

Numerical Dissipation Control in High Order Methods for Compressible Turbulence: Recent Development

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Further Numerical Method Development Are Needed

(Compressible Turbulence with Shocks)

21th- century has been an era in using rapidly developed high speed supercomputers to help understanding data limited flows

- **Computation is ahead of analysis**
Very large scale practical computations are increasingly used in poorly understood multiscale data limited complex nonlinear physical problems and in non-traditional fields.
- **When computations offer the ONLY way of generating the data limited simulations**
 - Too complex for analytical/mathematical means of analysis
 - Observable data are not available
 - Unfeasible by experimental means for any of a number of possible reasons
- **Production computational fluid dynamics (CFD) computer codes with methods that**
 - Were not developed to simulate compressible turbulence flows
 - Are not discretely physics-preserving methods
(e.g., entropy conserving, momentum conserving, and/or kinetic energy preserving)
 - Etc.

Objective

Proper Numerical Dissipation Control in High Order Methods would:

- Provide **accurate, stable & reliable DNS, LES & ILES** computations
- **Prevent incorrect computed Shock Speeds** for problems containing shock waves & stiff nonlinear source terms
- **Delineate** solutions of the discretized counterparts but not solutions of the governing equations
- **Prevent** numerical chaos/"turbulence" leading to **FALSE** predication of transition to turbulence

Typical Turbulence Numerical Simulations Involve:

DNS (Direct numerical simulations)

LES (Large eddy simulations)

ILES (Implicit large eddy simulations)

Turbulence Modeling

Challenges in Numerical Method Development

Nonlinear Instability:

Existing accurate schemes developed for rapidly developing flows usually SUFFER from **nonlinear instability for long time integration**

Numerical Stability & Accuracy: Conflicting Requirements for DNS & LES Simulations

- > Stable schemes usually contain more numerical dissipation than their higher accuracy counterparts
- > Numerical dissipation usually smears turbulent fluctuations
- > Proper amount of numerical dissipation is required for stability in the vicinity of discontinuities

Very high order methods without proper nonlinear stability control can numerically induced (on set) turbulence that is not physical turbulence:

See Yee 2001, Sections 6.3 – 6.4 & table 6.2 below for numerical bifurcation of transition to turbulence (false transition for a studied Reynold's #)

Efficiency & Scalability for High End Computing:

Low operations count & highly scalable accurate methods are required for computing this type of CPU intensive flows



Our New Development to address these challenges:

(Yee et al., Yee & Sjogreen, Sjogreen & Yee, Wang et al., Kotov et al., 2009-2023)

Added Issues for Certain Turbulent Flow Simulations

Difficult to Resolve All Scales:

Standard high order shock-capturing methods for rapid developing flows are too dissipative for turbulence simulations. As these methods need extremely fine grids & CPU Intensive for long time integration

Adaptive Numerical Dissipation Control:

Need smart flow sensor to locally control the amount of numerical dissipation needed at each grid point. If possible, leave the smooth flow portion free of numerical dissipation

Numerical Source Terms Treatments:

- > Well-balanced schemes are needed to preserve physical steady states exactly; e.g., Godunov symmetrizable MHD source term; forced turbulence source terms; turbulent modelling
- > Numerical dissipation & under-resolved grids lead to incorrect shock speeds if source term is stiff

Problems Containing a Wide Spectrum of Flow Speeds & Flow Types:

- > Forced compressible turbulence can initially start with shock-free turbulence but might develop moderate to strong shock waves at a later time (Kotov et al. JCP, 016)
- > Cannot be solved accurately with standard shock-capturing methods

Structure-Preserving High Order Methods:

To further minimize un-physically computed solutions, e.g., entropy conserving, momentum conserving, kinetic-energy preserving, etc., and combination of these high order structure-preserving numerical methods are needed

Sources of Nonlinearities

Due to Physics and/or Numerical Methods

Well Known Sources of Nonlinearities are Due to Physics:

Convection, diffusion, forcing, turbulence source terms, reacting flows, combustion related problems, or any combination of the above, etc.

Less Familiar Sources of Nonlinearities are Due to the Numerics -- Three Major Sources:

(1) Nonlinearities due to time discretizations -- discretized counterpart is **nonlinear in the time step**

- > Runge-Kutta methods. If fixed time steps are used, spurious steady-state or spurious asymptotic numerical solutions can occur, depending on the the initial condition (IC).
- > Linear multistep methods (LMMs) (Butcher 1987) are linear in the time step, and they do not exhibit spurious steady states. See Yee & Sweby (1991-1997) and references cited therein for the dynamics of numerics of standard time discretizations.

(2) Nonlinearities due to spatial discretizations -- discretized counterpart can be **nonlinear in the grid spacing and/or the scheme**

E.g., Total variation diminishing (TVD), essentially nonoscillatory (ENO) and weighted ENO (WENO) schemes. The resulting discretized counterparts are nonlinear (in the dependent variables) even though the governing equation is linear.

See Yee et al. (1989) references cited therein for the early forms of these schemes.

(3) Nonlinearities due to complex geometries, boundary interfaces, grid generation, grid refinements and grid adaptations

See, Yee & Sweby 1995 -- each of these procedures can introduce nonlinearities Even though the governing equation is linear.

Spurious Numerics Due to Source Terms

Phenomena occur in simple scalar case – 3D complex systems

Source Terms: Hyperbolic conservation laws with source terms

- > Most high order shock-capturing schemes are **NOT** well-balanced & produce **huge error**
- > High order WENO/Roe & their nonlinear filter counterparts are well-balanced for certain reacting flows **Our Work:** Wang et al. JCP papers 2010, 2011

Stiff Source Terms:

- > Numerical dissipation can result in **wrong** propagation speed of discontinuities for under-resolved grids if the source term is stiff LeVeque & Yee, 1990
- > This numerical issue has attracted much attention in the literature – last 27 years (Improvement can easily be obtained for a single reacting flow case)
- > A **New Sub-Cell Resolution Method** has been developed for stiff systems on **coarse** mesh
Our Work: Wang et al., JCP, 2012 ; CiCP 2016

Nonlinear Source Terms:

- > Occurrence of **spurious steady-state & discrete standing-wave** solutions – by the use of **fixed** grid spacings & time steps or grid adaptation Yee & Sweby, Yee et al., Griffiths et al., Lafon & Yee, 1990-2002

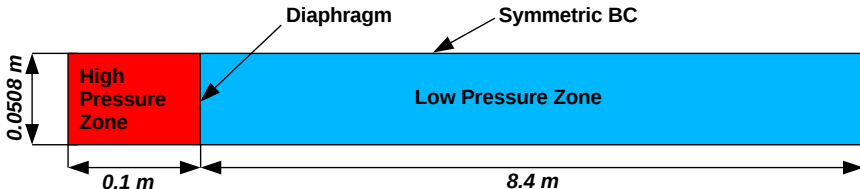
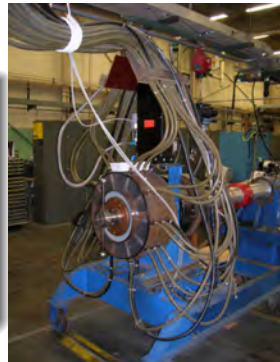
Stiff Nonlinear Source Terms with Discontinuities:

