Safe Tracking Control of an Uncertain Euler-Lagrange System with Full-State Constraints using Barrier Functions

Iman Salehi, Ghananeel Rotithor, Daniel Trombetta, Ashwin P. Dani

***Abstract*— This paper presents a novel, safe tracking control design method that learns the parameters of an uncertain Euler-Lagrange (EL) system online using adaptive learning laws. A barrier function (BF) is first used to transform the full-state constrained EL-dynamics into an equivalent unconstrained dynamics. An adaptive tracking controller is then developed along with the parameter update law in the transformed state space such that the states remain bounded for all time within a prescribed bound. A stability analysis is developed that considers the EL-dynamics' uncertainty, yielding a semi-globally uniformly ultimately bounded (SGUUB) tracking error and the parameter estimation error. The controller design is validated in simulations using a two-link planar manipulator. The results show the proposed method's ability to track the reference trajectory while remaining inside each of the predefined state bounds.**

I. Introduction

In many control engineering applications, maintaining system states within a prescribed bound is essential to satisfy the system safety property. For example, when a robot moves in a constrained space, it is crucial for the robot to satisfy requirements, such as the joint trajectories' boundedness, to carry out operations safely. The violation of constraints can lead to severe degradation of the robot's performance, unsafe behavior, and sometimes failure of the robot's components. In robotics applications such as construction, assembly/disassembly in a constrained space, for human-in-the-loop control applications [1], or distributed multi-robot control applications [2]–[4], restricting the motion of the robot to a constrained joint or state space is essential.

Barrier function (BF) is a commonly used approach to certify the forward invariance of a closed set with respect to a system model, which can be used to examine the system's safety property [5]–[7]. Control barrier functions (CBFs) are used for the synthesis of safety critical controllers [8]–[11]. Since CBF is a Lyapunov-based control design, a control Lyapunov function (CLF) and CBF are merged to synthesize stable and safe controllers by solving a quadratic program (QP) for cyber-physical systems in [12]–[14]. The robustness properties of the BFs and CBFs in the presence of disturbances and uncertainties are studied in [7]. CBFs are also used to design and synthesize controllers for hybrid systems for walking robot applications in [15]. Controllers that are synthesized using CBFs and CLFs assume that the system model is fully known.

In the context of system identification, a data-driven dynamic system model learning approach, which retains all its trajectories in a predefined constrained set, is developed in [16]. The learning approach uses active learning to select samples that are used to learn the model parameters by solving a constrained optimization problem. In [17], a safety critical-control synthesis using active set invariance is developed that uses a CBF methodology to enforce set invariance on systems in the presence of disturbances and uncertainties. The model identification and control design is achieved in traditional adaptive control, which requires a persistency of excitation (PE) condition to learn the system parameters. In [18], an adaptive CBF is proposed that ensures forward invariance of a closed set with respect to a nonlinear control-affine system with parameter uncertainty. The controller and the parameter update laws are computed by solving an optimization problem.

Barrier Lyapunov function (BLF) is another method that is used to control nonlinear systems with output and state constraints (cf. [19], [20]). By design, the value of a BLF grows to infinity whenever its argument approaches some predefined limits. In [19] and [21], a BLF is defined on the output tracking error to develop an adaptive controller for single-input and single-output (SISO) nonlinear systems in a strict-feedback form with constant and time-varying output constraints. The approach in [21] is extended to output tracking with partial state constraints in [22]. Using a similar BLF, in [23], an adaptive neural network with full-state feedback control that uses a Moore-Penrose pseudoinverse term in the control law design is developed for an uncertain robot dynamics with output constraints, and the signals of the closed-loop systems are proven to be SGUUB. In [24], [25], a BLF along with RL is used to solve a regulation problem for a SISO nonlinear systems in the Brunovsky form with full-state and input constraints.

In this paper, an adaptive tracking controller is developed for an uncertain nonlinear multi-input and multi-output (MIMO) system represented by the EL-dynamics with full-state constraints. In [24], [25], a similar BF is used to transform the full-state constrained system dynamics to an equivalent system dynamics with no explicit constraints. The BF is used on each state of the system, allowing for

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the transformation of the original MIMO system into a new unconstrained MIMO system. An adaptive controller is developed for the transformed dynamics, where the unknown model parameters are estimated using a gradient parameter adaptation law. The developed controller and update laws keep the state within an ultimate bound while remaining constrained inside the given bounds on the state from any initial conditions chosen within the bounds for the original and the transformed systems. A Lyapunov stability analysis is developed that yields SGUUB tracking error of the desired trajectory. The parameter estimates of the nonlinear system model remain bounded. The controller design is tested using a 2-link robot through numerical simulations. The developed controller's performance is compared with the controller's performance without the barrier function transform.

II. Preliminaries

In this section, the preliminaries of barrier function are briefly reviewed.

