



# **Shock Capturing via Limiting for High-Order Methods Including Discontinuous Galerkin**

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# Motivation

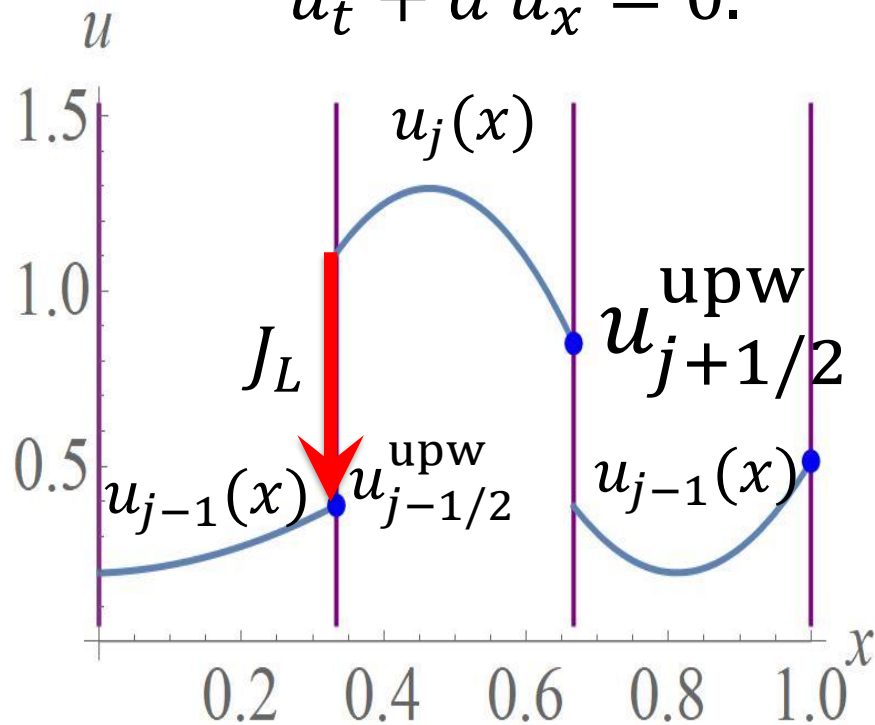
- High-order methods such as discontinuous Galerkin (DG), spectral, and Flux Reconstruction (FR) are garnering significant interest in CFD
- There are challenges to these methods for flows with shocks or discontinuities: the solutions can oscillate or lose accuracy near extrema

# Outline

- Discuss oscillations caused by a discontinuity (shocks) in an example
- Present a shock capturing method that suppresses oscillations near shocks and maintains accuracy at smooth regions
- Some numerical examples
- Conclusions and discussion

# DG via FR framework for Advection

$$u_t + a u_x = 0.$$



$$u_j(\xi) = \sum_{k=0}^p u_{j,k} L_k(\xi),$$

$u_{j,0}$  cell average value

$u_{j,1}$  linear mode

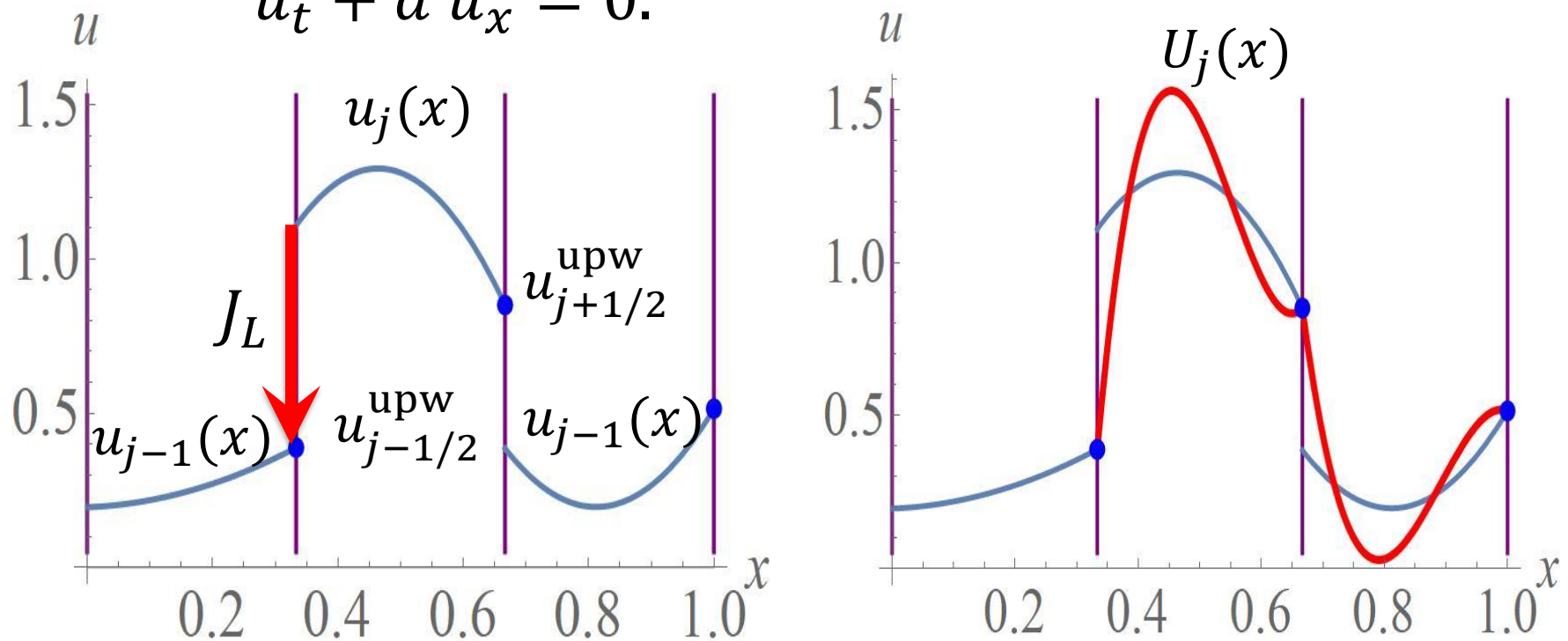
$u_{j,2}$  quadratic mode

$\sum_{k=0}^p u_{j,k}$  value at right interface

For DG2, the polynomial  $u_j(x)$  are quadratics (degree  $p = 2$ ).