- > **More complex spurious behavior**
- > **Forced Turbulence, numerical combustion, certain terms in turbulence modeling & reacting flows**
Yee et al., Yee & Sweby, Griffiths et al., Lafon & Yee, Kotov et al. 1990 – 2017

Experimental Setup

NASA Electric Arc Shock Tube (EAST) setup

- Chamber: **10 cm × 8.5 m**, window at **7 m**
- Shock velocity: **9 – 16 km/sec**
- Gas: **N₂ + O₂** driven by **He**
- Pressure after discharge: **1 – 27 atm**
- Driven gas initial pressure: **0.1 – 760 Torr**



Computational Model for EAST problem

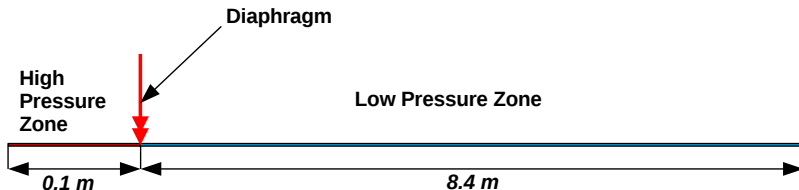
- Consider 1D / 2D reactive N-S equations, 13 species, one temperature energy model
- Guess initial conditions based on measured shock velocity
- Refine the mesh between shear & shock, growing mesh size in stream-wise direction
- Employ high-order non-equilibrium code ADPDIS3D¹ and MUTATION library²

Example of Tollmien-Schlichting-like instability complex patterns at the boundary layer

¹ **B. Sjögren & H.C. Yee, 2009.** *Variable high order multiblock overlapping grid methods for mixed steady and unsteady multiscale viscous flows. Commun. Comput. Phys. 5, 730-744*

² **M. Panesi, T. Magin, A. Bourdon, A. Bultel & O. Chazot, 2011.** *Study of electronically excited state populations of atoms and molecules predicted by means of a collisional-radiative model for the fire ii flight experiment. Journal of Thermophysics and Heat Transfer 25, 361-374*

1D EAST Problem Setup



13 species mixture:

e^- , He , N , O , N_2 , NO , O_2 , N_2^+ , NO^+ , N^+ , O_2^+ , O^+ , He^+ .

High Pressure Zone

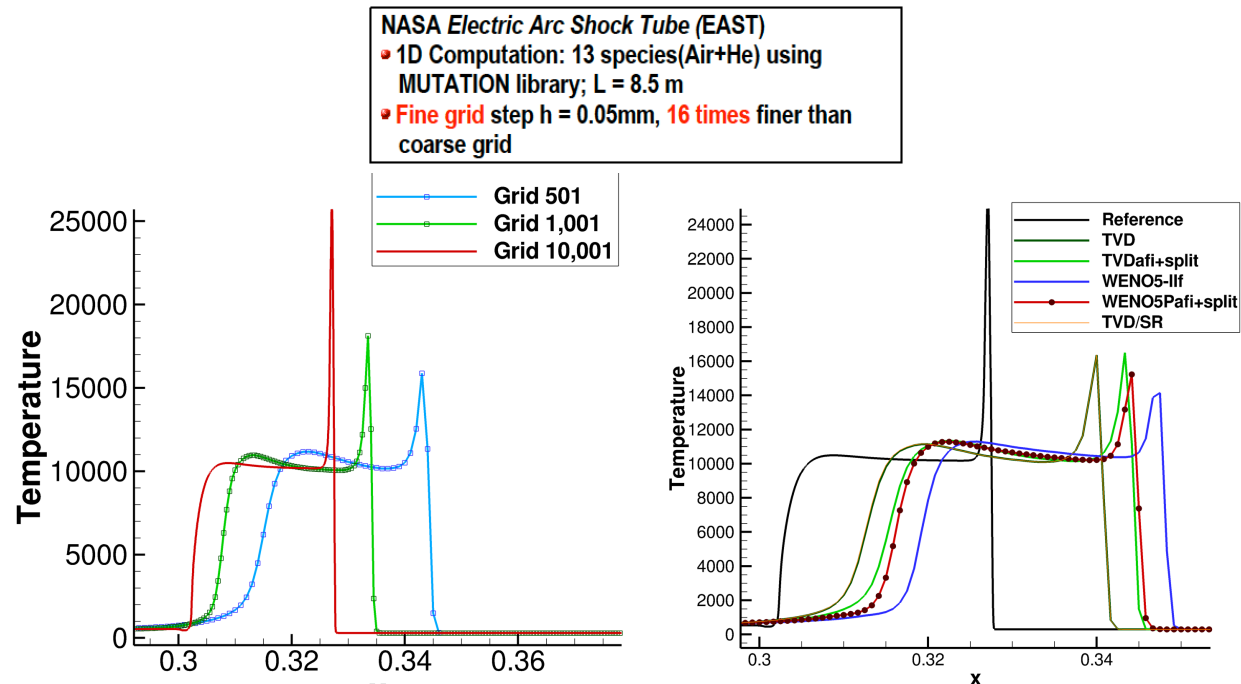
ρ	1.10546 kg/m^3
T	6000 K
p	12.7116 MPa
Y_{He}	0.9856
Y_{N_2}	0.0144

Low Pressure Zone

ρ	$3.0964 \times 10^{-4} \text{ kg/m}^3$
T	300 K
p	26.771 Pa
Y_{O_2}	0.21
Y_{N_2}	0.79

Stiff Source Terms: Wrong Discontinuity Locations

(Grid & method dependence of shock/shear locations)



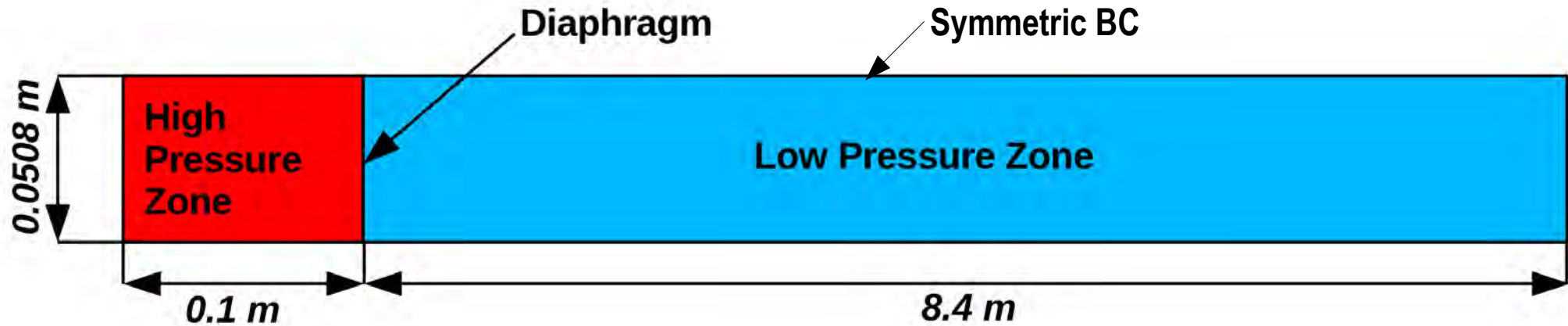
Flows without stiff source term: Computed locations of discontinuities are **independent** of the grid size or high-resolution shock-capturing methods