A. Barrier Function

Consider a continuous nonlinear dynamical system of the form

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), \quad \forall i = 1, \cdots, n-1, \\ \dot{x}_n(t) &= f\left(x(t)\right) + g\left(x(t)\right)u, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ are locally Lipschitz continuous nonlinear functions, $x(t) = [x_1(t) \ \cdots \ x_n(t)]^T \in \mathcal{X} \subset \mathbb{R}^n$, is the state of the system, and $u \in \mathbb{R}^m$ is the control input. The compact set $\mathcal{X} := \{x_i(t) \in \mathbb{R} : a_i < x_i(t) < A_i, \ \forall i = 1, \cdots, n\}$, where $a_i \in \mathbb{R}$ and $A_i \in \mathbb{R}$ are lower and upper bounds of each state which satisfy $a_i < 0 < A_i$, $\forall i = 1, \cdots, n$. Note that throughout this paper, for ease of notation, we abbreviate $x(t)$ by x, unless necessary for clarity.

The following logarithmic barrier function candidate $B(x_i; a_i, A_i)$: $\mathbb{R} \to \mathbb{R}$, is considered in this paper:

$$B(x_i; a_i, A_i) \triangleq \log\left(\frac{A_i}{a_i}\frac{a_i - x_i}{A_i - x_i}\right), \ \forall x_i \in (a_i, A_i), \tag{2}$$

where $\log(\cdot)$ is a natural logarithm. From (2), it can be seen that $\lim_{x_i \to a_i, A_i} B\left(x_i; a_i, A_i\right) = \infty$.

Using (2), the system can be transformed into a constrained state $s = [s_i \ \cdots \ s_n] \in \mathbb{R}^n$ as follows

$$s_i = B\left(x_i; a_i, A_i\right), \qquad \forall i = 1, \cdots, n \tag{3}$$

$$x_i = B^{-1}\left(s_i; a_i, A_i\right), \qquad \forall i = 1, \cdots, n \tag{4}$$

where

$$B^{-1}(s_i; a_i, A_i) = \frac{a_i A_i \left(e^{\frac{-s_i}{2}} - e^{\frac{s_i}{2}}\right)}{A_i e^{\frac{-s_i}{2}} - a_i e^{\frac{s_i}{2}}}. \tag{5}$$

Note that due to the monotonic characteristic of the natural logarithm the inverse of the barrier function (4) exists within the range of its definition.

Using the chain rule of differentiation, i.e., $\frac{dx_i}{dt} = \frac{\partial x_i}{\partial s_i}\frac{ds_i}{dt}$, where

$$\frac{\partial x_i}{\partial s_i} = \frac{A_i a_i^2 - a_i A_i^2}{a_i^2 e^{s_i} - 2a_i A_i + A_i^2 e^{-s_i}}, \tag{6}$$

and some algebraic manipulations result in the transformed state s_i, and it is given by

$$\begin{aligned} \dot{s}_i &= k_i(s_i) B^{-1}(s_{i+1}; a_{i+1}, A_{i+1}) \\ &= F_i\left(s_i, s_{i+1}\right), \quad \forall i = 1, \cdots, n-1, \end{aligned} \tag{7}$$

$$\dot{s}_n = k_n(s_n)\left(f(x) + g(x)u\right) = F_n(s) + G_n(s)u, \tag{8}$$

where

$$F_n(s) = k_n(s_n) f\left(\begin{bmatrix} B^{-1}(s_1) & \cdots & B^{-1}(s_n)\end{bmatrix}\right) \tag{9}$$

$$G_n(s) = k_n(s_n) g\left(\begin{bmatrix} B^{-1}(s_1) & \cdots & B^{-1}(s_n)\end{bmatrix}\right) \tag{10}$$

and $k_i(s_i) = \left(\frac{\partial x_i}{\partial s_i}\right)^{-1}$, $\forall i = 1, \cdots, n$. The constrained system in terms of s can be expressed in a compact form as follows

$$\dot{s} = F(s) + G(s)u, \tag{11}$$

where $F(s) = \left[F_1\left(s_1, s_2\right), \ \cdots \ , F_n(s)\right]^T$, $G(s) = \left[0, \ \cdots \ , 0, G_n(s)\right]^T$.

III. Problem Formulation and Solution Approach

A. Model Development

Consider the Euler-Lagrange (EL) dynamics

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G_r(q) = \tau, \tag{12}$$

where, $M(q) \in \mathbb{R}^{d\times d}$ denotes a generalized inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{d\times d}$ denotes a generalized centripetal-Coriolis matrix, $G_r(q) \in \mathbb{R}^d$ denotes a generalized gravity vector, $\tau = [\tau_1, \ \cdots \ , \tau_d]^T \in \mathbb{R}^d$ represents the generalized input control vector, and $q(t)$, $\dot{q}(t)$, $\ddot{q}(t) \in \mathbb{R}^d$ denote the link position, velocity, and acceleration vectors, respectively. The subsequent development is based on the assumption that all the states are observed, and that $M(q)$, $C(q, \dot{q})$, and $G_r(q)$, are unknown. The following properties, found in [26], [27], are also exploited in the subsequent development.

Property 1: The inertia matrix is positive definite, and satisfies the following inequality for any arbitrary vector $\xi \in \mathbb{R}^d$:

$$m_1 \|\xi\|^2 \leq \xi^T M(q) \xi \leq m_2 \|\xi\|^2, \tag{13}$$

where m_1 and m_2 are positive constants, and $\|\cdot\|$ represents the Euclidean norm.

