



Double Power-law Formation by Sequential Particle Acceleration

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Received 2023 November 26; revised 2024 February 15; accepted 2024 February 16; published 2024 April 1

Abstract

Spectral double power laws are common in solar high-energy phenomena such as flares and interplanetary energetic-electron events. However, the physical mechanism that produces the changes in power-law index within a single spectrum is unclear. We developed a fully analytical method of forming single power-law spectra from sequential acceleration of particles orbiting inside and hopping between simulated large-scale magnetic islands formed by flare reconnection. Here, we extend the analytical method to the formation of double power-law spectra by assuming sequential acceleration in two successive regions with different acceleration and particle-transport rates. The resulting spectral distribution is continuous and smooth, with a flattening at low energies, two power-law regions at mid-energies, and a steep rollover at high energies. The model provides analytical expressions for the spectral indices, all energy breaks, and normalization constants as functions of just three physical parameters of each acceleration region: (1) the energy gain in each accelerator, (2) the percentage of particles transferred between accelerators, and (3) the number of accelerators visited. One of the most salient predictions of our work is that the spectral index at high (low) energies is determined by the parameters of the first “seed” (second) acceleration region. By constructing the spectral distribution through an iterative analytical process, the evolution toward a double power law is easily characterized and explained. Our analytical model provides tools to interpret space- and ground-based observations from RHESSI, FOXSI, NuSTAR, Solar Orbiter/STIX, EOVSA, and future high-energy missions.

Unified Astronomy Thesaurus concepts: [Solar flares \(1496\)](#); [Solar magnetic reconnection \(1504\)](#); [Spectral index \(1553\)](#); [Solar flare spectra \(1982\)](#)

Supporting material: animations

1. Introduction

Particle acceleration is a ubiquitous process throughout the Universe, from stellar atmospheres to neutron stars to astrophysical jets emanating from supermassive black holes. In particular, the Sun is an excellent astrophysical laboratory for observing and understanding the fundamentals of particle acceleration in plasmas, relevant not only to cosmic plasmas but also to laboratory plasmas. The most impressive display of the Sun’s capability to produce energetic particles comes from solar eruptions. During these events, the stressed coronal magnetic field becomes unstable, and a large fraction of the free magnetic energy that was stored in the field is converted rapidly to other forms of energy, including kinetic, heat, radiation, and suprathermal-particle energies. Due to its proximity to Earth, the Sun is the only star for which we have remote imaging spectroscopy, which reveals the source locations of the emissions from accelerated particles, along with in situ coronal and heliospheric detections of the particles themselves. The observed photon and particle spectra from these observations provide crucial information about the particle acceleration process. However, despite several decades of solar observations, the main acceleration processes at work remain poorly understood.

Observed high-energy particle and photon spectra in solar eruptions are often well fitted with a broken or double power

law. However, it is not clear what produces the changes in power-law index within a single spectrum nor what determines the energies where such changes occur. For example, the nonthermal component of flare hard X-ray (HXR) emission is commonly fitted at high energies with a broken power law (Lin & Schwartz 1987; Smith & Orwig 1988; Dulk et al. 1992; Holman et al. 2003). A recent statistical study of peak HXR emission in 65 flares (Alaoui et al. 2019) found that 52 of the events had a high-energy photon spectrum with a downward (“knee”-shaped) break, 10 events had a single power law, and three events had an upward (“ankle”- or “elbow”-shaped) break. Elbow power laws with breaks at energies around 300–400 keV were observed in some giant flares by the Solar Maximum Mission, which could detect HXRs up to energies beyond 1 MeV (Dennis 1988; Vestrand 1988).

Breaks in the HXR emission spectrum may result from instrumental effects such as pulse pileup (Smith et al. 2002), HXR directivity effects, electron propagation effects, and/or the inherent shape of the accelerated electron spectrum. An example of an HXR directivity effect is the interplay between the anisotropy of the electron distribution and the observer’s direction (Petrosian 1973; Massone et al. 2004; Kontar et al. 2011; Jeffrey et al. 2020), although there is not yet strong proof of pitch-angle anisotropy in electron fluxes (Dickson & Kontar 2013; Casadei et al. 2017). Breaks with a small difference in spectral index between the high- and low-energy parts of the HXR spectrum ($\delta_{\text{high}} - \delta_{\text{low}} < 0.4$) can be explained by propagation effects (e.g., Su et al. 2011). However, many electron propagation effects can be ruled out for more prominent changes in spectral index at breaks (e.g.,



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Alaoui & Holman 2017). This includes the unusual case of Warmuth et al. (2009), who explained the observed strong break at 120 keV by invoking a high value of the low-energy cutoff. Due to all of these considerations, a double power-law distribution of the accelerated electrons may be more consistent with the observed HXR power laws than other explanations.

If instrumental or observational effects cannot explain HXR spectral breaks, then the accelerated electron spectrum must produce breaks in the photon spectrum when convolved with the relativistic electron-ion bremsstrahlung cross section. The HXR-emitting electron energy spectrum is commonly assumed to be a single or double power law (Holman et al. 2003; Brown et al. 2008; Kontar et al. 2011; Ireland et al. 2013), with a low-energy cutoff marking the beginning of the spectral flattening at low energies (Holman 2003; Sui et al. 2007; Brown et al. 2008; Ireland et al. 2013). This low-energy cutoff must exist to maintain a finite number and energy flux of electrons (Holman et al. 2003, 2011; Alaoui et al. 2019).

A photon single (double) power-law spectrum may be fitted by an electron single power law if the electron low-energy cutoff is smaller (larger) than the photon transition energy between the thermal and nonthermal components. However, a double power law in the distribution of emitting electrons always yields a photon spectrum with a broken power law, so long as the electron low-energy cutoff is smaller than the photon energy cutoff (e.g., Holman 2003; Warmuth & Mann 2016).

A double power-law electron spectrum could result from a single electron power law injected at the flare loop top and later modified by transport effects (e.g., interaction with the ambient plasma), the superposition of two independent populations of accelerated particles with different spectral indices, or an acceleration mechanism that directly injects an accelerated electron spectrum with energy breaks. We investigate the last option in the present paper.

Strong evidence for an acceleration mechanism that directly injects electron spectra with energy breaks comes from high-energy observations of flares with limb-occulted footpoints (Effenberger et al. 2017). In this case, the emission is close to the primary reconnection site, the place where the main acceleration process is believed to occur, and transport effects are minimal. Two studies of the same loop-top source observed at different times in a partially occulted flare indicate opposite shapes of the accelerated electron spectrum: Ning et al. (2019) found an elbow broken power law for the HXR-emitting electrons, while Chen et al. (2021) used a single knee distribution to describe the electron spectrum emitting both HXRs and microwaves. HXRs and microwaves sometimes show different spectral indices, but the similar time profiles of the emissions suggest that both originate in the same electron population (White et al. 2011; Asai et al. 2013). Whether both emissions originate from the same population remains an open question. A recent statistical study of temporal correlation of flare peaks concludes that HXR bremsstrahlung emissions and gyrosynchrotron microwaves are produced by the same flare-accelerated electron population or by closely linked acceleration mechanisms (Krucker et al. 2020). However, the small dynamic range of HXR observations to date make it difficult to probe the tenuous coronal acceleration region and the transport effects along flare arcades, in order to determine the dominant mechanisms.