Implication: The danger in trusting numerical simulation for problems with stiff source terms
Non-standard behavior of numerics observed in non-reacting flows

(Yee et al., Griffith et al., Wang et al., Kotov et al., 1990 - 2016)

2D EAST Problem (Viscous Nonequilibrium Flow)

NASA Electric Arc Shock Tube (EAST) – joint work with [Panesi](#), [Wray](#), [Prabhu](#)



13 Species mixture:

e^{-} , He , N , O , N_2 , NO , O_2 , N_2^{+} , NO^{+} , N^{+} , O_2^{+} , O^{+} , He^{+}

High Pressure Zone

ρ	1.10546 kg/m^3
T	6000 K
p	12.7116 MPa
Y_{He}	0.9856
Y_{N_2}	0.0144

Low Pressure Zone

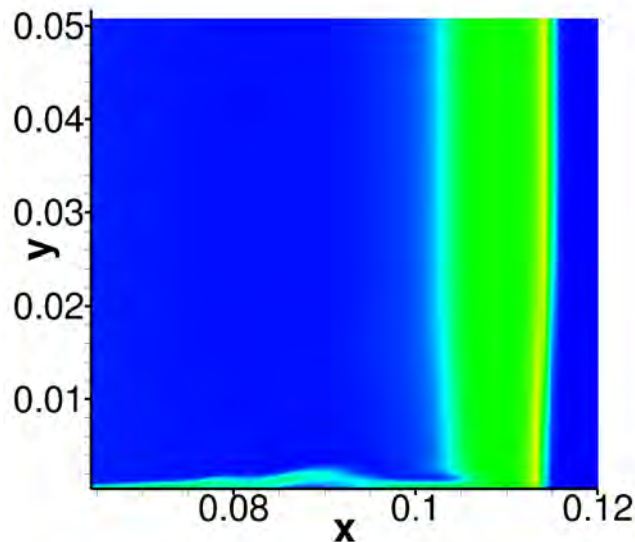
ρ	$3.0964\text{e}-4 \text{ kg/m}^3$
T	300 K
p	26.771 Pa
Y_{O_2}	0.21
Y_{N_2}	0.79

EAST: Temperature Computed at $t = 1.e-5$ s

Shock/Shear Locations Grid Dependence

TVD, CFL = 0.7

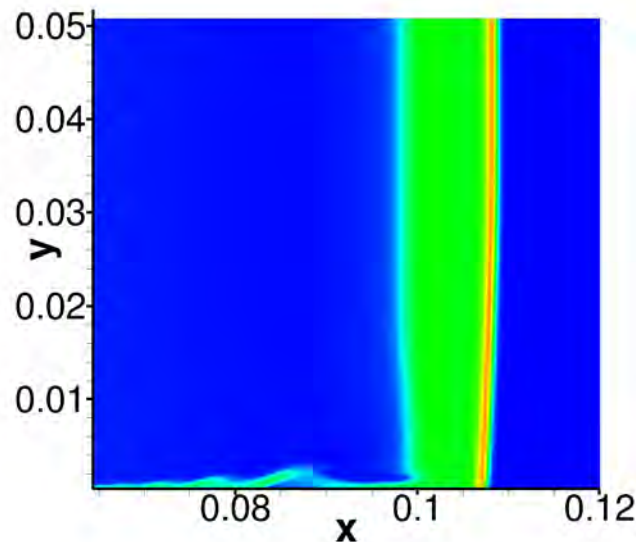
Tollmien-Schlichting-like instability complex patterns at the boundary layer



601x121

Uniform in x

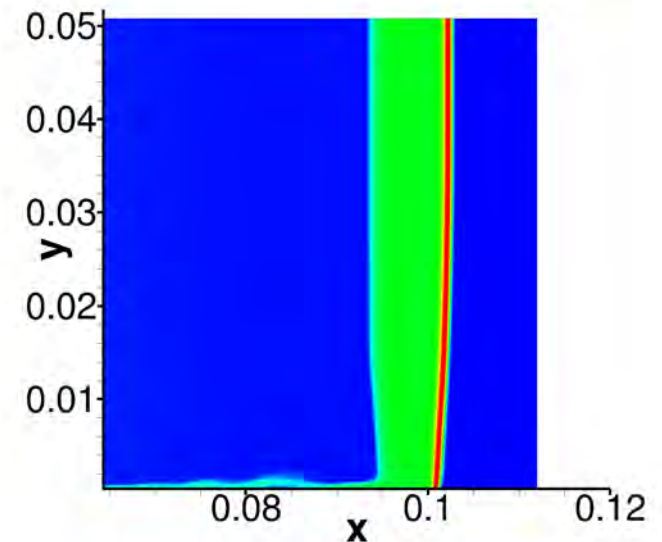
Tmax = 15,800 K



1201x121

Uniform in x

Tmax = 18,800 K



690x121

Cluster near shock in x

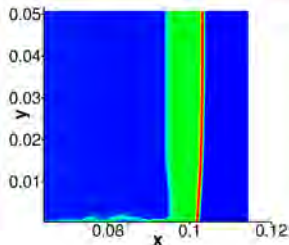
Tmax = 21,700 K

Fine grid $h = 0.05$ mm
Grid points needed for x -dimension: 170,000

Method Comparison

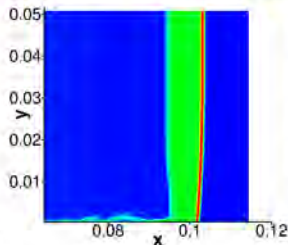
$t_{end} = 10^{-5} \text{ sec}$, grid 691×121 , cluster in X , $\Delta Y_{min} = 10^{-5}$

Tollmien-Schlichting-like instability complex patterns at the boundary layer



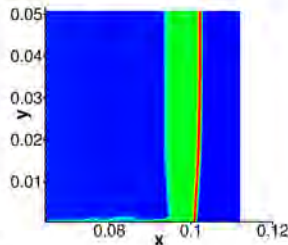
WENO5-LLF

CFL=0.2



WENO5P-LLF

CFL=0.4



TVD

CFL=0.7

Note 1: Different shock location: $X_{TVD} = 0.1024 \text{ m}$ & $X_{WENO} = 0.1034 \text{ m}$

Note 2: Different boundary layer: Different methods

**Physical Preserving (Structural Preserving) Methods
are Essential in Minimizing Spurious Numerics**

Selected Results on Different Nonlinear Behavior of Numerical Methods

(Spurious dynamics INDEPENDENTLY *introduced by* Spatial & Time *discretizations)*

Spurious Numerics Due to **Time Discretizations**:

- Occurrence of stable and unstable **spurious asymptotes ABOVE** the linearized stability limit of the scheme (for constant time steps)
- **Stabilization of unstable steady states** by implicit and semi-implicit methods
- Interplay of **initial data and time steps** on the occurrence of spurious asymptote
- Interference with the dynamics of the underlying implicit scheme by **procedures in solving the nonlinear algebraic equations** *(resulting from implicit discretizations of the continuum equations)*
- Dynamics of the **linearized implicit Euler** scheme solving the time-dependent equations to obtain steady states vs. **Newton's method solving the steady equation**
- Etc.