Remark 1: Since $M(q)$ is a symmetric positive definite matrix, it can be shown that $M^{-1}(q)$ is also a positive definite matrix, and its 2-norm is upper and lower bounded with known constants, i.e., $\underline{m} \leq \|M^{-1}(q)\| \leq \bar{m}$.

Property 2: The EL-dynamics in (12) are linearly parametrizable as follows

$$Y(q, \dot{q}, \ddot{q})\theta = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G_r(q), \tag{14}$$

where $Y : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is the regression matrix, and $\theta \in \mathbb{R}^m$ is the set of the unknown parameters.

Property 3: The norm of the centripetal-Coriolis can be upper bounded in the following manner:

$$\|C(q,\dot{q})\|_\infty \le \bar{C}\|\dot{q}\|, \tag{15}$$

where $\bar{C} \in \mathbb{R}$ denotes known positive bounding constant, and $\|\cdot\|_\infty$ denotes the induced infinity-norm of a matrix.

Let $x = [x_1, x_2]^T$, where $x_1 = q \in \mathbb{R}^d$, $x_2 = \dot{q} \in \mathbb{R}^d$, and the EL-dynamics in (12) can be written as follows

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \mathbf{f}(x) + \mathbf{g}(x)\tau, \end{aligned} \tag{16}$$

where $\mathbf{f} : \mathbb{R}^{2d} \to \mathbb{R}^d$, $\mathbf{g} : \mathbb{R}^{2d} \to \mathbb{R}^{d\times d}$ are nonlinear continuously differentiable functions, $\mathbf{f}(x) = M^{-1}(x_1)\big(-C(x_1,x_2)x_2 - G_r(x_1)\big)$, and $\mathbf{g}(x) = M^{-1}(x_1)$. With some algebraic manipulations, the EL-dynamics can be written into d separate first and second order dynamics:

$$\dot{x}_{1,j} = x_{2,j}, \tag{17}$$

$$\dot{x}_{2,j} = f_j(x) + g_j(x)\tau, \quad \forall j = 1,\cdots,d \tag{18}$$

where $f_j : \mathbb{R}^{2d} \to \mathbb{R}$, $g_j : \mathbb{R}^{2d} \to \mathbb{R}^{1\times d}$ are nonlinear continuously differentiable functions. Using the BF transformation described in Section II-A for the system in (17) and (18), a system equivalent to (11) can be formulated as follows

$$\dot{\mathfrak{s}} = \mathcal{F}(\mathfrak{s}) + \mathcal{G}(\mathfrak{s})\tau, \tag{19}$$

where $\mathfrak{s} = [\mathfrak{s}_1, \mathfrak{s}_2]^T \in \mathbb{R}^{2d}$, $\mathfrak{s}_1 = [s_{1,1},\cdots,s_{1,d}]^T$ and $\mathfrak{s}_2 = [s_{2,1},\cdots,s_{2,d}]^T$ are the constrained joint position and velocity vectors, respectively. Also, $\mathcal{F} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and $\mathcal{G} : \mathbb{R}^{2d} \to \mathbb{R}^{2d\times d}$ are given by

$$\mathcal{F}(\mathfrak{s}) = \begin{bmatrix} F_{1,1}(s_{1,1}, s_{2,1}) \\ \vdots \\ F_{1,d}(s_{1,d}, s_{2,d}) \\ F_{2,1}(\mathfrak{s}) \\ \vdots \\ F_{2,d}(\mathfrak{s}) \end{bmatrix}, \; \mathcal{G}(\mathfrak{s}) = \begin{bmatrix} 0_{d\times d} \\ G_{2,1}(\mathfrak{s}) \\ \vdots \\ G_{2,d}(\mathfrak{s}) \end{bmatrix}. \tag{20}$$

Assumption 1: The function $\mathcal{F} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is locally Lipschitz continuous, and there exists a positive constant $\bar{\mathcal{F}}$ such that for $\mathfrak{s} \in \mathcal{S}$, $\|\mathcal{F}(\mathfrak{s})\| < \bar{\mathcal{F}}\|\mathfrak{s}\|$, where $\mathcal{S} \subset \mathbb{R}^{2d}$ is a compact set containing the origin. Moreover, the system is assumed to be controllable over $\mathcal{S}$ with $\mathcal{G}(\mathfrak{s})$ being locally Lipschitz and bounded in $\mathcal{S}$, i.e., $\|\mathcal{G}(\mathfrak{s})\| < \bar{\mathcal{G}}$, where $\bar{\mathcal{G}}$ is a positive scalar.

Following (19), the EL-dynamics can be represented in the constrained space as follows

$$M(s_p)K_2^{-1}(\mathfrak{s}_2)\dot{\mathfrak{s}}_2 + C(s_p, s_v)K_1^{-1}(\mathfrak{s}_1)\mathfrak{s}_2 + G_r(s_p) = \tau, \tag{21}$$

where

$$\begin{aligned} s_p &= \left[B^{-1}(s_{1,1}),\ \cdots, B^{-1}(s_{1,d})\right]^T, \\ s_v &= \left[B^{-1}(s_{2,1}),\ \cdots, B^{-1}(s_{2,d})\right]^T, \end{aligned}$$

and

$$K_i(\mathfrak{s}_i) = \mathrm{diag}\left(k_{i,1}(s_{i,1}),\ \cdots, k_{i,j}(s_{i,j})\right), \tag{22}$$

with $k_{i,j}(s_{i,j}) = \frac{\partial B^{-1}(s_{i,j})}{\partial s_{i,j}}$, $\forall i = 1,2$ and $\forall j = 1,\cdots,d$.

