{i+1}$ and $\hat{\mathbf{x}}_k^i$ is greater than the specified iteration termination

tolerance, a new set of sigma points is generated using $\hat{\mathbf{x}}_k^{i+1}$ as the desired specified value. The state covariance $\mathbf{P}_k$ stays the same between iterations. Once the iteration termination criteria is reached, the final posterior estimate is the following

$$\begin{aligned}\hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_k^{i+1} \\ \hat{\mathbf{P}}_k &= (\mathbf{I} - \mathbf{K}_k^i \mathbf{H}_k^i) \bar{\mathbf{P}}_k (\mathbf{I} - \mathbf{K}_k^i \mathbf{H}_k^i)^T + \mathbf{K}_k^i \mathbf{R}_k \mathbf{K}_k^{i,T}\end{aligned}\quad (11)$$

where $\mathbf{I}$ is an identity matrix. The time step is then set to t_{k+1} and $\hat{\mathbf{x}}_k, \hat{\mathbf{P}}_k$ are augmented with process noise before creating sigma points for the next time step.

2.2 Iterative Divided Difference Filter (IDDF)

In addition to the IUKF, there is another nonlinear estimator that does not rely on linear approximations. This is the Divided Difference Filter, which is derived using a multidimensional extension of Stirling's interpolation formula [10, 11]. No derivatives are needed, only function evaluations subsequently used in interpolation formulas.

2.2.1 Stirling's Interpolation Formula and DDF Operators

Consider a general nonlinear function

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \quad (12)$$

It is assumed the random variable $\mathbf{x}$ has a Gaussian density with a mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P}_x$. The DDF begins by introducing a linear transformation of the states $\mathbf{x}$

$$\begin{aligned}\mathbf{z} &= \mathbf{S}_x^{-1} \mathbf{x} \\ \mathbf{y} &= \mathbf{f}(\mathbf{S}_x \mathbf{z}) = \tilde{\mathbf{f}}(\mathbf{z})\end{aligned}\quad (13)$$

where the transformation matrix $\mathbf{S}_x$ is defined as a square Cholesky factor of the state covariance matrix $\mathbf{P}_x$ such that $\mathbf{P}_x = \mathbf{S}_x \mathbf{S}_x^T$. This transformation makes the elements of $\mathbf{z}$ mutually uncorrelated with unit variance.

Applying the multidimensional Stirling interpolation formula about the new transformed mean, $\bar{\mathbf{z}}$, up to the second-order term can be approximated as

$$\mathbf{y} \approx \tilde{\mathbf{f}}(\bar{\mathbf{z}}) + \tilde{D}_{\Delta \mathbf{z}} \tilde{\mathbf{f}} + \frac{1}{2!} \tilde{D}_{\Delta \mathbf{z}}^2 \tilde{\mathbf{f}} \quad (14)$$

with the divided difference operators $\tilde{D}_{\Delta \mathbf{z}} \tilde{\mathbf{f}}$ and $\tilde{D}_{\Delta \mathbf{z}}^2 \tilde{\mathbf{f}}$ defined as [10, 11]

$$\begin{aligned}\tilde{D}_{\Delta \mathbf{z}} \tilde{\mathbf{f}} &= \frac{1}{h} \left(\sum_{p=1}^{n_x} \Delta z_p \mu_p \delta_p \right) \tilde{\mathbf{f}}(\bar{\mathbf{z}}) \\ \tilde{D}_{\Delta \mathbf{z}}^2 \tilde{\mathbf{f}} &= \frac{1}{h^2} \left[\sum_{p=1}^{n_x} (\Delta z_p)^2 \delta_p^2 + \sum_{p=1}^{n_x} \sum_{q=1, q \neq p}^{n_x} \Delta z_p \Delta z_q (\mu_p \delta_p)(\mu_q \delta_q) \right] \tilde{\mathbf{f}}(\bar{\mathbf{z}})\end{aligned}\quad (15)$$

with the following differential operators

$$\begin{aligned}\delta_p \tilde{\mathbf{f}}(\bar{\mathbf{z}}) &= \tilde{\mathbf{f}}\left(\bar{\mathbf{z}} + \frac{h}{2}\mathbf{s}_p\right) - \tilde{\mathbf{f}}\left(\bar{\mathbf{z}} - \frac{h}{2}\mathbf{s}_p\right) \\ \mu_p \tilde{\mathbf{f}}(\bar{\mathbf{z}}) &= \frac{1}{2} \left[\tilde{\mathbf{f}}\left(\bar{\mathbf{z}} + \frac{h}{2}\mathbf{s}_p\right) - \tilde{\mathbf{f}}\left(\bar{\mathbf{z}} - \frac{h}{2}\mathbf{s}_p\right) \right]\end{aligned}\tag{16}$$

where $\mathbf{s}_p$ is the p th column of the matrix $\mathbf{S}$ and h is an interval factor, set to $\sqrt{3}$ for Gaussian distributions [10].

An approximate mean of the output $\mathbf{y}$, its associated covariance, and the cross covariance can be obtained by applying the expectation operator $E[*]$ to the output estimate defined in (14) and the subsequent errors in the approximate mean.

2.2.2 Time Update

The following Cholesky factors are defined before calculating the divided differences

$$\begin{aligned}\mathbf{Q} &= \mathbf{S}_v \mathbf{S}_v^T \\ \mathbf{R} &= \mathbf{S}_w \mathbf{S}_w^T \\ \bar{\mathbf{P}} &= \bar{\mathbf{S}}_x \bar{\mathbf{S}}_x^T \\ \hat{\mathbf{P}} &= \hat{\mathbf{S}}_x \hat{\mathbf{S}}_x^T\end{aligned}\tag{17}$$

where $\mathbf{Q}$ is the process noise covariance, $\mathbf{R}$ is the measurement noise covariance, $\bar{\mathbf{P}}$ is the prior state covariance estimate, and $\hat{\mathbf{P}}$ is the posterior state covariance estimate. For the initial time step, $\hat{\mathbf{P}} = \bar{\mathbf{P}}$.

The time update for a second order approximation of a nonlinear system starts with computing the transformation matrices using the divided difference operators

$$\begin{aligned}\mathbf{S}_{x\hat{x}}^{(1)}(k+1) &= \{S_{x\hat{x}}^{(1)}(i, j)\} = \{(f_i(\hat{\mathbf{x}}_k + h\hat{\mathbf{s}}_{x,j}, \mathbf{u}_k, \mathbf{v}_k) - f_i(\hat{\mathbf{x}}_k - h\hat{\mathbf{s}}_{x,j}, \mathbf{u}_k, \mathbf{v}_k))/2h\} \\ \mathbf{S}_{x\hat{x}}^{(2)}(k+1) &= \{S_{x\hat{x}}^{(2)}(i, j)\} \\ &= \left\{ \frac{\sqrt{h^2 - 1}}{2h^2} (f_i(\hat{\mathbf{x}}_k + h\hat{\mathbf{s}}_{x,j}, \mathbf{u}_k, \mathbf{v}_k) \right. \\ &\quad \left. - f_i(\hat{\mathbf{x}}_k - h\hat{\mathbf{s}}_{x,j}, \mathbf{u}_k, \mathbf{v}_k) - 2f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, \mathbf{v}_k)) \right\} \\ \mathbf{S}_{xv}^{(1)}(k+1) &= \{S_{xv}^{(1)}(i, j)\} = \{(f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, \mathbf{v}_k + h\hat{\mathbf{s}}_{v,j}) - f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, \mathbf{v}_k - h\hat{\mathbf{s}}_{v,j}))/2h\} \\ \mathbf{S}_{xv}^{(2)}(k+1) &= \{S_{xv}^{(2)}(i, j)\} \\ &= \left\{ \frac{\sqrt{h^2 - 1}}{2h^2} (f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, \mathbf{v}_k + h\hat{\mathbf{s}}_{v,j}) \right. \\ &\quad \left. - f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, \mathbf{v}_k - h\hat{\mathbf{s}}_{v,j}) - 2f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, \mathbf{v}_k)) \right\}\end{aligned}\tag{18}$$

where the j subscript indicates the use of the j th column of the specified matrix. The same function evaluations are used to obtain the time-based update of the state.

$$\begin{aligned}
\bar{\mathbf{x}}_{k+1} &= \frac{h^2 - n_x - n_v}{h^2} \mathbf{f}(\hat{\mathbf{x}}_k, \mathbf{u}_k, \mathbf{v}_k) \\
&+ \frac{1}{2h^2} \sum_{p=1}^{n_x} (f_i(\hat{\mathbf{x}}_k + h\hat{\mathbf{s}}_{x,p}, \mathbf{u}_k, \mathbf{v}_k) + f_i(\hat{\mathbf{x}}_k - h\hat{\mathbf{s}}_{x,p}, \mathbf{u}_k, \mathbf{v}_k)) \\
&+ \frac{1}{2h^2} \sum_{p=1}^{n_v} (f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, v_k + h\mathbf{s}_{v,p}) + f_i(\hat{\mathbf{x}}_k, \mathbf{u}_k, v_k - h\mathbf{s}_{v,p}))
\end{aligned} \tag{19}$$

where n_x is the length of the states and n_v is the length of the process noise states.