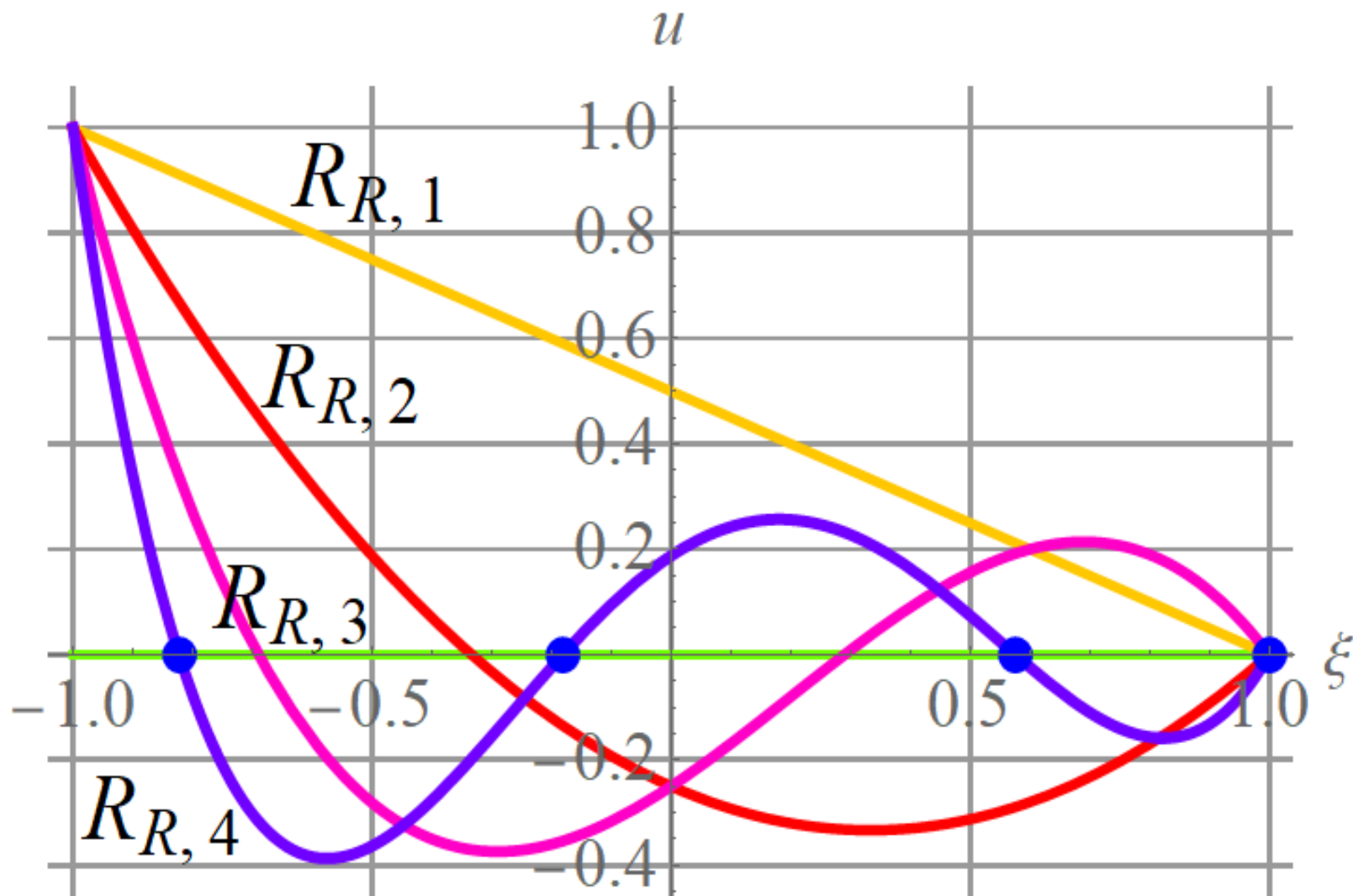
# DG via FR framework

$$u_t + a u_x = 0.$$

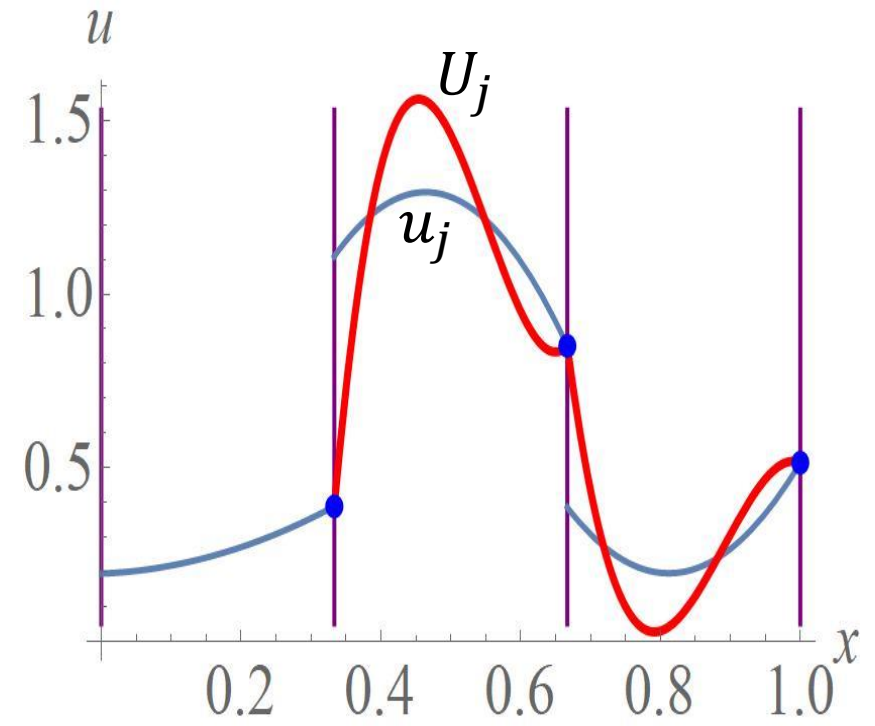
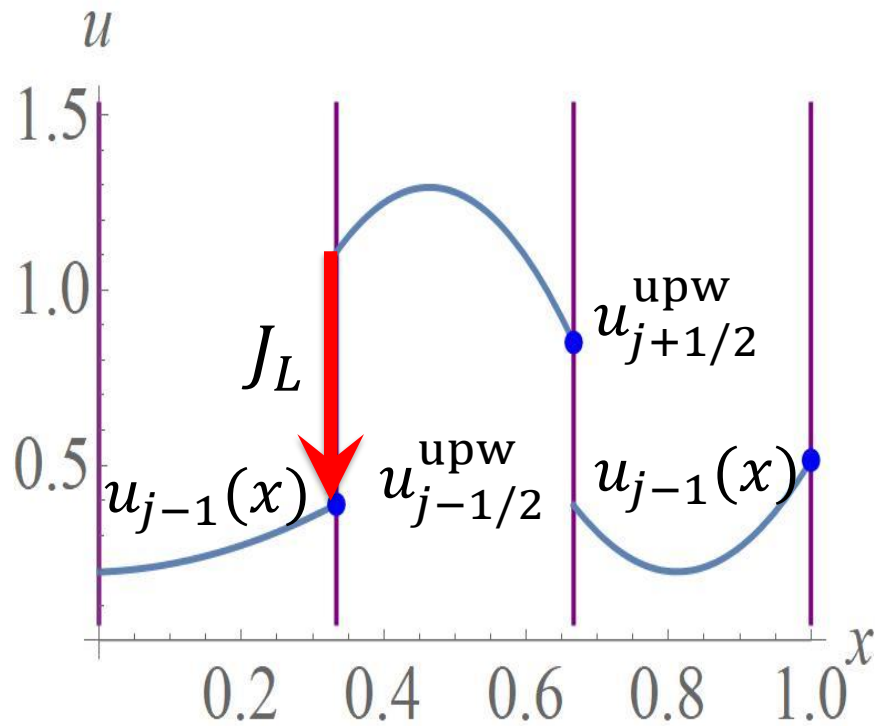


For DG2, the polynomial  $u_j(x)$  are quadratics (degree  $p = 2$ ).  
 $U_j(x)$  are cubics (degree  $p + 1$ ) defined by  $p + 2$  conditions:  
 $U_j(x_{j-1/2}) = u_{j-1/2}^{\text{upw}}$ , and  $U_j$  interpolates  $u_j$  at the  $p + 1 = 3$   
 right Radau points. (Thus,  $U_j(x_{j+1/2}) = u_{j+1/2}^{\text{upw}}$  ).

# Right Radau Polynomials



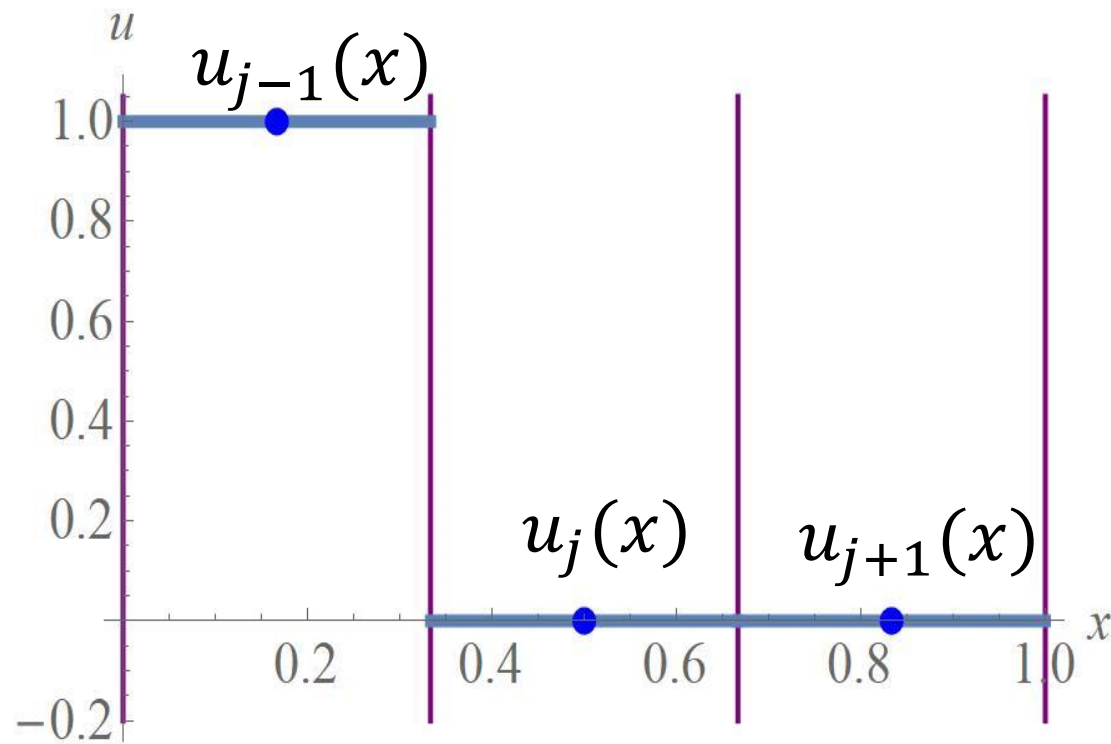
# DG via FR framework



$$U_j - u_j = \left[ u_{j-1/2}^{\text{upw}} - u_j(x_{j-1/2}) \right] R_{R,p+1}$$

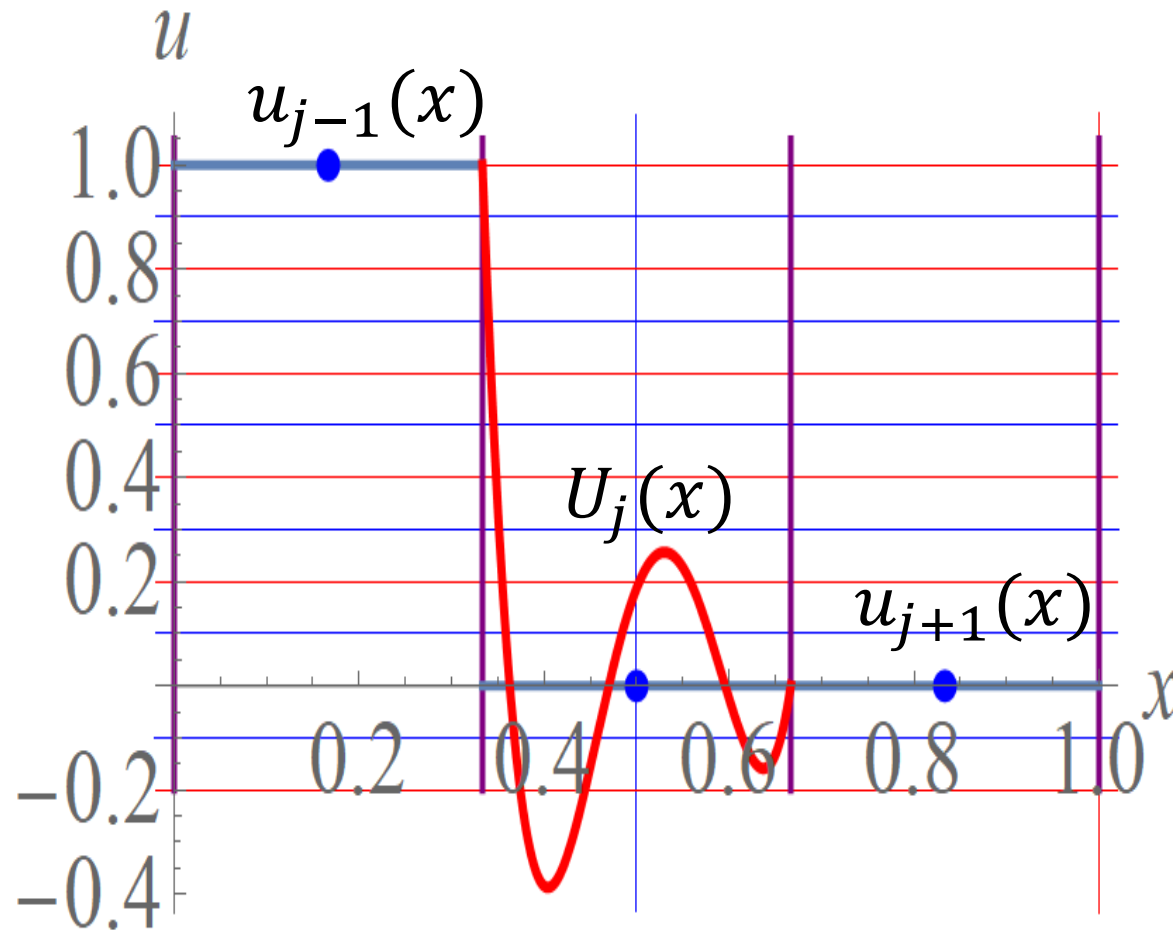
$$U_j = u_j + J_L R_{R,p+1}$$

# Behavior of DG Method at a Discontinuity



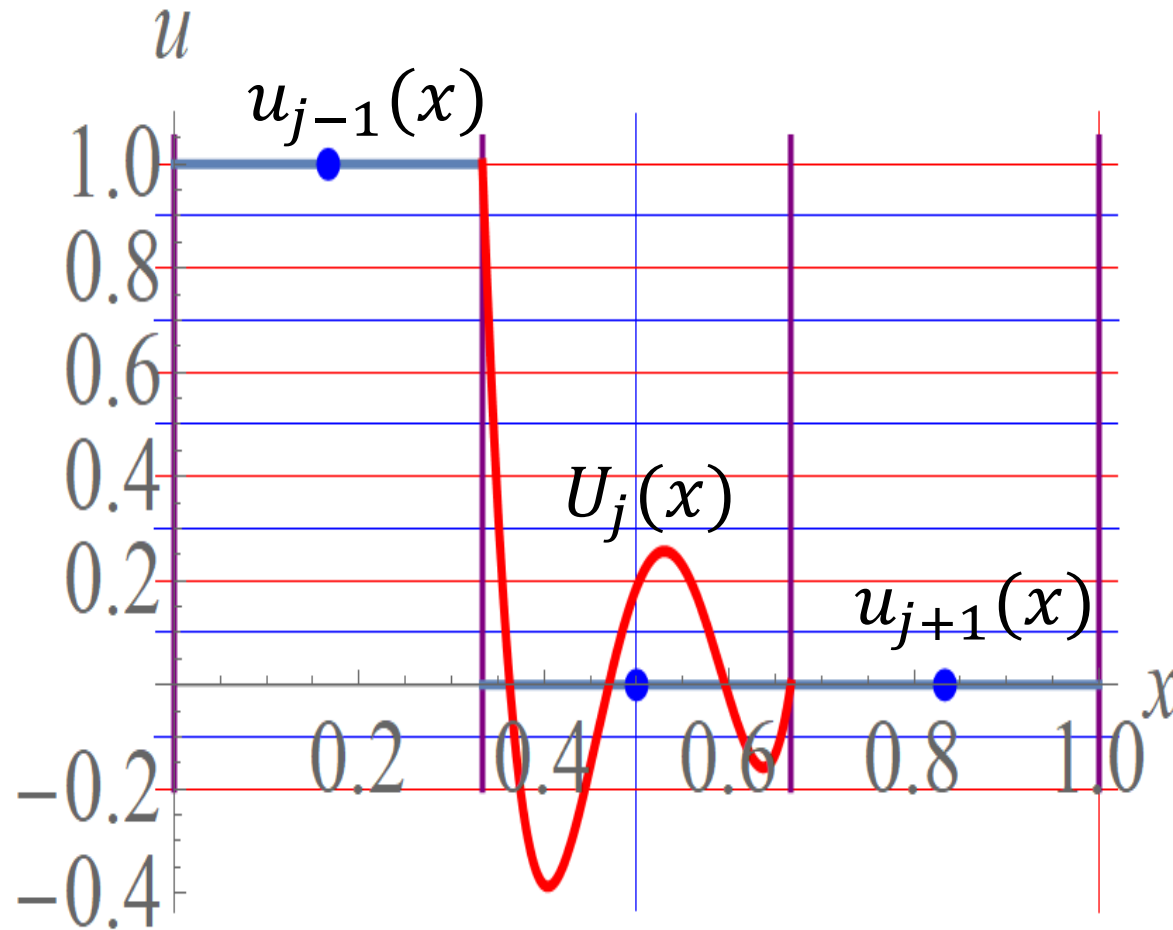
Solve  $u_t + a u_x = 0$ .