An injected broken-power-law electron spectrum is also favored by in situ observations of “prompt” solar energetic electrons (SEEs). All electron spectra from the set of prompt events observed over a wide energy range, from a few keV up to 300 keV and studied by Krucker et al. (2009), were well represented by a knee spectral shape. For these events, the inferred injection time at the Sun of the electrons escaping to the interplanetary medium coincides with the HXR or the Type III bursts at the Sun. Prompt events have scatter-free velocity dispersion (i.e., ballistic trajectories), which implies minimal transport effects and possibly a common acceleration mechanism (Krucker et al. 2008, 2009). On the other hand, Wu et al. (2023) showed delays of impulsive SEEs with respect to the HXRs and evidence of energy-dependent release for outward-propagating electrons. A delayed second acceleration episode or transport effects may cause spectral breaks in this case.

To the best of our knowledge, few proposed acceleration mechanisms can produce a break in the power-law spectrum of the accelerated particles. One flare acceleration process that may produce HXR spectral hardening at lower energies is diffusive shock acceleration of electrons at a finite-width flare termination shock (Li et al. 2013; Kong et al. 2019; Xu et al. 2022).

In this paper, we present a purely analytical model of sequential acceleration in two regions with different physical parameters that results in broken power laws. We show how the power laws are formed and broken over time, and present analytical expressions for all energy breaks, normalization constants, and spectral indices as functions of only a few physical parameters of the acceleration regions. This work is based on Guidoni et al. (2022, hereinafter referred to as GUID22), where the formation of single power laws was explained by an analytical model based on numerical simulations of flare reconnection. That model assumed that particles were sequentially energized in large-scale magnetic islands formed by intermittent flare reconnection; a fraction of particles sequentially “hops” between islands (denoted “accelerators” in GUID22 and below). The hopping mechanism between accelerators was unspecified, but possibilities include particle scattering and diffusion. More than one accelerator was needed to form a power law over several decades in energy because, over the lifetime of a single island, trapped particles (which are assumed to conserve their magnetic moment and parallel action) may increase their energy only by a modest factor in the range 1–4.5 (Guidoni et al. 2016, hereinafter referred to as GUID16). The main mechanism turned out to be betatron acceleration due to plasma compression from Lorentz forces acting on the highly bent, newly reconnected field lines of the islands; Fermi acceleration also contributed to energizing the particles (Borovikov et al. 2017).

The present work follows directly from the analytical model of GUID22. There, the energization in each accelerator was characterized by a single efficiency parameter; a small prescribed fraction of particles was assumed to hop between accelerators, and a finite number of hops occurred. Accelerators were assumed to be filled predominantly with Maxwellian particle distributions. Section 2 briefly reviews the single-power-law analytical results and discusses new results relevant to the present work, for which the mathematical derivations are given in Appendix A.

In this paper, the sequential acceleration process is the same as in GUID22, but in two regions characterized by independent

sets of parameters. The differences between regions could be ascribed to changing conditions within a single acceleration region or to particles being transported from one acceleration region to another without intervening changes in the electron distribution. Section 3 presents and discusses the main results of the model, for which the mathematical derivations are given in Appendix B.

2. Review of the Single-power-law Analysis

In the GUID22 analysis, the efficiency of an accelerator was parameterized by a factor $r \geq 1$, the ratio between the final and initial average energies of the accelerator's particle distribution. After this energy change occurred, a small number N_{tr} of the accelerator's total population N was transferred to the next accelerator. For simplicity, N_{tr} and N were assumed to be the same for all accelerators in the region. The parameter $t = N_{tr}/N$ characterized the transfer of particles between accelerators. The population of each accelerator in the sequence of acceleration episodes was assumed to be a Maxwellian plus the small transferred fraction of particles gained from a previous accelerator and minus the same small fraction lost to the next accelerator (N was conserved in the sequence). The process was repeated n times, with each accelerator increasing the average energy of its particle population by the factor r .

From this iterative acceleration process, we derived $f_f^{(n)}$, the final spectral distribution of an accelerator that has received a fraction of particles previously energized by n accelerators, as a linear combination of particle distribution f_j ,

$$f_f^{(n)}(\bar{E}) = \left(\frac{1-t}{t}\right) \sum_{j=1}^n t^j f_j + t^n f_n, \quad (1)$$

where $\bar{E} = E/(k_B T)$ is the particle energy E normalized to the temperature of the accelerator's region T (e.g., the temperature of the flare reconnection exhaust) multiplied by Boltzmann's constant k_B . $f_f^{(n)}$ and f_j are normalized with respect to N .

Each f_j models an initial distribution f_0 that has increased its average energy by a factor r through j cycles. These distributions are assumed to evolve independently from each other in a largely collisionless environment, but only a small fraction t is transferred into or out of a given accelerator in each cycle j . The functional form adopted for f_j (GUID22) is

$$f_j(\bar{E}, r, \alpha) = \frac{1}{\Gamma(\alpha+1)} \left(\frac{\bar{E}}{r^j}\right)^\alpha \frac{e^{-\bar{E}/r^j}}{r^j}, \quad (2)$$

where Γ is the complete gamma function. The functional form in Equation (2) was chosen because it describes two well-known distributions: the exponential distribution ($\alpha=0$), which best fitted the populations inside magnetic islands formed by intermittent reconnection (GUID16; GUID22), and the widely used Maxwellian distribution ($\alpha=1/2$). $\Gamma(\alpha+1)$ varies slowly for $\alpha \in (0, 1)$, over only about 13%, so the distributions are not greatly affected by this coefficient.

Examples of f_j are shown in Figure 1 for $\alpha=0$, $r=5$, and different values of j (annotated). As supplemental material for the current paper, we include the Python Jupyter notebook used to make all the figures of this paper and the animations described later. The user may explore the parameter space by plotting different distributions. It is straightforward to show that the envelope curve of functions f_j has slope equal to 1 (e.g., red dashed line in Figure 1) by finding the slope of lines

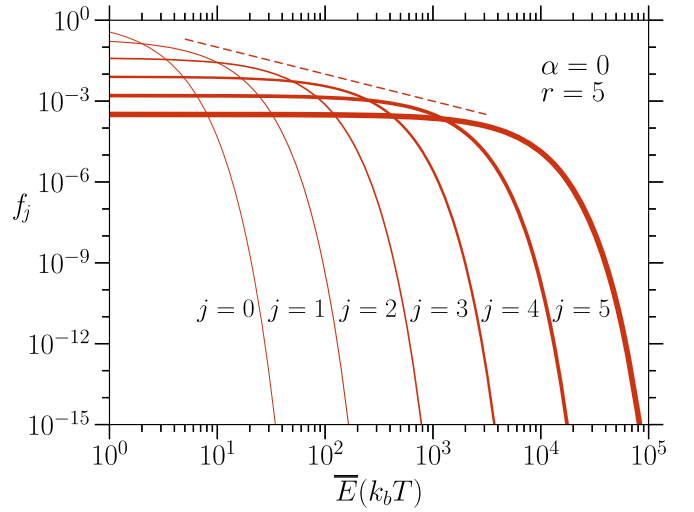


Figure 1. Examples of normalized distributions f_j (Equation (2)) for the annotated j , α , and r (red curves with increasing thickness as j increases). The red dashed straight line has slope equal to 1 and arbitrary intercept constant. The supplemental Python Jupyter notebook has the code to make this figure, and parameters can be changed to explore their effects.

connecting the intersections of consecutive curves f_j and f_{j+1} . This shows that the larger the j , the smaller (larger) its contribution to $f_f^{(n)}$ at lower (higher) energies.