An update is also applied to the state covariance by using the transformation matrices computed in 18 to compute the state covariance's Cholesky factor.

$$\begin{aligned}
\bar{\mathbf{S}}_x(k+1) &= \begin{bmatrix} \mathbf{S}_{x\hat{x}}^{(1)}(k+1) & \mathbf{S}_{xv}^{(1)}(k+1) & \mathbf{S}_{x\hat{x}}^{(2)}(k+1) & \mathbf{S}_{xv}^{(2)}(k+1) \end{bmatrix} \\
\bar{\mathbf{P}}(k+1) &= \bar{\mathbf{S}}_x \bar{\mathbf{S}}_x^T
\end{aligned} \tag{20}$$

2.2.3 Measurement Update

Similar to the beginning of the time update step, the measurement update begins with computing a new step of transformation matrices using the divided difference operators. An iterative component is also added here, with the i superscript indicating what changes with every iteration [27].

$$\begin{aligned}
\mathbf{S}_{y\bar{x}}^{(1)(i)}(k) &= \{S_{y\bar{x}}^{(1)(i)}(i, j)\} = \{(h_i(\bar{\mathbf{x}}_k^i + h\bar{\mathbf{s}}_{x,j}^i, \mathbf{w}_k) - h_i(\bar{\mathbf{x}}_k^i - h\bar{\mathbf{s}}_{x,j}^i, \mathbf{w}_k))/2h\} \\
\mathbf{S}_{y\bar{x}}^{(2)(i)}(k) &= \{S_{y\bar{x}}^{(2)(i)}(i, j)\} \\
&= \left\{ \frac{\sqrt{h^2 - 1}}{2h^2} (h_i(\bar{\mathbf{x}}_k^i + h\bar{\mathbf{s}}_{x,j}^i, \mathbf{w}_k) - h_i(\bar{\mathbf{x}}_k^i - h\bar{\mathbf{s}}_{x,j}^i, \mathbf{w}_k) - 2h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k)) \right\} \\
\mathbf{S}_{yw}^{(1)(i)}(k) &= \{S_{yw}^{(1)(i)}(i, j)\} = \{(h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k + h\mathbf{s}_{w,j}) - h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k - h\mathbf{s}_{w,j}))/2h\} \\
\mathbf{S}_{yw}^{(2)(i)}(k) &= \{S_{yw}^{(2)(i)}(i, j)\} \\
&= \left\{ \frac{\sqrt{h^2 - 1}}{2h^2} (h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k + h\mathbf{s}_{w,j}) - h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k - h\mathbf{s}_{w,j}) - 2h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k)) \right\}
\end{aligned} \tag{21}$$

An estimate of the output is obtained in a similar manner as the time-based update for the states

$$\begin{aligned}
\bar{\mathbf{y}}_k^i &= \frac{h^2 - n_x - n_w}{h^2} \mathbf{h}(\bar{\mathbf{x}}_k^i, \mathbf{w}_k) \\
&+ \frac{1}{2h^2} \sum_{p=1}^{n_x} [h_i(\bar{\mathbf{x}}_k^i + h\bar{\mathbf{s}}_{x,p}^i, \mathbf{w}_k) + h_i(\bar{\mathbf{x}}_k^i - h\bar{\mathbf{s}}_{x,p}^i, \mathbf{w}_k)] \\
&+ \frac{1}{2h^2} \sum_{p=1}^{n_w} [h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k + h\mathbf{s}_{w,p}) + h_i(\bar{\mathbf{x}}_k^i, \mathbf{w}_k - h\mathbf{s}_{w,p})]
\end{aligned} \tag{22}$$

where n_w is the length of the measurement noise state. Using the matrices defined in equation 21, the output covariance and cross covariance can be computed as

$$\begin{aligned} \mathbf{S}_y^i(k) &= \begin{bmatrix} \mathbf{S}_{y\bar{x}}^{(1)(i)}(k) & \mathbf{S}_{yw}^{(1)(i)}(k) & \mathbf{S}_{y\bar{x}}^{(2)(i)}(k) & \mathbf{S}_{yw}^{(2)(i)}(k) \end{bmatrix} \\ \mathbf{P}_y^i(k) &= \mathbf{S}_y^i(\mathbf{S}_y^i)^T \\ \mathbf{P}_{xy}(k) &= \bar{\mathbf{S}}_x^i(k) \mathbf{S}_{y\bar{x}}^i(k)^T \end{aligned} \quad (23)$$

The a posteriori estimate of the state and covariance is given by the iterative update,

$$\begin{aligned} \mathbf{K}_k^i &= \mathbf{P}_{xy}^i(k) [\mathbf{S}_y^i(k) \mathbf{S}_y^i(k)^T]^{-1} \\ \mathbf{H}_k^i &= (\mathbf{P}_{xy}^i(k))^T \mathbf{P}_k^{-1} \\ \hat{\mathbf{x}}_k^{i+1} &= \bar{\mathbf{x}}_k^i + \mathbf{K}_k^i (\mathbf{y}_k - \bar{\mathbf{y}}_k^i - \mathbf{H}_k^i (\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_k^i)) \\ \hat{\mathbf{S}}_x^{i+1}(k) &= \left[\bar{\mathbf{S}}_x(k) - \mathbf{K}_k^i \mathbf{S}_{yx}^{(1)(i)}(k) \quad \mathbf{K}_k^i \mathbf{S}_{yw}^{(1)(i)}(k) \quad \mathbf{K}_k^i \mathbf{S}_{yx}^{(2)(i)}(k) \quad \mathbf{K}_k^i \mathbf{S}_{yw}^{(2)(i)}(k) \right] \\ \hat{\mathbf{P}}_k &= \hat{\mathbf{S}}_x^i(\hat{\mathbf{S}}_x^i)^T \end{aligned} \quad (24)$$

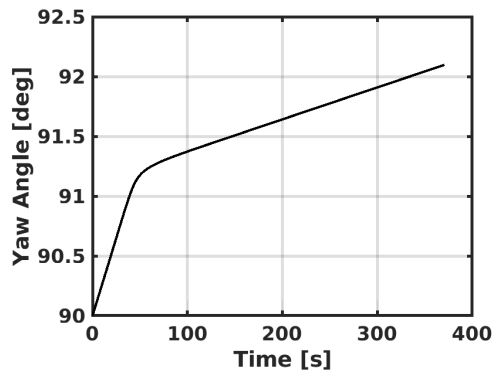
The iterations are repeated until the difference between $\hat{\mathbf{x}}_k^{i+1}$ and $\hat{\mathbf{x}}_k^i$ is less than a specified iteration termination tolerance, or the maximum number of iterations is reached.

3 Case Studies

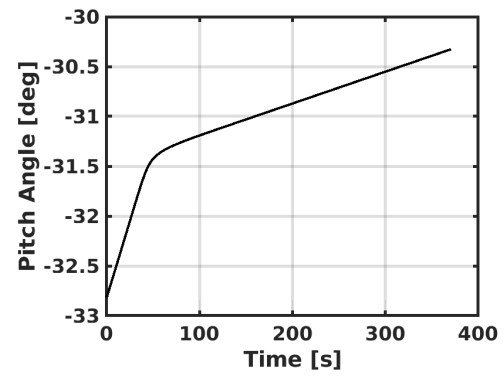
Several test cases were investigation using the Iterated SPKF implementations for sample trajectory reconstruction problems. The first case uses a simulated data set based on a Program to Optimize Simulated Trajectories II (POST2) [28] trajectory simulation of an atmospheric entry vehicle. The second test case is an application to a sounding rocket flight test with real flight data.

3.1 Simulation Test Case

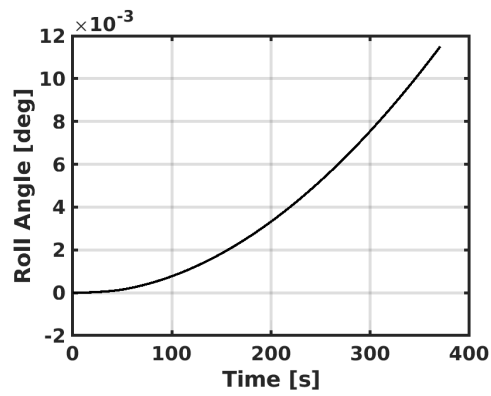
The simulation test case involves an atmospheric entry trajectory of a spherical body with a constant drag coefficient. The sensor data includes an IMU, GPS, two Radar tracks, and balloon or otherwise a priori observations of atmospheric winds, density, and pressure. The truth trajectory from the simulation is shown in Figures 1–3. The IMU accelerations are shown in Figure 4, note that the side acceleration and all components of the angular rates are zero. The three components of wind are also zero. Radar measurements are shown in Figure 5. Figure 5(a) shows the two station location relative to the vehicle ground track. Figures 5(b)–5(d) show the range, azimuth, and elevation data for both tracking stations.