# Behavior of **DG3** Method at a Discontinuity



Solve  $u_t + a u_x = 0$ . Set  $U_j = R_{R,4}$ .

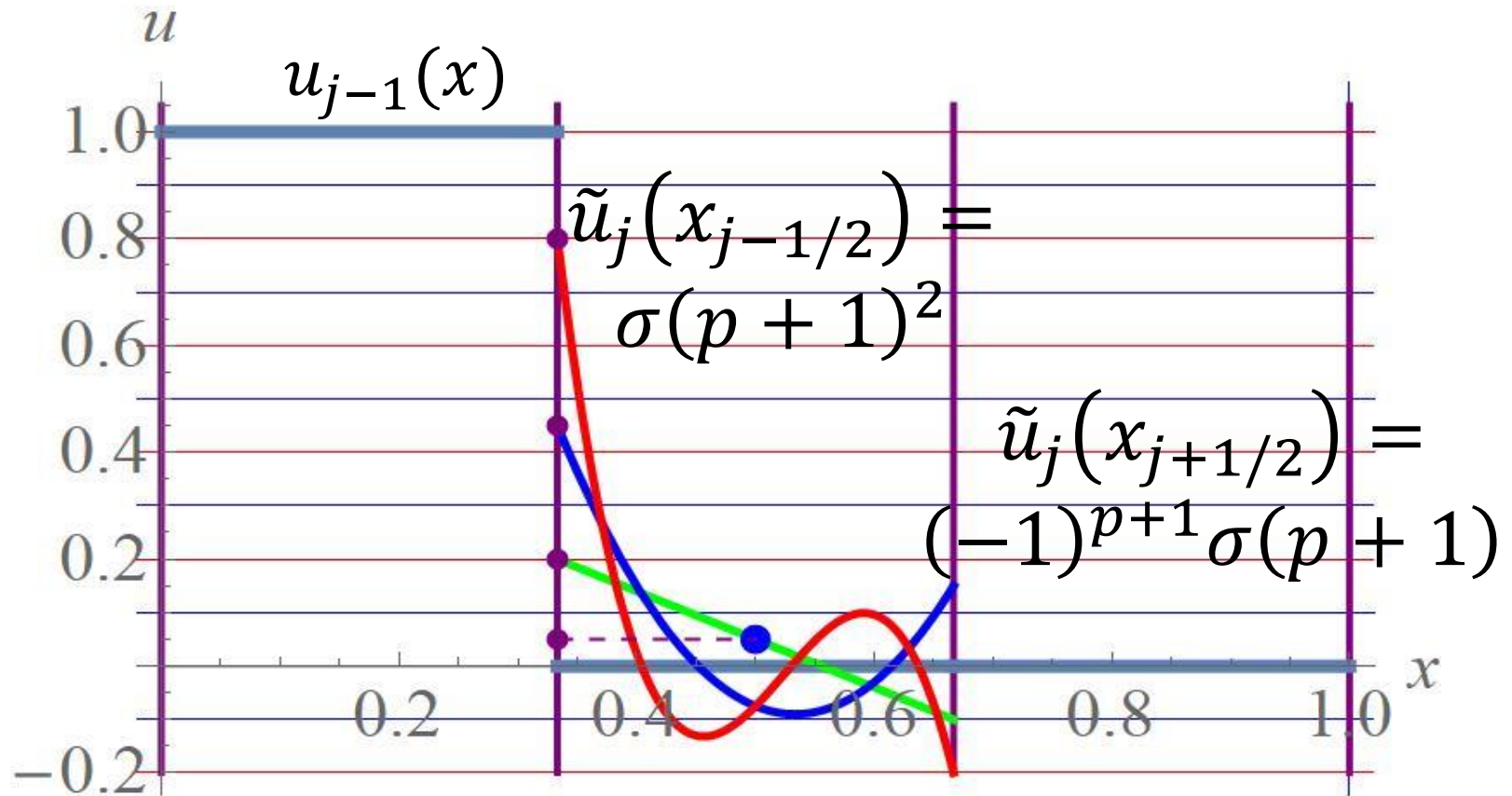
# Behavior of DG3 Method at a Discontinuity



$$u_t + a u_x = 0.$$

$$\text{Solution } \tilde{u}_j = -2\sigma(R_{R,4})'$$

# Behavior of DG Method at a Discontinuity



$$\tilde{u}_j = -\sigma \{ -L_0 + 3L_1 - 5L_2 \dots + (-1)^{k+1} (2k+1)L_k \}$$

# Limiting

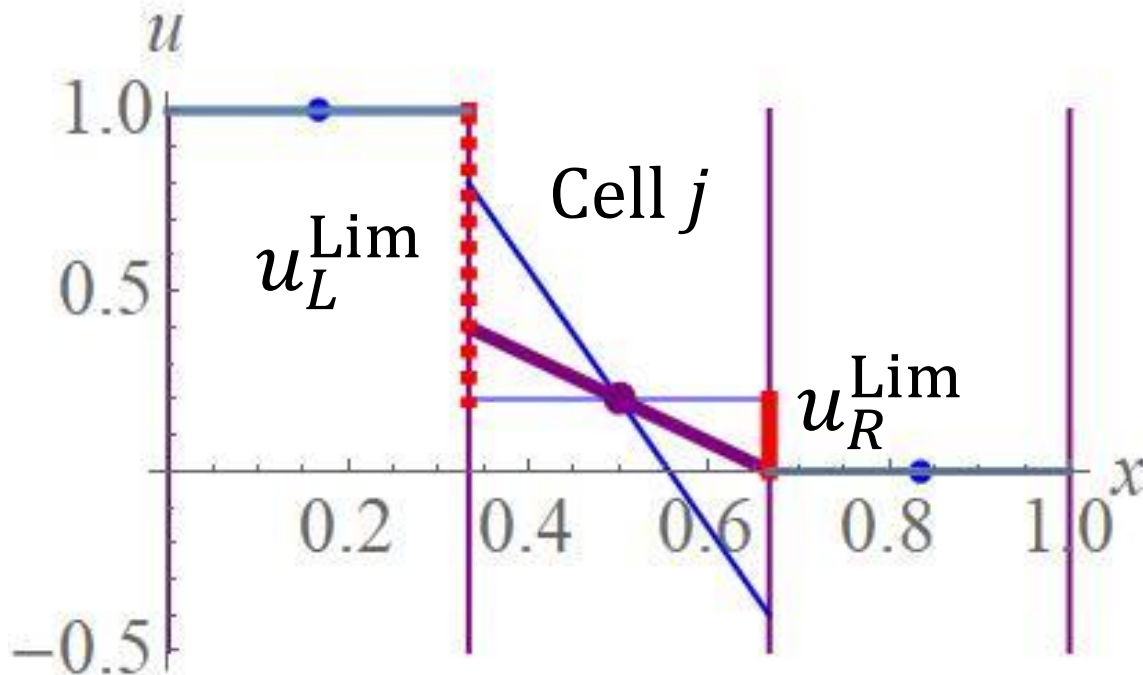
- The solution after an Euler forward time step (i.e., one stage of RK) is considered the data  $u_{j,k}$  (might be oscillatory).
- We wish to impose constraints and reduce  $|u_{j,k}|$ ,  $k \geq 1$ , to suppress oscillations (the cell average quantities  $u_{j,0}$  should not be altered)

# Limiting

For cell  $j$ , using the neighboring cell averages, set

$$u_L^{\text{Lim}} = |u_{j-1,0} - u_{j,0}|, \quad u_R^{\text{Lim}} = |u_{j+1,0} - u_{j,0}|, \\ u^{\text{Lim}} = \min(u_L^{\text{Lim}}, u_R^{\text{Lim}}).$$

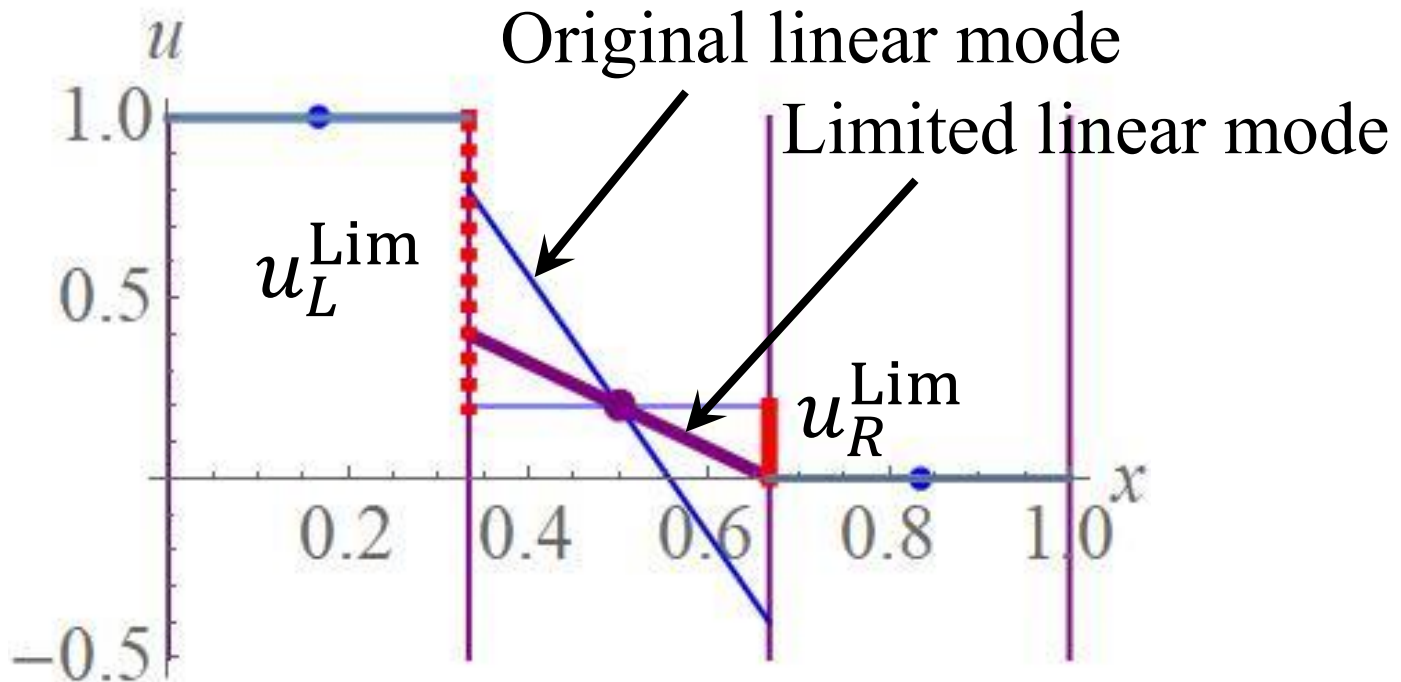
Constraint for linear mode:  $u_{j,1} \leq u^{\text{Lim}}$



# Limiting

- With  $u_1^{\text{Lim}} = u^{\text{Lim}} = \min(u_L^{\text{Lim}}, u_R^{\text{Lim}})$ , we require
$$|u_{j,1}| \leq u_1^{\text{Lim}}.$$

Set  $u_{j,1}^{\text{Ltd}} = \text{sign}(u_{j,1}) \min(|u_{j,1}|, u_1^{\text{Lim}})$ .



