Beyond Guesswork: Code Verification for Acoustic Duct Mode Prediction

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Spring 2024 NASA Acoustics Technical Working Group
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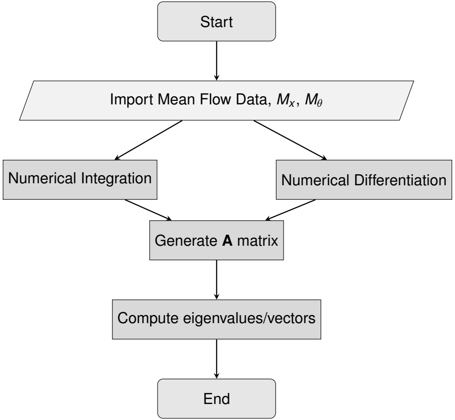
Section 1

Basis of Technical Work

Code Implementation



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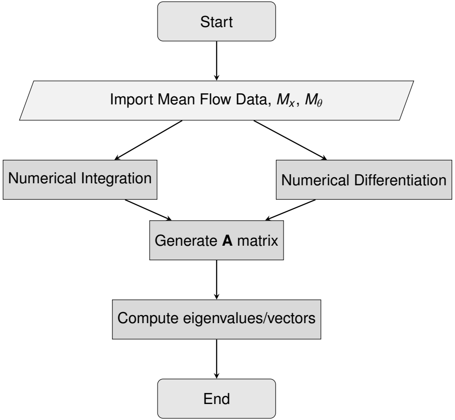

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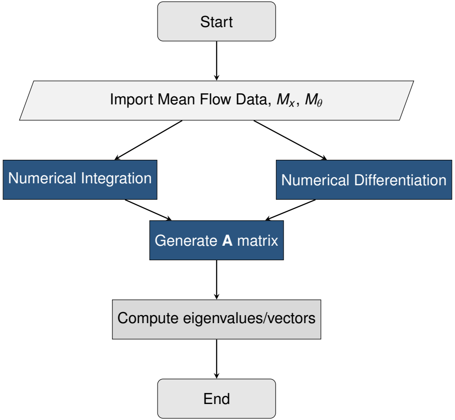

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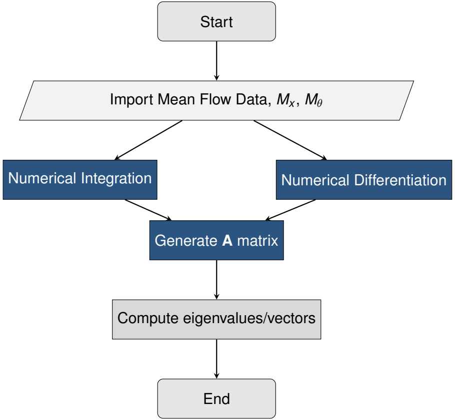

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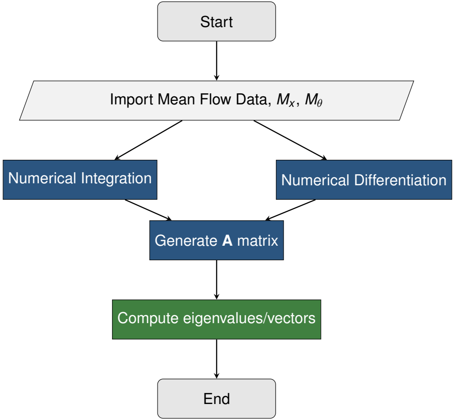

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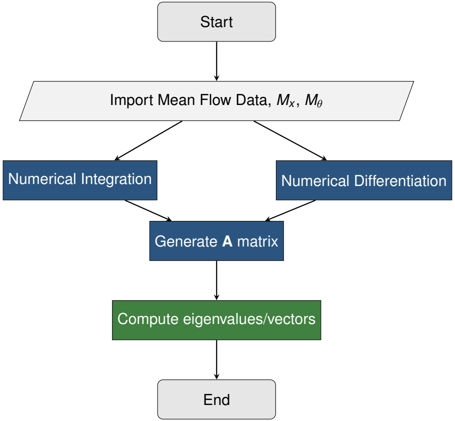