As required by the model, the average energy of f_j , $\langle \bar{E} \rangle_j = \Gamma(\alpha+2)r^j/\Gamma(\alpha+1)$ (see Table 3 in GUID22), is incremented by a factor r when $j \rightarrow j+1$ (one cycle). In the Maxwellian case, for example, each cycle increases the temperature of the distribution from T to rT . Each cycle maintains the total number of particles of f_j , N . Each accelerator in the cycle shifts the basis functions of $f_f^{(n)}$ to the right (increased energy) and downward (only a small fraction of the basis distribution is transferred), resulting in the final power law.

Although the individual distributions f_j may significantly increase their average energy after several cycles, the average energy of the final distribution $f_f^{(n)}$, $\langle \bar{E} \rangle_n$, is changed only slightly for small t :

$$\frac{\langle \bar{E} \rangle_f}{\langle \bar{E} \rangle_1} = 1 + t(r-1) \frac{1 - (tr)^{n-1}}{1 - tr}. \quad (3)$$

This is a restatement of Equation (13) in GUID22.

As expected, more efficient accelerators (larger r) result in a $f_f^{(n)}$ with a harder spectrum. This is demonstrated with the examples shown in Figure 2. The curves are plotted using the parameters r , t , α , and n shown in the figure's color-coded annotations. Both curves have the same t and α , but the red one assumes fewer, more efficient accelerators. The chosen parameters for the figure are similar to those shown in Figure 9 of GUID22, where the accelerator efficiencies were those of simulated magnetic islands formed by reconnection, t was chosen to yield spectral indices consistent with those inferred from high-energy flare observations, and n resulted in similar power-law energy ranges for both distributions over several orders of magnitude.

GUID22 showed that, at mid-range energies between a low-energy break (or cutoff) $\bar{E}_{leb}$ and a high-energy break $\bar{E}_{heb}$, $f_f^{(n)}$

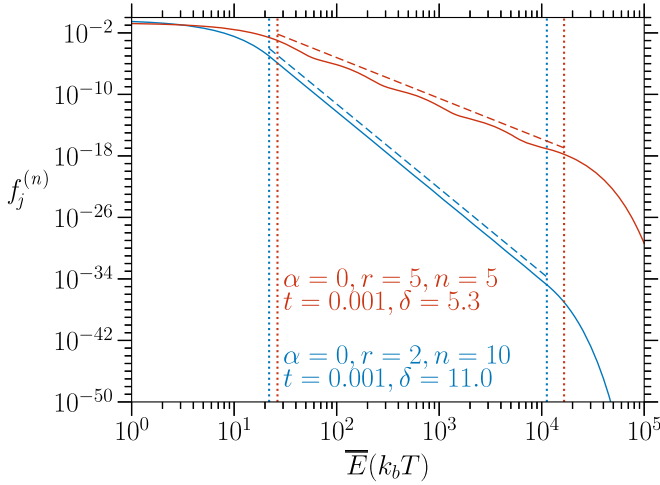


Figure 2. Examples of normalized energy spectra (solid curves). Each curve corresponds to Equation (1) for the color-coded annotated efficiency r and total number of visited accelerators n . Both curves were generated with the same transfer factor $t = 0.001$ and exponential basis function ($\alpha = 0$). Color-coded dashed curves show the analytical power law $C(E)^{-\delta}$ (Equations (5) and (8)), shifted upward by a factor of 10 for visual clarity. The spectral index δ of each power law (Equation (5)) is annotated in color. Low-energy breaks (Equation (6)) and high-energy breaks (Equation (7)) are shown with color-coded dotted vertical lines. The supplemental Python Jupyter notebook contains the code to make this Figure.

obeys an approximate power law

$$f_f^{(n)}(\bar{E}) \approx C(\bar{E})^{-\delta} \quad (4)$$

in the interval $\bar{E}_{\text{leb}} \lesssim \bar{E} \lesssim \bar{E}_{\text{heb}}$. For an overview of the importance of determining low- and high-energy cutoffs in microwave and HXR spectra of solar flares, see Holman (2003).

The quantities δ , $\bar{E}_{\text{leb}}$, and $\bar{E}_{\text{heb}}$ have simple analytical expressions as functions of the acceleration region's physical parameters r , t , and n :

$$\delta = 1 - \frac{\log t}{\log r}, \quad (5)$$

$$\bar{E}_{\text{leb}} \approx (\alpha + \delta)r, \quad (6)$$

$$\bar{E}_{\text{heb}} \approx (\alpha + \delta)r^n. \quad (7)$$

Note that $\delta > 1$ because $t < 1$ and $r > 1$. The larger r or t , the harder the spectrum. The spectral index and the low-energy break do not change with the number of accelerators, but the energy range of the power law increases with this number. At energies below $\bar{E}_{\text{leb}}$, the flattening of $f_f^{(n)}$ is mostly determined by the first few terms on the right side of Equation (1). The last terms of that equation determine the high-energy exponential-like rollover of the distribution (steeply decreasing profile at $\bar{E} \gtrsim \bar{E}_{\text{heb}}$ in Figure 2).

In Appendix A of the present paper, we show that the power-law normalization constant C can be expressed as

$$C = \left(\frac{1-t}{t \ln r} \right) \frac{\Gamma(\alpha + \delta)}{\Gamma(\alpha + 1)}. \quad (8)$$

Equations (5)–(8) are the foundational equations for the double power-law model in Section 3. Color-coded dashed lines in Figure 2 show Equation (4) with δ and C from Equations (5) and (8), respectively, shifted vertically for visual aid. These

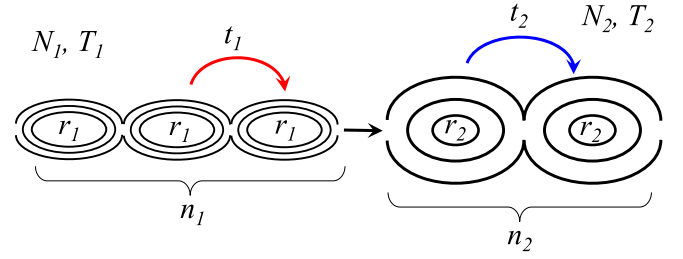


Figure 3. Sketch of two acceleration regions with annotated parameters of the model. Subscript 1 indicates the early (seed) acceleration region and subscript 2 the later acceleration region. Each region ($i = 1$ or 2) is filled with a bath of accelerators assumed to have predominantly Maxwellian populations and characterized by the average physical parameters: temperature T_i , number of particles per accelerator N_i , transport factor t_i , accelerator efficiency r_i , and number of accelerators visited n_i . Although the two regions are close to each other in this sketch, they do not need to be spatially or temporally contiguous.

lines are plotted between energies $\bar{E}_{\text{leb}}$ (Equation (6), color-coded dotted vertical lines to the left) and $\bar{E}_{\text{heb}}$ (Equation (7), color-coded dotted vertical lines to the right). In the supplemental Python notebook, the user can change the Boolean variable controlling the shift of the dashed lines in Figure 2 to see how well they overlap with the solid lines.