# Limiting for All Modes

- For all modes, we require

$$\sum_{k=1}^p |u_{j,k}| \leq u^{\text{Lim}} = u_1^{\text{Lim}}.$$

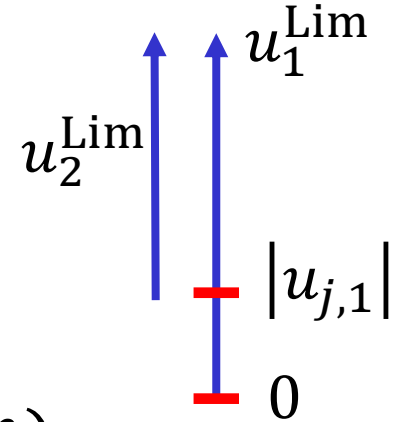
$$u_{j,1}^{\text{Ltd}} = \text{sign}(u_{j,1}) \min(|u_{j,1}|, u_1^{\text{Lim}}).$$

For  $k \geq 2$ , set

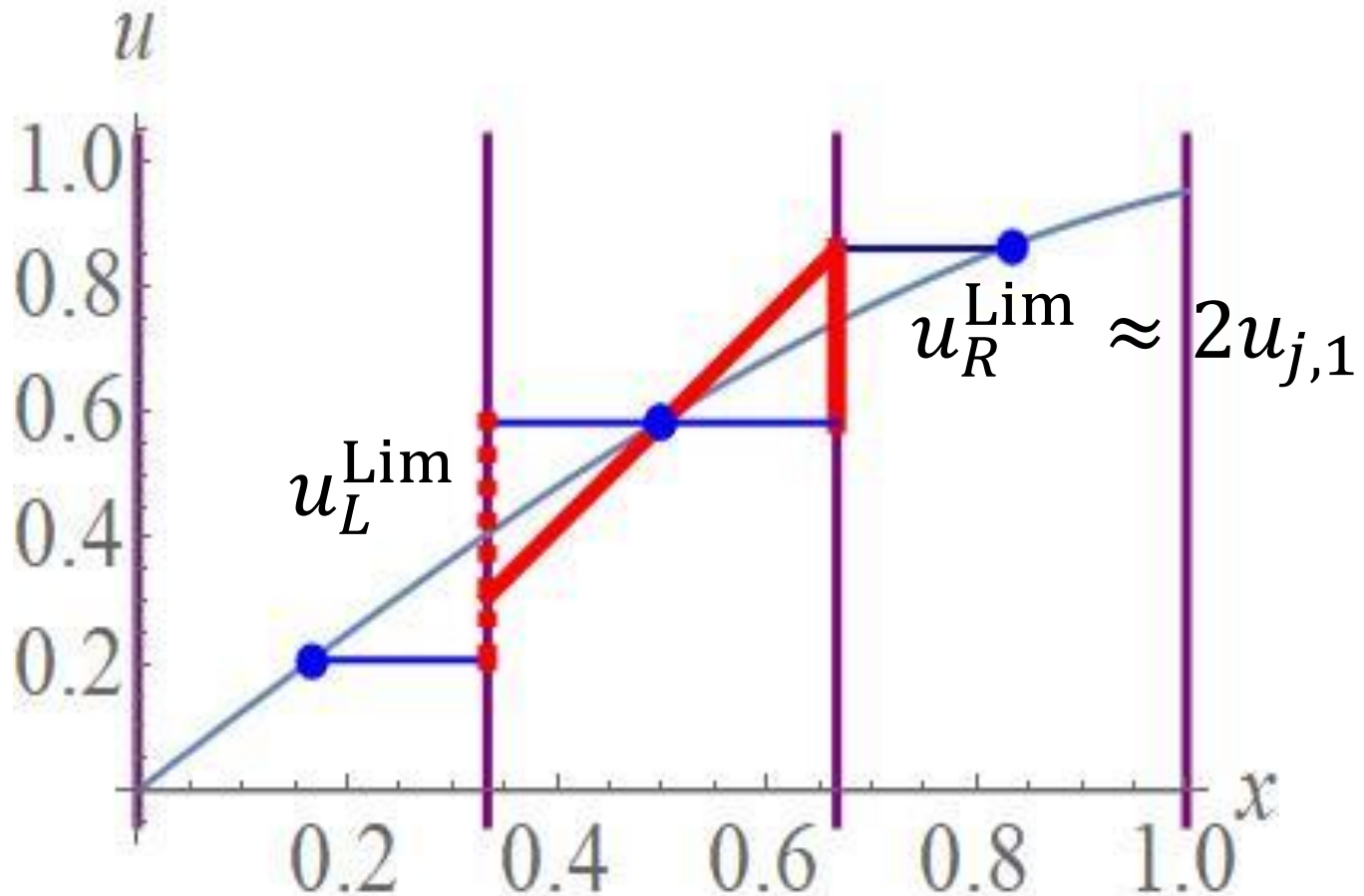
$$u_k^{\text{Lim}} = u_{k-1}^{\text{Lim}} - |u_{j,k-1}^{\text{Ltd}}|$$

And

$$u_{j,k}^{\text{Ltd}} = \text{sign}(u_{j,k}) \min(|u_{j,k}|, u_k^{\text{Lim}}).$$

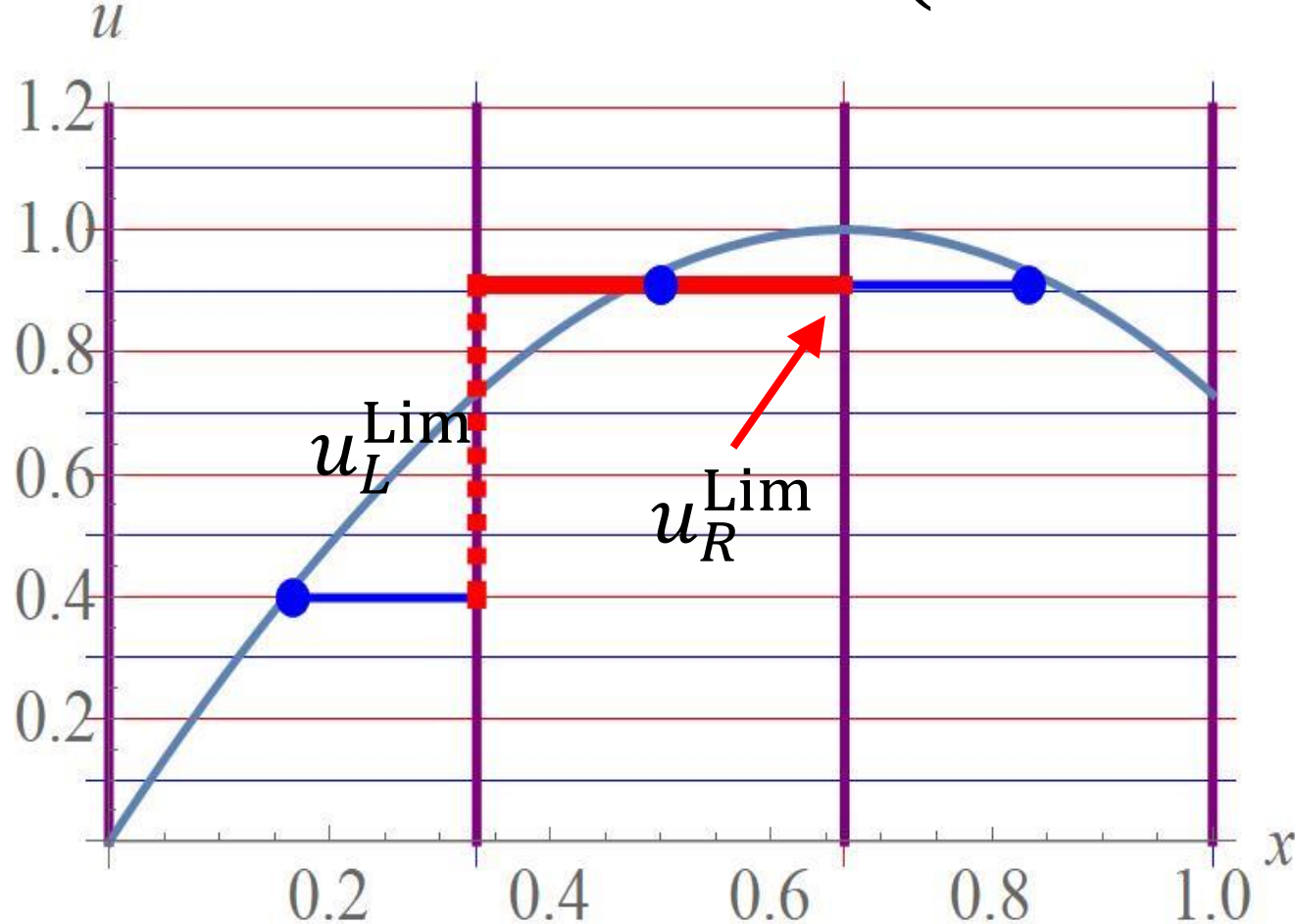


# At a Smooth Region with Nonzero Slope



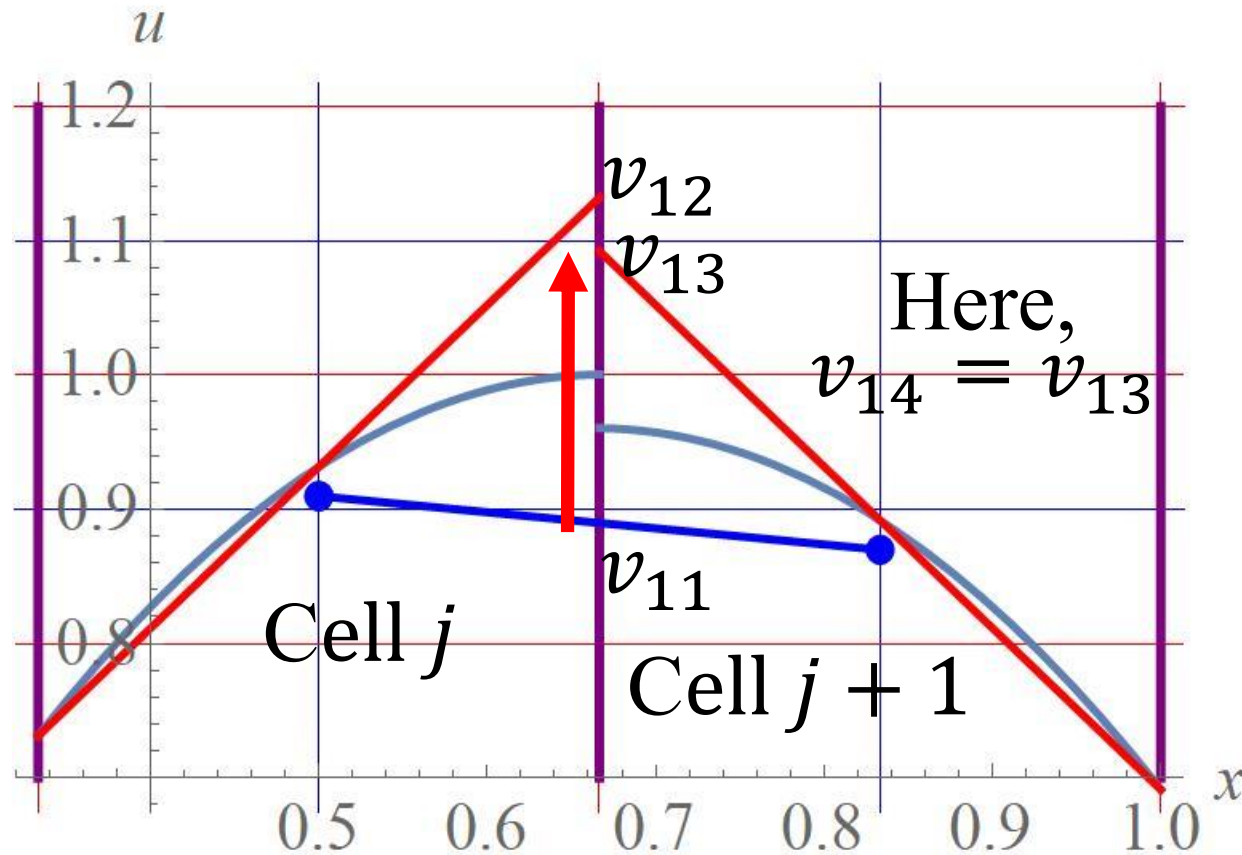
$u^{\text{Lim}}$  provides lots of room, and leave the data alone,  
i.e., limiting has no effect