Spurious Numerics Due to **Spatial Discretizations**:

- Convergence problems & spurious behavior of high-resolution shock-capturing methods
- **Numerically induced & suppressed** (spurious) **chaos & numerically induced chaotic transients**
- Spurious dynamics generated by grid adaptations
- **False Numerical Prediction of Transition to Turbulence**
- **Wrong propagation speed of discontinuities** for problems containing stiff source terms, etc.

Certain Structure-Preserving or Combination of Methods

Can also Minimizing Spurious Solution & the Used of Numerical Dissipation

Skew-Symmetric Splitting of Inviscid Flux Derivatives (Improve nonlinear stability for high order central schemes)

- **Entropy splitting:** **Semi-conservative** splitting for **shock-free turbulence**
(Olsson & Oliger 1994, Yee et al. 1999-2007, Sandham et al. 2002-present)
- **Natural Splitting:** Linearized Euler & Non-conservative Systems
- **Splitting to Preserve Discrete Momentum and/or Energy:**
(Arakawa 1966, Blaisdell et al. 1996, Mansour 1980, Ducros et al. 2000, Pirozzoli 2009, Sjogreen et al. 2017)
- **Ducros et al. Type Conservative Splitting:** **Euler & MHD** (Sjogreen et al. 2017)
- **Generalized Skew-Symmetric Splitting:** 3-parameter family (Pirozzoli 2009)

Preprocessing Step: Improve stability of classical central scheme

Replacing high order classical central approximation of the inviscid flux derivative

→ High order approximation of their split form counterpart

Selected Illustration

- **Spurious Numerics Due to Stiff Nonlinear Source Terms**
- **Physical Preserving High Order Numerical Methods for Turbulence with Shocks**
E.g., *Entropy Conserving, Momentum Conserving, Kinetic Energy Preserving, Positivity Preserving, Skew-Symmetric Splitting to Improve Nonlinear Stability without ADD Numerical Dissipation*

Comparison Among Seven 8th-Order Methods

- **ECLOG**: Tadmor-type entropy conserving method using the Tadmor entropy function $E_L = -\rho \log(p\rho^{-\gamma})$.
- **ECLOGKP**: Tadmor-type entropy conserving method using the Tadmor entropy function with Ranocha's kinetic energy preserving modification [32].
- **ECHKP**: Entropy conserving using the Harten class of entropy functions $E_H = -\frac{\gamma+\alpha}{\gamma-1}\rho(p\rho^{-\gamma})^{\frac{1}{\alpha+\gamma}}$ [50, 23]. In its base form also satisfies Ranocha's kinetic energy preservation condition (KEP); only one variant for this method [43, 32].
- **KGP**: Kennedy-Gruber-Pirozzoli (KGP) skew-symmetric splitting of the inviscid flux derivative that is kinetic energy preserving [26, 31, 13].

Comparison Among **Seven** 8th-Order Methods

(Cont.)

- **ESSW**: Entropy split with Ducros et al. splitting but switch to regular central near discontinuities [46].
- **DSKP**: Ducros Split with KEP.
- **DS**: Momentum conserving Ducros et al. skew-symmetric split of the inviscid flux derivative [14].

3D Taylor-Green vortex

(Inviscid & Viscous Shock-Free Turbulence)

Computational Domain: 2π square cube, 64^3 grid.
(Reference solution on 256^3 grid)

Initial condition

$$\begin{aligned}\rho &= 1, \\ p &= 100 + ([\cos(2z) + 2][\cos(2x) + \cos(2y)] - 2)/16, \\ u_x &= \sin x \cos y \cos z \\ u_y &= -\cos x \sin y \cos z \\ u_z &= 0.\end{aligned}$$

Initial turbulent Mach number: $M_{t,0} = 0.042$

Final time: $t = 10$

Viscous case

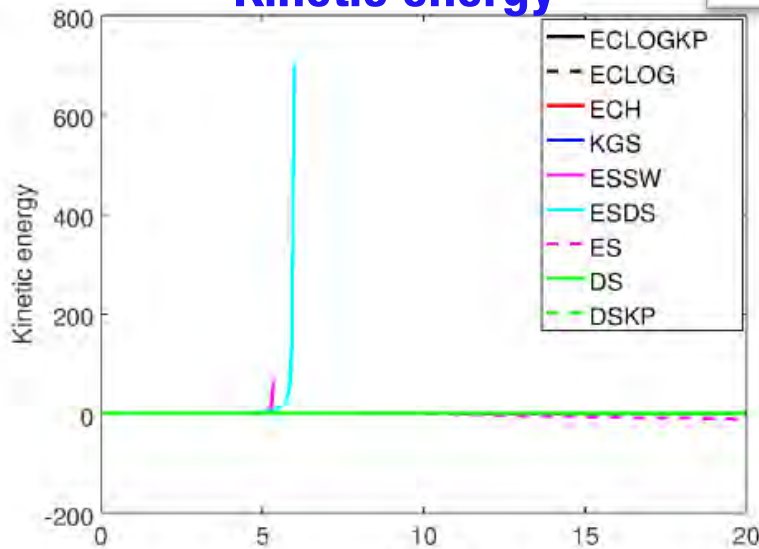
$$\begin{aligned}\mu / \mu_{ref} &= (T / T_{ref})^{3/4} \\ \mu_{ref} &= 0.005, T_{ref} = 1, Re_0 = 2040\end{aligned}$$