Assumption 2: The term $K_i(\mathfrak{s}_i)$ defined in (22) is positive definite, and its 2-norm is upper and lower bounded by known positive constants, i.e., $\underline{k}_i \le \|K_i(\mathfrak{s}_i)\| \le \bar{k}_i$, $\forall i = 1,2$.

Lemma 1: Given the term $K_i(\mathfrak{s}_i)$ defined in (22) with

$$k_{i,j}(s_{i,j}) = \frac{\left(a_{i,j}^2 e^{s_{i,j}} - 2a_{i,j}A_{i,j} + A_{i,j}^2 e^{-s_{i,j}}\right)}{A_{i,j}a_{i,j}^2 - a_{i,j}A_{i,j}^2}, \ \forall i = 1,2, \ \forall j{=}1,\cdots,d,$$

the 2-norm of its inverse, $K_i^{-1}(\mathfrak{s}_i)$, can be upper bounded by a positive constant $\bar{\kappa}_i$, i.e., $\|K_i^{-1}(\mathfrak{s}_i)\| \le \bar{\kappa}_i$, $\forall i = 1,2$.

Proof: The 2-norm of $K_i^{-1}(\mathfrak{s}_i) = \mathrm{diag}\left(k_{i,1}^{-1}(s_{i,1}),\ \cdots, k_{i,d}^{-1}(s_{i,d})\right)$ can be upper bounded because $k_{i,j}^{-1}(s_{i,j})$ is bounded, that is

$$\lim_{s_{i,j}\to\infty} \frac{A_{i,j}a_{i,j}^2 - a_{i,j}A_{i,j}^2}{\left(a_{i,j}^2 e^{s_{i,j}} - 2a_{i,j}A_{i,j} + A_{i,j}^2 e^{-s_{i,j}}\right)} = 0,$$

which implies that 2-norm of $K_i^{-1}(\mathfrak{s}_i)$ can be upper bounded by a positive constant $\bar{\kappa}_i$. ■

Now, using Property 2, the EL-dynamics in (21) can be linearly parameterized, and it is given by

$$MK_2^{-1}\dot{\mathfrak{s}}_2 + CK_1^{-1}\mathfrak{s}_2 + G_r = Y_1(s_p, s_v, \mathfrak{s}_1, \mathfrak{s}_2, \dot{\mathfrak{s}}_2)\,\theta, \tag{23}$$

where $Y_1 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is the regression matrix. Note that in (23), and henceforth the parameter dependency of the elements in the EL-dynamics are dropped for brevity.

Lemma 2: Suppose that there exist a controller that tracks the desired trajectory for the system given in (21). Then, the same controller can also track the desired trajectory of the original system in (12) given that the initial state of the system $x(0) = x_0 \in \mathcal{X}$.

Proof: See proof of ([19], Lemma 1) ■

Lemma 2 proves that if the initial state is within the prescribed bound, a control law can be designed for the full-state constrained system such that it satisfies the tracking objective of the original system.

B. Control Development

In this subsection, an adaptive control technique is used to identify the parameters of an uncertain system and track the desired joint position $\mathfrak{s}_1^{des}(t) : \mathbb{R}^+ \to \mathbb{R}^d$ and joint velocity $\mathfrak{s}_2^{des}(t) : \mathbb{R}^+ \to \mathbb{R}^d$ trajectories.

Assumption 3: The signals $\mathfrak{s}_1^{des}$, $\mathfrak{s}_2^{des}$, $\dot{\mathfrak{s}}_2^{des}$ are uniformly continuous and bounded such that $\|\mathfrak{s}_1^{des}\| \le \bar{\mathfrak{s}}_1^{des}$, $\|\mathfrak{s}_2^{des}\| \le \bar{\mathfrak{s}}_2^{des}$, $\|\dot{\mathfrak{s}}_2^{des}\| \le \bar{\dot{\mathfrak{s}}}_2^{des}$, where $\bar{\mathfrak{s}}_1^{des}$, $\bar{\mathfrak{s}}_2^{des}$, and $\bar{\dot{\mathfrak{s}}}_2^{des}$ are known positive constants.

To this end, consider the following tracking control input design

$$\tau = \hat{M}K_2^{-1}a + \hat{C}K_1^{-1}v + \hat{G}_r - \beta K_2 r, \tag{24}$$

where $(\hat{\cdot})$ denotes the parameter estimates and β is a positive scalar. Signals a, v, r are given by

$$a = \dot{\mathfrak{s}}_2^{des} - \Lambda\tilde{\mathfrak{s}}_2, \quad (25)$$
$$v = \mathfrak{s}_2^{des} - \Lambda\tilde{\mathfrak{s}}_1, \quad (26)$$
$$r = \tilde{\mathfrak{s}}_2 + \Lambda\tilde{\mathfrak{s}}_1, \quad (27)$$

where $\tilde{\mathfrak{s}}_1 \triangleq \mathfrak{s}_1 - \mathfrak{s}_1^{des}$ and $\tilde{\mathfrak{s}}_2 \triangleq \mathfrak{s}_2 - \mathfrak{s}_2^{des}$ are position and velocity tracking errors, respectively. $\Lambda \in \mathbb{R}^{d\times d}$ is a positive definite diagonal matrix, and its 2-norm is upper bounded by a known positive constant, i.e., $\|\Lambda\| \leq \bar{\Lambda}$.

