

Hamiltonian control for coupled bending-torsion motion of a swept wing with R. T. Jones' approximation

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This paper presents distributed Hamiltonian control design approach for a Lagrangian infinite-dimensional system describing the coupled bending-torsion motion of a swept wing in unsteady airflow, which is taken into account using Theodorsen's complex valued function of reduced frequency. An implementable flutter suppressing control surface deflection commands are obtained using modes separation and R.T. Jones's approximation, and were validated in desktop simulations for flight conditions beyond the flutter airspeed.

I. Equation of Coupled Bending-Torsion Motion

A coupled bending-torsion motion of a swept wing in an unsteady airflow can be described as [1]

$$\begin{aligned} \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + \rho A e_{cg} \frac{\partial^2 \theta(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial w(x,t)}{\partial x} \right) &= \cos \Lambda \left(\bar{l}(x,t) + \sin \Lambda \frac{\partial \bar{m}(x,t)}{\partial x} \right) + l_\delta(x) \delta(t) + l_g(x,t) \\ \rho I_{xx} \frac{\partial^2 \theta(x,t)}{\partial t^2} + \rho A e_{cg} \frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left(GJ \frac{\partial \theta(x,t)}{\partial x} \right) &= \cos^2 \Lambda \bar{m}(x,t) + m_\delta(x) \delta(t) + m_g(x,t), \end{aligned} \quad (1)$$

where $w(x,t)$ is the bending displacement and $\theta(x,t)$ is the torsional rotation, $\rho A(x)$ is the wing running mass, $\rho I_{xx}(x)$ is the wing polar inertia, $e_{cg}(x)$ is the offset of the center of gravity of the wing from the elastic axis, $EI(x)$ is the bending stiffness, $GJ(x)$ is the torsional stiffness, $l(x,t)$ and $m(x,t)$ are respectively the unsteady aerodynamic lift force and pitching moment, expressed as

$$\begin{aligned} \bar{l}(x,t) &= C(k) c_{L_\alpha} q_\infty c(x) \left(\theta(x,t) \cos \Lambda - \sin \Lambda \frac{\partial w(x,t)}{\partial x} + \frac{e_c(x) \cos \Lambda}{V_\infty} \frac{\partial \theta(x,t)}{\partial t} - \frac{1}{V_\infty} \frac{\partial w(x,t)}{\partial t} \right) \\ &+ \frac{\pi \rho_\infty c^2(x)}{4} \left(V_\infty \cos \Lambda \frac{\partial \theta(x,t)}{\partial t} + e_m(x) \frac{\partial^2 \theta(x,t)}{\partial t^2} \cos \Lambda - \frac{\partial^2 w(x,t)}{\partial t^2} \right) \\ \bar{m}(x,t) &= C(k) c_{L_\alpha} q_\infty c(x) e(x) \left(\theta(x,t) \cos \Lambda - \frac{\partial w(x,t)}{\partial x} \sin \Lambda + \frac{e_c(x)}{V_\infty} \frac{\partial \theta(x,t)}{\partial t} \cos \Lambda - \frac{1}{V_\infty} \frac{\partial w(x,t)}{\partial t} \right) \\ &- \frac{\pi \rho_\infty c^2(x)}{4} \cos \Lambda \left(\left(e_m^2(x) + \frac{c^2(x)}{32} \right) \frac{\partial^2 \theta(x,t)}{\partial t^2} - \frac{e_m(x)}{\cos \Lambda} \frac{\partial^2 w(x,t)}{\partial t^2} + V_\infty e_c(x) \frac{\partial \theta(x,t)}{\partial t} \right), \end{aligned} \quad (2)$$

c_{L_α} is given as

$$c_{L_\alpha} = \frac{2\pi \cos \Lambda}{\sqrt{1 - M_\infty^2 \cos^2 \Lambda} + \frac{c(x)}{L}}, \quad (3)$$

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M_∞ is the Mach number and L is the wing span, $l_\delta(x)$ and $m_\delta(x)$ are respectively the unsteady control force and control moment sensitivities, given as

$$\begin{aligned} c_{L_\delta}(x) &= \frac{1}{\pi} c_{L_\alpha} \cos \Lambda \left(\arccos(1 - 2c_f) + 2\sqrt{c_f(1 - c_f)} \right) \\ c_{m_\delta}(x) &= -2 \cos \Lambda (1 - c_f) \sqrt{c_f(1 - c_f)} \\ l_\delta(x) &= C(k) c_{L_\delta}(x) q_\infty c(x) \cos^2 \Lambda \\ m_\delta(x) &= q_\infty c^2(x) (c_{m_\delta}(x) + e(x) C(k) c_{L_\delta}(x)) \cos^3 \Lambda, \end{aligned} \quad (4)$$