The normalized number of nonthermal particles, $\bar{N}_{\text{nt}}$, is expressed analytically by replacing $\bar{E}_{\text{leb}}$, $\bar{E}_{\text{heb}}$, and C (Equations (6)–(8)) in the integral

$$\begin{aligned} \bar{N}_{\text{nt}} &= \frac{N_{\text{nt}}}{N} = \int_{\bar{E}_{\text{leb}}}^{\bar{E}_{\text{heb}}} C \bar{E}^{-\delta} d\bar{E} \\ &= \frac{(1-t)(1-t^{n-1})}{-\ln t} \frac{\Gamma(\alpha + \delta)}{\Gamma(\alpha + 1)} (\alpha + \delta)^{1-\delta} \\ &\approx \left(\frac{1-t}{-\ln t} \right) \frac{\Gamma(\alpha + \delta)}{\Gamma(\alpha + 1)} (\alpha + \delta)^{1-\delta}. \end{aligned} \quad (9)$$

$\bar{N}_{\text{nt}}$ is weakly dependent on the number of accelerators visited because $t^{n-1} \ll 1$ (assuming that a small number of particles is transferred between accelerators). This limit was used to obtain the last approximate equality in the equation above.

The distribution $f_f^{(n)}$ oscillates about the overall power law with a small amplitude that depends on the parameters of the acceleration region (the analytical expression of this amplitude can be estimated from Equation (A7) of Appendix A in this paper). The waviness of $f_f^{(n)}$ can be seen in the red curve in Figure 2. On the other hand, the oscillations of the blue curve are almost imperceptible to the naked eye.

3. Double Power Law

Because the reconnection process evolves throughout a solar eruption, in time and space, we must evaluate the effects of changing the model parameters during the acceleration interval. Therefore we consider the simplest case with increased complexity: a sequential acceleration method, similar to the one described in Section 2, but in two regions with different parameters. We assume that particles are immersed first in a region filled with accelerators characterized by parameters $r = r_1$ and $t = t_1$ (the seed region), and then in a second region filled with accelerators characterized by parameters $r = r_2$ and $t = t_2$. Figure 3 sketches the two acceleration regions with their corresponding model parameters.

In Section 2, the distributions were normalized to the average number of particles in each accelerator of the single

acceleration region. In the present section, all distributions are normalized with respect to a generic normalization factor N , and for each region $i = 1, 2$, we define the ratio of the average number of particles per accelerator with respect to N as $p_i = N_i/N$. Similarly, the energies are normalized to a generic temperature T , as each region may have a different background temperature; then we define $h_i = T_i/T$. With these definitions, the renormalization of distribution f_j in region i with respect to N and T , $f_j^{(i)}$, is written as

$$f_j^{(i)}(\bar{E}, r, \alpha) = \frac{p_i}{\Gamma(\alpha + 1)} \left(\frac{\bar{E}}{h_i r^j} \right)^\alpha \frac{e^{-\bar{E}/(h_i r^j)}}{h_i r^j}. \quad (10)$$

The normalized number of particles in $f_j^{(i)}$ is p_i . For simplicity, all figures in this section are plotted with $p_i = 1$, $h_i = 1$, and $\alpha_i = 0$ for both regions, but the analysis is performed for general values of these parameters.

If a distribution function $f_j^{(1)}$ increases its average energy r_2 times in region 2, for example, the shape of its final distribution function is described by Equation (10), but with each occurrence of the factor r^j in the equation multiplied by r_2 . With this in mind, we generalize the independent distributions f_j in Equation (10) with the following basis function that describes a generic number of cycles j and k in regions 1 and 2, respectively:

$$f_{jk}^{(i)}(\bar{E}, r_1, r_2, \alpha_i) = \frac{p_i}{\Gamma(\alpha_i + 1)} \left(\frac{\bar{E}}{h_i r_1^j r_2^k} \right)^{\alpha_i} \frac{e^{-\bar{E}/(h_i r_1^j r_2^k)}}{h_i r_1^j r_2^k} \quad (11)$$

$$= \frac{p_i}{h_i r_2^k} f_j(\bar{E}/h_i r_2^k, r_1, \alpha_i) \quad (12)$$

$$= \frac{p_i}{h_i r_1^j} f_k(\bar{E}/h_i r_1^j, r_2, \alpha_i). \quad (13)$$

$f_{jk}^{(i)}$ has the same functional form as f_j in Equation (2). Equations (12) and (13) (expressions also used in Appendix B) show that the joint distribution f_{jk} can be written in terms of the single distributions f_j and f_k , except for the normalization factors in temperature, number of particles, and efficiency.

The average energy of particles in $f_{jk}^{(i)}$, $\langle \bar{E} \rangle_{jk} = \Gamma(\alpha + 2) r_1^j r_2^k h_i / \Gamma(\alpha + 1)$, increases by a factor r_1 if the particles of the distribution visit an accelerator in region 1 ($j \rightarrow j + 1$) and by r_2 if they visit an accelerator in region 2 ($k \rightarrow k + 1$). As j or k increases, $f_{jk}^{(i)}$ has more particles at high energies and fewer at low energies (similar to the behavior shown in Figure 1). Table 1 summarizes the parameters of the model.

We assume that particles visit n_1 accelerators in region 1 initially, so their distribution will be the same as that in Equation (1) (except that the normalization is now with respect to N_1 , instead of N), which is expressed as a function of the generalized distributions in Equation (11) by setting $k = 0$:

$$f_f^{(n_1)} = \left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{j0}^{(1)} + t_1^{n_1} f_{n_1 0}^{(1)}. \quad (14)$$

We call $f_f^{(n_1)}$ the *seed population*, as it occurs first in the overall acceleration process. Its normalized number of particles is p_1 because the sequential acceleration process assumes conservation of the average number of particles inside each accelerator of a given region.

Table 1
Model Parameters for Each Accelerator i

Symbol	Definition
N	Normalization constant: number of particles
T	Normalization constant: temperature
$\bar{E}$	$= E/(k_B T)$, normalized energy
N_i	Average number of particles per accelerator
T_i	Background temperature
p_i	$= N_i/N$, normalized number of particles per accelerator
h_i	$= T_i/T$, normalized temperature of the region
α_i	Type of basis distribution (Equation (2) or (11)) (i.e., $\alpha = 0$: exponential or $\alpha = 1/2$: Maxwellian)
r_i	$= \langle \bar{E} \rangle_{j+1} / \langle \bar{E} \rangle_j$, accelerator efficiency (average energy gain in one cycle)
t_i	$= N_{tr}/N_i$, fraction of particles transferred between accelerators, where N_{tr} = number of particles transferred between accelerators
n_i	Number of accelerators visited by particles

If we assume that a small fraction of the seed population enters an accelerator in region 2 and that particles in region 2 are sequentially accelerated in n_2 accelerators with the same small fraction of particles transferred between accelerators, as in a single power law, the final distribution is

$$\begin{aligned} f_f^{(n_1, n_2)} &= \left(\frac{1 - t_2}{t_2} \right) \sum_{k=1}^{n_2} t_2^k f_{0k}^{(2)} \\ &+ t_2^{n_2} \frac{p_2}{p_1} \left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{jn_2}^{(1)} \\ &+ t_2^{n_2} t_1^{n_1} \frac{p_2}{p_1} f_{n_1 n_2}^{(1)}. \end{aligned} \quad (15)$$

The detailed sequential acceleration process that leads to the above equation is described in Appendix B. The factor p_2/p_1 is needed because t_2 is normalized with respect to N_2 . The total number of particles of $f_f^{(n_1, n_2)}$ is p_2 , which was conserved for all acceleration cycles in region 2.