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Goal: Code Verification by the Method of Manufactured and Exact Solutions (MMS/MES)

Verification Vs Validation

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This work is focused code verification through the order-of-accuracy (OOA) test

Section 2

Verification of Numerical Integration

Numerical Integration for the Speed of Sound

Defining Error and Order of Accuracy

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α is the formal order-of-accuracy (OOA) of the numerical scheme which dictates the expected rate of convergence as grid spacing is decreased.

Verification of Numerical Integration

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$$\widetilde{A}_{\text{analytic}} = 1 - (\gamma - 1) \int_{\tilde{r}}^{1} \frac{M_\theta}{\tilde{r}} d\tilde{r} \tag{4}$$

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Code Verification by the MMS

To verify the numerical integration an analytic function is chosen i.e. manufactured for M_θ

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$$\tilde{A}_{MS} = \frac{1}{24}\left(\begin{array}{c} \tanh\left(\frac{\tilde{r}}{3} - \frac{1}{3}\right) + \tanh\left(\frac{\tilde{r}}{3} - \frac{31}{120}\right) + \\ \tanh\left(\frac{\tilde{r}}{3} - \frac{11}{60}\right) + \tanh\left(\frac{\tilde{r}}{3} - \frac{13}{120}\right) + \tanh\left(\frac{\tilde{r}}{3} - \frac{1}{30}\right) \end{array}\right) + \frac{853}{880} \tag{6}$$

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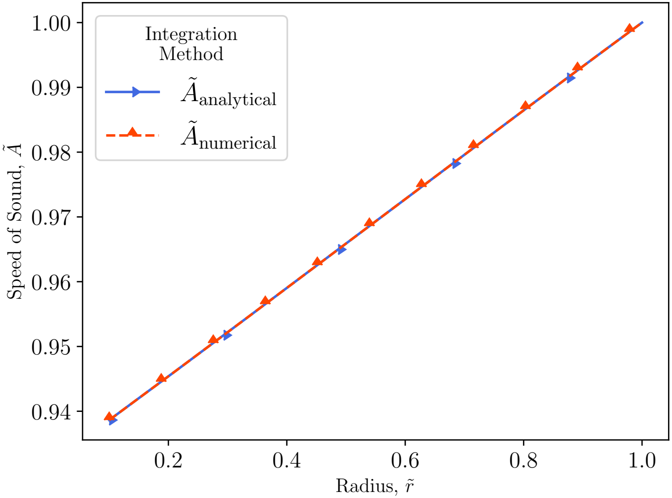
This is one possible analytic solution to the speed of sound, but this solution is *manufactured*, hence the subscript *MS*

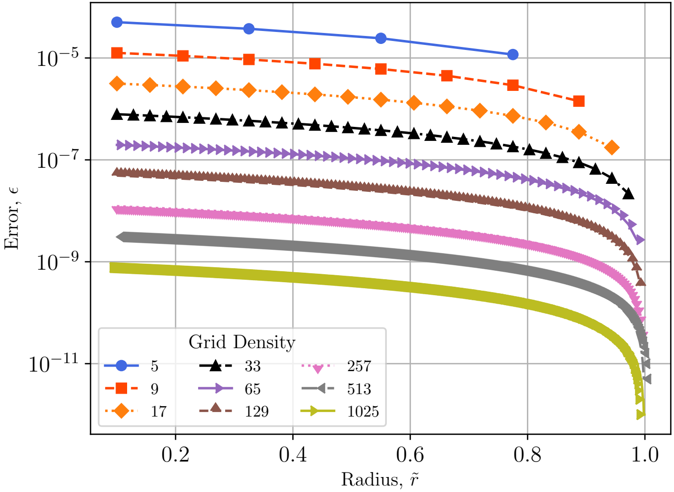
Verification of Numerical Integration



Visually looks identical..