$\delta(t)$ is the vector of distributed control surface deflections, c_f is the actuator's support, $l_g(x, t)$ is aerodynamic lift due to an atmospheric gust encounter, $m_g(x, t)$ is the aerodynamic pitching moment due to gust, Λ is the sweep angle of the elastic axis, $C(k)$ is the Theodorsen's complex valued function of reduced frequency $k = \frac{\omega c(x)}{2V_\infty}$ [2], ρ_∞ is the air density, V_∞ is the airspeed, q_∞ is the dynamic pressure, $c(x)$ is the wing chord, $e(x)$ is the offset of the quarter-chord point on the wing from the elastic axis, $e_m(x)$ is the offset of the mid-chord point from the elastic axis, and $e_c(x)$ is the offset of the three-quarter chord point from the elastic axis.

The aerodynamic lift and pitching moment due to wind gust are modeled as

$$\begin{aligned} l_g(x, t) &= C(k) c_{L_\alpha} q_\infty c(x) \cos \Lambda \frac{\pi}{180} \sqrt{1 - \frac{x^2}{L^2}} \sin \omega_g t \\ m_g(x, t) &= C(k) c_{L_\alpha} q_\infty c(x) e(x) \cos^2 \Lambda \frac{\pi}{180} \sqrt{1 - \frac{x^2}{L^2}} \sin \omega_g t, \end{aligned} \quad (5)$$

where ω_g is the gust frequency.

The unsteady aerodynamic lift and pitching moment are expressed as

$$l(x, t) = \cos \Lambda \bar{l}(x, t) = l_\theta \theta + l_{\theta_t} \frac{\partial \theta}{\partial t} + l_{\theta_{tt}} \frac{\partial^2 \theta}{\partial t^2} + l_{w_x} \frac{\partial w}{\partial x} + l_{w_t} \frac{\partial w}{\partial t} + l_{w_{tt}} \frac{\partial^2 w}{\partial t^2} \quad (6)$$

$$m_x(x, t) = \Lambda \bar{m}(x, t) \cos^2 = m_\theta \theta + m_{\theta_t} \frac{\partial \theta}{\partial t} + m_{\theta_{tt}} \frac{\partial^2 \theta}{\partial t^2} + m_{w_x} \frac{\partial w}{\partial x} + m_{w_t} \frac{\partial w}{\partial t} + m_{w_{tt}} \frac{\partial^2 w}{\partial t^2}, \quad (7)$$

where

$$\begin{aligned} l_\theta &= C(k) c_{L_\alpha} q_\infty c \cos^2 \Lambda \\ l_{\theta_t} &= \frac{\pi \rho_\infty c^2}{4} V_\infty \cos^2 \Lambda \left[\frac{2e_c C(k) c_{L_\alpha}}{\pi c} + 1 \right] \\ l_{\theta_{tt}} &= \frac{\pi \rho_\infty c^2}{4} e_m \cos^2 \Lambda \\ l_{w_x} &= -C(k) c_{L_\alpha} q_\infty c \sin \Lambda \cos \Lambda \\ l_{w_t} &= -\frac{C(k) c_{L_\alpha} q_\infty c}{V_\infty} \cos \Lambda \\ l_{w_{tt}} &= -\frac{\pi \rho_\infty c^2}{4} \cos \Lambda \\ m_\theta &= C(k) c_{L_\alpha} q_\infty c e \cos^3 \Lambda \\ m_{\theta_t} &= \frac{\pi \rho_\infty c^2}{4} V_\infty e_c \cos^3 \Lambda \left[\frac{2e C(k) c_{L_\alpha}}{\pi c} - 1 \right] \\ m_{\theta_{tt}} &= -\frac{\pi \rho_\infty V_\infty c^2}{4} \left(e_m^2 + \frac{c^2}{32} \right) \cos^3 \Lambda \\ m_{w_x} &= -C(k) c_{L_\alpha} q_\infty c e \sin \Lambda \cos^2 \Lambda \\ m_{w_t} &= -\frac{C(k) c_{L_\alpha} q_\infty c e}{V_\infty} \cos^2 \Lambda \\ m_{w_{tt}} &= \frac{\pi \rho_\infty c^2 e_m}{4} \cos^2 \Lambda \end{aligned}$$