# At a Smooth Extremum (Zero Slope)



$u_R^{\text{Lim}}$  provides no room, and limiting flattens the data

# At a Smooth Extremum (Slope near Zero)



$$v_{11} = 0.5(u_{j,0} + u_{j+1,0}),$$

$$v_{12} = u_{j,0} + u_{j,1} - 2u_{j,2}$$

$$v_{13} = u_{j+1,0} - u_{j+1,1} - 2u_{j+1,2}$$

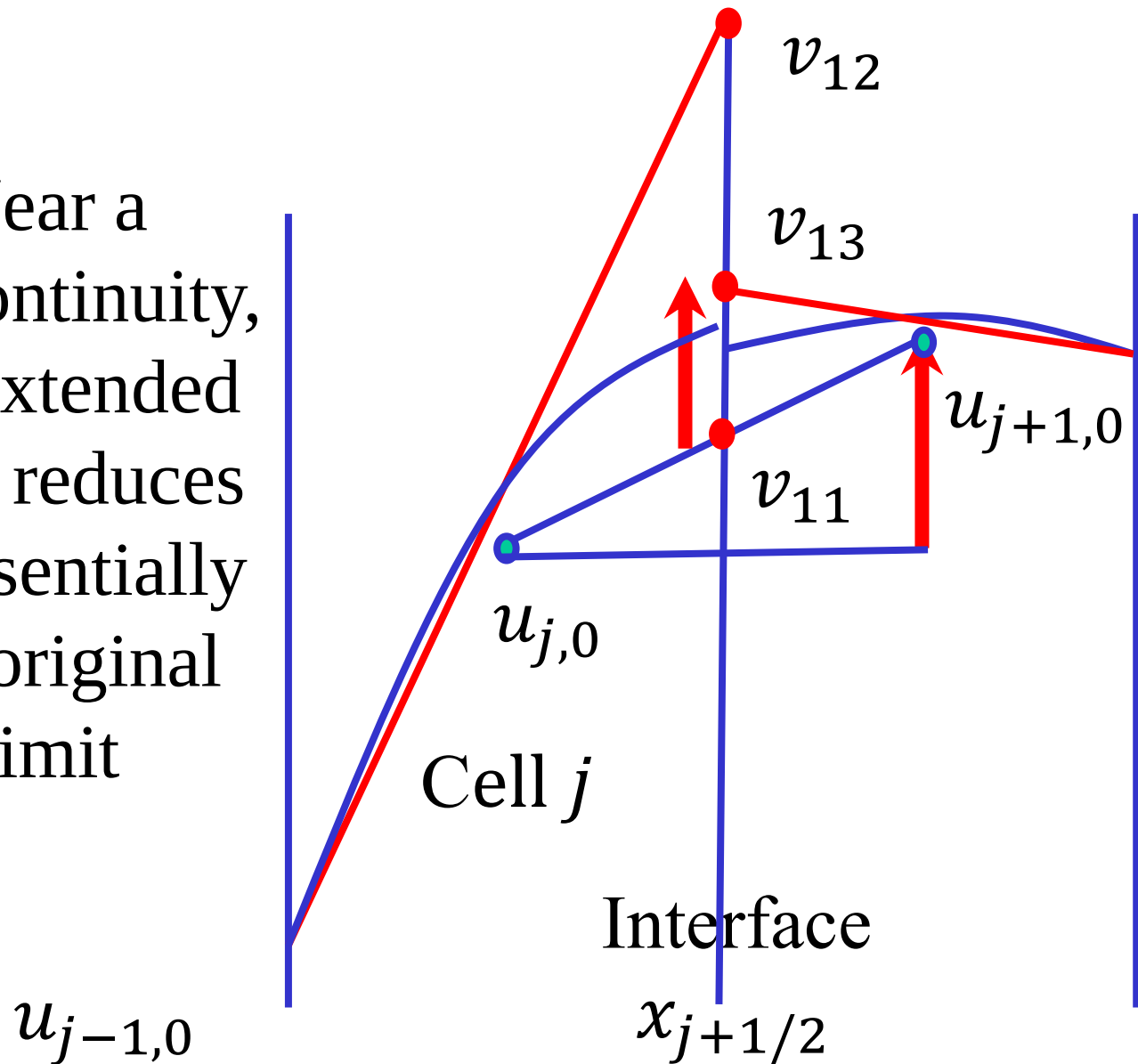
$$v_{14} = \text{median}(v_{11}, v_{12}, v_{13})$$

At a Smooth Extremum (Slope near Zero)

$$u_R^{\text{XLim}} = \text{Max}(|u_{j+1,0} - u_{j,0}|, |v_{14} - u_{j,0}|)$$

# Near a Discontinuity

Near a Discontinuity, the extended limit reduces to essentially the original limit



# Limiting Algorithm

$$\left\{ \begin{array}{l} v_{11} = \frac{1}{2} (u_{j,0} + u_{j+1,0}); \quad v_{12} = u_{j,0} + u_{j,1} - 2u_{j,2}; \\ v_{13} = u_{j+1,0} - u_{j+1,1} - 2u_{j+1,2}; \quad v_{14} = \text{median}(v_{11}, v_{12}, v_{13}); \\ u_R^{\text{XLim}} = \text{Max}(|u_{j+1,0} - u_{j,0}|, |v_{14} - u_{j,0}|); \end{array} \right.$$

$$\left\{ \begin{array}{l} v_{21} = \frac{1}{2} (u_{j,0} + u_{j-1,0}); \quad v_{22} = u_{j,0} - u_{j,1} - 2u_{j,2}; \\ v_{23} = u_{j-1,0} + u_{j-1,1} - 2u_{j+1,2}; \quad v_{24} = \text{median}(v_{21}, v_{22}, v_{23}); \\ u_L^{\text{XLim}} = \text{Max}(|u_{j-1,0} - u_{j,0}|, |v_{24} - u_{j,0}|); \end{array} \right.$$

$$u_1^{\text{Lim}} = u^{\text{XLim}} = \min(u_L^{\text{XLim}}, u_R^{\text{XLim}});$$

$$\left\{ \begin{array}{l} \text{For } k = 1, \dots, p, \quad \text{set } u_{j,k}^{\text{Ltd}} = \text{sign}(u_{j,k}) \min(|u_{j,k}|, u_k^{\text{Lim}}). \\ u_{k+1}^{\text{Lim}} = u_k^{\text{Lim}} - |u_{j,k}^{\text{Ltd}}| \end{array} \right.$$

# Detecting “Good Cells”

- **Gap Criteria**

Define quadratic content in cell  $j$ ,  $c_j^Q = |u_{j,1}| + |u_{j,2}|$

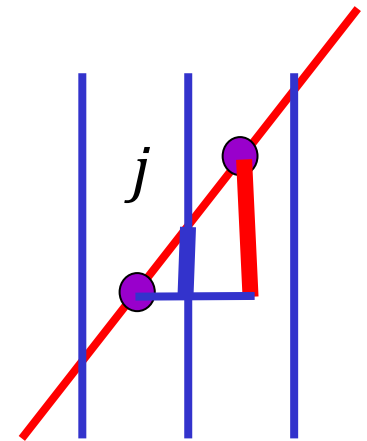
$$\text{Max} \left[ |u_{j-1}(1) - u_j(-1)|, |u_{j+1}(-1) - u_j(1)| \right] \leq \frac{1}{5} c_j^Q + 10^{-3}.$$

- **Curvature Criteria (Second Mode)**

$$\frac{4}{5} \leq \frac{u_{j-1,2}}{u_{j,2}} \leq \frac{5}{4} \quad \text{and} \quad \frac{4}{5} \leq \frac{u_{j+1,2}}{u_{j,2}} \leq \frac{5}{4}.$$

- **Slope Criteria** ( $u_{j,1} \approx \frac{1}{2} u^{\text{Lim}}$ )

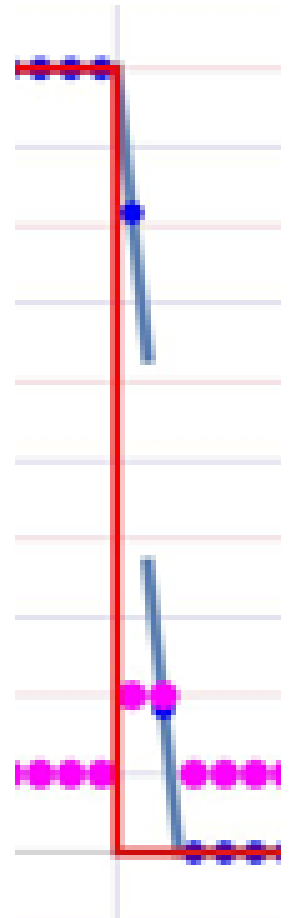
$$|u_{j,1}| \leq \frac{3}{4} u^{\text{Lim}} + 10^{-4}.$$



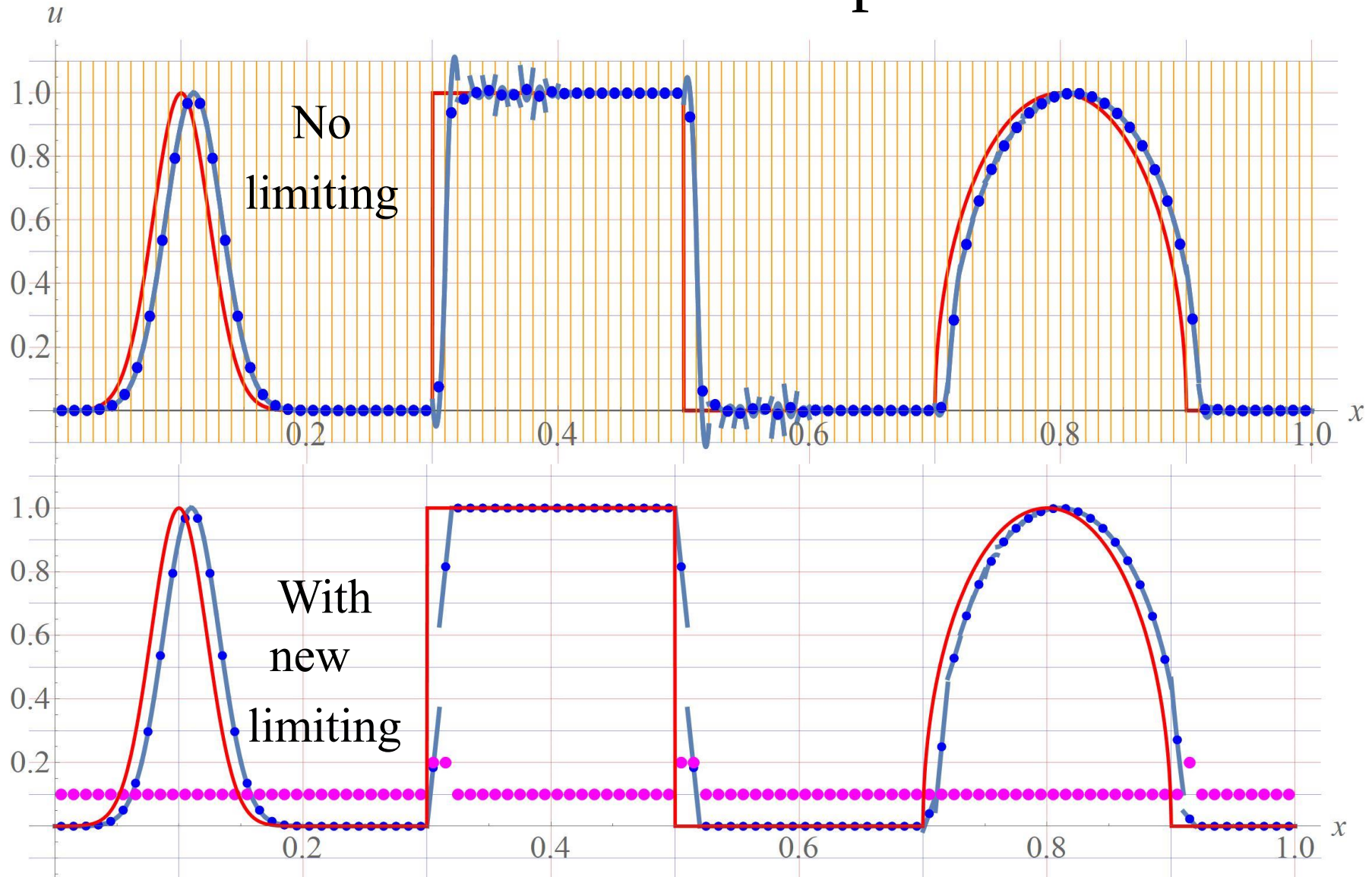
# Numerical Examples

In all numerical examples, the dots represent the cell average values, and the curves represent the polynomial solution.