The first step of the OOA test is to compute the error

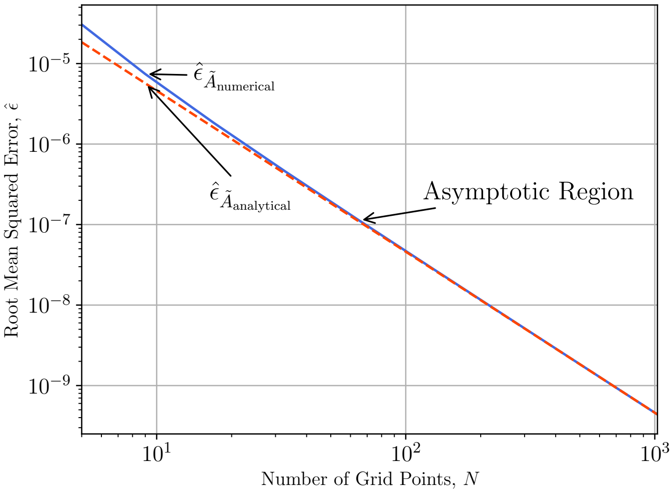
Verification of Numerical Integration



Value of error indicates accuracy

The root mean square of the error, (L_2 norm) is then used to compute the *observed* OOA

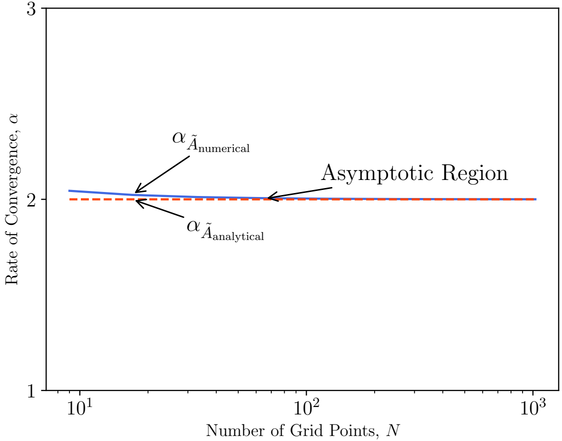
Verification of Numerical Integration



A straight line of expected slope is superimposed

Convergence starts in the asymptotic region at $\approx$ 100 grid points

Verification of Numerical Integration



The observed rate of convergence (OOA) approaches the formal value

The numerical integration is now verified via MMS

Section 3

Verification of Numerical Differentiation

Verification of Numerical Differentation and Matrix Construction

The Governing Equations

- A given mean flow is used to establish an eigenvalue problem with the Linearized Euler Equations

$$[A]\chi = \lambda[B]\chi \tag{7}$$

where χ and λ contain the eigenvalues (axial wavenumbers) and vectors (radial pressure modes)

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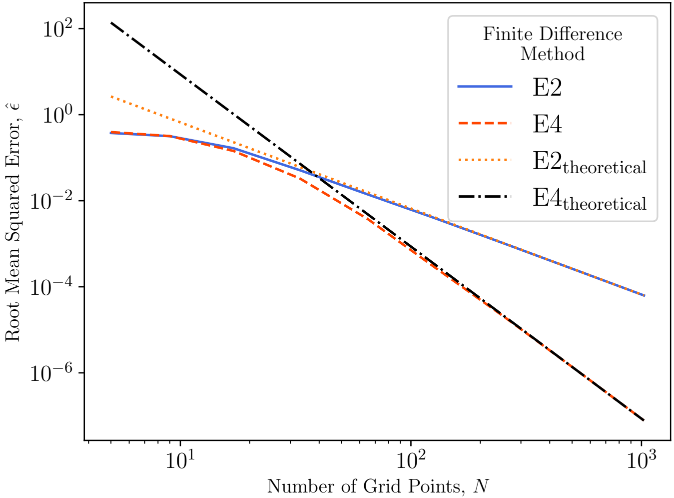
- Manufactured solutions were generated using a summation of tangents to verify OOA of a second and fourth order scheme

