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Implementing a VLBI time delay model for Earth-orbiting satellites: partial derivatives and verification

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Abstract

This document describes the partial derivatives for commonly estimated parameters including antenna positions, satellite position, and satellite velocity from a near-field VLBI delay model for Earth-orbiting satellites. This model was presented in the Journal of Geodesy by Jaron & Nothnagel (2019). In this context, we present a streamlined version of the near-field VLBI delay. From this simplified model, we deduce a delay rate expression and calculate partial derivatives, maintaining only the terms with significant impact on the computed derivatives' magnitude. To verify the accuracy of the simplified model and the partial derivatives computed from it, we have created a simple simulation in Matlab of an Earth-orbiting satellite at the altitude of a typical Global Navigation Satellite Systems (GNSS) satellite. From this simulation, we compare the simplified and original delays, and we verify the magnitude and direction of the partial derivatives against the numerically computed derivatives from the original delay model. The partial derivatives and simplified VLBI delay model detailed here are implemented in Fortran in the open-source library VTD.

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1 Nomenclature

1.1 Symbols

τ	time delay of arrival
γ	Lorentz factor
μ	standard gravitational parameter
c	speed of light
t	time
$\mathbf{x}_i$	position vector of object i
$\mathbf{v}_i$	velocity vector of object i
$\mathbf{a}_i$	acceleration vector of object i
$\mathbf{x}_{ij}$	position difference from object i to object j

1.2 Acronyms

ECEF	Earth-centered, Earth-fixed
ECI	Earth-centered, inertial
VLBA	very long baseline array
VLBI	very long baseline interferometry

2 Introduction

Jaron and Nothnagel (2019) introduced a model of near-field VLBI delay for an Earth-orbiting satellite. To correctly estimate antenna positions using this model, the partial derivatives of the delay and delay rate with respect to each of the two antenna positions are necessary. In this memo, we will retain terms that have an effect on the path delay at the picosecond level. We will also cover the partial derivatives with respect to satellite position and velocity.

The partial derivatives and simulation laid out in this memo have been coded in Matlab and are available upon request. We have also implemented the code in the open-source Fortran library VTD. We have verified the Fortran implementation by interfacing the subroutine in *vtd_jn2019.f* with the described Matlab simulation.

3 Simplifying the delay expressions

Jaron and Nothnagel introduce Equation 1 as the total path delay from antenna 2 to antenna 1. The multiplicative term $1 - L_G$ transforms from Geocentric Coordinate Time (TCG) to Terrestrial Time (TT).

$$\tau = (\Delta t_2 + \Delta t_0)(1 - L_G) \tag{1}$$

The individual quantities Δt_0 and Δt_2 are given by Equations 2 and 3.

$$\begin{aligned}\Delta t_0 =& \gamma_0^2 \left[\frac{\mathbf{x}_{01} \cdot \mathbf{v}_0}{c^2} - t_{g\ 01} \right] \\ & - \sqrt{\gamma_0^4 \left[\frac{\mathbf{x}_{01} \cdot \mathbf{v}_0}{c^2} - t_{g\ 01} \right]^2 + \gamma_0^2 \left[\frac{x_{01}^2}{c^2} - t_{g\ 01}^2 \right]}\end{aligned} \tag{2}$$

$$\Delta t_2 = -\gamma_2^2 \left[\frac{\mathbf{x}_{02}\cdot\mathbf{v}_2}{c^2} - t_{g\ 02}\right] - \sqrt{\gamma_2^4 \left[\frac{\mathbf{x}_{02}\cdot\mathbf{v}_2}{c^2} - t_{g\ 02}\right]^2 + \gamma_2^2 \left[\frac{x_{02}^2}{c^2} - t_{g\ 02}^2\right]} \tag{3}$$