Each summation on the right side of Equation (15) corresponds to a single power law (same functional shape as Equation (1) or (14)), with the first (second) summation being dominant at low (high) energies. Examples of $f_f^{(n_1, n_2)}$ are shown with a solid black curve in Figures 4 (“knee” distribution) and 5 (“elbow” distribution), for the annotated parameters. In Figure 4, t_1 , r_1 , and n_1 (t_2 , r_2 , and n_2) are the same as those for the single power law plotted with a solid blue (red) line in Figure 2. In Figure 5, the parameters of each region have been reversed. In the case of a knee (elbow) spectrum, the seed region is softer (harder) because it has accelerators that are less (more) efficient.

The seed power law is formed first and then shifted to the right (increased energy) and downward (a small fraction of particles is transferred in each cycle) as it is cycled in region 2, with negligible contribution at low energies. The bulk of distribution $f_f^{(n_1, n_2)}$ is initially composed of particles in a Maxwellian distribution with temperature T_2 , which later sequentially forms the low-energy power law, almost independent of what the seed population is doing. Two animations that graphically show the formation of the distributions in Figures 4 and 5 as each acceleration cycle occurs are available online. In the animations the final distribution is shown with a red solid

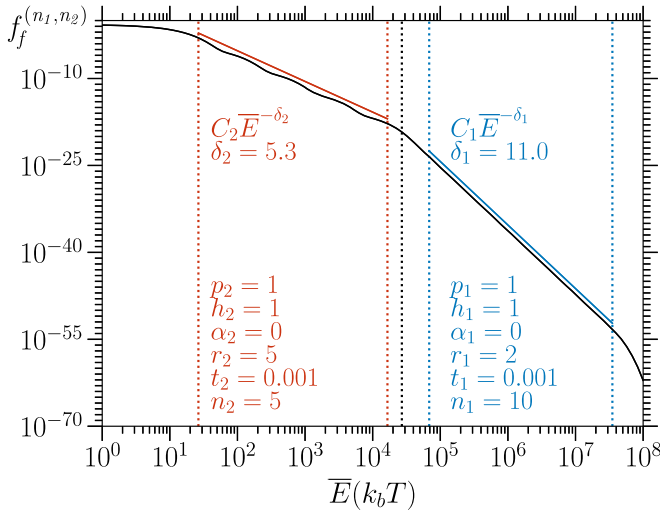


Figure 4. Example of a normalized knee energy spectrum ($\delta_2 < \delta_1$) for the annotated parameters (solid black curve, Equation (15)). Blue and red solid lines show the analytical power laws associated with the first acceleration region (Equations (20) and (21)) and the second acceleration region (Equations (16) and (17)), respectively, shifted upward by a factor of 10 for visual clarity. The energy breaks of each power-law region (Equations (22), (23), (18), and (19)) are shown with color-coded vertical dotted lines. The vertical dotted black line shows $\bar{E}_{\text{mid}}$ (Equation (24)). The supplemental Python Jupyter notebook has the code to make this figure. The online animation shows the evolution of the particle distribution as each discrete acceleration cycle occurs in each region, culminating in the final distribution shown in the figure. The animation's size is 44 KB and its duration is 3 s.

(An animation of this figure is available.)

line, and for reference dashed blue curves show the initial distribution in each acceleration region (cycle “1”).

From the discussion in Appendix B, the properties of the power law at low energies are completely determined by the parameters of region 2 (Equations (B5)–(B8)), namely

$$\delta_2 = 1 - \frac{\log t_2}{\log r_2}, \quad (16)$$

$$C_2 \approx \left(\frac{1 - t_2}{t_2 \ln r_2} \right) \frac{\Gamma(\alpha_2 + \delta_2)}{\Gamma(\alpha_2 + 1)} p_2 h_2^{\delta_2 - 1} \quad (17)$$

$$\bar{E}_{\text{leb}}^{(2)} \approx (\alpha_2 + \delta_2) h_2 r_2, \quad (18)$$

$$\bar{E}_{\text{heb}}^{(2)} \approx (\alpha_2 + \delta_2) h_2 r_2^{n_2}. \quad (19)$$

Equations (16)–(19) have the same functional forms as Equations (5)–(8) for a single power law, except for coefficients related to normalization factors. The power law $C_2 \bar{E}^{-\delta_2}$ is shown with a red solid line in Figures 4 and 5, where it has been plotted between $\bar{E}_{\text{leb}}^{(2)}$ and $\bar{E}_{\text{heb}}^{(2)}$ and shifted vertically for visual clarity. In Figure 4 (Figure 5), δ_2 , $\bar{E}_{\text{leb}}^{(2)}$, and $\bar{E}_{\text{heb}}^{(2)}$ are the same as those of the red (blue) curve in Figure 2 because all parameters are the same for this region (and we considered $h_2 = 1$).

The power law at high energies has a spectral index that only depends on the properties of region 1 (seed), but its energy breaks and normalization constant depend on the parameters of both regions (also from Equations (B5)–(B8)):

$$\delta_1 = 1 - \frac{\log t_1}{\log r_1}, \quad (20)$$

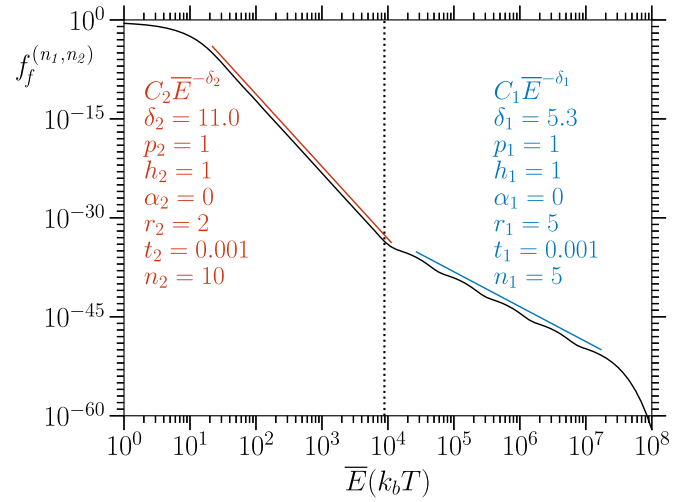


Figure 5. Example of a normalized elbow energy spectrum ($\delta_2 > \delta_1$) for the annotated parameters (solid black curve, Equation (15)). Blue and red solid lines show the analytical power laws associated with the first acceleration region (Equations (20) and (21)) and the second acceleration region (Equations (16) and (17)), respectively, shifted upward by a factor of 10 for visual clarity and plotted between the energy breaks of each power-law region (Equations (22), (23), (18), and (19)). The vertical dotted black line shows $\bar{E}_{\text{mid}}$ (Equation (24)). The supplemental Python Jupyter notebook has the code to make this figure. The animation shows the evolution of the particle distribution as each discrete acceleration cycle occurs in each region, culminating in the final distribution shown in the figure. The animation's size is 47 KB and duration is 3 s.