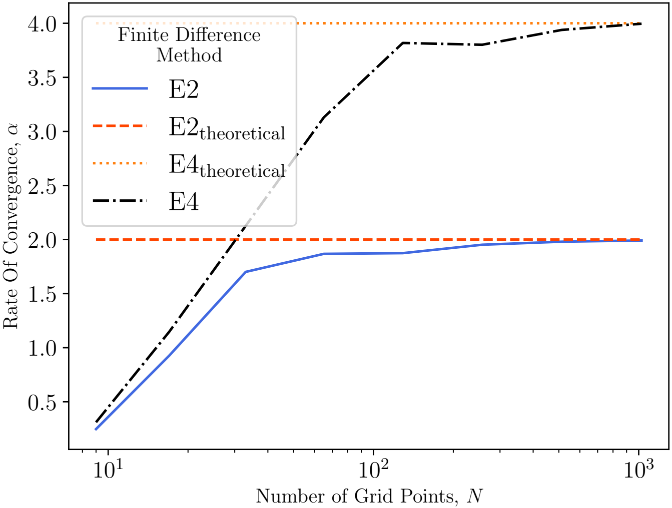
Verification of Numerical Differentation and Matrix Construction



Two straight lines of expected slope for each scheme is superimposed

Similarly, convergence is observed the asymptotic region at ≈ 100 grid points

Verification of Numerical Differentation and Matrix Construction



The observed rate of convergence (OOA) approaches the formal value

The finite differencing scheme and matrix construction have been verified with MMS

Section 4

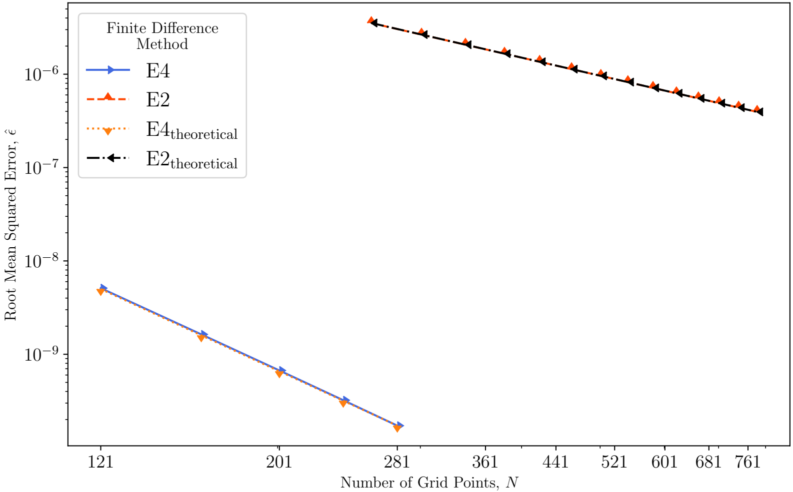
Verification of Eigenvalues and vectors

Verification of Eigenvalues and vectors

Verification by the Method of Exact Solutions

- There are exact solutions for the modal content in simplified mean flows.
- Code verification by the MES was performed using a uniform axial flow in a hard-walled annular duct