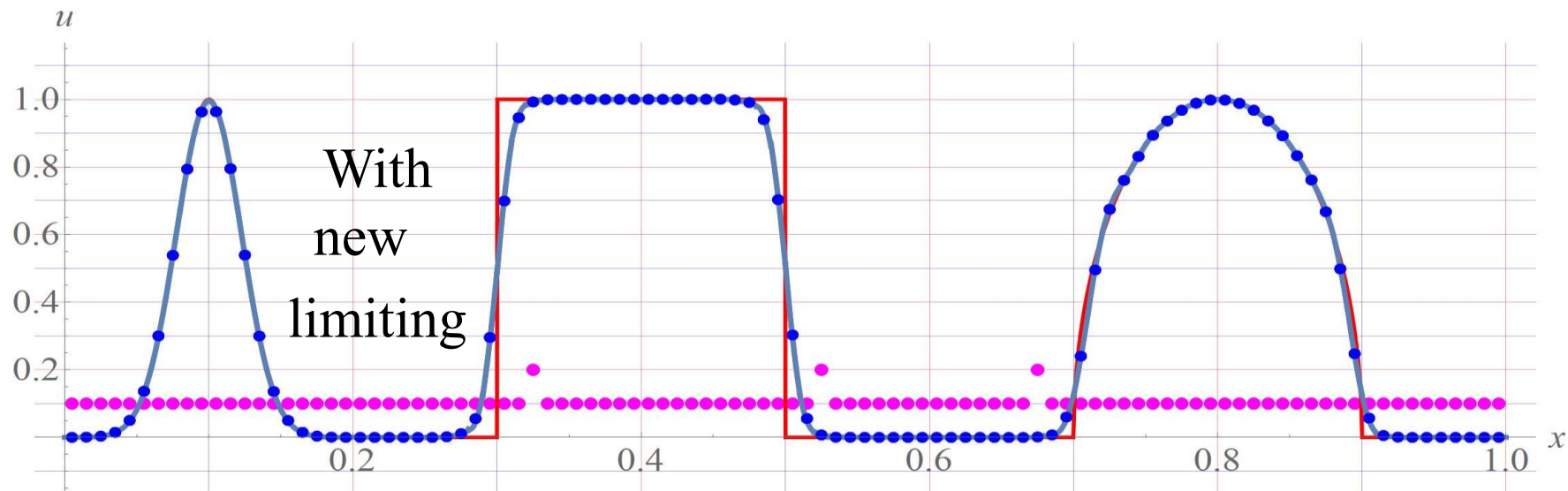
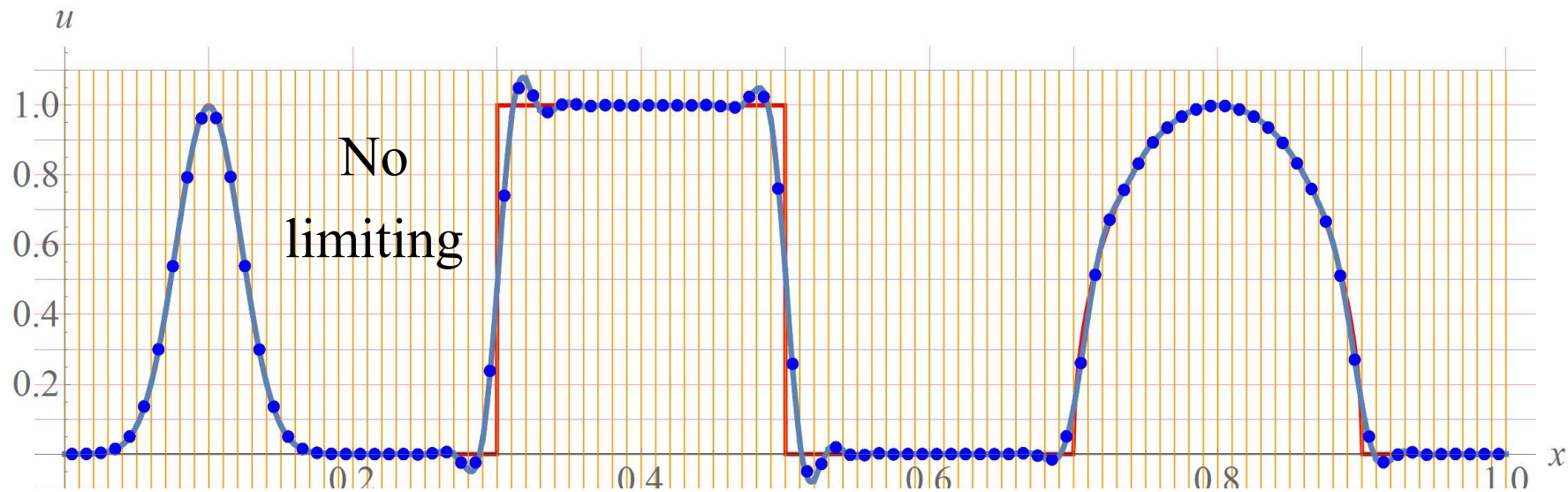
The magenta dots represent either “good cells” where no limiting is required, or “troubled cells” where limiting is needed (when they are bumped up).



# Numerical Examples

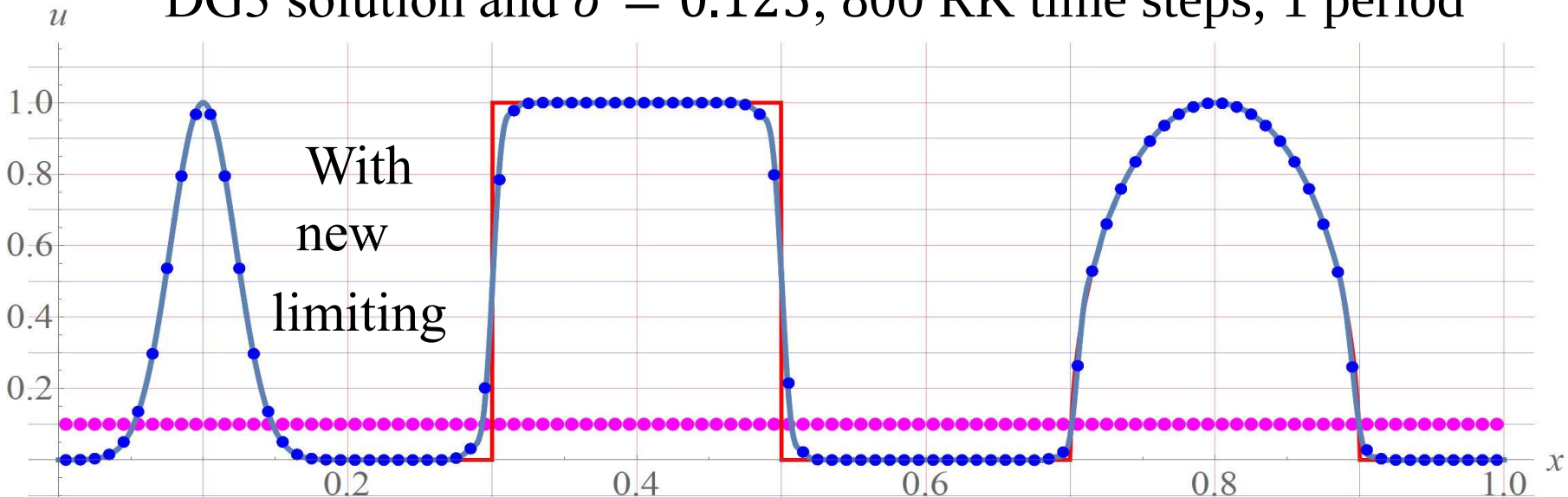
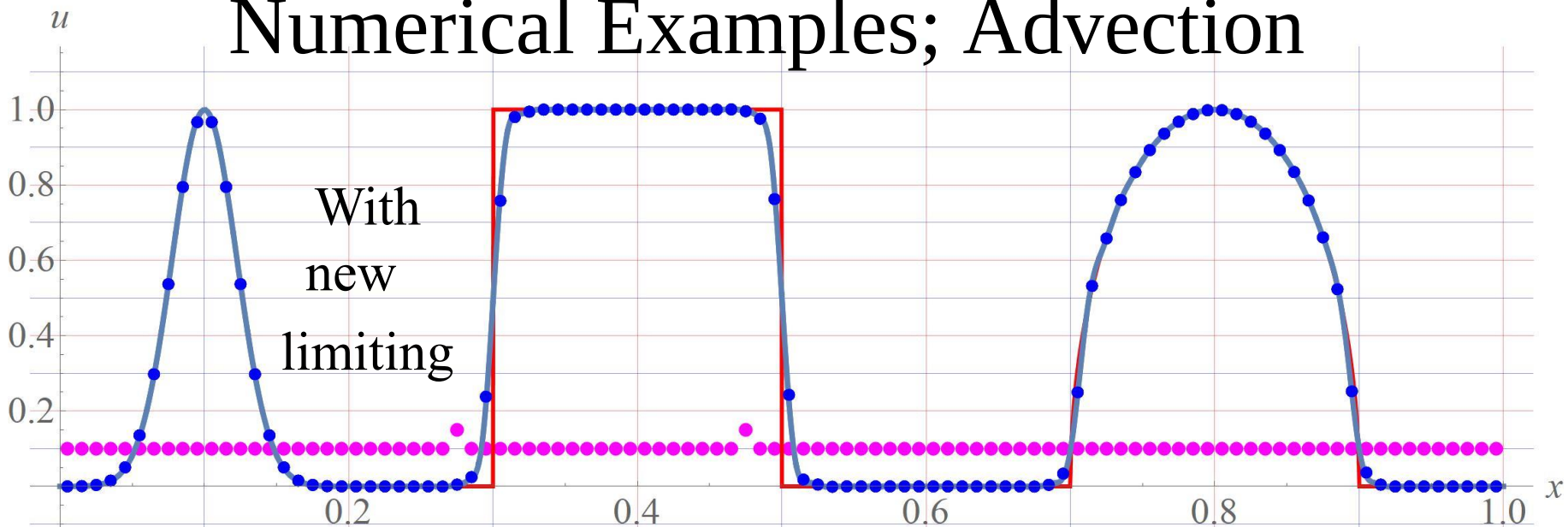