(An animation of this figure is available.)

$$C_1 \approx \left(\frac{1 - t_1}{t_1 \ln r_1} \right) \frac{\Gamma(\alpha_1 + \delta_1)}{\Gamma(\alpha_1 + 1)} p_2 h_1^{\delta_1 - 1} r_2^{n_2(\delta_1 - \delta_2)}, \quad (21)$$

$$\bar{E}_{\text{leb}}^{(1)} \approx (\alpha_1 + \delta_1) h_1 r_1 r_2^{n_2}, \quad (22)$$

$$\bar{E}_{\text{heb}}^{(1)} \approx (\alpha_1 + \delta_1) h_1 r_1^{n_1} r_2^{n_2}. \quad (23)$$

Similar to the low-energy power law, Equations (20)–(23) have the same functional forms as Equations (5)–(8) for a single power law, except for coefficients related to normalization factors. The power law $C_1 \bar{E}^{-\delta_1}$ is shown with a blue solid line in Figures 4 and 5, where it has been plotted between $\bar{E}_{\text{leb}}^{(1)}$ and $\bar{E}_{\text{heb}}^{(1)}$ and shifted vertically for visual clarity. In Figure 4 (Figure 5), δ_1 is the same as in the blue (red) curve in Figure 2 because r and t are the same for this region.

The expressions for the energy breaks in both regions dictate that each cycle of sequential acceleration in a given region extends the energy range of the power law in that region without changing the spectral indices (the same was true for single power laws such as those in Figure 2). These energies are plotted in Figure 4 with color-coded vertical dotted lines. Note that the expression for $\bar{E}_{\text{leb}}^{(1)}$ is less accurate in the case of a double power law than for a single power law, because the distribution does not flatten below this energy. Similarly, $\bar{E}_{\text{heb}}^{(2)}$ is not as well defined as in the single-power-law case, because the distribution does not roll over sharply after this energy. On the other hand, the location of the break between power-law regions, $\bar{E}_{\text{mid}}$, is accurately defined as the energy where the two power laws converge, namely

$$\bar{E}_{\text{mid}} = \left(\frac{C_1}{C_2} \right)^{1/(\delta_1 - \delta_2)} \propto r_2^{n_2}. \quad (24)$$

This energy is shown with a black dotted vertical line in Figures 4 and 5. The location of this break is proportional to $r_2^{n_2}$. The more accelerators that are visited in the second region, the higher the middle-energy break. After n_2 acceleration cycles in region 2, the seed power law has been shifted to the right in energy by the same factor (from Equations (22) and (23)).

As shown in Equations (16)–(23), measuring spectral indices and energy breaks (and even normalization constants) may offer information about the physical parameters of each acceleration region. For example, if we make the assumptions that $p_1 = p_2 = 1$ and $h_1 = h_2 = 1$, and choose $\alpha_1 = \alpha_2 = 0$ or $\alpha_1 = \alpha_2 = 0.5$, measuring temperatures, spectral indices, and low-energy breaks from each power-law region could, in principle, be used to estimate efficiencies and transport factors for both regions, as well as n_2 . n_1 can also be inferred if $\bar{E}_{\text{heb}}^{(1)}$ is observed.

4. Conclusions

We have extended our previous first-principles, fully analytical method of forming electron distributions with single power laws (GUID22) to form double power laws, by assuming a sequential acceleration process in two consecutive but distinct regions. Particles are sequentially energized in a region with accelerators that increase the average energy of particle distributions by a modest amount, and a prescribed average fraction of particles sequentially “hops” between accelerators. The resulting spectral functions are continuous and smooth over the entire range of energy, with a flattening at low energies, two power-law regions joined across a middle break, and a steep rollover at high energies.

The model provides interpretations and analytical expressions for the energy breaks, normalization constants, and spectral indices as functions of a few physical parameters of each region: the ratio of particles transferred between accelerators, the energy gain in each accelerator, and the number of accelerators visited. We also show that several of the acceleration regions’ parameters can be inferred from observed spectral indices and energy breaks. For example, Equations (5) and (6) provide an observational method to estimate r and t for a single power-law spectrum. The same applies to the second acceleration region producing a broken power-law spectrum. Furthermore, if $\bar{E}_{\text{heb}}$ is also observationally determined, n (or its lower bound) can also be constrained. The simplicity of the present analytical model allows for easy conceptual understanding of how to form and break power laws as a function of physically meaningful parameters, and for the construction of the particle distributions as each cycle of a sequential acceleration occurs.

The most important conclusion of this work is that, for accelerated electrons with double power-law spectra, the spectral index at higher energies is determined solely by the parameters of the initial acceleration region (seed population), while the index at lower energies is set by the parameters of the second region. The location of the break between the power laws is controlled by how many accelerators are visited in the second region. The seed power law is just shifted to higher energies without changing its spectral index, as small portions of this distribution are transferred to new accelerators in the second acceleration region. Meanwhile, the bulk of the second-region particles undergo the standard power-law formation

process described in GUID22. Therefore, a knee (elbow) break occurs when the seed acceleration region is softer (harder) than the second region. This difference in spectral indices could be due to changing conditions in the acceleration region (e.g., if considering magnetic islands as accelerators, different fields may be reconnected, resulting in magnetic islands with different sizes or guide fields) or to particles being transported unaffected to a second acceleration site. Our work indicates that the typical HXR knee spectrum found in flares may correspond to a soft accelerated electron spectrum formed before a hard spectrum in the acceleration region. On the other hand, if microwaves and HXRs are assumed to originate from the same electron population (still an open question), their characteristic elbow distribution would indicate a hard-to-soft progression in the acceleration region.

It is worth noting that our model does not directly address spatial scales or timescales for particle escape and transport between accelerators as it abstracts the consequences of a detailed physical model into its three parameters. To prove or disprove our model, dedicated comprehensive simulations that couple self-consistently the macroscopic and microscopic populations (e.g., Drake et al. 2019; Arnold et al. 2021; Shou et al. 2021) or Parker-type transport theories (e.g., Zank et al. 2014; Zhao et al. 2018, 2019; Adhikari et al. 2019; Le Roux et al. 2019), applied to coronal conditions, are needed. For example, these simulations and theories could determine whether a broken power law can be formed with two consecutive acceleration regions within solar flare timescales. Additionally, they could constrain the few physical parameters of our model and estimate their dependence on relevant variables, such as advection, diffusion, scattering, magnetic field strength and direction, etc. Addressing these issues in detail is beyond the scope of the current article, but is being studied by others.

Our analytical model is independent of the type of acceleration, as long as the acceleration occurs sequentially and small fractions of particles are transferred between accelerators. Our model assumes that the increase in average energy inside the accelerators, the ratio of particles transferred between accelerators, and the number of accelerators visited are uniform. It does not consider more complex scenarios, e.g., energy- or charge/mass-dependent parameters. However, more complex conditions clearly are not necessary to explain single or double power laws. These simplifications were made for the benefit of easy interpretation and generalization, but it is likely that different acceleration mechanisms may be parameterized by different combinations of parameter values and then ultimately tested by observations. We hope that our simple method to form broken power laws provides an incentive for dedicated simulations or models that can test different sequential acceleration mechanisms in two different consecutive acceleration regions. Such work is beyond our ambition here, but it could be a useful focus of future work because understanding the physics of particle transport and acceleration is certainly the overarching goal.