Eigenvalue (Axial Wavenumber) Comparison



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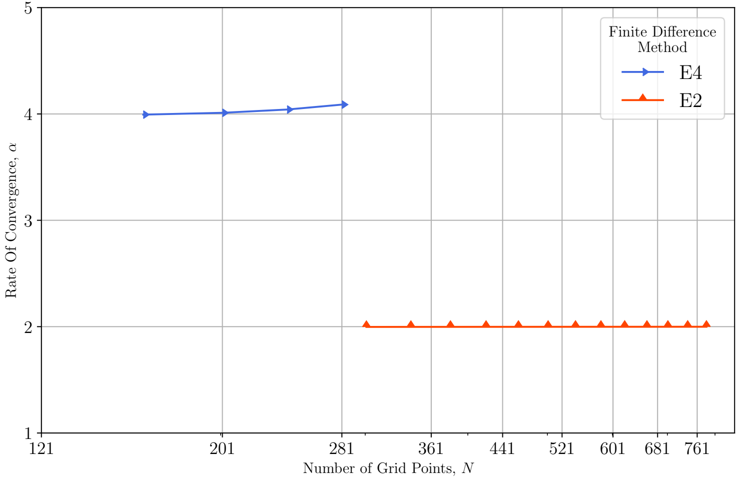

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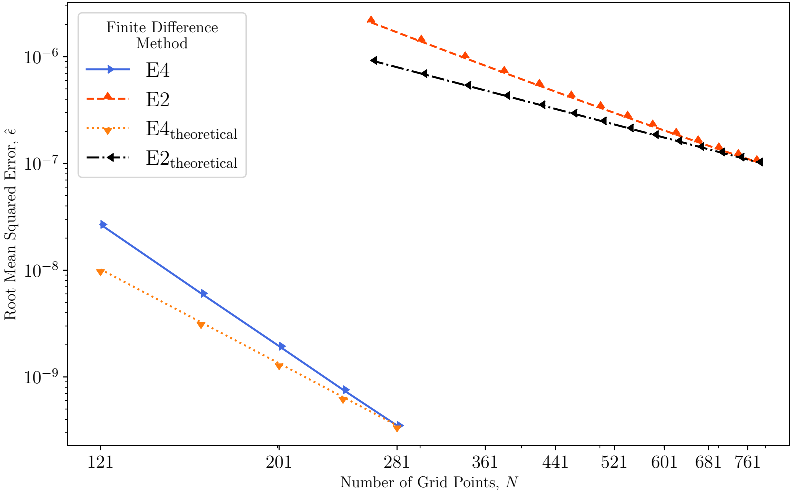

The imposed lines are not parallel to the computed norms..

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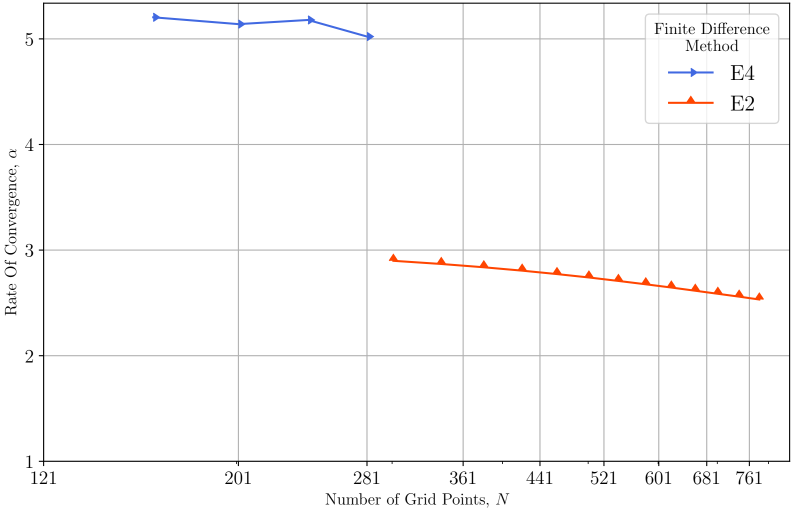
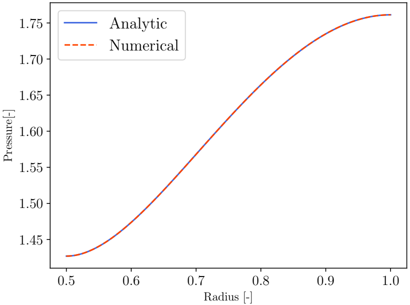
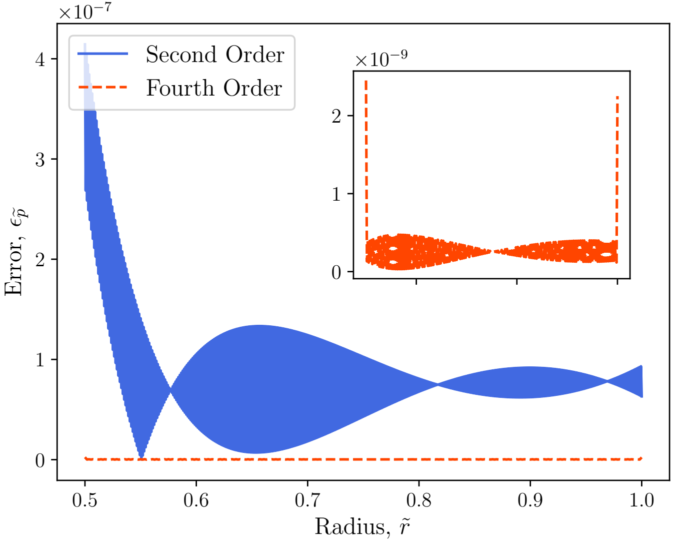