In terms of the linear parameterization of the EL-dynamics, i.e., Property 2, the control input (24) can be rewritten as

$$\tau = Y_2\left(s_p, s_v, K_2^{-1}(\mathfrak{s}_2)a, K_1^{-1}(\mathfrak{s}_1)v\right)\hat{\theta} - \beta K_2(\mathfrak{s}_2)r, \quad (28)$$

where $Y_2 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is the regression matrix. Substituting (24) in the EL-dynamics (21) yields the following closed-loop error dynamics given by

$$MK_2^{-1}\dot{r} + CK_1^{-1}r + \beta K_2 r = Y_2\tilde{\theta}, \quad (29)$$

where $\tilde{\theta} = \hat{\theta} - \theta$ is the parameter estimation error. The parameter $\hat{\theta}$ update rule is given by

$$\dot{\hat{\theta}} = \text{proj}\left(-\Gamma^{-1}Y_2^T K_2 r\right), \quad (30)$$

where $\Gamma \in \mathbb{R}^{m\times m}$ is a diagonal and positive definite matrix, and $\text{proj}(\cdot)$ is a standard projection operator that ensures the parameter estimates are bounded, i.e., $\underline{\theta} \leq \hat{\theta} \leq \overline{\theta}$ (for further details see [26]).

Remark 2: The parameter estimation error $\tilde{\theta}$ is bounded and uniformly continuous since $\hat{\theta}$ evolves according to the update law in (30).

C. Stability Analysis

To facilitate the following development of the Lyapunov stability analysis, let $\zeta : [0, \infty) \to \mathbb{R}^{2d+m}$ denotes the composite state vector, i.e., $\zeta(t) \triangleq \left[r^T(t), \mathfrak{s}_1^T(t), \tilde{\theta}^T(t)\right]^T$. Let $\lambda_{min}\{\cdot\}$ and $\lambda_{max}\{\cdot\}$ denote the minimum and maximum eigenvalues of its argument.

Theorem 1: The controller and parameter update laws defined in (28) and (30) ensure SGUUB tracking of the desired state trajectories, provided the following sufficient conditions,

$$\gamma_1 > 2\left(1 + \gamma_5 + \gamma_7\right), \qquad \gamma_3 > 2\left(1 + \gamma_5 + \gamma_6\right), \quad (31)$$

are satisfied, where

$$\begin{aligned} \gamma_1 &= \beta\underline{m}\ \underline{k}_2^2 & \gamma_5 &= \beta\bar{\lambda}\bar{k}_2^2\bar{m} \\ \gamma_2 &= \lambda_{max}\left\{\Lambda^T\Lambda\right\}\bar{\alpha} & \gamma_6 &= \left(\gamma_2 + \gamma_4\right)\bar{\mathfrak{s}}_2^{des} \\ \gamma_3 &= \gamma_1\lambda_{min}\left\{\Lambda^T\Lambda\right\} & \gamma_7 &= \left(\bar{\alpha} + \gamma_4\right)\bar{\mathfrak{s}}_2^{des} \\ \gamma_4 &= \bar{\Lambda}\bar{\alpha} & \bar{\alpha} &= \bar{k}_2\bar{m}\bar{C}\bar{\kappa}_1^2 \end{aligned}$$

Proof: Consider a quadratic Lyapunov function candidate $V : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subset \mathbb{R}^{2d+m}$ satisfying $V(0) = 0$ of the following form

$$V(\zeta) = \frac{1}{2}r^T r + \tilde{\mathfrak{s}}_1^T\tilde{\mathfrak{s}}_1 + \frac{1}{2}\tilde{\theta}^T\Gamma\tilde{\theta}. \quad (32)$$

The Lyapunov candidate can be bounded by

$$\lambda_{min}\{P\}\|e\|^2 + b_1 \leq V(\zeta) \leq \lambda_{max}\{P\}\|e\|^2 + b_2, \quad (33)$$

where $P = \begin{bmatrix} \frac{3}{2}I_n & \frac{1}{2}\Lambda \\ \frac{1}{2}\Lambda & \frac{1}{2}I_n \end{bmatrix}$, b_1 and b_2 are known positive bounding constants, and $e(t) \in \mathbb{R}^{2d}$ is defined as

$$e(t) \triangleq \begin{bmatrix} \tilde{\mathfrak{s}}_1^T(t) & \tilde{\mathfrak{s}}_2^T(t) \end{bmatrix}^T. \quad (34)$$

Utilizing (29), the time derivative of the Lyapunov function (32) can be expressed as

$$\begin{aligned} \dot{V}(\zeta) = &- r^T\left(K_2M^{-1}\left(CK_1^{-1} + \beta K_2\right)\right)r \\ &+ 2\tilde{\mathfrak{s}}_1^T\tilde{\mathfrak{s}}_2 + \tilde{\theta}^T\left(\Gamma\dot{\hat{\theta}} + Y_2^T M^{-T}K_2 r\right) \end{aligned} \quad (35)$$