(a) Yaw

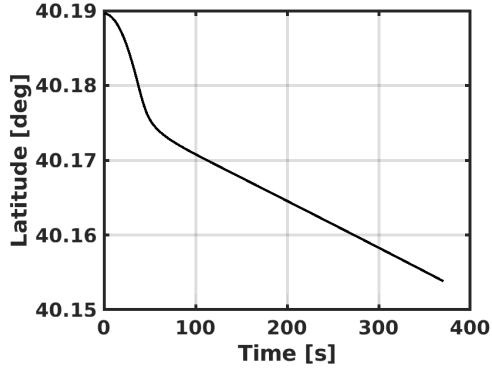


(b) Pitch

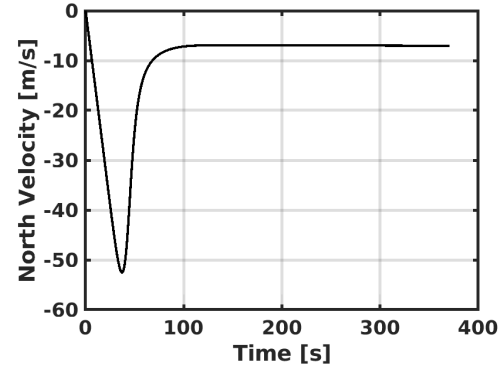


(c) Roll

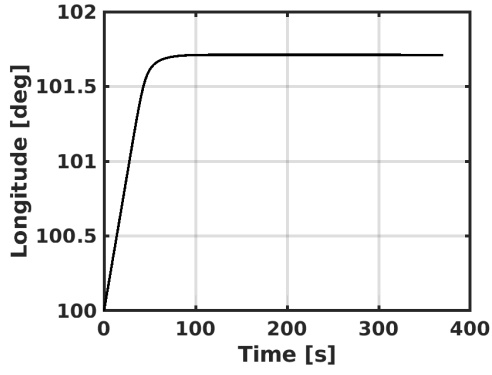
Figure 1: Truth Trajectory: Attitude



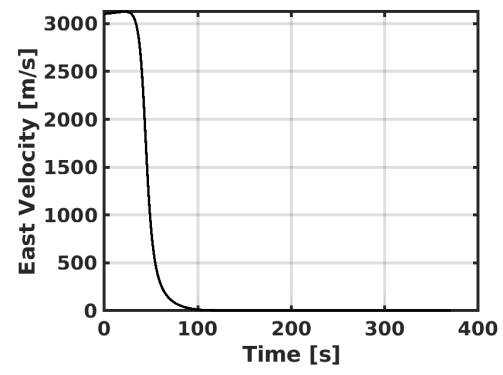
(a) Latitude



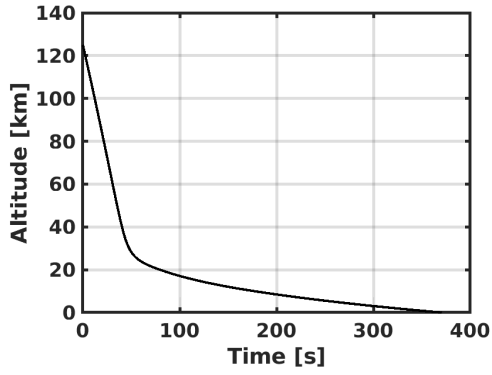
(b) North Velocity



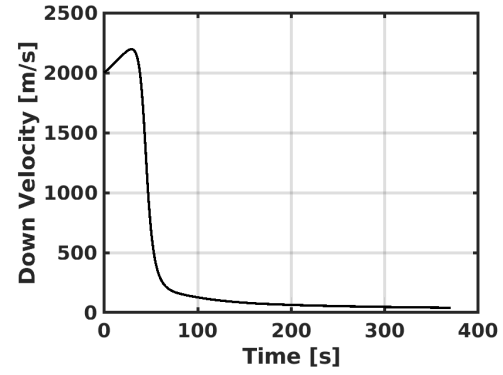
(c) Longitude



(d) East Velocity



(e) Altitude



(f) Down Velocity

Figure 2: Truth Trajectory: Position and Velocity

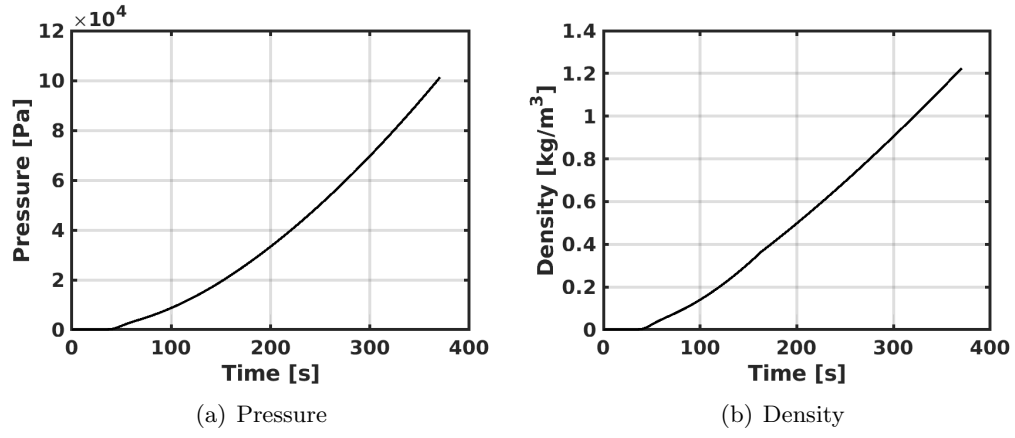


Figure 3: Truth Trajectory: Atmosphere

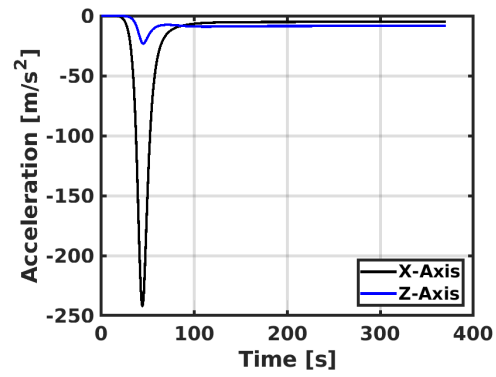


Figure 4: Sensed Acceleration

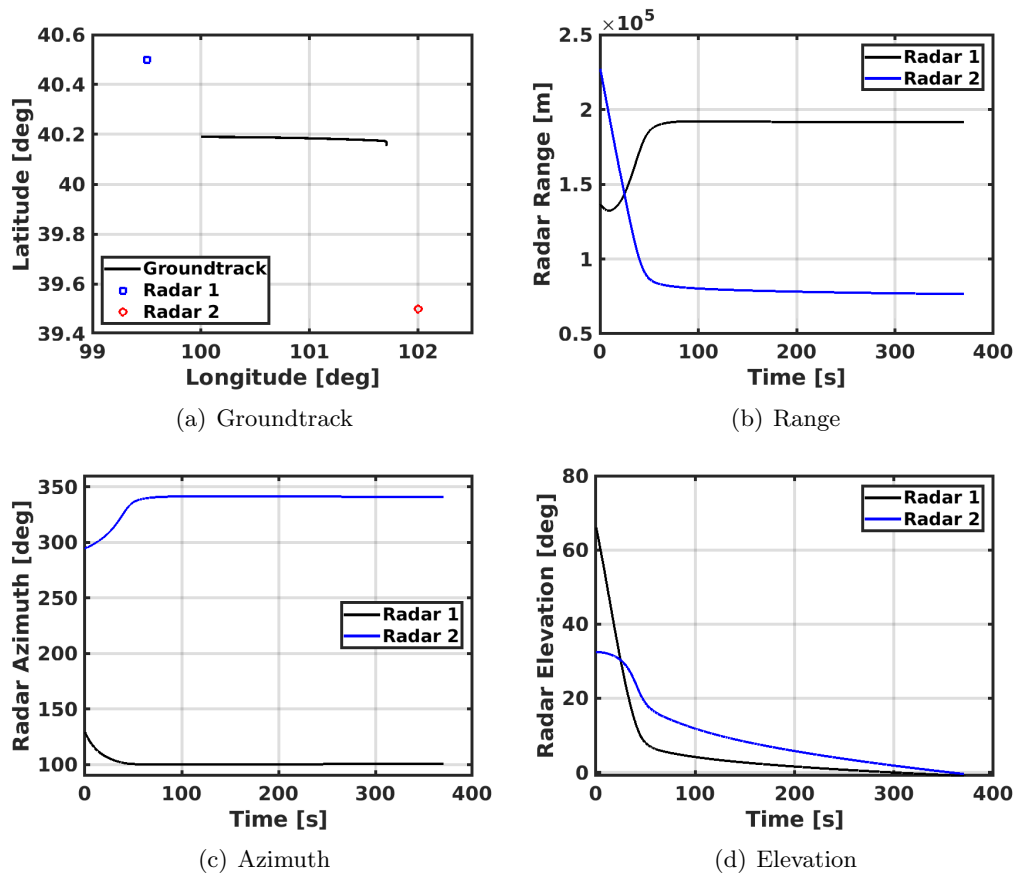
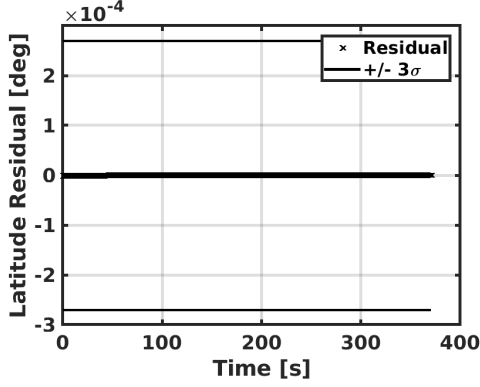