The Lorentz factor for antenna i, γ_i, is given by,

$$\gamma_i^2 = \frac{1}{1 - \frac{v_i^2}{c^2}}. \tag{4}$$

Introducing $\beta_i = \frac{v_i^2}{c^2}$, we can take $\gamma_i \approx 1 + \frac{1}{2}\beta_i$ and $\gamma_i^2 \approx (1-\beta_i)^{-1}$. Take $\frac{(\mathbf{x}_{01}\cdot\mathbf{v}_0)^2}{c^4} < \frac{x_{01}^2 v_0^2}{c^4} = \beta_0 \frac{x_{01}^2}{c^2}$. Since β_0 is small, γ_0^4 can be omitted. Similarly, $t_{g\ 01}$ is small, so its square can be omitted. The expression under the square root in Equation 2 can therefore be reduced to $\frac{1}{c}\gamma_0\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot\mathbf{v}_0)^2}{c^2}}$. Equation 2 then reduces to,

$$\Delta t_0 = \frac{\mathbf{x}_{01}\cdot\mathbf{v}_0}{c^2} - \frac{1}{c}\gamma_0\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot\mathbf{v}_0)^2}{c^2}} - t_{g\ 01} \tag{5}$$

Similarly, Equation 3 can be reduced to,

$$\Delta t_2 = -\frac{\mathbf{x}_{02}\cdot\mathbf{v}_2}{c^2} + \frac{1}{c}\gamma_0\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot\mathbf{v}_2)^2}{c^2}} + t_{g\ 02}. \tag{6}$$

with

$$\mathbf{x}_{01} = \bar{\mathbf{x}}_0(t_1) - \mathbf{x}_1(t_1) = \mathbf{v}_0(t_1 - \tilde{\delta}_1)\tilde{\delta}_1 + \mathbf{x}_0(t_1 - \tilde{\delta}_1) - \mathbf{x}_1(t_1), \tag{7}$$

and

$$\tilde{\delta}_1 = \frac{\|\mathbf{x}_1(t_1) - \mathbf{x}_0(t_1)\|}{c}. \tag{8}$$

The term $\mathbf{x}_{02}$ in Equation 6 is given by,

$$\begin{aligned}\mathbf{x}_{02} &= \bar{\mathbf{x}}_0(t_1) - \bar{\mathbf{x}}_2(t_1) + [\mathbf{v}_0(t_1) - \mathbf{v}_2(t_1)]\Delta t_0\\ &= \mathbf{v}_0(t_1 - \tilde{\delta}_1)\tilde{\delta}_1 + \mathbf{x}_0(t_1 - \tilde{\delta}_1) + \mathbf{v}_2(t_1 + \tilde{\tau})\tilde{\tau} - \mathbf{x}_2(t_1 + \tilde{\tau}) + [\mathbf{v}_0(t_1) - \mathbf{v}_2(t_1)]\Delta t_0,\end{aligned} \tag{9}$$

where

$$\tilde{\tau} = \tilde{\delta}_2 - \tilde{\delta}_1, \text{ and } \tilde{\delta}_2 = \frac{\|\mathbf{x}_2(t_1) - \mathbf{x}_0(t_1)\|}{c}. \tag{10}$$

The gravitational delays $t_{g\ 01}$ and $t_{g\ 02}$ are approximately given by,

$$t_{g\ 01} \approx \frac{2\mu}{c^3}\ln\frac{x_0 + x_1 + x_{01}}{x_0 + x_1 - x_{01}}, \tag{11}$$

and

$$t_{g\ 02} \approx \frac{2\mu}{c^3}\ln\frac{x_0 + x_2 + x_{02}}{x_0 + x_2 - x_{02}}. \tag{12}$$

These delays will be treated as constant and ignored in deriving both partial derivatives and delay rates.