Substituting the expression for $\dot{\hat{\theta}}$ from the update law (30) and the filtered tracking error (27) into (35) yields

$$\begin{aligned} \dot{V}(\zeta) = &- \tilde{\mathfrak{s}}_2^T\left[K_2M^{-1}\left(CK_1^{-1} + \beta K_2\right)\right]\tilde{\mathfrak{s}}_2 \\ &- \tilde{\mathfrak{s}}_1^T\Lambda^T\left[K_2M^{-1}\left(CK_1^{-1} + \beta K_2\right)\right]\Lambda\tilde{\mathfrak{s}}_1 \\ &- 2\tilde{\mathfrak{s}}_1^T\Lambda^T\left[K_2M^{-1}\left(CK_1^{-1} + \beta K_2\right)\right]\tilde{\mathfrak{s}}_2 + 2\tilde{\mathfrak{s}}_1^T\tilde{\mathfrak{s}}_2 \\ &- \tilde{\theta}^T\left(Y_2^T(\mathbb{I} + M^{-T})K_2\left(\tilde{\mathfrak{s}}_2 + \Lambda\tilde{\mathfrak{s}}_1\right)\right). \end{aligned} \quad (36)$$

The term $\left(\mathbb{I} + M^{-T}\right)$ in (36) can be upper bounded using the triangle inequality given by $\|\mathbb{I} + M^{-T}\| \leq 1 + \bar{m} = \bar{\mathcal{M}}$. Furthermore, utilizing the bounding property of EL-dynamics (15), Remark 1, and Lemma 1, (36) can be upper bounded as

$$\begin{aligned} \dot{V}(\zeta) \leq\ &\bar{\alpha}\|\mathfrak{s}_2\|\|\tilde{\mathfrak{s}}_2\|^2 - \gamma_1\|\tilde{\mathfrak{s}}_2\|^2 + \gamma_2\|\mathfrak{s}_2\|\|\tilde{\mathfrak{s}}_1\|^2 - \gamma_3\|\tilde{\mathfrak{s}}_1\|^2 \\ &+ 2\gamma_4\|\mathfrak{s}_2\|\|\tilde{\mathfrak{s}}_1\|\|\tilde{\mathfrak{s}}_2\| + 2\gamma_5\|\tilde{\mathfrak{s}}_1\|\|\tilde{\mathfrak{s}}_2\| + 2\|\tilde{\mathfrak{s}}_1\|\|\tilde{\mathfrak{s}}_2\| \\ &+ \gamma_8\|\tilde{\mathfrak{s}}_2\| + \gamma_9\|\tilde{\mathfrak{s}}_1\|, \end{aligned} \quad (37)$$

where $\gamma_8 = \bar{Y}_2\bar{\mathcal{M}}\bar{k}_2\bar{\tilde{\theta}}$ and $\gamma_9 = \gamma_8\lambda_{max}\{\Lambda\}$ in which $\bar{Y}_2$ and $\bar{\tilde{\theta}}$ denote positive bounding constants on the regression matrix Y_2 and $\tilde{\theta}$, respectively. Completing the squares in (37) and rearranging the terms yield

$$\begin{aligned} \dot{V}(\zeta) \leq &- \left(\frac{\gamma_1}{2} - \bar{\alpha}\|\mathfrak{s}_2\|\right)\|\tilde{\mathfrak{s}}_2\|^2 - \left(\frac{\gamma_3}{2} - \gamma_2\|\mathfrak{s}_2\|\right)\|\tilde{\mathfrak{s}}_1\|^2 \\ &+ 2\left(\gamma_4\bar{\alpha}\|\mathfrak{s}_2\| + \gamma_5 + 1\right)\|\tilde{\mathfrak{s}}_1\|\|\tilde{\mathfrak{s}}_2\| + \delta, \end{aligned} \quad (38)$$

where $\delta = \frac{\gamma_8^2}{2\gamma_1} + \frac{\gamma_9^2}{2\gamma_3}$. Using Young's inequality, the third term in (38) can be upper bounded as

$$\begin{aligned} 2\left(\gamma_4\|\mathfrak{s}_2\| + \gamma_5 + 1\right)\|\tilde{\mathfrak{s}}_1\|\|\tilde{\mathfrak{s}}_2\| \leq &\left(\gamma_4\|\mathfrak{s}_2\| + \gamma_5 + 1\right) \\ &\times\left(\|\tilde{\mathfrak{s}}_1\|^2 + \|\tilde{\mathfrak{s}}_2\|^2\right). \end{aligned} \quad (39)$$