The computed slopes are higher than expected

Is *Swirl* converging to the right solution?

Examining the Error in the Radial Pressure Mode





Nonphysical oscillations and boundary errors are dominating the solution

Applying Filters to the Radial Pressure Modes

Will filters/artificial dissipation yield the expected OOA?

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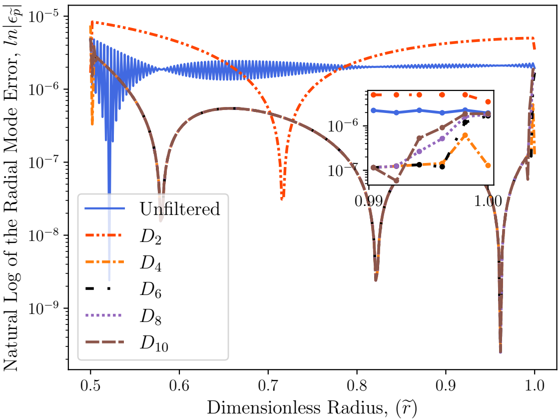
The OOA was recomputed after applying Kennedy and Carpenter (K&C) and Rigby filters ($D2 - D10$)

Eigenvector (Radial Pressure Mode) Error With Various Orders of Filters - Second Order Study



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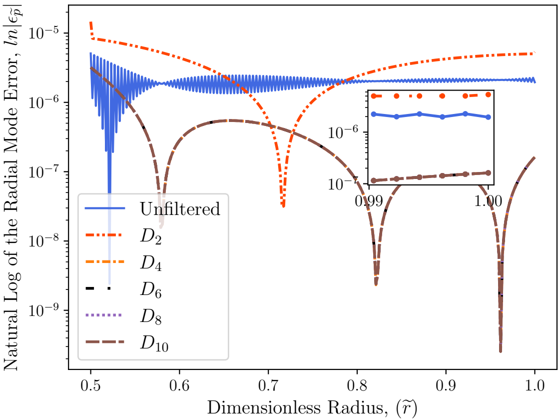

Rigby

Both filters mitigate oscillations, Rigby is more aggressive on the boundaries

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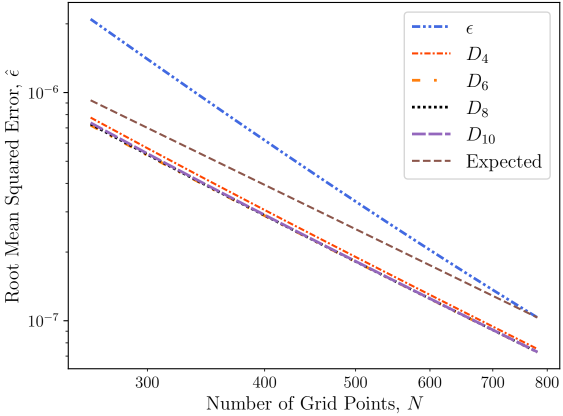