# Numerical Examples



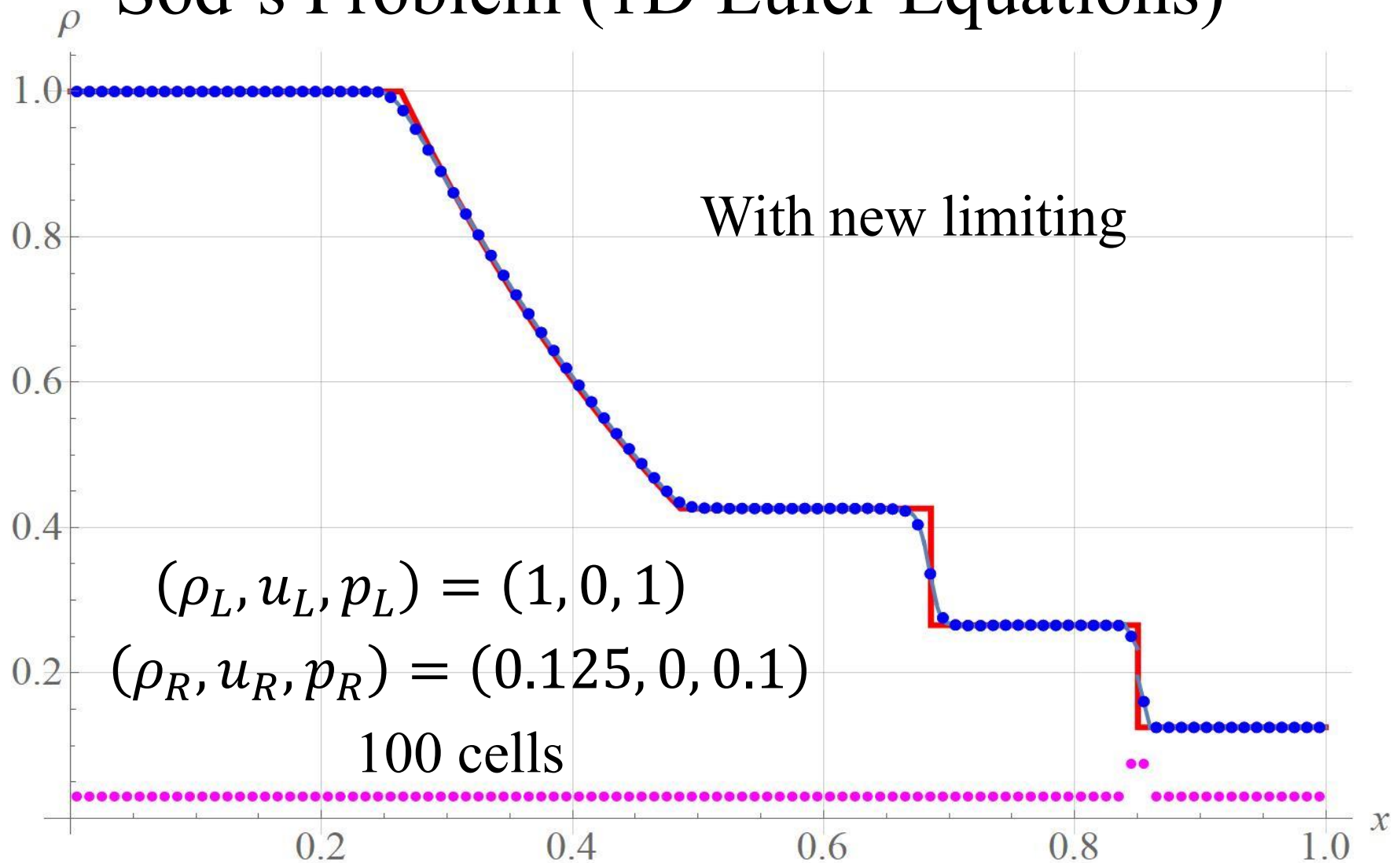
DG2 solution,  $\sigma = 0.2$ , 500 RK time steps, 1 period

# Numerical Examples; Advection



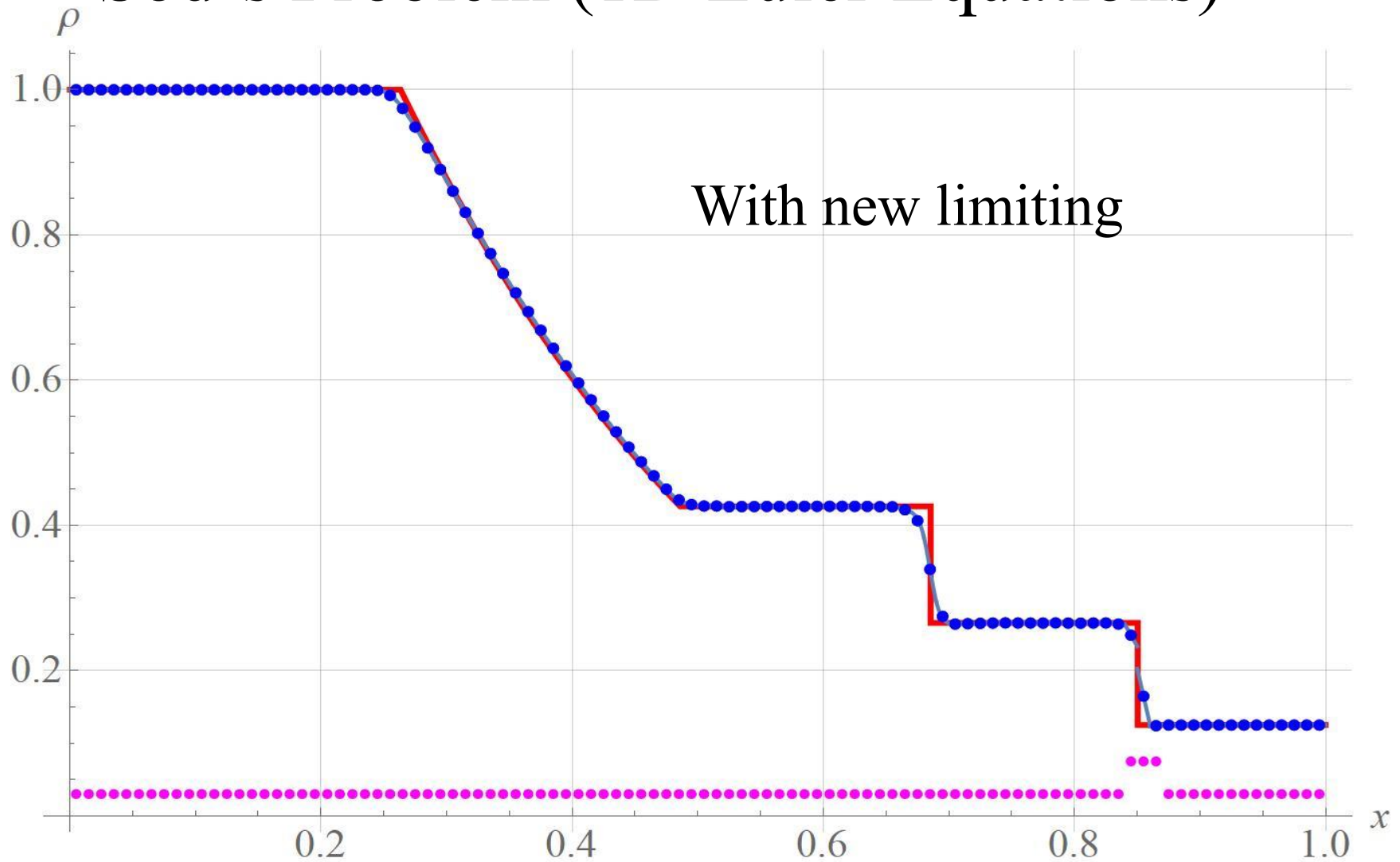
DG4 solution and  $\sigma = 1/12$ ; 1200 RK time steps; 1 period

# Sod's Problem (1D Euler Equations)



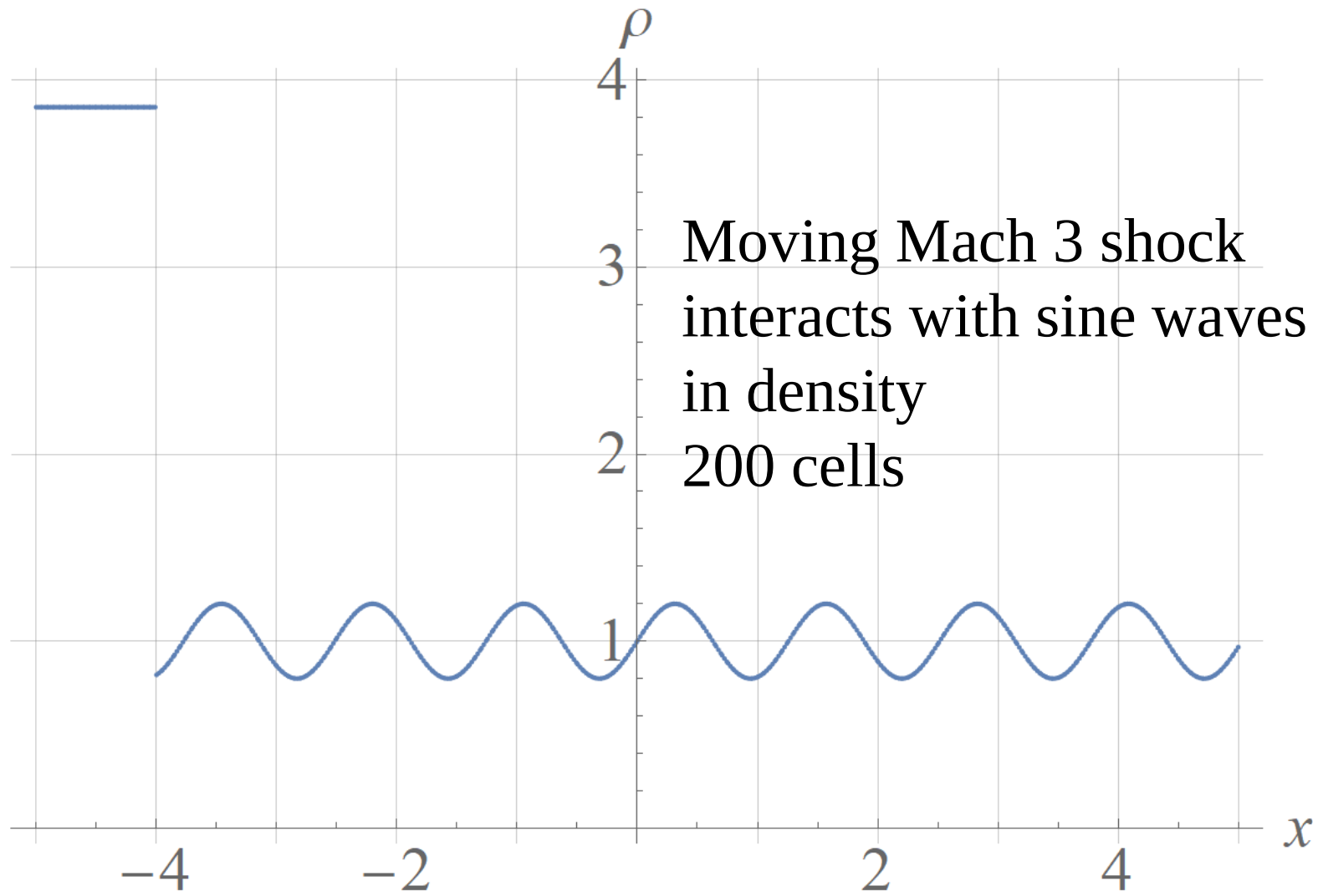
DG2 with  $\Delta t = 0.001$ , 200 time steps, final time 0.2