Expressing $\|\mathfrak{s}_2\| = \|\tilde{\mathfrak{s}}_2 + \mathfrak{s}_2^{des}\|$, and using triangle inequality along with Assumption 3, the expression in (38) can be further bounded as

$$\begin{aligned} \dot{V}(\zeta) \leq &- \left(\frac{\gamma_3}{2} - \left(\gamma_2 + \gamma_4\right)\|\tilde{\mathfrak{s}}_2\| - \gamma_6 - \gamma_5 - 1\right)\|\tilde{\mathfrak{s}}_1\|^2 \\ &- \left(\frac{\gamma_1}{2} - \left(\bar{\alpha} + \gamma_4\right)\|\tilde{\mathfrak{s}}_2\| - \gamma_7 - \gamma_5 - 1\right)\|\tilde{\mathfrak{s}}_2\|^2 + \delta. \end{aligned} \quad (40)$$

By letting $\mathfrak{c}_1 = \frac{\gamma_3}{2} - \gamma_6 - \gamma_5 - 1$ and $\mathfrak{c}_2 = \frac{\gamma_1}{2} - \gamma_7 - \gamma_5 - 1$, the upper bound on the derivative of the Lyapunov function (40) can be rewritten and simplified further as

$$\dot{V}(\zeta) \leq -\left(\mathfrak{c} - \rho\|\tilde{\mathfrak{s}}_2\|\right)\|e\|^2 + \delta, \tag{41}$$

where $\mathfrak{c} \triangleq \min\{\mathfrak{c}_1, \mathfrak{c}_2\}$; hence $\mathfrak{c}$ is positive if $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are chosen according to the sufficient conditions given by (31). Also, $\rho \triangleq \max\{(\gamma_2 + \gamma_4), (\bar{\alpha} + \gamma_4)\}$; and the negative term in (41) dominates only if the positive term involving $\|e\|^2$ is negative definite, i.e., $\|\tilde{\mathfrak{s}}_2\| < \frac{\mathfrak{c}}{\rho}$.

Now, let $\eta = \mathfrak{c} - \rho\|\tilde{\mathfrak{s}}_2\|$ and substituting in for $\|e\|^2$ from (33), yield

$$\dot{V}(\zeta) \leq -\frac{\eta}{\lambda_{max}\{P\}}V + \epsilon, \quad \forall \|\tilde{\mathfrak{s}}_2\| < \frac{\mathfrak{c}}{\rho}, \tag{42}$$

where $\epsilon = \frac{\eta b_2}{\lambda_{max}\{P\}} + \delta$. The solution to the linear differential inequality in (42) can be obtained using the Comparison lemma, lemma 3.4 of [28], and it is given by

$$V(\zeta(t)) \leq V(0)\exp\left(-\frac{\eta}{\lambda_{max}\{P\}}t\right) + \frac{\lambda_{max}\{P\}\epsilon}{\eta}\left[1 - \exp\left(-\frac{\eta}{\lambda_{max}\{P\}}t\right)\right], \tag{43}$$

where it is defined on the following domain

$$\mathcal{D} \triangleq \left\{\zeta \in \mathbb{R}^{2d+m} \middle| \|\zeta\| \leq \frac{\mathfrak{c}}{\rho}\right\}. \tag{44}$$

It can be seen from (43) and (32) that $e(t) \in \mathcal{L}_\infty$. Using the standard signal chasing, Assumption 3, and Remark 2, it can be concluded that the designed controller ensured semi-globally uniformly ultimately bounded tracking of the desired trajectory. ■

IV. Numerical Evaluations

In this section, the controller and adaptive laws developed in (24) and (30) are simulated for a two-link robot planar manipulator, with dynamics shown in (45), where c_1, c_2, c_{12} denote $\cos(q_1)$, $\cos(q_2)$, and $\cos(q_1 + q_2)$ respectively, $\sin_2$ denotes $\sin(q_2)$, and g is the gravitational constant.

$$\underbrace{\begin{bmatrix}\theta_1 + 2\theta_2 c_2 & \theta_3 + \theta_2 c_2\\ \theta_3 + \theta_2 c_2 & \theta_3\end{bmatrix}}_{M(q)}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} + \underbrace{\begin{bmatrix}-\theta_2 \sin_2 \dot{q}_2 & -\theta_2 \sin_2(\dot{q}_1 + \dot{q}_2)\\ \theta_2 \sin_2 \dot{q}_1 & 0\end{bmatrix}}_{C(q,\dot{q})} \cdot \begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \underbrace{\begin{bmatrix}\theta_4 g c_1 + \theta_5 g c_{12}\\ \theta_5 g c_{12}\end{bmatrix}}_{G_r(q)} = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} \tag{45}$$

The nominal values of the parameter vector $\theta = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]^T$ are

$$\theta_1 = 0.325 \text{ kg.m}^2 \quad \theta_3 = 0.217 \text{ kg.m}^2$$
$$\theta_2 = 0.240 \text{ kg.m}^2 \quad \theta_4 = 2.4 \text{ kg.m} \quad \theta_5 = 1.0 \text{ kg.m}$$

The desired trajectory is selected as

$$q_{d_1} = \left(1 + 2e^{-2t}\right) sin(t), \quad q_{d_2} = \left(1 + 5e^{-t}\right) cos(t).$$