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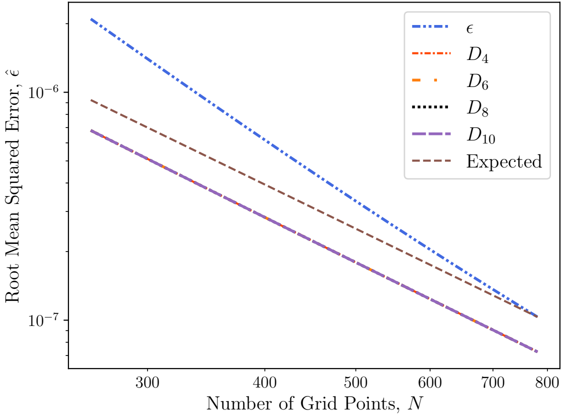

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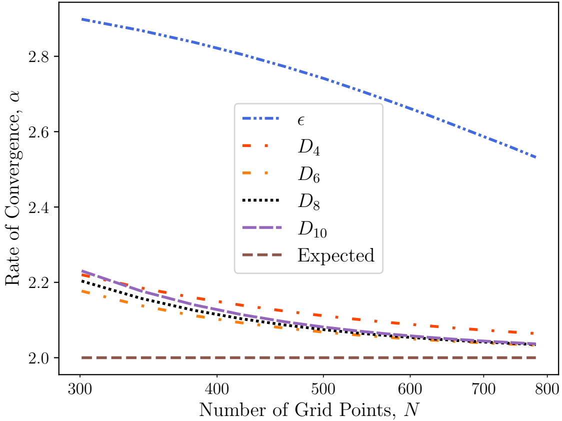

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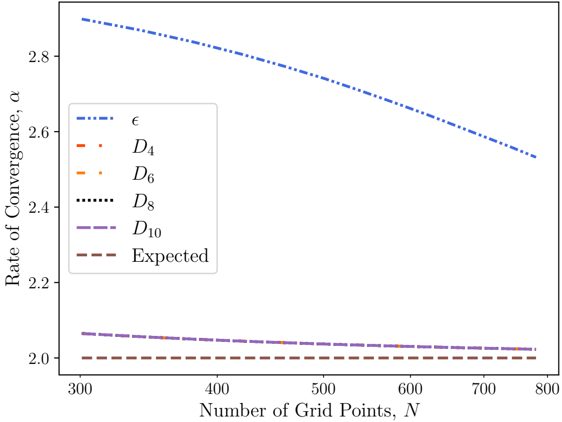

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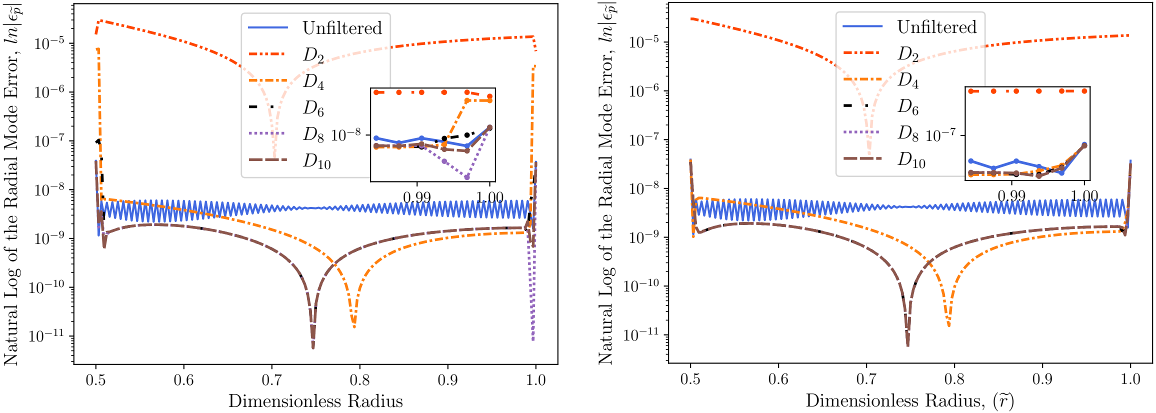