4 Deriving the delay rate

The delay rate is the time derivative of the delay in Equation 1:

$$\dot{\tau} = (\Delta\dot{t}_0 + \Delta\dot{t}_2)(1 - L_G). \tag{13}$$

The time derivative of $\Delta\dot{t}_0$ in reduced form as in Equation 5 is given by,

$$\Delta\dot{t}_0 = \frac{\dot{\mathbf{x}}_{01}\cdot\mathbf{v}_0 + \mathbf{x}_{01}\cdot\mathbf{a}_0}{c^2} - \frac{\dot{\gamma}_0}{c}\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot\mathbf{v}_0)^2}{c^2}} - \frac{\gamma_0}{c}\frac{\dot{\mathbf{x}}_{01}\cdot\mathbf{x}_{01} + \frac{\mathbf{x}_{01}\cdot\mathbf{v}_0}{c^2}(\dot{\mathbf{x}}_{01}\cdot\mathbf{v}_0 + \mathbf{x}_{01}\cdot\mathbf{a}_0)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot\mathbf{v}_0)^2}{c^2}}}. \tag{14}$$

The time derivative of the Lorentz factor γ_i is,

$$\dot{\gamma}_i = \frac{\frac{2v_i a_i}{c^2}}{1 - v_i^2/c^2} = \frac{2v_i a_i}{c^2 - v_i^2} \approx \frac{2v_i a_i}{c^2} \tag{15}$$

Note that it is important to use the non-approximated form, as $\frac{d}{dt}(1 - \frac{1}{2}\frac{v_i^2}{c^2}) = \frac{-v_i a_i}{c^2}$, which is different in sign and magnitude. This term can be neglected without a significant loss of precision, however, as $\frac{v_0 a_0}{c^2} \approx \frac{(3\cdot 10^3)(1)}{(3\cdot 10^8)^2} = \frac{1}{3}\cdot 10^{-14}$. Simplifying,

$$\Delta\dot{t}_0 = \frac{\dot{\mathbf{x}}_{01}\cdot\mathbf{v}_0 + \mathbf{x}_{01}\cdot\mathbf{a}_0}{c^2} - \frac{\gamma_0}{c}\frac{\dot{\mathbf{x}}_{01}\cdot\mathbf{x}_{01} + \frac{\mathbf{x}_{01}\cdot\mathbf{v}_0}{c^2}(\dot{\mathbf{x}}_{01}\cdot\mathbf{v}_0 + \mathbf{x}_{01}\cdot\mathbf{a}_0)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot\mathbf{v}_0)^2}{c^2}}}, \tag{16}$$

where

$$\dot{\mathbf{x}}_{01} = \mathbf{a}_0(t_1 - \tilde{\delta}_1)\tilde{\delta}_1 + \mathbf{v}_0(t_1 - \tilde{\delta}_1)\dot{\tilde{\delta}}_1 + \mathbf{v}_0(t_1 - \tilde{\delta}_1) - \mathbf{v}_1, \tag{17}$$

and

$$\dot{\tilde{\delta}}_1 = \frac{(\mathbf{v}_1 - \mathbf{v}_0)\cdot(\mathbf{x}_1 - \mathbf{x}_0)}{c\|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{18}$$

Similarly, the time derivative of $\Delta\dot{t}_2$ as shown in simplified form in Equation 6 is given by,

$$\Delta\dot{t}_2 = -\frac{\dot{\mathbf{x}}_{02}\cdot\mathbf{v}_2 + \mathbf{x}_{02}\cdot\mathbf{a}_2}{c^2} + \frac{\gamma_2}{c}\frac{\dot{\mathbf{x}}_{02}\cdot\mathbf{x}_{02} + \frac{\mathbf{x}_{02}\cdot\mathbf{v}_2}{c^2}(\dot{\mathbf{x}}_{02}\cdot\mathbf{v}_2 + \mathbf{x}_{02}\cdot\mathbf{a}_2)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot\mathbf{v}_2)^2}{c^2}}}, \tag{19}$$

where

$$\begin{aligned}\dot{\mathbf{x}}_{02} = {}& \mathbf{a}_0(t_1 - \tilde{\delta}_1)\tilde{\delta}_1 + \mathbf{v}_0(t_1 - \tilde{\delta}_1)\dot{\tilde{\delta}}_1 + \mathbf{v}_0(t_1 - \tilde{\delta}_1) - \mathbf{v}_1 \\ & + \mathbf{a}_2(t_1 + \tilde{\tau})\tilde{\tau} + \mathbf{v}_2(t_1 + \tilde{\tau})\dot{\tilde{\tau}} + (\mathbf{a}_0 - \mathbf{a}_2)\Delta t_0 + (\mathbf{v}_0 - \mathbf{v}_2)\Delta\dot{t}_0,\end{aligned} \tag{20}$$