3D Taylor-Green Vortex

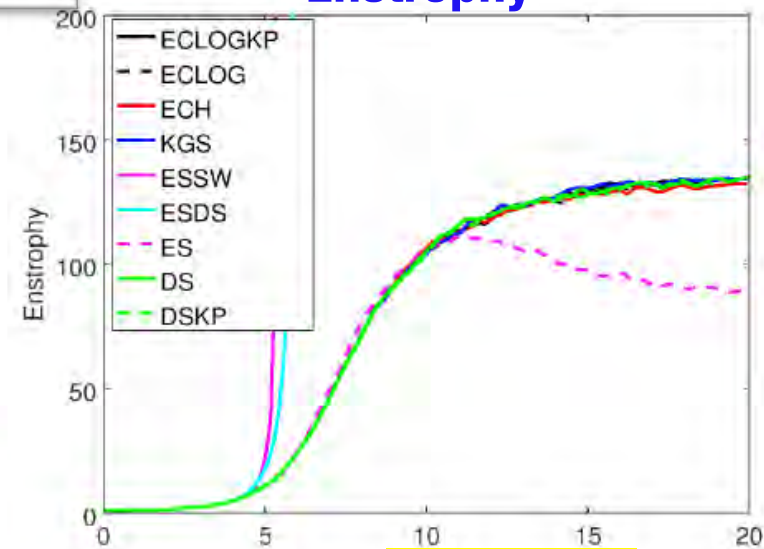
Nine 8th-order method comparison, 64³ grid

$$\beta = 2$$

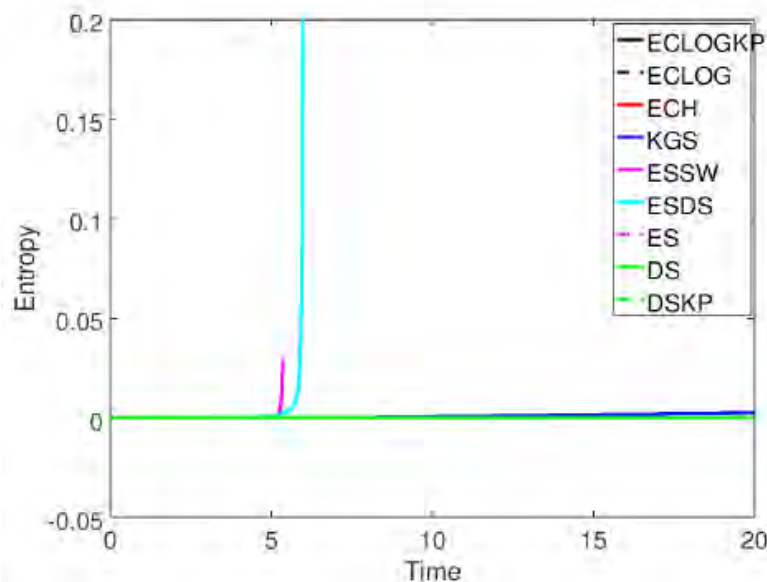
Kinetic energy



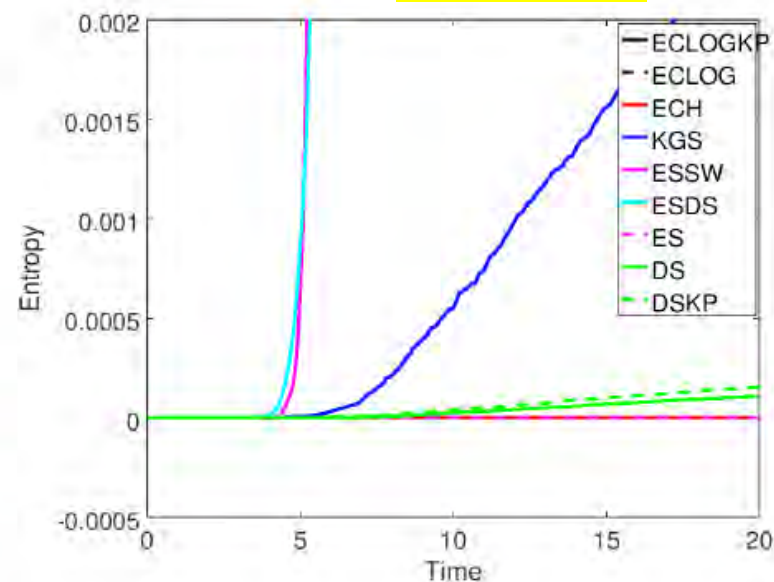
Enstrophy



Entropy

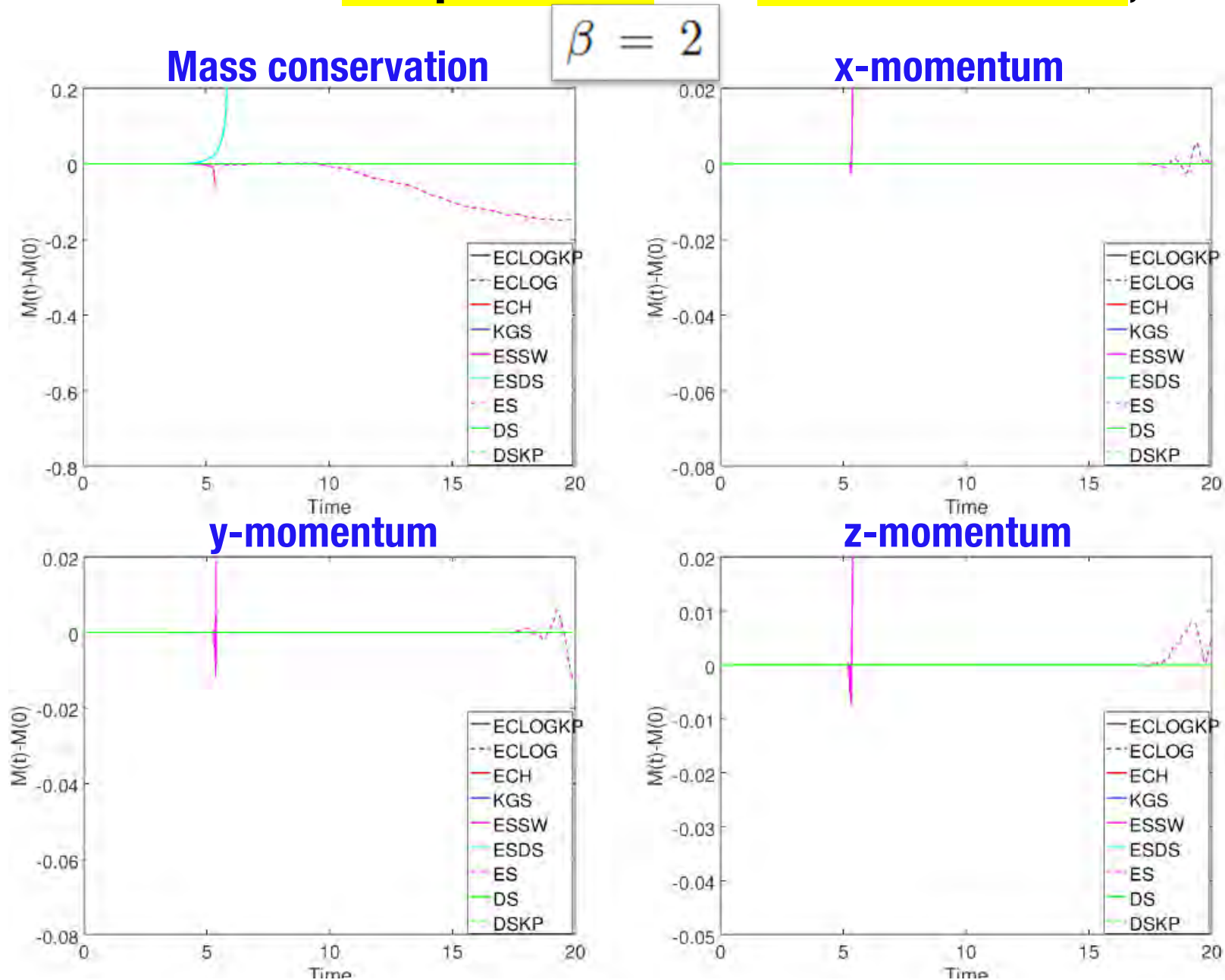


Entropy (closed up)