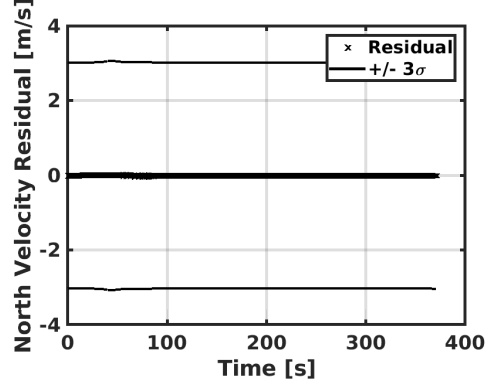
Figure 5: Radar Measurements

3.1.1 Filter Residuals

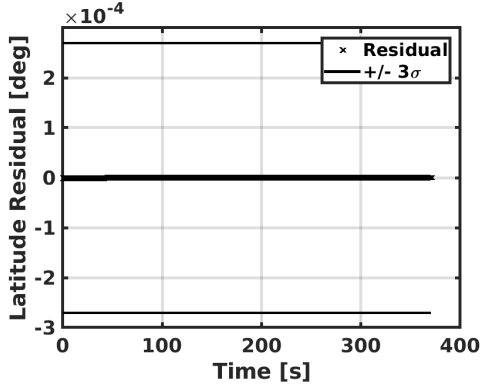
The measurement residuals are shown in Figures 6–8. Each figure shows the residuals for the EKF, UKF, and DDF cases. In each case, the residuals indicate that the filter solutions match the measurement data and thus the filters are performing correctly. Note that only a subset of all residuals are shown here. The remaining residuals show similar performance.



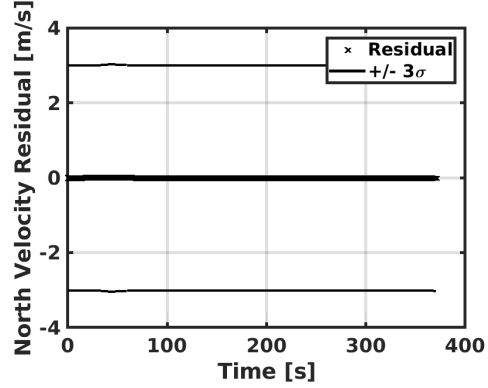
(a) Latitude: EKF



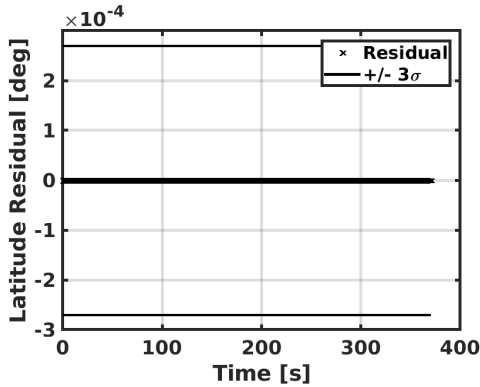
(b) North Velocity: EKF



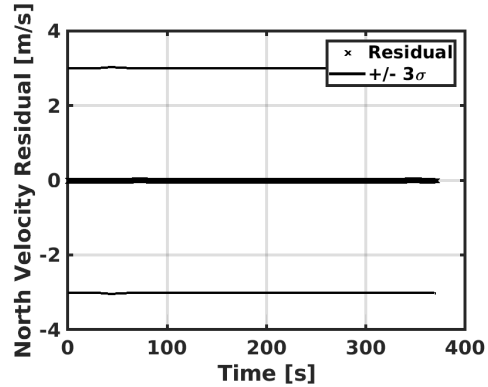
(c) Latitude: UKF



(d) North Velocity: UKF

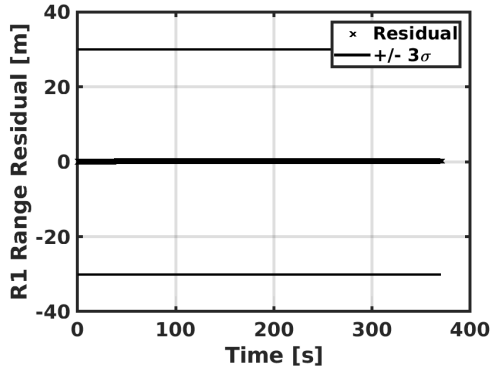


(e) Latitude: DDF

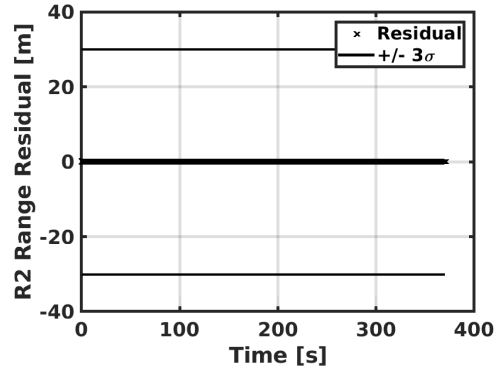


(f) North Velocity: DDF

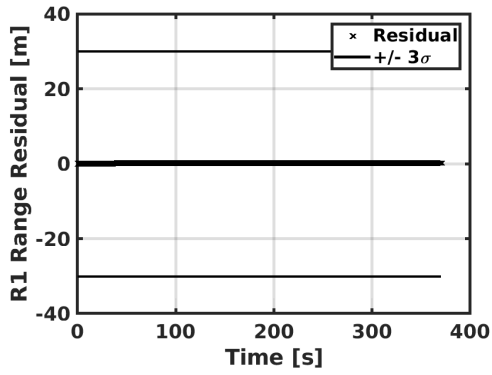
Figure 6: Latitude and North Velocity Residuals