So long as the particle spectrum that exits region 1 is not modified before entering region 2, the two regions do not have to be spatially or temporally contiguous. The first acceleration region could be located at or near the reconnection site, followed later by sequential acceleration at a separate (perhaps turbulent) site. For example, this could explain why a positive correlation was found between the estimated high-energy

spectral index of a set of SEE events with double power-law spectra measured by WIND/3DP at 1 au and the estimated power-law index of HXR-producing electrons in the associated west-limb HXR flares (Wang et al. 2021).

Furthermore, the process described in this manuscript for acceleration in two regions can be easily generalized to as many acceleration regions as desired by following the steps described in the Appendix, each region producing a power law with a spectral index that only depends on the efficiency and transport factor of that region. For example, this may explain the existence of triple power-law fits (plus a thermal component) for some HXR photon spectra (e.g., da Silva & Valio 2021).

We provide a supplemental self-contained Python Jupyter notebook with all functions required to reproduce the figures in this paper, with the exception of Figure 3. This notebook is available on figshare: doi:10.57912/25308256.

Acknowledgments

We thank the referee for reviewing our paper and providing useful comments. Our work was supported in part through the following NASA grants: 80NSSC21K0817 to American University for S.E.G.; 80NSSC23K0448 to the University of Maryland for M.A.; and an H-ISFM award to Goddard Space Flight Center for C.R.D. and J.T.K.

Appendix A

Single Power-law Normalization Constant and Oscillation Amplitude

In this appendix, we show the calculations that lead to the analytical expression of the power-law normalization constant C (Equation (8)) by taking a few steps beyond the calculations in GUID22.

We showed that, for energies between the low-energy break, $\bar{E}_{\text{leb}}$ (Equation (6)), and the high-energy break, $\bar{E}_{\text{heb}}$ (Equation (7)), the first n terms of $f_f^{(n)}$ (Equation (1)) can be approximated as

$$\begin{aligned} f_f^{(n)} &\approx \left(\frac{1-t}{t}\right) \sum_{j=1}^n t^j f_j(\bar{E}, r, \alpha) \\ &\approx \left(\frac{1-t}{t}\right) \frac{\bar{E}^{-\delta}}{\Gamma(\alpha+1)} g_f^\infty(x, \tau, \alpha), \end{aligned} \quad (\text{A1})$$

where $x = \log \bar{E}$, $\tau = \log r$, and $g_f^\infty(x, \tau, \alpha)$ is an infinite sum of equally shaped and horizontally shifted pulse-like functions $g(x) = e^{-10^x} 10^{x(\alpha+\delta)}$, namely

$$\begin{aligned} g_f^\infty(x, \tau, \alpha) &= \sum_{j=-\infty}^{\infty} g(x - j\tau) \\ &= \sum_{k=-\infty}^{\infty} e^{i2\pi kx/\tau} \mathcal{F}_k. \end{aligned} \quad (\text{A2})$$

$g_f^\infty(x, \tau, \alpha)$ is a periodic function, with period τ , that can be expanded in a complex Fourier series (second equality above) with coefficients $\mathcal{F}_k$:

$$\mathcal{F}_k = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} g_f^\infty(x, \tau, \alpha) e^{-i2\pi kx/\tau} dx \quad (\text{A3})$$

$$= \int_{-\infty}^{\infty} g(y\tau) e^{-i2\pi ky} dy \quad (\text{A4})$$

$$= \frac{1}{\tau \ln 10} \int_0^\infty e^{-z} z^{(\alpha+\delta-1-[i2\pi k/\tau \ln 10])} dz \quad (\text{A5})$$

$$= \frac{1}{\tau \ln 10} \Gamma(\alpha + \delta - [i2\pi k/\tau \ln 10]). \quad (\text{A6})$$

In GUID22 the integral in Equation (A3) was not explicitly written, and only $\mathcal{F}_0$ was calculated numerically; here we present the general analytical solution for all $\mathcal{F}_k$. The integral in Equation (A4) results from the change of variable $\tau y = x - j\tau$; the integral in Equation (A5) results from the change of variable $10^{\tau y} = z$ and the definition of function $g(x)$. The complex conjugate of $\mathcal{F}_k$, $\mathcal{F}_k^*$, is equal to $\mathcal{F}_{-k}$ because $\Gamma(z) = \Gamma(z^*)$. Therefore, the Fourier sum in Equation (A2) is expressed as

$$\begin{aligned} g_f^\infty(x, \tau, \alpha) &= g_f^\infty(x', \tau', \alpha) \\ &= \mathcal{F}_0 + \frac{2}{\tau'} \sum_{k=1}^{\infty} \left\{ \cos\left(\frac{2\pi k}{\tau'} x'\right) \Re \left[\Gamma\left(\alpha + \delta - \frac{2\pi k}{\tau'} i\right) \right] \right. \\ &\quad \left. - \sin\left(\frac{2\pi k}{\tau'} x'\right) \Im \left[\Gamma\left(\alpha + \delta - \frac{2\pi k}{\tau'} i\right) \right] \right\}, \end{aligned} \quad (\text{A7})$$

where $x' = \ln \bar{E}$, $\tau' = \ln r$, i is the imaginary unit, and $\Re$ and $\Im$ indicate the real and imaginary parts of a complex number, respectively. $g_f^\infty(x', \tau', \alpha)$ oscillates about its average value $\mathcal{F}_0$ with period τ' . The amplitude of the oscillation is modulated by the real and imaginary parts of $\Gamma(\alpha + \delta - i2\pi k/\tau')$, which decrease in magnitude with k .

$f_f^{(n)}$ (Equation (A1)) then oscillates about the overall power law with an amplitude that depends on the amplitude of g_f^∞ . This is visible in Figure 2, where the oscillations in the red curve have larger amplitudes than those in the blue curve.

From Equation (A1), we see that the power-law normalization constant C can be expressed as

$$C \approx \left(\frac{1-t}{t}\right) \frac{\mathcal{F}_0}{\Gamma(\alpha+1)} = \left(\frac{1-t}{t \ln r}\right) \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha+1)}. \quad (\text{A8})$$

Q.E.D.

Appendix B

Double Power-law Method

In this appendix, we show how double power laws are formed by sequential acceleration in two different regions. We assume that the seed population from region 1 described by Equation (14) is transported to region 2, where a new sequential acceleration of the type described by GUID22 occurs. Region 2 consists of a bath of predominantly Maxwellian accelerators with temperature T_2 .

To start the sequential acceleration, the first distribution of the sequence $f_i^{(n,1)}$ is assumed to receive a small fraction t_2 of particles from the seed population and lose the same fraction of its original population. Then

$$\begin{aligned} f_i^{(n,1)} &= (1 - t_2) f_{00}^{(2)} + t_2 \frac{p_2}{p_1} f_f^{(n)} = (1 - t_2) f_{00}^{(2)} \\ &\quad + t_2 \frac{p_2}{p_1} \left[\left(\frac{1-t_1}{t_1}\right) \sum_{j=1}^n t_1^j f_{j0}^{(1)} + t_1^{n_1} f_{n_1 0}^{(1)} \right]. \end{aligned} \quad (\text{B1})$$

The factor p_2/p_1 is needed because t_2 is normalized with respect to N_2 . The second superscript in $f_i^{(n_1,1)}$ indicates the cycle number of the sequential acceleration process. We counted the above transfer as the first acceleration episode ($k=1$) of the region.