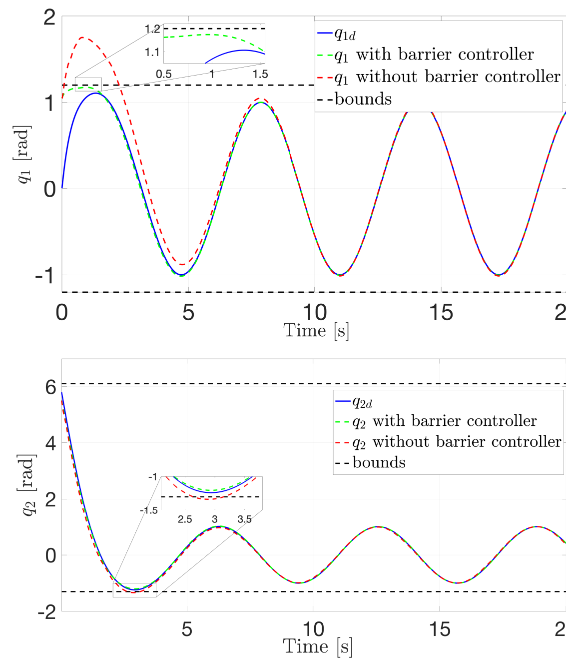

Fig. 1. Evolution of the joint angles for the planar robot simulation using an adaptive law with and without BF.

The objective is to track the desired joint trajectory provided that the model parameters are unknown while the state $Q = [q, \ \dot{q}]^T$ satisfies the following constraints,

$$q_1 \in (-1.2, 1.2) \qquad \dot{q}_1 \in (-1.3, 3.1)$$
$$q_2 \in (-1.3, 6.1) \qquad \dot{q}_2 \in (-5.5, 1.1)$$

To this end, the barrier function formulation presented in Section II is used along with the adaptive control developed in Section III.

The feedback and adaptation gains, i.e., β and Λ, and Γ are selected based on the gain conditions presented in Theorem 1. The results of the simulation are shown in Figs. 1-3. The joints position evolution $q_1(t)$ and $q_2(t)$ of a two degrees-of-freedom planar robot using an adaptive law with and without BF are shown in Fig. 1. It can be observed from Fig. 1 that when the adaptive law with BF is used, the estimated trajectories are blocked from crossing over the boundaries that are set for each of the joints. The position and velocity estimation errors are depicted in Fig. 2. From Figs. 1 and 2, it is clear that the tracking error asymptotically converges to zero, and, because the Lyapunov candidate does not contain any terms that are negative definite in $\tilde{\theta}$, the parameter estimation does not converge but it does remain bounded. Boundedness of the parameter estimation errors can be seen in Fig. 3.

V. Conclusion

An online safe tracking controller for an uncertain multi-input and multi-output robotic system with full-state constraints is developed. With no prior knowledge of the system parameters, a barrier function was used to transform the full-state constrained EL-dynamics into an equivalent unconstrained system. A controller is developed on the

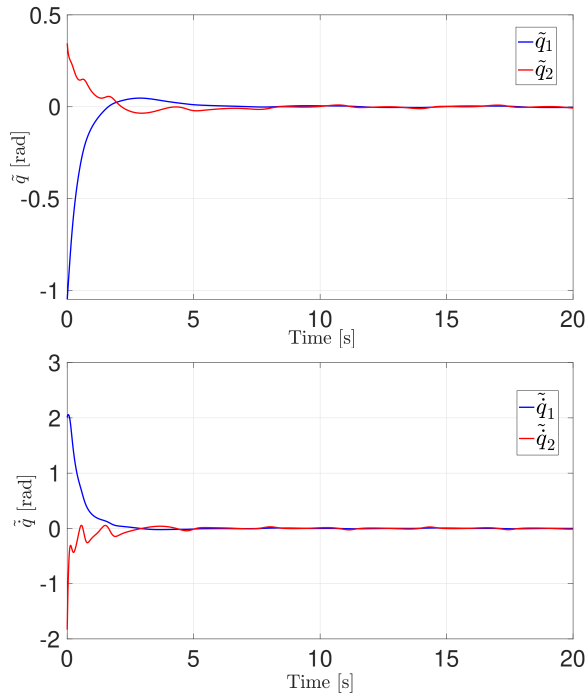

Fig. 2. Evolution of the position and velocity joint angles estimation errors for the planar robot simulation using an adaptive law with BF.

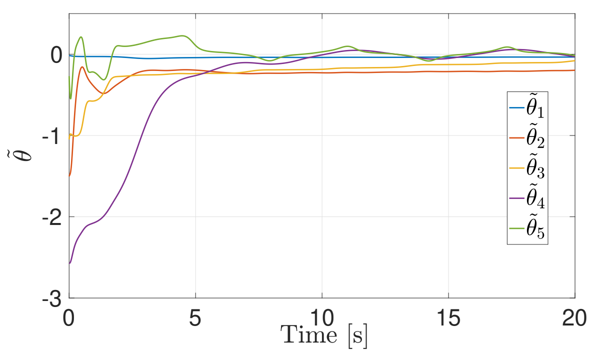

Fig. 3. Evolution of the parameter estimation error for the planar robot simulation.

transformed system that tracks the desired trajectories of the original system. An adaptive law was designed resulting in the solutions of the constrained dynamics to remain inside a pre-specified safe region with guarantees on the boundedness of the parameter estimation error and SGUUB of tracking errors. The simulation results validate the boundedness of estimated parameters and that of the tracking errors.

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