3D Taylor-Green Vortex

Mass Conservation Comparison of Nine 8th-order Methods, 64³ grid



2D Alfven Wave Test Case

$$\begin{aligned}\rho &= 1 \quad p = 1 \quad \mathbf{v} = A(-\sin \alpha \sin \phi, \cos \alpha \sin \phi, \cos \phi) \\ \mathbf{B} &= (\cos \alpha - A \sin \alpha \sin \phi, \sin \alpha + A \cos \alpha \sin \phi, A \cos \phi) \quad . \\ \phi &= 2\pi(x \cos \alpha + y \sin \alpha + t)\end{aligned}$$

where $\alpha = 30^\circ$ and $A = 0.1$ are chosen with periodic boundary conditions

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The computational domain is 1.3546×2.1915

The flow is solved up to time 300 until all methods eventually diverge.

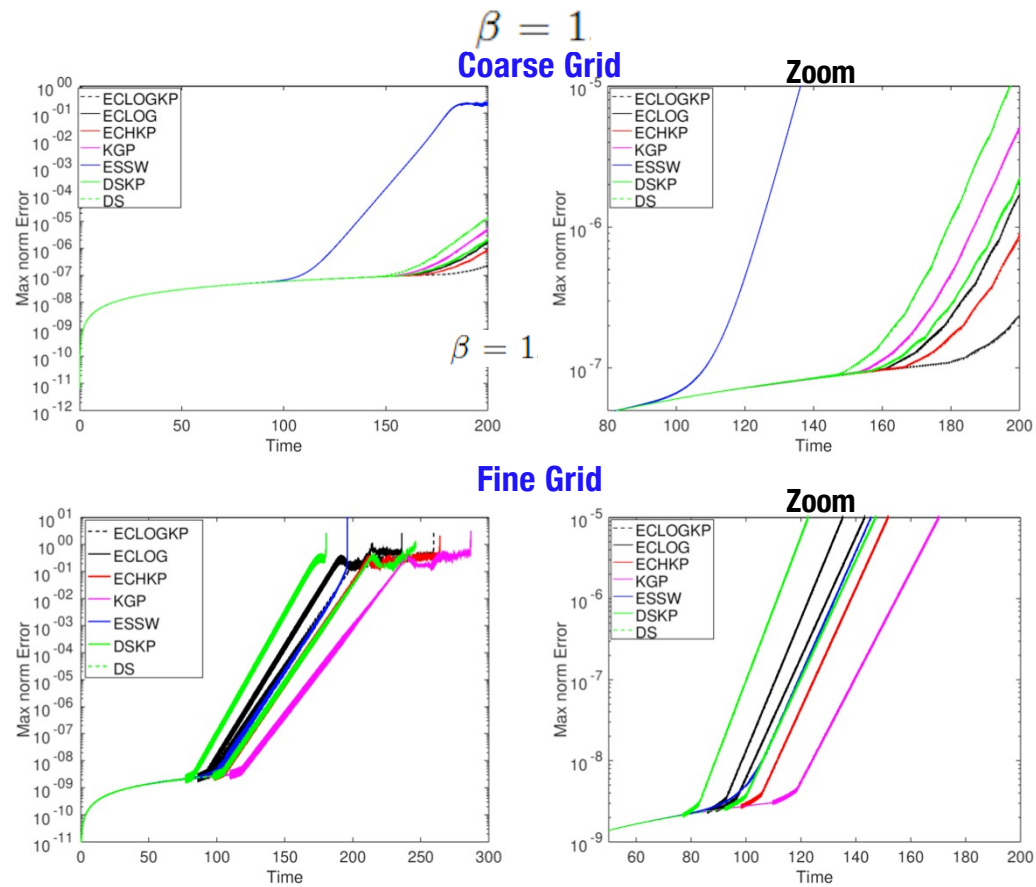
$$\Delta x = \Delta y = 0.022$$

$$\Delta x = \Delta y = 0.011$$

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Alfven Wave (Max-Norm Error vs. Time)

(Comparison Among Seven 8th-order Methods)



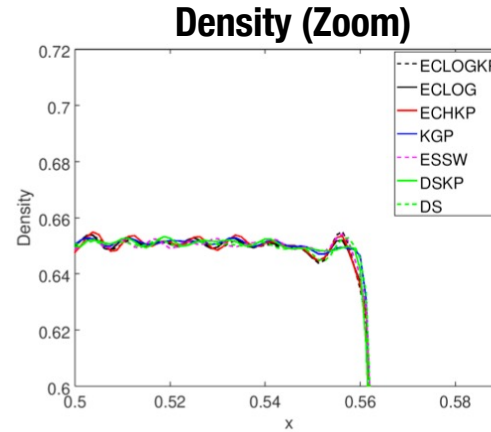
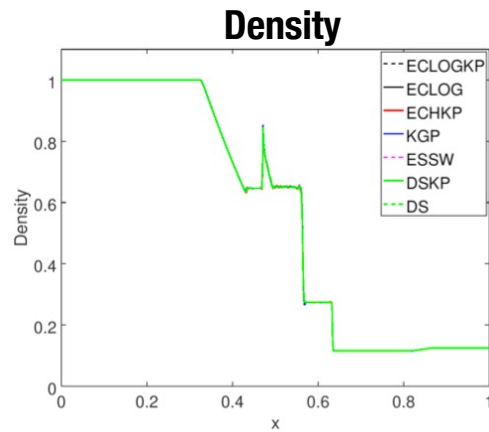
Brio-Wu MHD Shock Tube Problem

$$(\rho_L, u_L, p_L, B^{(x)}, B^{(y)}, B^{(z)}) = (1, 0, 1, 0.75, 1, 0)$$

$$(\rho_R, u_R, p_R, B^{(x)}, B^{(y)}, B^{(z)}) = (0.125, 0, 0.1, 0.75, -1, 0).$$