After population $f_i^{(n_1,1)}$ is cycled through one accelerator in region 2, its final population is

$$f_f^{(n_1,1)} = (1 - t_2)f_{01}^{(2)} + t_2 \frac{p_2}{p_1} \left[\left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{j1}^{(1)} + t_1^{n_1} f_{n_1}^{(1)} \right]. \quad (\text{B2})$$

In going from Equation (B1) to Equation (B2), we increased by 1 the second subscript of each generalized distribution, indicating an energy increase by a factor r_2 in each basis function. However, the total average energy of $f_i^{(n_1,1)}$ (as well as its number of particles) after being cycled to become $f_f^{(n_1,1)}$ is essentially unchanged.

A small fraction t_2 of particles in $f_f^{(n_1,1)}$ is then transferred to a second accelerator. The initial and final distributions in this case are

$$\begin{aligned} f_i^{(n_1,2)} &= (1 - t_2)f_{00}^{(2)} + t_2 f_f^{(n_1,1)} \\ &= (1 - t_2)f_{00}^{(2)} + t_2 \left\{ (1 - t_2)f_{01}^{(2)} + t_2 \frac{p_2}{p_1} \left[\left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{j1}^{(1)} + t_1^{n_1} f_{n_1}^{(1)} \right] \right\} \\ f_f^{(n_1,2)} &= (1 - t_2)f_{01}^{(2)} + t_2 \left\{ (1 - t_2)f_{02}^{(2)} + t_2 \frac{p_2}{p_1} \left[\left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{j2}^{(1)} + t_1^{n_1} f_{n_2}^{(1)} \right] \right\} \\ &= (1 - t_2)f_{01}^{(2)} + t_2(1 - t_2)f_{02}^{(2)} + t_2^2 \frac{p_2}{p_1} \left[\left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{j2}^{(1)} + t_1^{n_1} f_{n_2}^{(1)} \right]. \end{aligned}$$

The distribution in Equation (B2) is already normalized to N_2 ; therefore, there is no need to introduce another factor p_2/p_1 .

The above expressions can be generalized to n_2 cycles, namely

$$\begin{aligned} f_i^{(n_1, n_2)} &= (1 - t_2) \sum_{k=0}^{n_2-1} t_2^k f_{0k}^{(2)} + t_2^{n_2} \frac{p_2}{p_1} \\ &\quad \times \left[\left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{jn_2-1}^{(1)} + t_1^{n_1} f_{n_1(n_2-1)}^{(1)} \right] \\ f_f^{(n_1, n_2)} &= \left(\frac{1 - t_2}{t_2} \right) \sum_{k=1}^{n_2} t_2^k f_{0k}^{(2)} + t_2^{n_2} \frac{p_2}{p_1} \\ &\quad \times \left[\left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_{jn_2}^{(1)} + t_1^{n_1} f_{n_1 n_2}^{(1)} \right]. \end{aligned} \quad (\text{B3})$$

It is easy to see that Equation (B3) has two power-law regions by replacing the generalized independent distributions

$f_{jk}^{(i)}$ with Equations (12) and (13):

$$\begin{aligned} f_f^{(n_1, n_2)} &= \frac{p_2}{h_2} \left(\frac{1 - t_2}{t_2} \right) \sum_{k=1}^{n_2} t_2^k f_k(\bar{E}/h_2, r_2, \alpha_2) + \left(\frac{t_2}{r_2} \right)^{n_2} \\ &\quad \times \frac{p_1 p_2}{h_1 p_1} \left[\left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_j(\bar{E}/[h_1 r_2^{n_2}], r_1, \alpha_1) \right. \\ &\quad \left. + t_1^{n_1} f_{n_1}(\bar{E}/[h_1 r_2^{n_2}], r_1, \alpha_1) \right] \\ &= \left(\frac{t_1}{r_1} \right)^0 \frac{p_2}{h_2} \left(\frac{1 - t_2}{t_2} \right) \sum_{k=1}^{n_2} t_2^k f_k(\bar{E}/[h_2 r_1^0], r_2, \alpha_2) \\ &\quad + \left(\frac{t_2}{r_2} \right)^{n_2} \frac{p_2}{h_1} \left(\frac{1 - t_1}{t_1} \right) \sum_{j=1}^{n_1} t_1^j f_j(\bar{E}/[h_1 r_2^{n_2}], r_1, \alpha_1) \\ &\quad + \left(\frac{t_2}{r_2} \right)^{n_2} \frac{p_2}{h_1} t_1^{n_1} f_{n_1}(\bar{E}/[h_1 r_2^{n_2}], r_1, \alpha_1). \end{aligned} \quad (\text{B4})$$

The two summations on the right side of the last equality in Equation (B4) were rewritten to explicitly show the same structure as in the middle expression of Equation (A1), but with a renormalized energy that is expressed as $\bar{E} = \bar{E}/[h_i r_i^{c_i}]$, with $i \neq i'$, i and $i' = 1$ or 2 , $c_1 = 0$, and $c_2 = n_2$. The independent distributions composing one summation have negligible weight at the energies where the independent distributions of the other

summation have their greatest weight. Therefore, each summation corresponds to a power law between a pair of energy breaks. Following the same steps as in Appendix A and in GUID22, but by defining $x = \log \bar{E}$ and $\tau = \log r_i$, it is straightforward to show that each power law has the following properties:

$$\delta_i = 1 - \frac{\log t_i}{\log r_i}, \quad (\text{B5})$$

$$\bar{E}_{\text{leb}}^{(i)} \approx (\alpha_i + \delta_i) r_i \quad \text{or} \quad \bar{E}_{\text{leb}}^{(i)} \approx (\alpha_i + \delta_i) h_i r_i r_i^{c_i'}, \quad (\text{B6})$$

$$\bar{E}_{\text{heb}}^{(i)} \approx (\alpha_i + \delta_i) r_i^{n_i} \quad \text{or} \quad \bar{E}_{\text{heb}}^{(i)} \approx (\alpha_i + \delta_i) h_i r_i^{n_i} r_i^{c_i'}, \quad (\text{B7})$$

$$\begin{aligned} \frac{C_i}{p_2} &\approx \left(\frac{1 - t_i}{t_i \ln r_i} \right) \frac{\Gamma(\alpha_i + \delta_i)}{\Gamma(\alpha_i + 1)} h_i^{\delta_i-1} (t_i' r_i'^{\delta_i-1})^{c_i'} \\ &= \left(\frac{1 - t_i}{t_i \ln r_i} \right) \frac{\Gamma(\alpha_i + \delta_i)}{\Gamma(\alpha_i + 1)} h_i^{\delta_i-1} (r_i'^{\delta_i-\delta_i'})^{c_i'}. \end{aligned} \quad (\text{B8})$$

In the last three equations, the extra factors to the right with respect to the single power-law equations come from the extra factors in $\bar{E}$. For example, when calculating the normalization

constant, we need to take into account that the factor $\bar{E}^{-\delta}$ in Equation (A1) should be $\bar{E}^{-\delta}$ in Equation (B4). Q.E.D.

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