and

$$\dot{\tilde{\tau}} = \dot{\tilde{\delta}}_2 - \dot{\tilde{\delta}}_1 = \frac{(\mathbf{v}_2 - \mathbf{v}_0)\cdot(\mathbf{x}_2 - \mathbf{x}_0)}{c\|\mathbf{x}_2 - \mathbf{x}_0\|} - \frac{(\mathbf{v}_1 - \mathbf{v}_0)\cdot(\mathbf{x}_1 - \mathbf{x}_0)}{c\|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{21}$$

5 Deriving and testing the antenna position partial derivatives

5.1 Delay partial derivatives

In the following expressions, unless otherwise stated, functions of time are evaluated at time t_1. The partial derivative of the VLBI time delay τ with respect to antenna position i is given by,

$$\frac{\partial \tau}{\partial \mathbf{x}_i} = \left(\frac{\partial \Delta t_0}{\partial \mathbf{x}_i} + \frac{\partial \Delta t_2}{\partial \mathbf{x}_i} \right) (1 - L_G). \tag{22}$$

From Equation 5, the partial derivative of Δt_0 with respect to antenna position 1 is,

$$\frac{\partial \Delta t_0}{\partial \mathbf{x}_1} = \frac{1}{c^2} \frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_0 - \frac{\gamma_0}{c} \frac{\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{x}_{01} + \frac{\mathbf{x}_{01} \cdot \mathbf{v}_0}{c^2} (\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_0)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01} \cdot \mathbf{v}_0)^2}{c^2}}}, \tag{23}$$

where

$$\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} = \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_1} - \mathbf{I} \tag{24}$$

$\mathbf{I}$ is the 3 by 3 identity matrix, and

$$\frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_1} = \frac{\partial}{\partial \mathbf{x}_1} \frac{1}{c} \sqrt{(\mathbf{x}_1 - \mathbf{x}_0) \cdot (\mathbf{x}_1 - \mathbf{x}_0)} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{c \|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{25}$$

Similarly, the partial derivative of Δt_2 with respect to antenna position 2 is,

$$\frac{\partial \Delta t_2}{\partial \mathbf{x}_2} = -\frac{1}{c^2} \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2} \cdot \mathbf{v}_2 + \frac{\gamma_2}{c} \frac{\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2} \cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02} \cdot \mathbf{v}_2}{c^2} (\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2} \cdot \mathbf{v}_2)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02} \cdot \mathbf{v}_2)^2}{c^2}}}, \tag{26}$$

where

$$\begin{aligned} \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2} &= \mathbf{v}_2(t_1 + \tilde{\tau}) \otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_2} - \frac{\partial \mathbf{x}_2(t_1 + \tilde{\tau})}{\partial \mathbf{x}_2} \\ &\approx \mathbf{v}_2(t_1 + \tilde{\tau}) \otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_2} - \mathbf{I}, \end{aligned} \tag{27}$$

and

$$\frac{\partial \tilde{\tau}}{\partial \mathbf{x}_2} = \frac{\partial \tilde{\delta}_2}{\partial \mathbf{x}_2} = \frac{\mathbf{x}_2 - \mathbf{x}_0}{c \|\mathbf{x}_2 - \mathbf{x}_0\|}. \tag{28}$$

The cross-partial derivatives $\frac{\partial \Delta t_0}{\partial \mathbf{x}_2}$ and $\frac{\partial \Delta t_2}{\partial \mathbf{x}_1}$ are given by,

$$\frac{\partial \Delta t_2}{\partial \mathbf{x}_1} = -\frac{1}{c^2} \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_2 + \frac{\gamma_2}{c} \frac{\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02} \cdot \mathbf{v}_2}{c^2} (\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_2)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02} \cdot \mathbf{v}_2)^2}{c^2}}}, \tag{29}$$