# Sod's Problem (1D Euler Equations)

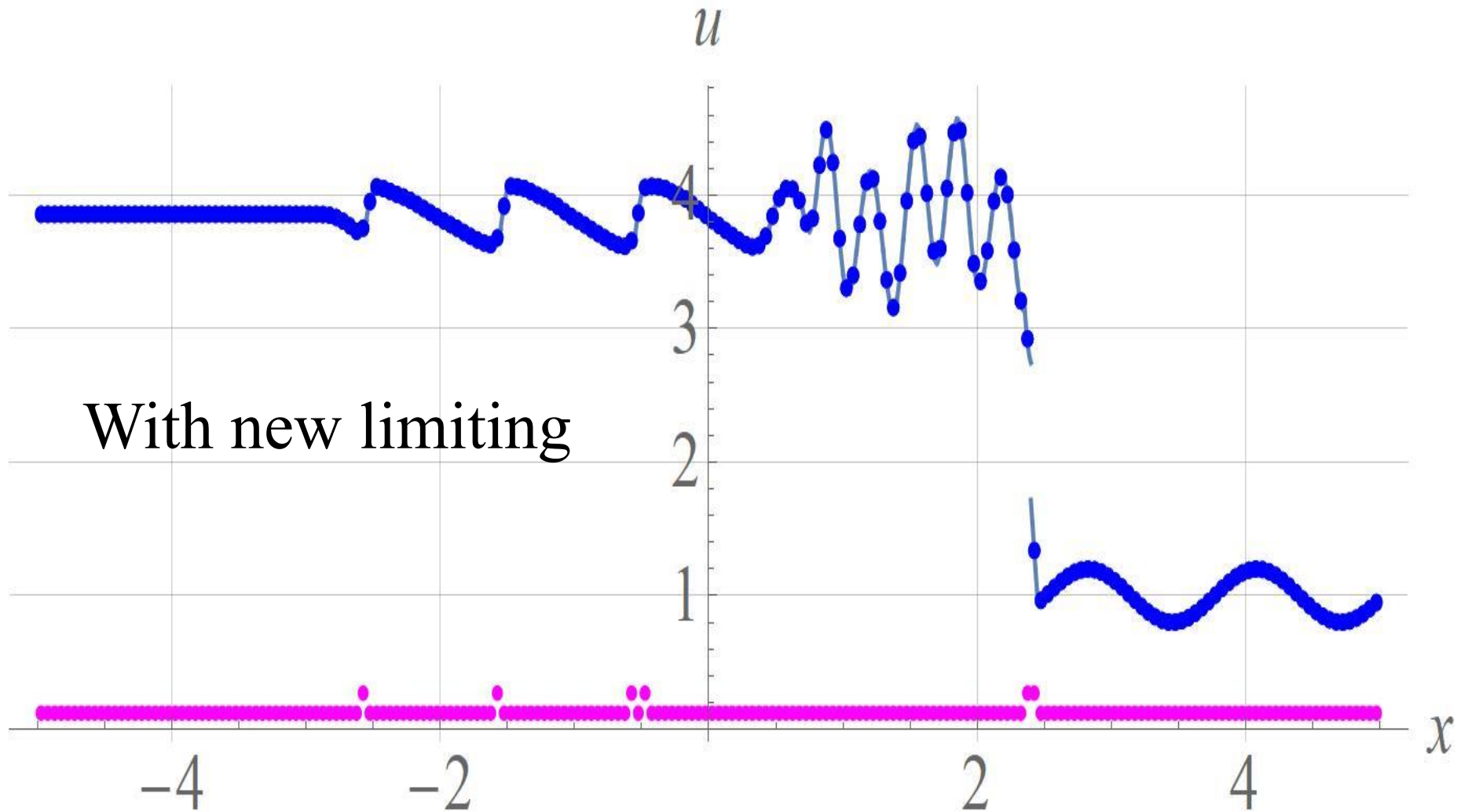


DG3 solution,  $\Delta t = 0.000625$ , 320 time steps

# Shu's Problem (1D Euler Equations)

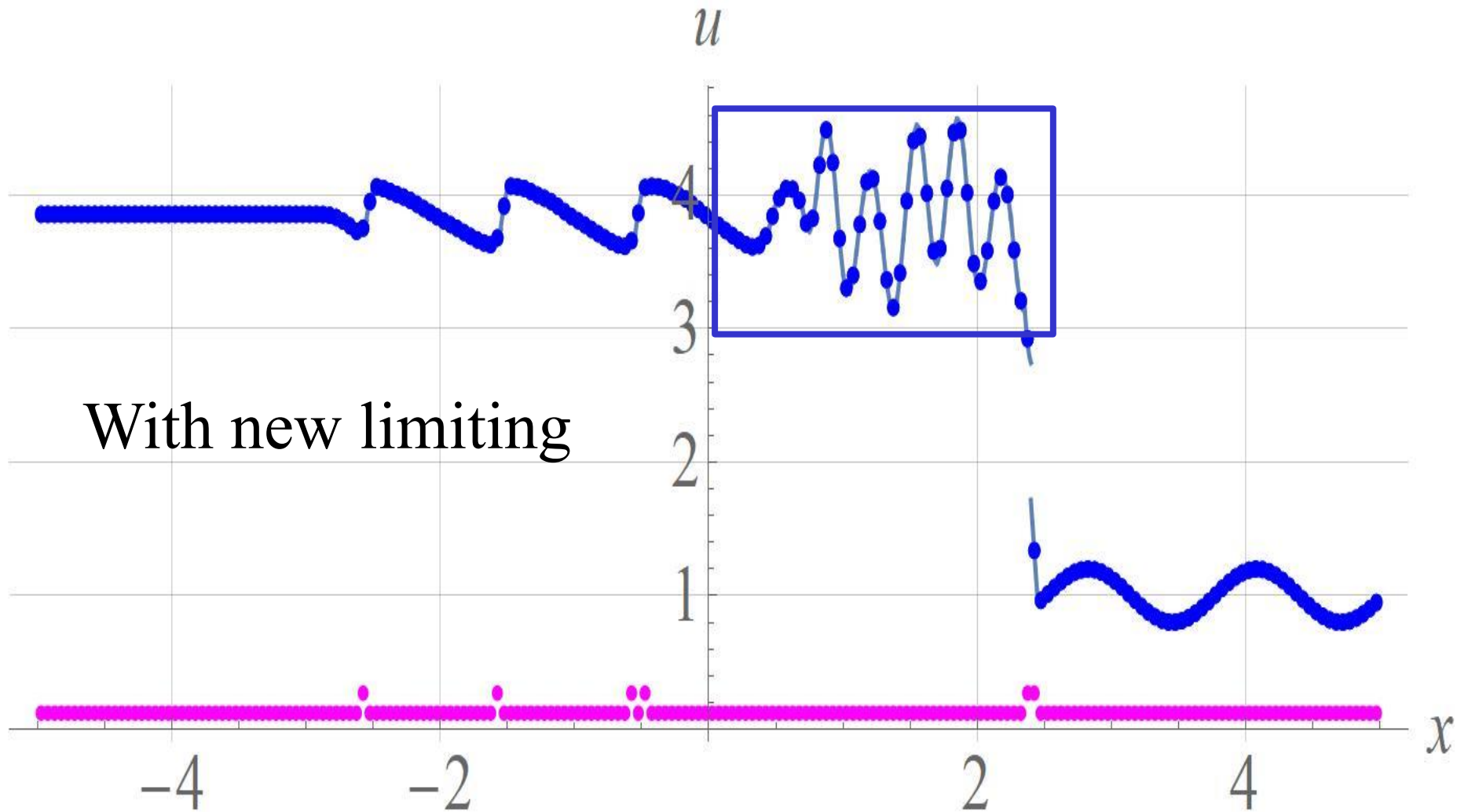


# Shu's Problem (1D Euler Equations)



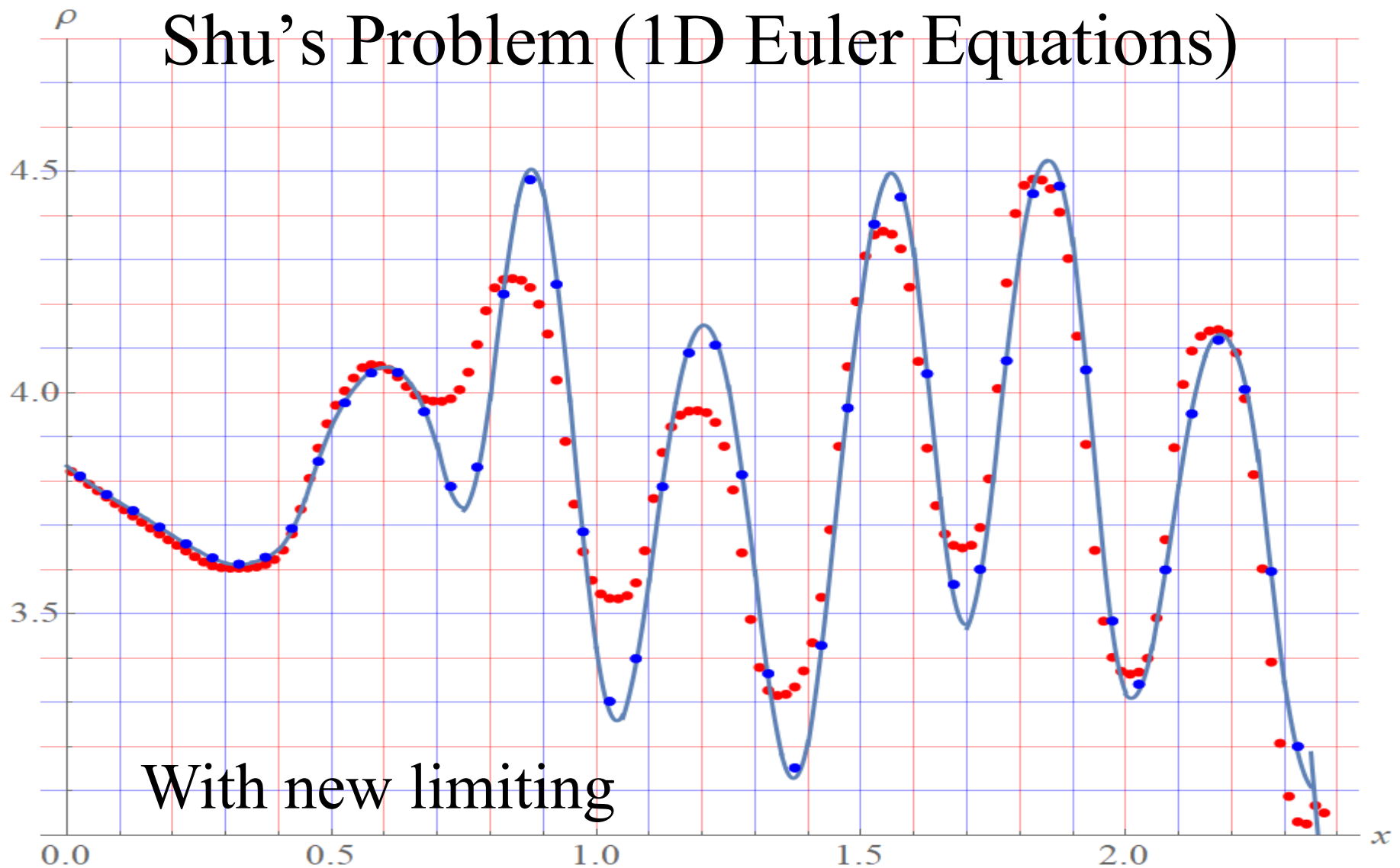
DG2 after 860 time steps, each of size 0.002093

# Shu's Problem (1D Euler Equations)



DG2 after 860 time steps, each of size 0.002093

# Shu's Problem (1D Euler Equations)



With new limiting

Close-up of DG2 solution. Red dots: Standard  
Second-order Shock-capturing solution



# Summary

- Discussed oscillations caused by a discontinuity using derivative of Radau polynomial
- Presented (a) a shock capturing method that suppresses oscillations near shocks and maintains accuracy at smooth regions and (b) a detection technique
- Numerical examples demonstrating the effectiveness of the new method

## Future Research

- Additional 2D and 3D tests
- Extension to unstructured (triangular) mesh



Thank you