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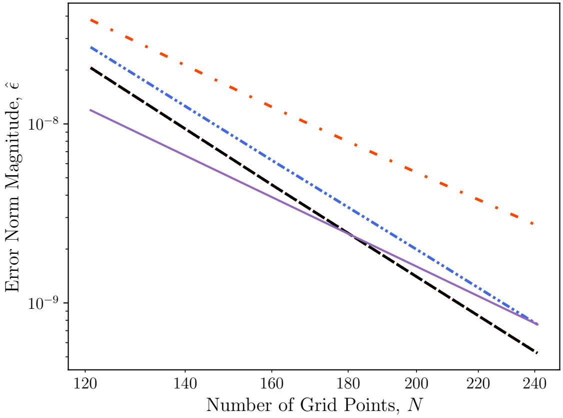

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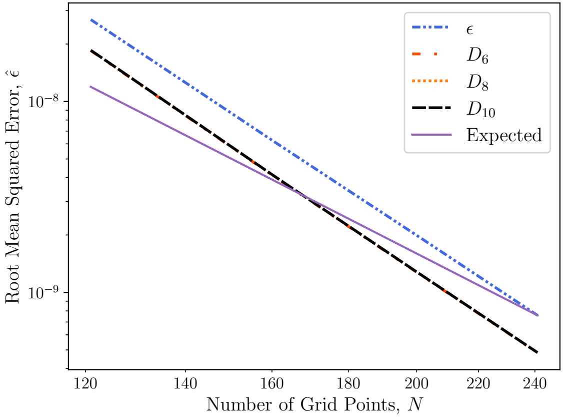

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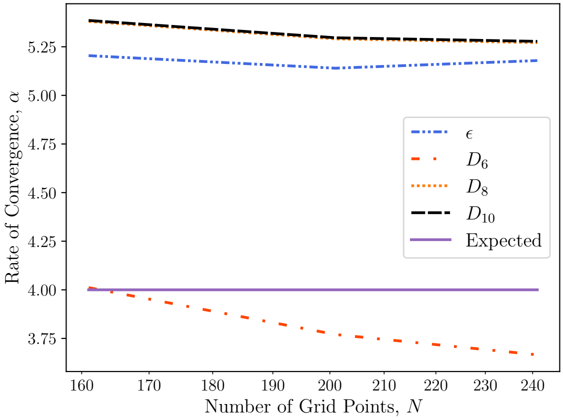

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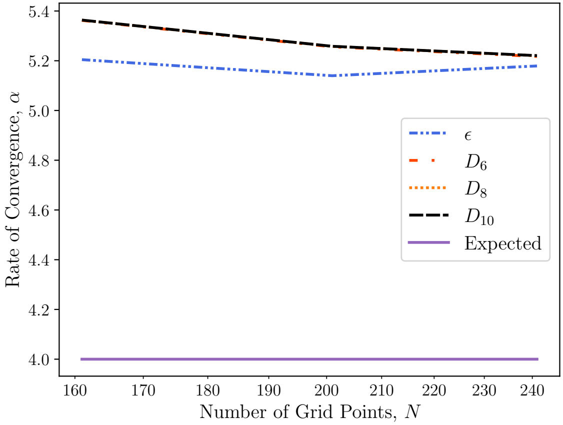

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By using both MMS/MES on multiple code components, a path to improvements to the numerical approximations and BC implementation are outlined

Questions?

References I

K. A. Kousen. Eigenmode analysis of ducted flows with radially dependent axial and swirl components. In *CEAS/AIAA Joint Aeroacoustics Conference, 1 st, Munich, Germany*, pages 1085–1094, 1995.