Denoting the structural and aerodynamic mass matrices as

$$m_s(x) = \rho \begin{bmatrix} A(x) & e_{cg} A(x) \\ e_{cg} A(x) & I_{xx} \end{bmatrix}, \quad m_a(x) = \begin{bmatrix} -l_{w_{tt}} - m_{w_{tt}} \tan \Lambda \frac{\partial}{\partial x} & -l_{\theta_{tt}} - m_{\theta_{tt}} \tan \Lambda \frac{\partial}{\partial x} \\ -m_{w_{tt}} & -m_{\theta_{tt}} \end{bmatrix}, \quad (8)$$

the system (1) can be expressed as

$$m(x) \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left(K_1(x) \frac{\partial u(x,t)}{\partial x} \right) + \frac{\partial^2}{\partial x^2} \left(K_2(x) \frac{\partial^2 u(x,t)}{\partial x^2} \right) = q_c(x,t) + b(x) \delta(t) + g(x,t), \quad (9)$$

where $m(x) = m_s(x) + m_a(x)$, $u(x,t) = [w(x,t) \ \theta(x,t)]^\top$,

$$q_c(x,t) = \begin{bmatrix} l_\theta \theta + l_{\theta_t} \frac{\partial \theta}{\partial t} + l_{w_x} \frac{\partial w}{\partial x} + l_{w_t} \frac{\partial w}{\partial t} + \tan \Lambda \left[m_\theta \frac{\partial \theta}{\partial x} + m_{\theta_t} \frac{\partial^2 \theta}{\partial t \partial x} + m_{w_x} \frac{\partial^2 w}{\partial x^2} + m_{w_t} \frac{\partial^2 w}{\partial t \partial x} \right] \\ m_\theta \theta + m_{\theta_t} \frac{\partial \theta}{\partial t} + m_{w_x} \frac{\partial w}{\partial x} + m_{w_t} \frac{\partial w}{\partial t} \end{bmatrix}$$

represents aerodynamic lift and pitching moment portions independent of bending and torsion accelerations, $b(x) = [l_\delta^\top(x) \ m_\delta^\top(x)]^\top$, $g(x,t) = [l_g(x,t) \ m_g(x,t)]^\top$, and

$$K_1(x) = \begin{bmatrix} 0 & 0 \\ 0 & GJ(x) \end{bmatrix}, \quad K_2(x) = \begin{bmatrix} EI(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad (10)$$

We notice that in this formulation $q_c(x,t)$ and $b(x)$ are complex valued due to presence of $C(k)$ in their expressions.

II. Hamiltonian Control with R. T. Jones Approximation

For the system (9) Hamiltonian control was designed in Refs. [3–5] in weak form as

$$\delta_w(t) = -B_w^* (B_w B_w^*)^{-1} \int_0^L \Theta(x) \left[q_c(x,t) + cm(x) \frac{\partial u}{\partial t} + \mu \frac{\partial}{\partial x} \left(K_1(x) \frac{\partial u}{\partial x} \right) - \mu \frac{\partial^2}{\partial x^2} \left(K_2(x) \frac{\partial^2 u}{\partial x^2} \right) \right] dx,$$

where $\Theta(x)$ is a matrix of bases functions, and a complex valued matrix B_w is defined as

$$B_w = \int_0^L \Theta(x) b(x) dx, \quad (11)$$

and in the strong form as

$$\delta_s(t) = -B_s^* (B_s B_s^*)^{-1} \int_0^L \frac{\partial u^\top(x,t)}{\partial t} \left[q_c(x,t) + cm(x) \frac{\partial u}{\partial t} + \mu \frac{\partial}{\partial x} \left(K_1(x) \frac{\partial u}{\partial x} \right) - \mu \frac{\partial^2}{\partial x^2} \left(K_2(x) \frac{\partial^2 u}{\partial x^2} \right) \right] dx,$$

where a complex valued matrix B_s is defined as

$$B_s = \int_0^L \frac{\partial u^\top(x,t)}{\partial t} b(x) dx, \quad (12)$$

without specifying how these complex valued control inputs are implemented.