and

$$\frac{\partial \Delta t_0}{\partial \mathbf{x}_2} = \mathbf{0}. \tag{30}$$

The expression $\dfrac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1}$ in Equation 29 is given by,

$$\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} = \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_1} + \mathbf{v}_2(t_1 + \tilde{\tau}) \otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_1} + (\mathbf{v}_0 - \mathbf{v}_2) \otimes \frac{\partial \Delta t_0}{\partial \mathbf{x}_1}, \tag{31}$$

where

$$\frac{\partial \tilde{\tau}}{\partial \mathbf{x}_1} = -\frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_1} = -\frac{\mathbf{x}_1 - \mathbf{x}_0}{c\|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{32}$$

5.2 Delay rate partial derivatives

The delay rate partial derivative with respect to antenna i is given by,

$$\frac{\partial \dot{\tau}}{\partial \mathbf{x}_i} = \left(\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{x}_i} + \frac{\partial \Delta \dot{t}_2}{\partial \mathbf{x}_i} \right) (1 - L_G). \tag{33}$$

To simplify the computation of the delay rate partial derivatives, we will use the approximation $\mathbf{v}_{\mathrm{ECI}} \approx \omega \times \mathbf{r}_{\mathrm{ECI}}$, where $\omega = [0, 0, \dfrac{2\pi}{86164.0916}]$. We will also assume that $\mathbf{a}_{\mathrm{ECI}} = -2\omega \times \omega \times \mathbf{r}_{\mathrm{ECI}}$. The partial derivative of the time derivative of Δt_0 with respect to the position of antenna 1 is given by,

$$\begin{aligned}
\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{x}_1} = {} & \frac{1}{c^2} \left(\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_0 + \frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{a}_0 \right) \\
& + \frac{\gamma_0}{c} \frac{\left(\dfrac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{x}_{01} + \dfrac{\mathbf{x}_{01} \cdot \mathbf{v}_0}{c^2} \left(\dfrac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_0 \right) \right) \left(\dot{\mathbf{x}}_{01} \cdot \mathbf{x}_{01} + \dfrac{\mathbf{x}_{01} \cdot \mathbf{v}_0}{c^2} (\dot{\mathbf{x}}_{01} \cdot \mathbf{v}_0 + \mathbf{x}_{01} \cdot \mathbf{a}_0) \right)}{\left(x_{01}^2 + \dfrac{(\mathbf{x}_{01} \cdot \mathbf{v}_0)^2}{c^2} \right)^{\frac{3}{2}}} \\
& - \frac{\gamma_0}{c} \frac{\dfrac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{x}_{01} + \dot{\mathbf{x}}_{01} \cdot \dfrac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} + \dfrac{1}{c^2} \left(\dfrac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_0 \right) (\dot{\mathbf{x}}_{01} \cdot \mathbf{v}_0 + \mathbf{x}_{01} \cdot \mathbf{a}_0)}{\sqrt{x_{01}^2 + \dfrac{(\mathbf{x}_{01} \cdot \mathbf{v}_0)^2}{c^2}}} \\
& - \frac{\gamma_0}{c} \frac{\dfrac{\mathbf{x}_{01} \cdot \mathbf{v}_0}{c^2} \left(\dfrac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_0 + \dfrac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_1} \cdot \mathbf{a}_0 \right)}{\sqrt{x_{01}^2 + \dfrac{(\mathbf{x}_{01} \cdot \mathbf{v}_0)^2}{c^2}}}.
\end{aligned} \tag{34}$$

The partial derivative of the time derivative of the position difference between source 0 and antenna 1 with respect to the position of antenna 1 is given by the matrix expression,

$$\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_1} = \mathbf{a}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_1} + \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_1} - \frac{\partial \mathbf{v}_1}{\partial \mathbf{x}_1}. \tag{35}$$