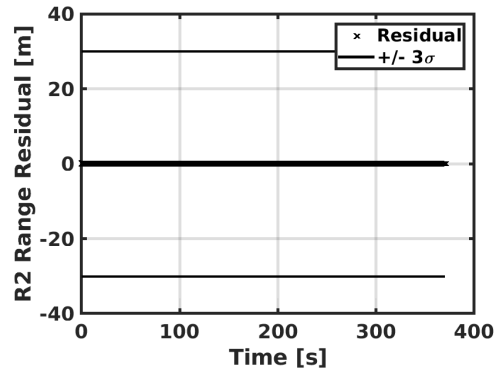
(a) Radar 1 Range: EKF



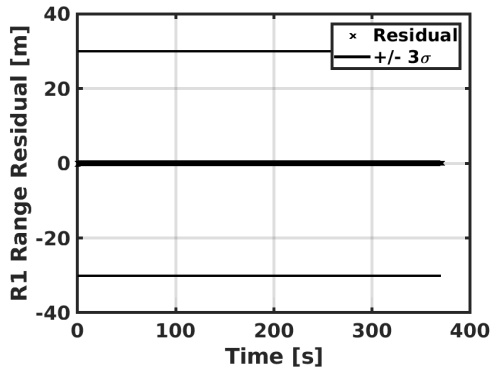
(b) Radar 2 Range: EKF



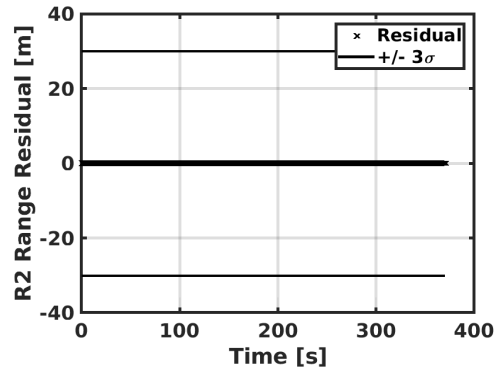
(c) Radar 1 Range: UKF



(d) Radar 2 Range: UKF

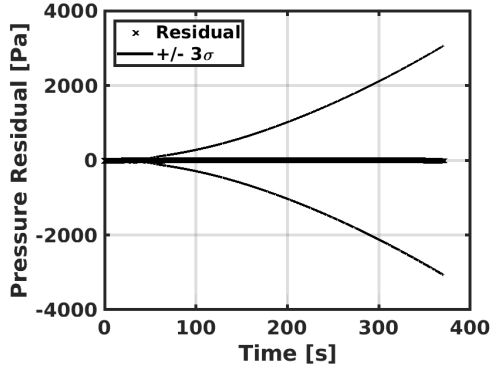


(e) Radar 1 Range: DDF

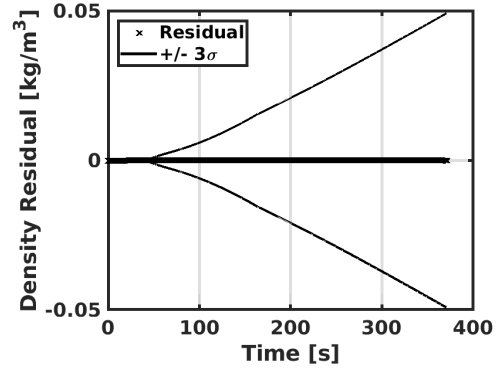


(f) Radar 2 Range: DDF

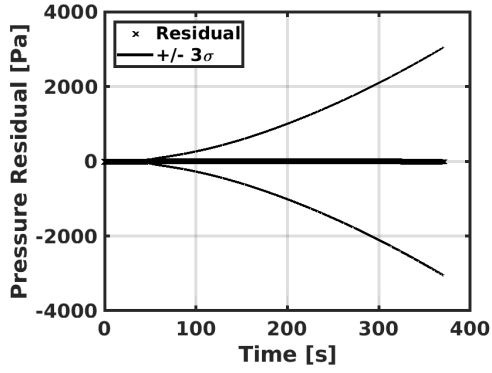
Figure 7: Radar Range Residuals



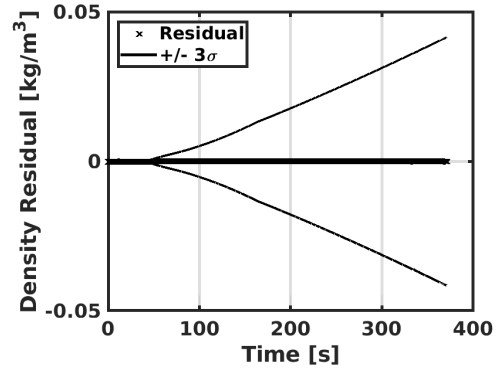
(a) Pressure: EKF



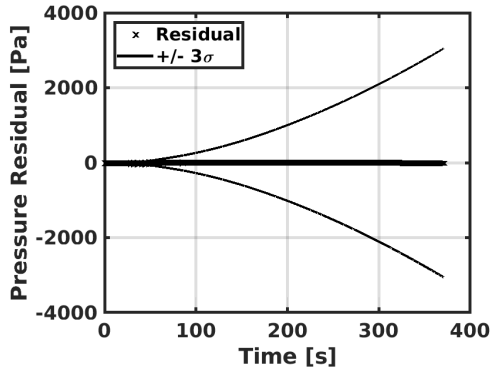
(b) Density: EKF



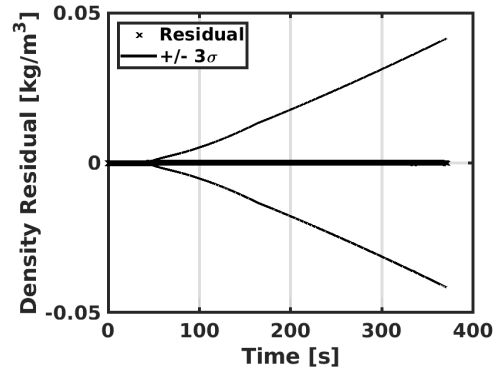
(c) Pressure: UKF



(d) Density: UKF



(e) Pressure: DDF



(f) Density: DDF

Figure 8: Pressure and Density Residuals

3.1.2 Filter Uncertainties

Comparison of the filter uncertainties are shown in Figures 9–12. The figures show the 3σ uncertainties for the EKF, UKF, and DDF cases. The UKF and DDF performs similarly. In both cases, the sigma-point filters exhibit a smaller uncertainty than that of the EKF. This result is due to the second-order accuracy of the sigma-point filters.

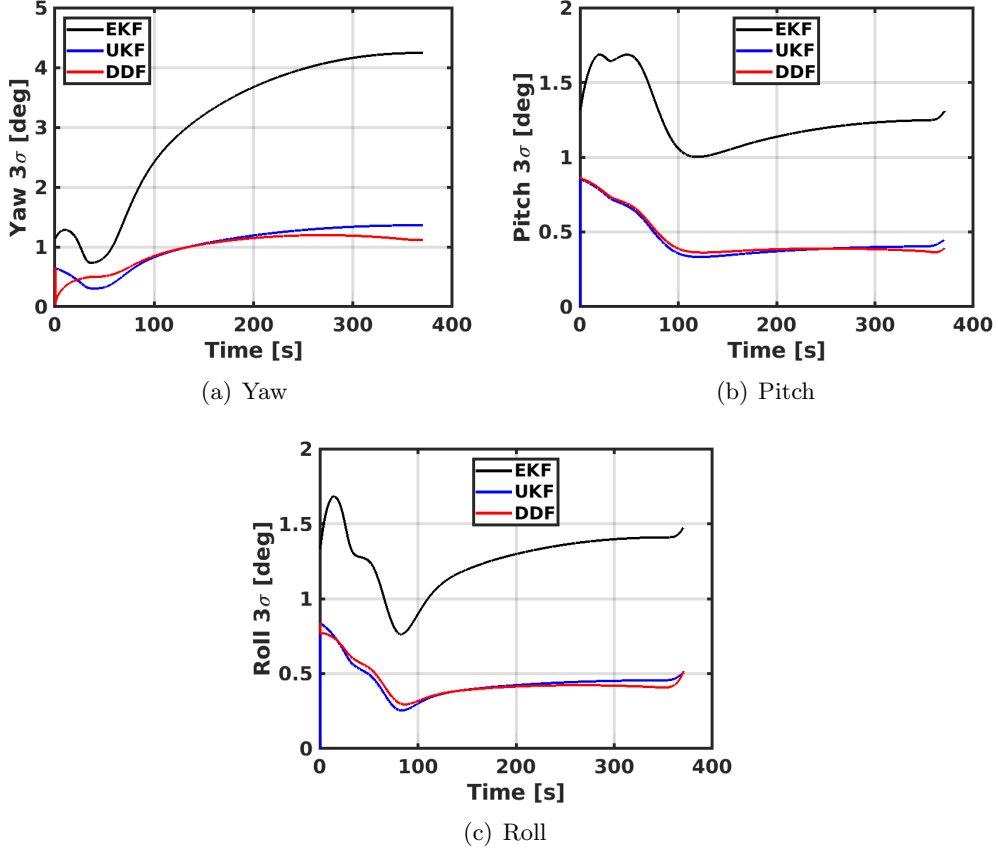
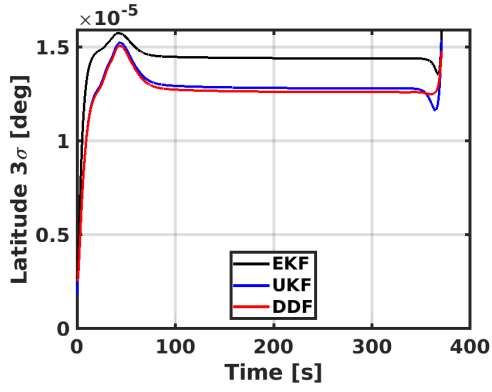
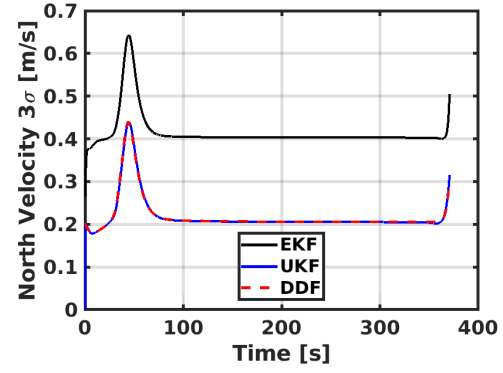