To handle this problem, here we use the R. T. Jones proposed approximation of the Theodorsen's function [6]

$$C(k) \approx \hat{C}(\bar{s}) = \frac{0.5\bar{s}^2 + a_1\bar{s} + a_2}{\bar{s}^2 + a_3\bar{s} + a_2}, \quad (13)$$

where $\bar{s} = \frac{s}{a}$ is a dimensionless Laplace transform variable with $a = \frac{2V_\infty}{c}$, and $a_1 = 0.2808$, $a_2 = 0.01365$, and $a_3 = 0.3455$. Therefore, for any time dependent function $f(t)$ we can write

$$\begin{aligned} C(k)f(t) &\approx \hat{C}(\bar{s})f(t) = 0.5f(t) + G_1(\bar{s}) [f(t)] \\ C(k)\dot{f}(t) &\approx \hat{C}(\bar{s})\dot{f}(t) = 0.5\dot{f}(t) + a_4 a f(t) + a G_2(\bar{s}) [f(t)], \end{aligned} \quad (14)$$

where

$$G_1(\bar{s}) = \frac{a_4\bar{s} + 0.5a_2}{\bar{s}^2 + a_3\bar{s} + a_2}, \quad G_2(\bar{s}) = \frac{a_5\bar{s} + a_6}{\bar{s}^2 + a_3\bar{s} + a_2}$$

are stable transfer functions with $a_4 = a_1 - 0.5a_3 = 0.10805$, $a_5 = 0.5a_2 - a_1a_3 + 0.5a_3^2 = -0.030506275$, $a_6 = -a_1a_2 + 0.5a_2a_3 = -0.0014748825$.

To this end we write the system (9) as

$$m(x) \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left(K_1(x) \frac{\partial u(x,t)}{\partial x} \right) + \frac{\partial^2}{\partial x^2} \left(K_2(x) \frac{\partial^2 u(x,t)}{\partial x^2} \right) = q_c(x,t) + v(x,t) + g(x,t), \quad (15)$$

where $v(x,t) \equiv b(x)\delta(t)$ is regarded as a virtual control. It was shown in Refs. [3–5] that the system (15) is stable if the time derivative of Hamiltonian function is non positive

$$\dot{H}(t) = \int_0^L \frac{\partial u^\top(t,x)}{\partial t} [q_c(x,t) + v(x,t) + g(x,t)] dx \leq 0, \quad (16)$$

which is achieved if we define $v(x,t)$ such that

$$\frac{\partial u^\top(t,x)}{\partial t} [q_c(x,t) + v(x,t) + g(x,t)] \leq 0. \quad (17)$$

Obviously, $v(x,t)$ can be defined as

$$v(x,t) = -q_c(x,t) - g(x,t) + \mu \frac{\partial u(t,x)}{\partial t}. \quad (18)$$

To define the actual control surface deflections, we represent the equation $v(x,t) = b(x)\delta(t)$ using approximation (13) as follows

$$v(x,t) = (b_0(x) + b_1(x)C(k))\delta(t) = (b_0(x) + 0.5b_1(x))\delta(t) + b_1(x)\delta_f(t), \quad (19)$$

where $\delta_f(t)$ is the output of the system $G_1(s)$ with the input $\delta(t)$ and can be computed from the following realization of $G_1(s)$

$$\begin{aligned} \dot{s}_1(t) &= s_2(t) \\ \dot{s}_2(t) &= -a_2a^2s_1(t) - a_3as_2(t) + \delta(t) \\ \delta_f(t) &= 0.5a_2a^2s_1(t) + a_4as_2(t). \end{aligned} \quad (20)$$

However, this virtual control still has complex values. In order to represent it in an implementable form, we first separate the variables using Galiorkin's method, then apply approximations of $C(k)$.

Let $w(x,t) = \sum_{i=1}^{N_w} \phi_i(x)p_i(t) \equiv \Phi^\top(x)P(t)$, where $\phi_i(x)$ are the bending mode-shape functions defined as

$$\phi_i(x) = \cos \frac{\mu_m i}{L} + \gamma_i \cosh \frac{\mu_i x}{L}, \quad \sin \mu_i + \cos \mu_i \tanh \gamma_i = 0, \quad \gamma_i = \frac{\cos \mu_i}{\cosh \mu_i} \quad (21)$$

for symmetric modes and

$$\phi_i(x) = \sin \frac{\mu_m i}{L} + \gamma_i \sinh \frac{\mu_i x}{L}, \quad -\sin \mu_i + \cos \mu_i \tanh \gamma_i = 0, \quad \gamma_i = \frac{\sin \mu_i}{\sinh \mu_i} \quad (22)$$

for anti-symmetric modes, and $\theta(x,t) = \sum_{i=1}^{N_\theta} \psi_i(x)q_i(t) \equiv \Psi^\top(x)Q(t)$, where $\psi_i(x)$ are the torsion mode-shape functions defined as $\psi_i(x) = \cos \frac{i\pi x}{L}$ for symmetric modes and $\psi_i(x) = \cos \frac{(2i-1)\pi x}{L}$ for anti-symmetric modes.