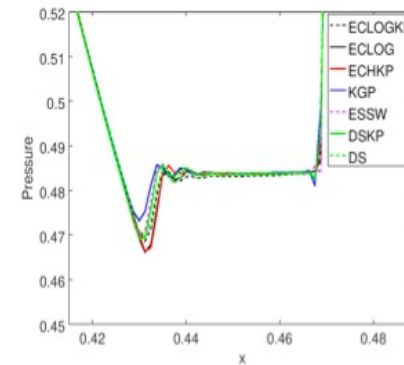
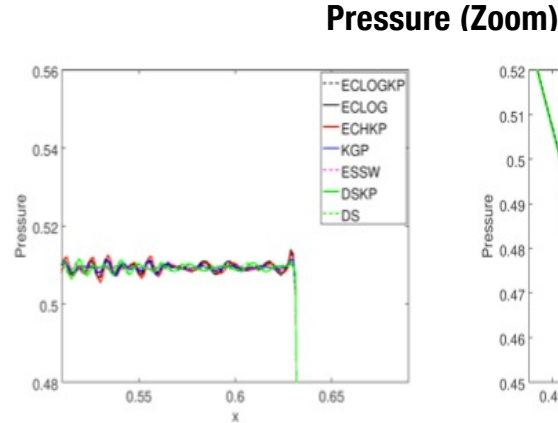
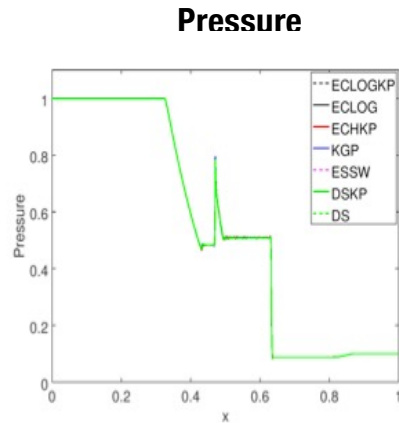
Brio-Wu MHD Shock Tube Problem

(Comparison Among Seven Methods + 7th-order nonlinear filter)



Very difficult test case

Standard methods are only stable using 1st & 2nd-order methods with low accuracy & higher spurious oscillations on zoomed regions



Brief Summary

GAS dynamics: Classical, DRP and Pade' Spatial discretizations

- All split methods can improve nonlinear stability for **smooth flows**
(Without added numerical dissipation in certain flows)
- All nonlinear filter versions of split methods can improve stability & accuracy for DNS & LES
- All split methods provide similar stability & accuracy improvement
(Pade' methods are most CPU intensive & not friendly in parallelization for multi-D flows)
- All split methods are **less CPU intensive** than the **Tadmor-type entropy conserving** methods

Plasma: Classical & Pade' spatial discretizations

- Both split methods can improve nonlinear stability in general for **smooth flows**
but are MHD equations dependent (Pade' methods are most CPU intensive & not friendly in parallelization for multi-D flows)
- Nonlinear filter version of split schemes can improve stability & accuracy for flows with discontinuities **but are MHD equations dependent**
- High order entropy conserving methods (centered or nonlinear filter version) can provide **different stability & accuracy improvement**, depending on the **forms of the MHD equations & the choice of entropy fluxes**

Key Overview References Related to Our Work

- Yee, H.C. and Sweby, P.K. (1997), ``Dynamics of Numerics & Spurious Behaviors in CFD Computations,’’ Keynote paper, 7th ISCFD Conference, Sept. 15-19, 1997, Beijing, China}, RIACS Technical Report 97.06, June 1997.
- Yee, H.C. and Sjogreen, B., (2001c) ``Building Blocks for Reliable Complex Nonlinear Numerical Simulations,’’ Turbulent Flow Computation, (Eds. D. Drikakis & B. Geurts), Kluwer Academic Publisher; also RIACS Technical Report, Dec. 2001.
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- Sjogreen, B., Yee, H.C., Kotov, D. and Kristsuk, A.G. (2020) Skew-symmetric splitting for multiscale gas dynamics and {MHD} turbulence flows. J. Sci. Comput.}, 83 (43) doi: 10.1007/s10915-020-01225-w.
- Yee, H. C., Sjogreen, B. (2022): Recent Advancement of Entropy Split Methods for Compressible Gas Dynamics and MHD. J. Appl. Math. Comput., A special issue in the Journal of Applied Mathematics and Computation (ACM) on Hyperbolic PDE in computational physics: Advanced Mathematical Models and Structure-Preserving Numerics.

High Order Entropy Conservative Methods

(One way to improve nonlinear stability & minimize added numerical dissipation)

- Numerical solutions satisfy additional discretized conservation law
- Low order entropy conservative methods with linear numerical dissipation for shock-capturing require further accuracy improvement

(Tadmor 1984 – gas dynamics, Janhunen 2000 – MHD, Winters & Gassner 2016 – MHD)

- High order entropy conservative methods for central schemes

(Fjordholm et al. 2012 – ENO, Carpenter et al. 2013-2016,
Sjogreen & Yee 2016, 2017 – central + nonlinear filter, gas dynamics & MHD)

Plasma (Hypersonic Flows):

Four forms of the MHD equations to be considered

- > Conservative form
- > Godunov/Powell symmetrizable form (non-conservative)
- > Janhunen form: (Div B) terms not included in the gas dynamics part of the equations
- > Brackbill & Barnes form

Three forms of the entropy fluxes to be considered

(Winter & Gassner 2016, Chandrasheka & Klingenberg 2016, Sjogreen & Yee 2016-2017)

Major Stumbling Blocks In Genuinely Nonlinear Studies

(Unlike linear model equations used for conventional stability and accuracy considerations in time-dependent PDEs, there are no equivalent unique nonlinear model equations for nonlinear hyperbolic and parabolic PDEs for fluid dynamics)

- On one hand, a numerical method behaving in a certain way **for a particular nonlinear PDE might exhibit a different behavior for a different nonlinear PDE** even though the PDEs are of the same type.
- On the other hand, even for **simple nonlinear model PDEs** with known solutions, **the discretized counterparts can be extremely complex, depending on the numerical methods.**
- Except in special cases, there is **no general theory at the present time to characterize the various nonlinear behaviors of the underlying discretized counterparts.**

Note:

- “**Discretized counterparts**” is used to mean the finite difference equations resulting from finite discretizations of the underlying PDEs.
- “**Dynamics**” is used loosely to mean the dynamical behavior of nonlinear dynamical systems (continuum or discrete)
- “**Numerics**” is used loosely to mean the numerical methods and procedures for solving dynamical systems.
- In the study of the “**Dynamics of Numerics**”, we usually assume the continuum (governing equations) is **nonlinear** (*unless the*

Approaches

(Consider the PDE & Its Discretized Counterparts as SEPARATE Dynamical Systems & Analyze them)

- Based on Knowledge gained for nonlinear model problems with known analytical solutions
(Available theory not able to fully analyze nonlinear behavior of numerics for the nonlinear Euler & Navier-Stokes Eqns.)
- How well numerical methods can mimic the solution behavior of the underlying **highly nonlinear** PDE model for finite time steps & grid spacings *(not as they approach zero).*
- Identify & explain some of the possible sources & remedies of numerical uncertainties in practical computations.