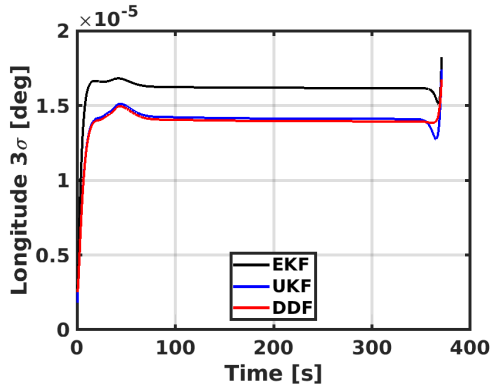
Figure 9: Attitude 3σ Uncertainties



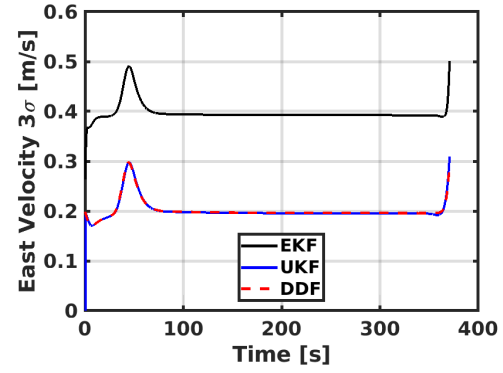
(a) Latitude



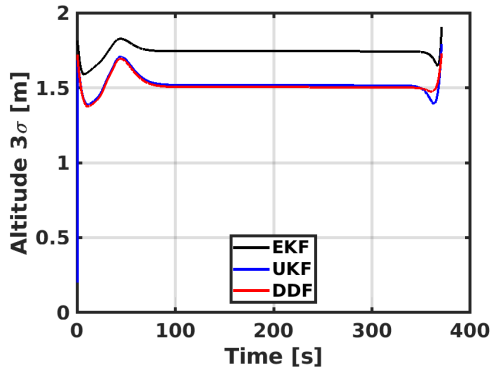
(b) North Velocity



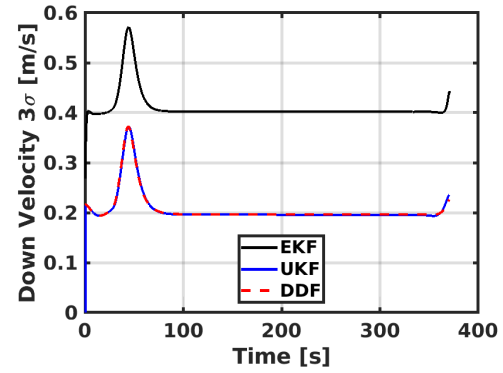
(c) Longitude



(d) East Velocity

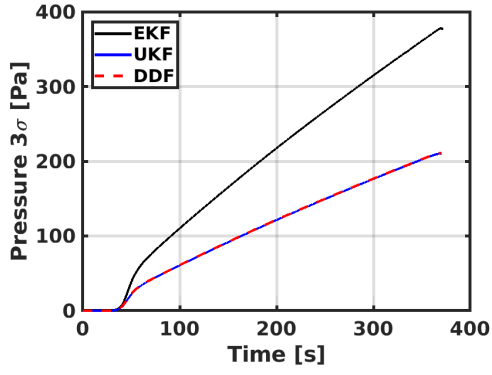


(e) Altitude

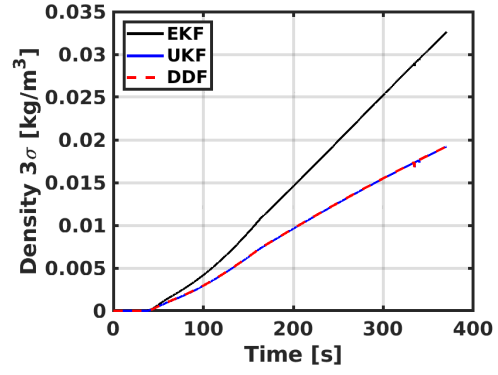


(f) Down Velocity

Figure 10: Position and Velocity 3σ Uncertainties

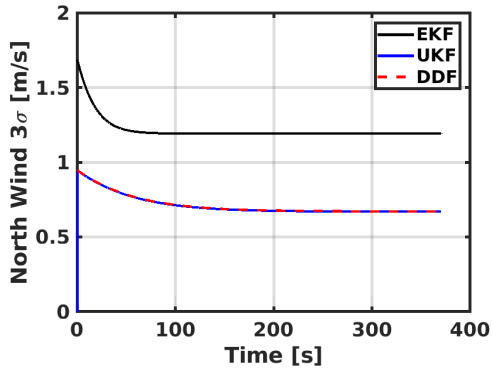


(a) Pressure

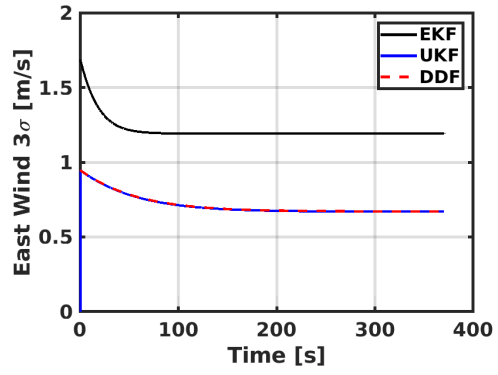


(b) Density

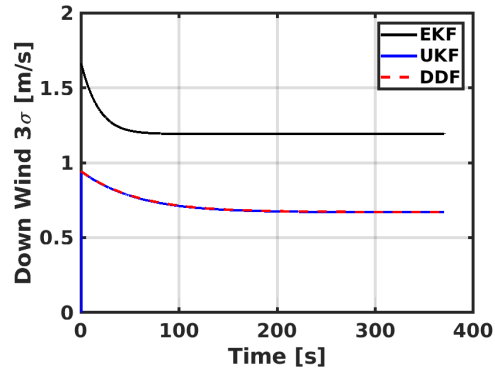
Figure 11: Atmosphere 3σ Uncertainties



(a) North Wind



(b) East Wind



(c) Down Wind

Figure 12: Wind 3σ Uncertainties

3.2 Flight Data Test Case

The third test flight Advanced Supersonic Parachute Inflation Research Experiment (ASPIRE) sounding rocket was chosen as an example test case using real flight data [29]. This flight was a test of the Mars 2020 parachute system, which was a strengthened disk-gap-band type parachute similar to that used on Mars Science Laboratory, but manufactured using stronger materials to withstand a higher load. The test flight took place at Wallops Island on September 7th, 2018. The measured flight data includes accelerations and rates from an IMU, position and velocity from GPS, radar tracks from 4 ground stations, and weather balloon data aided by meteorological analysis from the Goddard Earth Observing System Model, Version 5 (GEOS-5) to provide the atmosphere. Additional details of the test flight, instrumentation, and trajectory reconstruction based on the IEKF can be found in [29].

Since this is real data from an actual flight test, there is no known truth trajectory to compare the filter results against. The results of the ISPKF cases can be compared to the previous reconstruction based on the IEKF, and analysis of residuals can be used to infer observations about the filter performance.

3.2.1 Trajectory Estimates

The overall trajectories generated by the three filters are nearly identical in large scale. Several trajectory variables are shown in Fig. 13. The trajectory estimates are similar but the second-order filters have smaller uncertainties, following the observations made in the previous section with the simulated sample problem. Other trajectory variables that are not shown in Fig. 13 follow the same trend. Note that these plots only show the first 200 s of the trajectory to focus on the experiment phase [29].