Separating the variables we obtain

$$M\ddot{U}(t) + (C_{a0} + C_{a1}C(k))\dot{U}(t) + (K + K_{a1}c(k))U(t) = F(t) + G(t), \quad (23)$$

where $U(t) = [P^\top(t) \ Q^\top(t)]^\top$, and $M, C_a, K, K_a, B_a, G(t)$ are computed as usual, and $F(t) = (B_{a0} + 0.5B_{a1})\delta(t) + B_{a1}\delta_f(t)$ is the expression of virtual control after separation of variables.

Let

$$Y(t) = G_1(s) [U(t)], \quad Z(t) = G_2(s) [U(t)].$$

Then we obtain the following system of $3n$ second order ordinary differential equations

$$\begin{aligned} M\ddot{U}(t) + (C_{a0} + 0.5C_{a1})\dot{U}(t) + (K + 0.5K_{a1} + a_4aC_{a1})U(t) + K_{a1}Y(t) + aC_{a1}Z(t) &= F(t) + G(t) \\ \ddot{Y}(t) + a_3a\dot{Y}(t) + a_2a^2Y(t) &= a_4a\dot{U}(t) + 0.5a_2a^2U(t) \\ \ddot{Z}(t) + a_3a\dot{Z}(t) + a_2a^2Z(t) &= a_5a\dot{U}(t) + a_6a^2U(t), \end{aligned} \quad (24)$$

We notice that $6n$ order system (24) with output $U(t)$ and control $F(t)$ has a relative degree of $2n$. Therefore it has an internal dynamics of order $4n$. Setting $U(t) \equiv 0$ and $\dot{U}(t) \equiv 0$ results in the exponentially stable zero-dynamics

$$\begin{aligned} \ddot{Y}(t) + a_3a\dot{Y}(t) + a_2a^2Y(t) &= 0 \\ \ddot{Z}(t) + a_3a\dot{Z}(t) + a_2a^2Z(t) &= 0, \end{aligned} \quad (25)$$

therefore we can ignore the internal dynamics and design the control input $F(t)$ only for the first equation in (24), which according to (18) is defined as

$$F = (C_{a0} + 0.5C_{a1})\dot{U}(t) + (0.5K_{a1} + a_4aC_{a1})U(t) + K_{a1}Y(t) + aC_{a1}Z(t) - G(t) - \mu\dot{U}(t), \quad (26)$$

resulting into exponentially stable system

$$\begin{aligned} M\ddot{U}(t) + k\dot{U}(t) + KU(t) &= 0 \\ \ddot{Y}(t) + a_3a\dot{Y}(t) + a_2a^2Y(t) &= a_4a\dot{U}(t) + 0.5a_2a^2U(t) \\ \ddot{Z}(t) + a_3a\dot{Z}(t) + a_2a^2Z(t) &= a_5a\dot{U}(t) + a_6a^2U(t), \end{aligned} \quad (27)$$

The actual control surface deflection commands are found from the equation

$$(B_{a0} + 0.5B_{a1})\delta(t) + B_{a1}\delta_f(t) = F(t) \quad (28)$$

as

$$\delta(t) = (B_{a0} + 0.5B_{a1})^{-1} \left[F(t) - B_{a1} \left(0.5a_2a^2S_1(t) + a_4aS_2(t) \right) \right]. \quad (29)$$

It can be observed that a bounded control exists if the system

$$\begin{aligned} \dot{S}_1(t) &= S_2(t) \\ \dot{S}_2(t) &= \left(-a_2a^2I - 0.5a_2a^2(B_{a0} + 0.5B_{a1})^{-1}B_{a1} \right) S_1(t) + \left(-a_3aI - a_4a(B_{a0} + 0.5B_{a1})^{-1}B_{a1} \right) S_2(t). \end{aligned} \quad (30)$$

is stable, which can be easily check for a given wind configuration and flight condition.