Spurious Numerics: Solution of Discretized Counterparts but NOT SOLUTION of the PDE

Methods of Lines (MOL) Approach

In General, MOL Approach is Easier to Analyze than Mix Time & Space Approaches:

- Spurious Numerics due to Time Discretizations are Easier to Overcome by, e.g., Adaptive Time Step Control
- Spurious Numerics due to Spatial Discretizations are More Difficult to Overcome even by, e.g., grid refinement, adaptation and/or multigrid approaches

Knowledge Gained from Spurious Numerics Due to Spatial Discretizations Leading to the Need:

- Adaptive Numerical Dissipation Controls for Long Time Integration of Unsteady Flows: Direct Numerical Simulations (DNS), Large Eddy Simulations (LES), Simplectic Dynamical Systems, etc.
- Physical Preserving (Structural Preserving) Methods
- Improvement in nonlinear Stability & Accuracy by High Order Skew Symmetric Splitting of the Inviscid Flux Derivatives
- Numerical Treatment of Source Terms and Nonlinear Stiff Source Terms
- Etc.

Our Approach to Improve Nonlinear Stability & Accuracy (for Long Time Integration of Multiscale Compressible Turbulence for gas dynamics & MHD)

- **Skew-Symmetric Splitting** of the inviscid flux derivative (before the application of non-dissipative centered schemes for nonlinear stability) Yee & Sjogreen, Sjogreen & Yee, 2016-2018
- **DRP** (Dispersion Preservation-Relation) & compact schemes as **alternatives** to split version of classical high order central schemes Yee & Sjogreen, Sjogreen & Yee, 2017-2022
- **High-Order Entropy Conservative Numerical Fluxes** with entropy satisfying properties
 - Numerical solution satisfies an additional discretized conservation law Sjogreen & Yee, 2016-2018
- Standard high order **Linear Filters** are to be **replaced by** high order **Nonlinear Filters**
Yee et al., Yee & Sjogreen, Sjogreen & Yee, Kotov et al. (1999-2017)
- **Smart Flow Sensors** to provide locations & appropriate amount of numerical dissipation needed Yee & Sjogreen, Kotov et al. (2009-2016)
Yee et al. high-order nonlinear filter schemes with smart local flow sensors
- **Nonlinear Dynamics** is utilized to complement the traditional linearized stability theory (Yee & Sweby, Yee et al., Griffiths et al., Lafon & Yee, Wang et al., Kotov et al. 1990- 2015)
 - Minimize numerically induced false transition to turbulence
 - Minimize numerical instability due to long time integration of turbulent flows
 - Minimize numerically induced standing wave solutions
 - Minimize wrong shock speeds

Schematic of Possible Spurious Bifurcation (Assume a certain physical transition; same IC & BC)

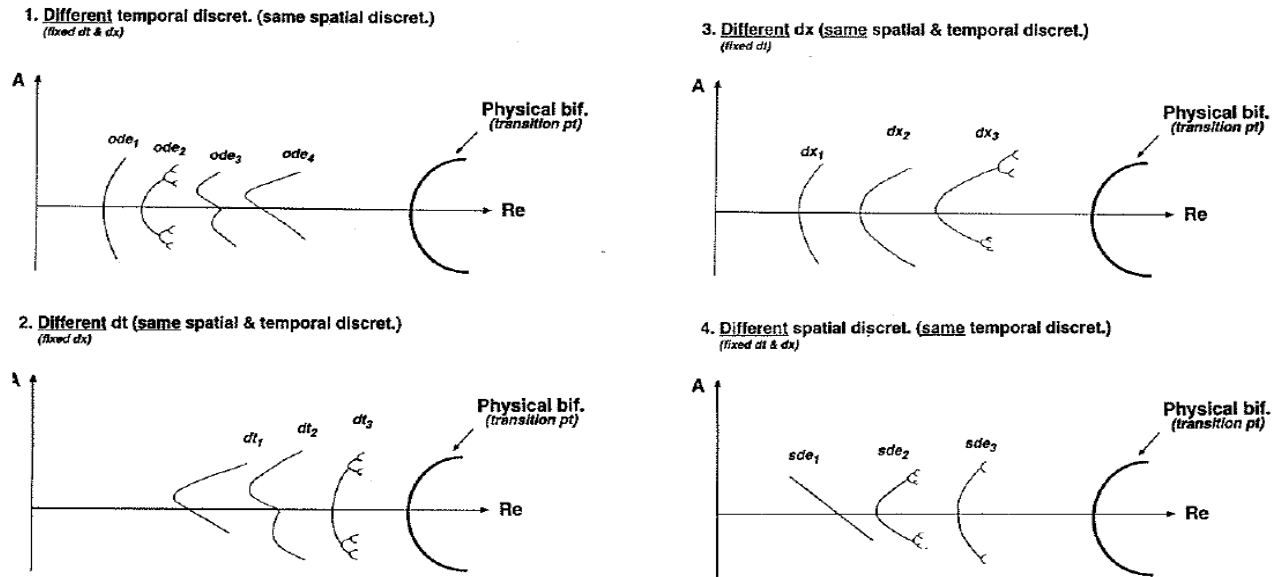


Figure 6.2. Schematic of possible spurious bifurcation for constant time steps and grid spacings. (1) Different temporal discretizations ode_1 , ode_2 , ode_3 and ode_4 (same spatial discretization and the same constant dt and dx). (2) Different constant time steps dt_1 , dt_2 , dt_3 and dt_4 (same temporal and spatial discretizations, and the same constant dx). (3) Different constant grid spacings dx_1 , dx_2 , dx_3 and dx_4 (same spatial and temporal discretizations, and the same constant dt). (4) Different spatial discretizations sde_1 , sde_2 and sde_3 (same temporal discretization and the same constant dt).

2D Isentropic Vortex Convection

(Inviscid, pure convection of the IC)

Computational Domain: Period BC, $0 \leq x \leq 18, 0 \leq y \leq 18$

Initial Condition:

$$\rho(x, y) = \left(1 - \frac{(\gamma - 1)\beta^2}{8\gamma\pi^2} e^{1-r^2}\right)^{\frac{1}{\gamma-1}}$$

$$u(x, y) = u_\infty - \frac{\beta(y - y_0)}{2\pi} e^{(1-r^2)/2}$$

$$v(x, y) = v_\infty + \frac{\beta(x - x_0)}{2\pi} e^{(1-r^2)/2}$$

$$p(x, y) = \rho(x, y)^\gamma,$$

$$r^2 = x^2 + y^2, \beta = 5, \gamma = 1.4, u_\infty = 1, v_\infty = 0.$$

2D Isentropic Vortex Convection

Nine 8th-order method comparison, 201² grid

$$\beta = 2$$