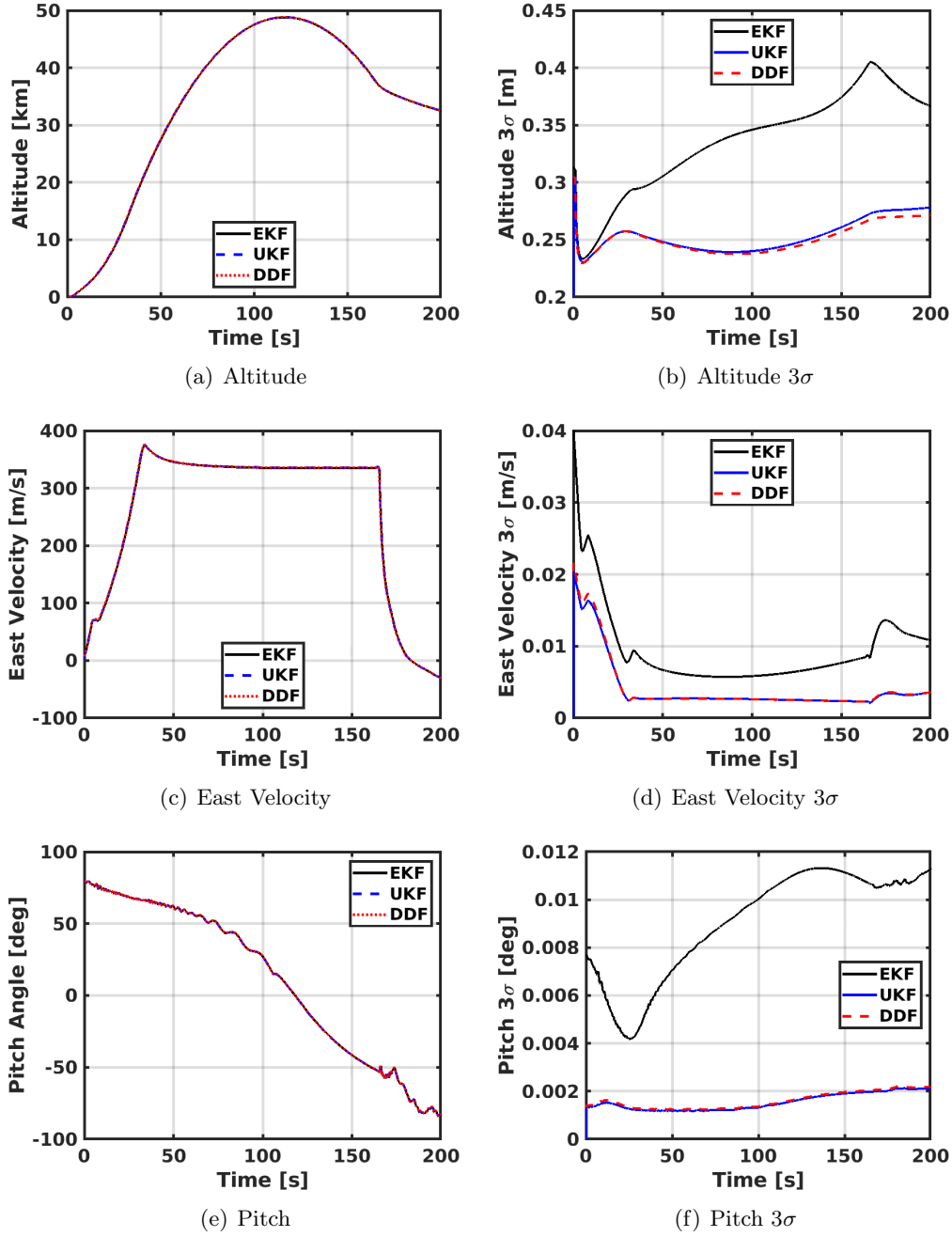
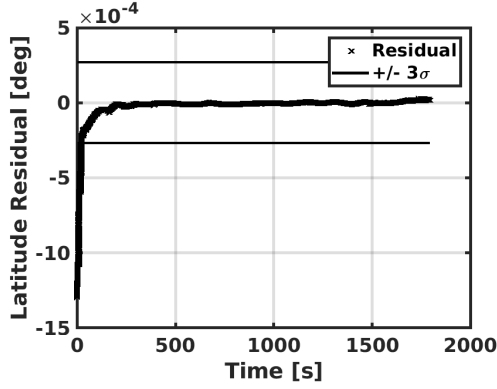


Figure 13: Trajectory and Uncertainties

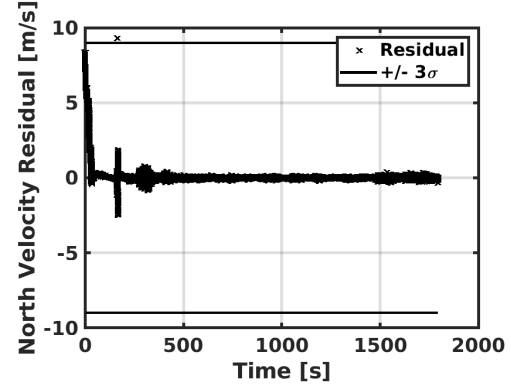
3.2.2 Filter Residuals

Residuals between the GPS position and velocity measurements are shown in Figs. 14, 15, and 16. Recall that there is no truth trajectory to compare to since this data is from an actual flight test experiment. The residuals are, in general, similar to each other demonstrating a reasonable fit to the measured data. The radar range

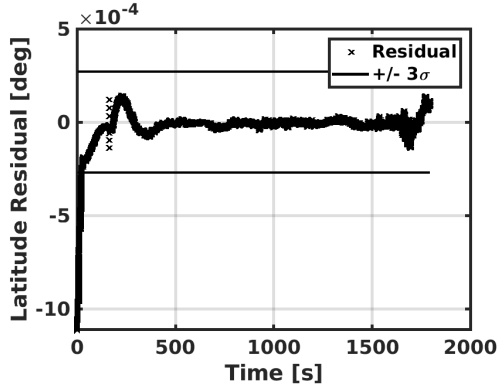
residuals indicate a similar conclusion and so are not shown here.



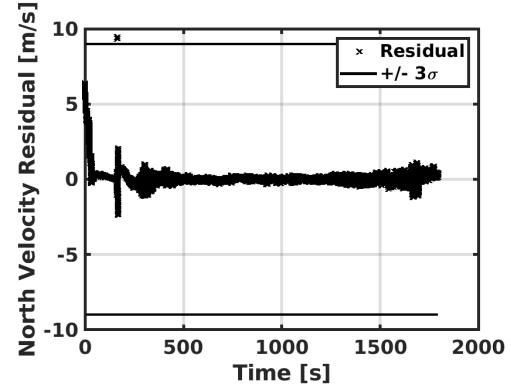
(a) Latitude: EKF



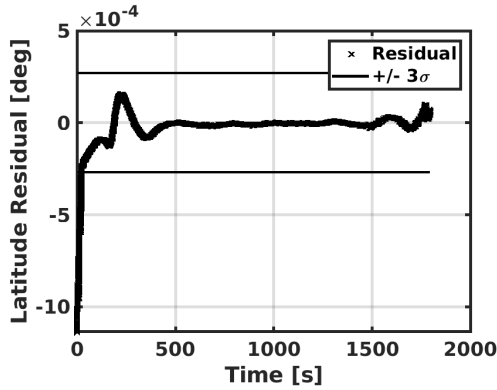
(b) North Velocity: EKF



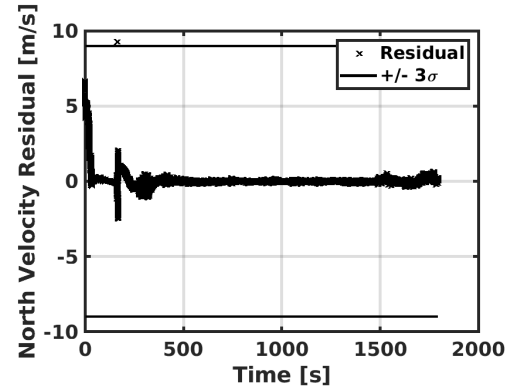
(c) Latitude: UKF



(d) North Velocity: UKF

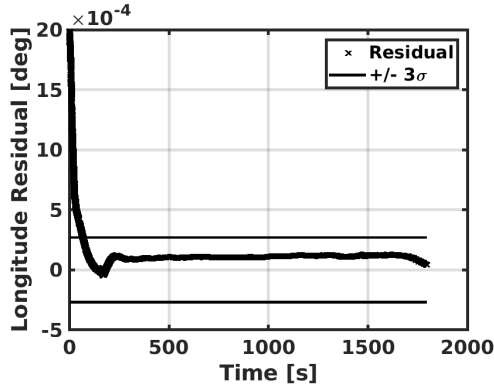


(e) Latitude: DDF

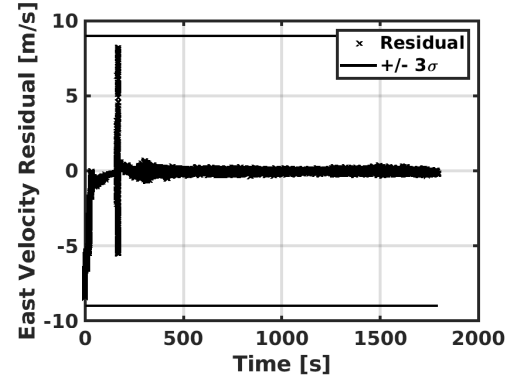


(f) North Velocity: DDF

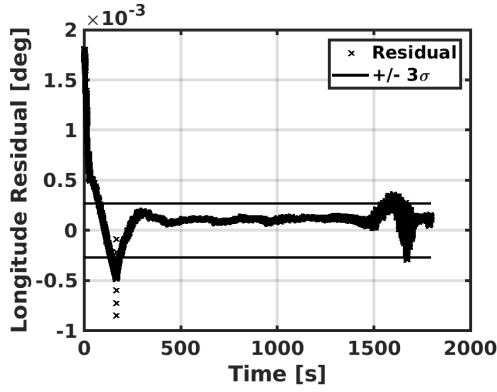
Figure 14: Latitude and North Velocity Residuals