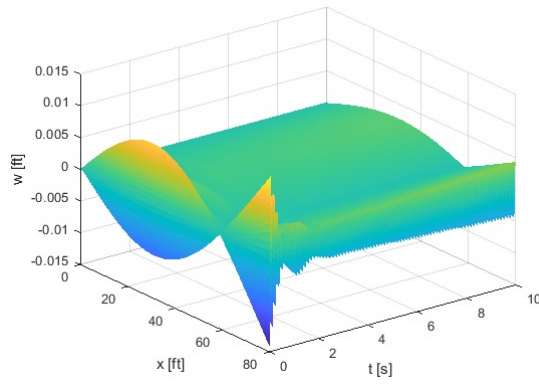
III. Simulation Results

We simulate the coupled bending-torsion motion of a wing with length of $L = 80 \text{ ft}$ and sweep angle of $\Lambda = 20 \text{ deg}$. Five control surfaces of equal length of $0.2L$ are distributed along the wing. A flutter corresponding to the second torsion mode at a frequency of 50.64 rad/sec occurs at an airspeed of $V_f = 376 \text{ ft/sec}$, implying instability at and above V_f even without a wind gust [4].

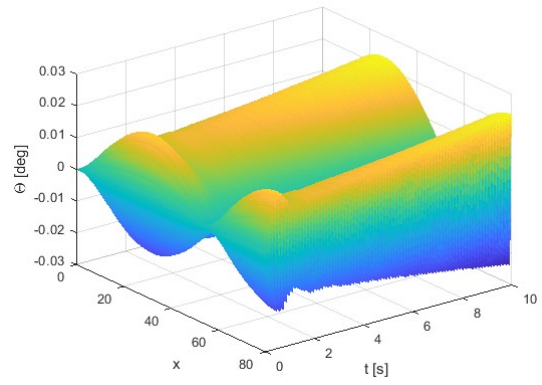
Figure 1 shows the nominal open-loop response of the wing in bending and torsion at 400 ft/sec just beyond the flutter point without a suppression control and wind gust in the first 10 seconds, when only real part of is retained. As can be seen, the motion of the wing in bending and torsion is slowly divergent.

A flutter suppression control is designed using 29 with the velocity feedback gain set to $\mu = 1$. Figure 2 displays the closed-loop behavior of the wing. The corresponding control surface deflections are presented in Figure 3, where no saturation limits are applied.

Next we apply a wind gust of 10 times models in (5) taking into account the with R. T. Jones' approximation of $C(k)$. The corresponding wind behavior and control surface deflections are respectively presented in Figures 4 and 5.

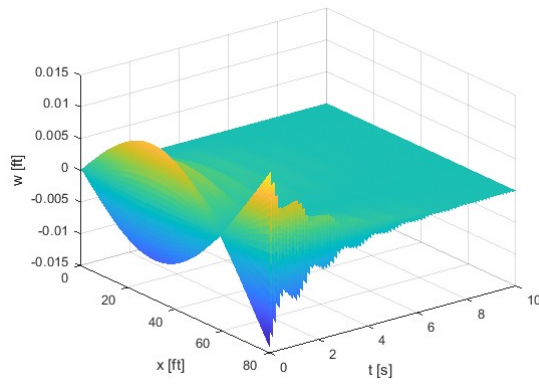


(a) Wing bending deflection

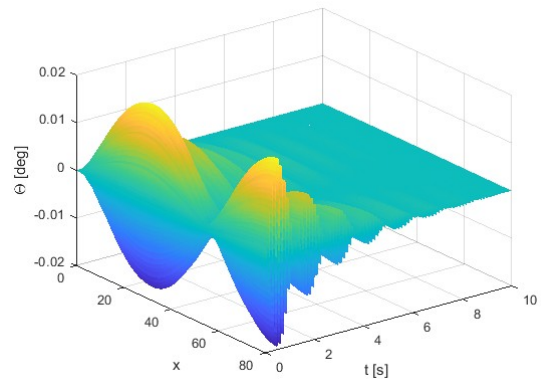


(b) Wing torsional twist

Fig. 1 Open-loop system response without flutter suppression control and wind gust