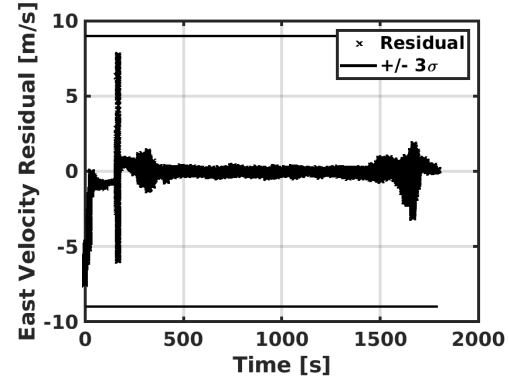
(a) Longitude: EKF



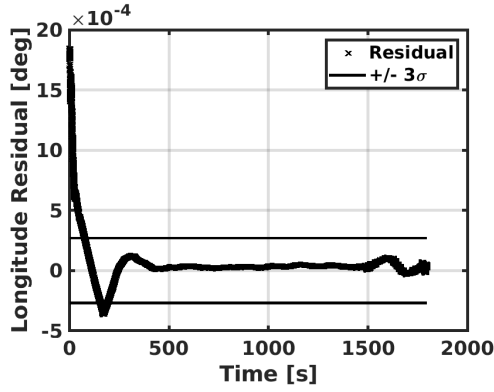
(b) East Velocity: EKF



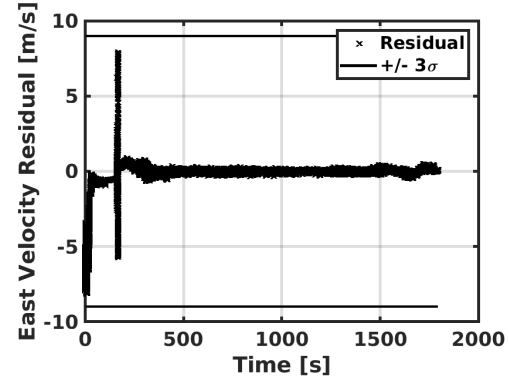
(c) Longitude: UKF



(d) East Velocity: UKF

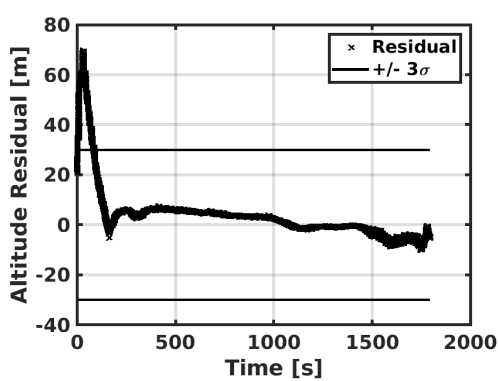


(e) Longitude: DDF

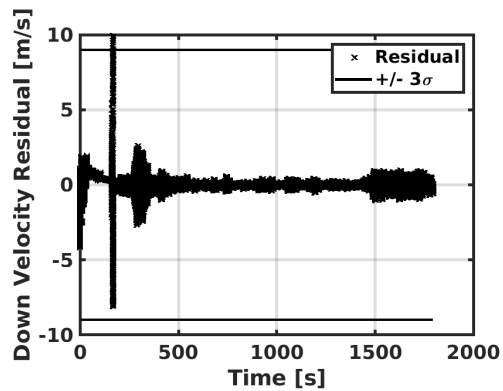


(f) East Velocity: DDF

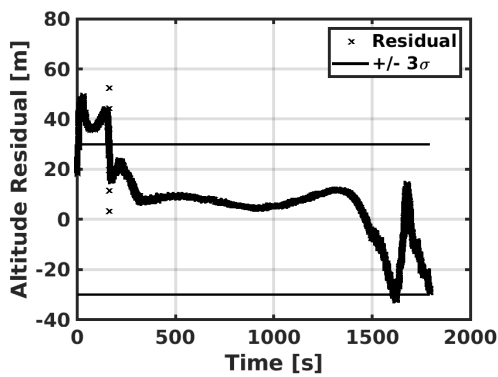
Figure 15: Longitude and East Velocity Residuals



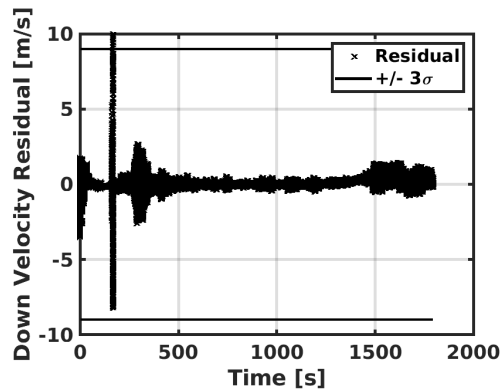
(a) Altitude: EKF



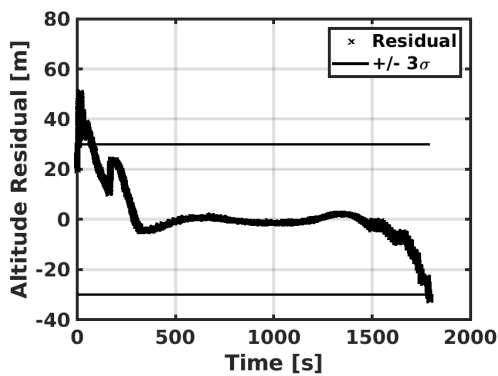
(b) Down Velocity: EKF



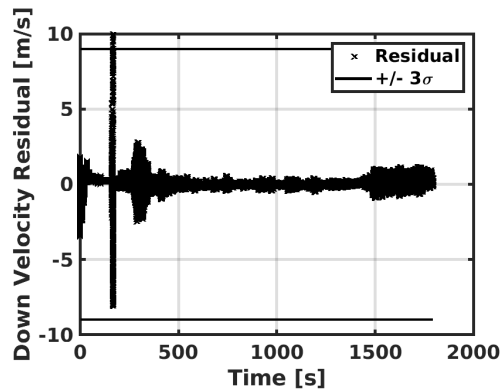
(c) Altitude: UKF



(d) Down Velocity: UKF



(e) Altitude: DDF



(f) Down Velocity: DDF

Figure 16: Altitude and Down Velocity Residuals

4 Summary

Two types of second-order sigma-point Kalman filters have been implemented into the New Statistical Trajectory Estimation Program software for solving trajectory reconstruction problems. The two filters are the Unscented Kalman Filter and the Divided Difference Filter. Both filters are implemented with an iterative measurement update step. The results of a simulation-based test case and a real flight data test case show that the filters perform better than the existing first-order filtering approach by providing a reduction in the state estimate uncertainties. The reduction in state estimate uncertainties of the sigma-point Kalman filters arises as a result of the second-order accuracy expansions of the process and measurement models that are used for the prediction and correction steps in the filter, respectively.

These new filters implemented for solving trajectory reconstruction problems are anticipated to be useful for problems with poor initial conditions, where the traditional first-order methods can exhibit convergence problems. Additionally, these new filter implementations are expected to be useful for problems with sparse measurement data where the second-order process model approximations offers an improved covariance prediction between the measurements.

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REPORT DOCUMENTATION PAGE					Form Approved OMB No. 0704-0188	
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1. REPORT DATE (DD-MM-YYYY) 01-03-2024		2. REPORT TYPE Technical Memorandum		3. DATES COVERED (From - To)		
4. TITLE AND SUBTITLE Iterated Sigma-Point Kalman Filtering for Trajectory Reconstruction				5a. CONTRACT NUMBER		
				5b. GRANT NUMBER		
				5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S) Sophia Vedvik, Christopher D. Karlgaard				5d. PROJECT NUMBER		
				5e. TASK NUMBER		
				5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Langley Research Center Hampton, Virginia 23681-2199				8. PERFORMING ORGANIZATION REPORT NUMBER L-		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001				10. SPONSOR/MONITOR'S ACRONYM(S) NASA		
				11. SPONSOR/MONITOR'S REPORT NUMBER(S) NASA/TM-20240001235		
12. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified-Unlimited Subject Category Availability: NASA STI Program (757) 864-9658						
13. SUPPLEMENTARY NOTES An electronic version can be found at http://ntrs.nasa.gov .						
14. ABSTRACT Two second-order Kalman filtering methods are added to the New Statistical Trajectory Estimation Program (NewSTEP). The two filters are of the class of Sigma-Point Kalman Filters, namely the Unscented Kalman Filter and the Divided Difference Filter. Both filters are implemented with an iterative measurement update step. Results from simulation test case and a flight data test case are provided to show the filter performance. The results indicate that the new second-order filters produce trajectory estimates with reduced uncertainties when compared to the existing first-order filtering method.						
15. SUBJECT TERMS						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON	
a. REPORT	b. ABSTRACT	c. THIS PAGE			STI Information Desk (help@sti.nasa.gov)	
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