(a) Wing bending deflection



(b) Wing torsional twist

Fig. 2 Closed-loop system response without wind gust

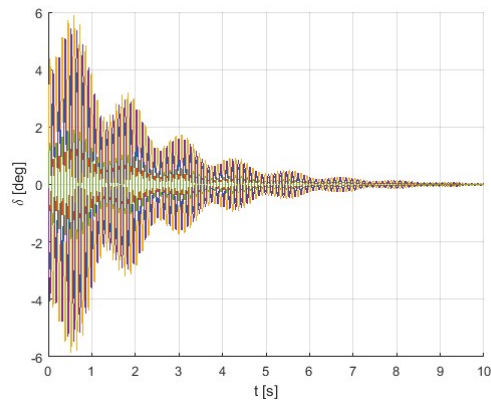
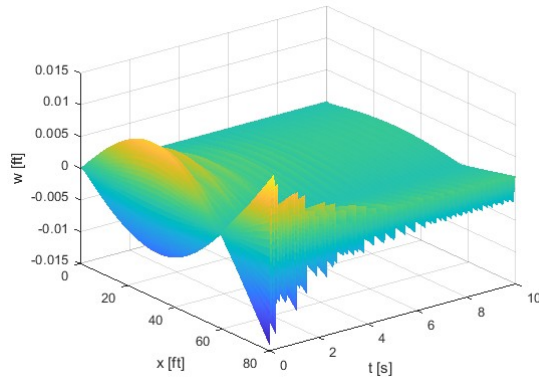
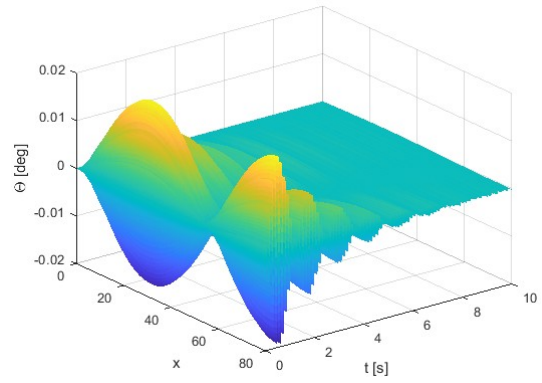


Fig. 3 Control surface deflections without wind gust



(a) Wing bending deflection



(b) Wing torsional twist

Fig. 4 Closed-loop system response with wind gust

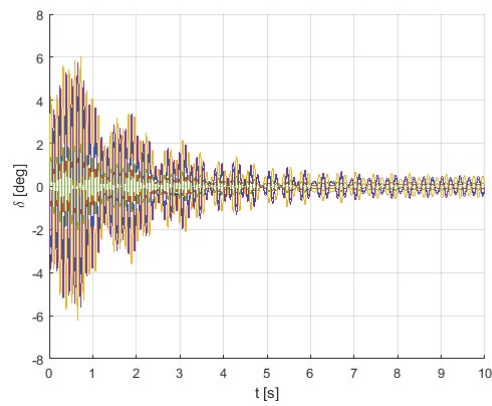


Fig. 5 Control surface deflections with wind gust

Acknowledgment

The authors would like to acknowledge the funding support of this work by NASA Advanced Air Transport Technology project.

References

- [1] N. Nguyen, E. Ting, D. Nguyen, and K. Trinh. Flutter Analysis of Mission-Adaptive Wing with Variable Camber Continuous Trailing Edge Flap. *55th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, AIAA-2014-0839, January 2014.*
- [2] T. Theodorsen. General Theory of Aerodynamic Instability and the Mechanism of Flutter. *NACA Report No. 496, 1935.*
- [3] N. Nguyen, K. Hashemi, E. Arabi and T. Yucelen. Model-Reference Adaptive Control of Distributed Lagrangian Infinite-Dimensional Systems Using Hamilton's Principle. *AIAA Guidance, Navigation, and Control Conference, AIAA-2020-0840, January 2020.*
- [4] N. Nguyen. Hamiltonian Adaptive and Optimal Control of Lagrangian Infinite-Dimensional Systems with Gust Load Alleviation Application. *AIAA Guidance, Navigation, and Control Conference, AIAA-2021-1129, January 2021.*
- [5] N. Nguyen. Distributed Optimal Control of Lagrangian Infinite-Dimensional Systems with Kalman Filter Estimation. *AIAA Guidance Navigation and Control Conference, AIAA-2022-2027, January 2022.*
- [6] R. T. Jones. The unsteady lift of a wing of finite aspect ratio. *NACA Report No. 681, 1940.*