Using the assumption $\mathbf{v}_{\text{ECI}} \approx \omega \times \mathbf{r}_{\text{ECI}}$,

$$\frac{\partial \mathbf{v}_1}{\partial \mathbf{x}_1} = \omega \times \mathbf{I}, \tag{36}$$

and the partial derivative of the time derivative of the light travel time with respect to antenna 1 position is given by,

$$\frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_1} = \frac{\frac{\partial \mathbf{v}_1}{\partial \mathbf{x}_1} \cdot (\mathbf{x}_1 - \mathbf{x}_0)}{c\|\mathbf{x}_1 - \mathbf{x}_0\|} + \frac{\mathbf{v}_1 - \mathbf{v}_0}{c\|\mathbf{x}_1 - \mathbf{x}_0\|} - \frac{(\mathbf{v}_1 - \mathbf{v}_0) \cdot (\mathbf{x}_1 - \mathbf{x}_0)}{c\|\mathbf{x}_1 - \mathbf{x}_0\|^3}(\mathbf{x}_1 - \mathbf{x}_0). \tag{37}$$

Similarly, the partial derivative of the time derivative of the delay Δt_2 with respect to antenna 1 position is given by,

$$\begin{aligned}
\frac{\partial \Delta \dot{t}_2}{\partial \mathbf{x}_1} = &-\frac{1}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_2 + \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{a}_2\right) \\
&- \frac{\gamma_2}{c}\frac{\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02} \cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_2\right)\right)\left(\dot{\mathbf{x}}_{02} \cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02} \cdot \mathbf{v}_2}{c^2}(\dot{\mathbf{x}}_{02} \cdot \mathbf{v}_2 + \mathbf{x}_{02} \cdot \mathbf{a}_2)\right)}{\left(x_{02}^2 + \frac{(\mathbf{x}_{02} \cdot \mathbf{v}_2)^2}{c^2}\right)^{\frac{3}{2}}} \\
&+ \frac{\gamma_2}{c}\frac{\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{x}_{02} + \dot{\mathbf{x}}_{02} \cdot \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} + \frac{1}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_2\right)(\dot{\mathbf{x}}_{02} \cdot \mathbf{v}_2 + \mathbf{x}_{02} \cdot \mathbf{a}_2)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02} \cdot \mathbf{v}_2)^2}{c^2}}} \\
&+ \frac{\gamma_2}{c}\frac{\frac{\mathbf{x}_{02} \cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{v}_2 + \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_1} \cdot \mathbf{a}_2\right)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02} \cdot \mathbf{v}_2)^2}{c^2}}}
\end{aligned} \tag{38}$$

where

$$\frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{x}_1} = -\frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_1} \tag{39}$$

and

$$\begin{aligned}
\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_1} = &\,\mathbf{a}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\mathbf{x}_1} + \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \dot{\tilde{\delta}}_1}{\mathbf{x}_1} + \mathbf{a}_2(t_1 + \tilde{\tau}) \otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_1} \\
&+ \mathbf{v}_2(t_1 + \tilde{\tau}) \otimes \frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{x}_1} + (\mathbf{a}_0 - \mathbf{a}_2)\frac{\partial \Delta t_0}{\partial \mathbf{x}_1} + (\mathbf{v}_0 - \mathbf{v}_2)\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{x}_1}.
\end{aligned} \tag{40}$$

The delay Δt_0 does not depend on antenna position 2, so $\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{x}_2} = \mathbf{0}$. The partial derivative of the time derivative of delay Δt_2 with respect to antenna position 2, $\frac{\partial \Delta \dot{t}_2}{\partial \mathbf{x}_2}$, is a bit more complicated due to the dependence of $\mathbf{v}_2$ and $\mathbf{a}_2$ on $\mathbf{x}_2$:

$$
\begin{aligned}
\frac{\partial \Delta \dot{t}_2}{\partial \mathbf{x}_2} = &-\frac{1}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{v}_2 + \dot{\mathbf{x}}_{02}\cdot \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2} + \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{a}_2 + \mathbf{x}_{02}\cdot \frac{\partial \mathbf{a}_2}{\partial \mathbf{x}_2}\right) \\
&+ \frac{\gamma_2}{c\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}}}\cdot \\
&\left(\frac{-\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{v}_2 + \mathbf{x}_{02}\cdot \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2}\right)\right)\left(\dot{\mathbf{x}}_{02}\cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}(\dot{\mathbf{x}}_{02}\cdot \mathbf{v}_2 + \mathbf{x}_{02}\cdot \mathbf{a}_2)\right)}{\left(x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}\right)}\right. \\
&+ \frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{x}_{02} + \dot{\mathbf{x}}_{02}\cdot \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2} + \frac{1}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{v}_2 + \mathbf{x}_{02}\cdot \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2}\right)(\dot{\mathbf{x}}_{02}\cdot \mathbf{v}_2 + \mathbf{x}_{02}\cdot \mathbf{a}_2) \\
&\left. + \frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{v}_2 + \dot{\mathbf{x}}_{02}\cdot \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2} + \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_2}\cdot \mathbf{a}_2 + \mathbf{x}_{02}\cdot \frac{\partial \mathbf{a}_2}{\partial \mathbf{x}_2}\right)\right),
\end{aligned} \tag{41}
$$

where

$$
\begin{aligned}
\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_2} = &\frac{\partial \mathbf{a}_2(t_1+\tilde{\tau})}{\partial \mathbf{x}_2}\tilde{\tau} + \mathbf{a}_2(t_1+\tilde{\tau})\otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_2} + \frac{\partial \mathbf{v}_2(t_1+\tilde{\tau})}{\partial \mathbf{x}_2}\dot{\tilde{\tau}} \\
&+ \mathbf{v}_2(t_1+\tilde{\tau})\otimes \frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{x}_2} - \frac{\partial \mathbf{v}_2(t_1+\tilde{\tau})}{\partial \mathbf{x}_2} - \frac{\partial \mathbf{a}_2}{\partial \mathbf{x}_2}\Delta t_0 - \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2}\Delta \dot{t}_0,
\end{aligned} \tag{42}
$$

and

$$
\frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{x}_2} = \frac{\partial \dot{\tilde{\delta}}_2}{\partial \mathbf{x}_2} = \frac{\frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2}\cdot(\mathbf{x}_2 - \mathbf{x}_0)}{c\|\mathbf{x}_2 - \mathbf{x}_0\|} + \frac{\mathbf{v}_2 - \mathbf{v}_0}{c\|\mathbf{x}_2 - \mathbf{x}_0\|} - \frac{(\mathbf{v}_2 - \mathbf{v}_0)\cdot(\mathbf{x}_2 - \mathbf{x}_0)}{c\|\mathbf{x}_2 - \mathbf{x}_0\|^3}(\mathbf{x}_2 - \mathbf{x}_0). \tag{43}
$$

The partial derivatives of velocity and acceleration with respect to the antenna position $\mathbf{x}_2$ are approximate expressions accounting for only Earth rotation:

$$
\frac{\partial \mathbf{v}_2(t_1+\tilde{\tau})}{\partial \mathbf{x}_2} \approx \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2} = \omega \times \mathbf{I}, \tag{44}
$$

$$
\frac{\partial \mathbf{a}_2(t_1+\tilde{\tau})}{\partial \mathbf{x}_2} \approx \frac{\partial \mathbf{a}_2}{\partial \mathbf{x}_2} = -2\omega \times \omega \times \mathbf{I}. \tag{45}
$$

5.3 Testing the model

To test these expressions, we have created a toy model of a satellite rotating a spherical Earth in MATLAB. Its Keplerian elements are given by,

a (km)	e	i (deg)	Ω (deg)	ω (deg)	M_0
26846.6	0.01	25	30	0	0

Table 1: Keplerian elements for the tested orbit.

In addition, we have defined two antenna positions at real locations of collected data at Fort Davis, Texas. FD-VLBA is the Fort Davis, TX VLBA antenna.

Antenna	X (m)	Y (m)	Z (m)
1 (DBR231)	-1330750.4861	-5328117.3845	3236420.5290
2 (FD-VLBA)	-1324009.4522	-5332181.9440	3231962.3358

Table 2: ECEF positions used for the two antennas

The paths of the satellite and the two antennas in Earth-Centered Inertial (ECI) space are shown in Figure 1.

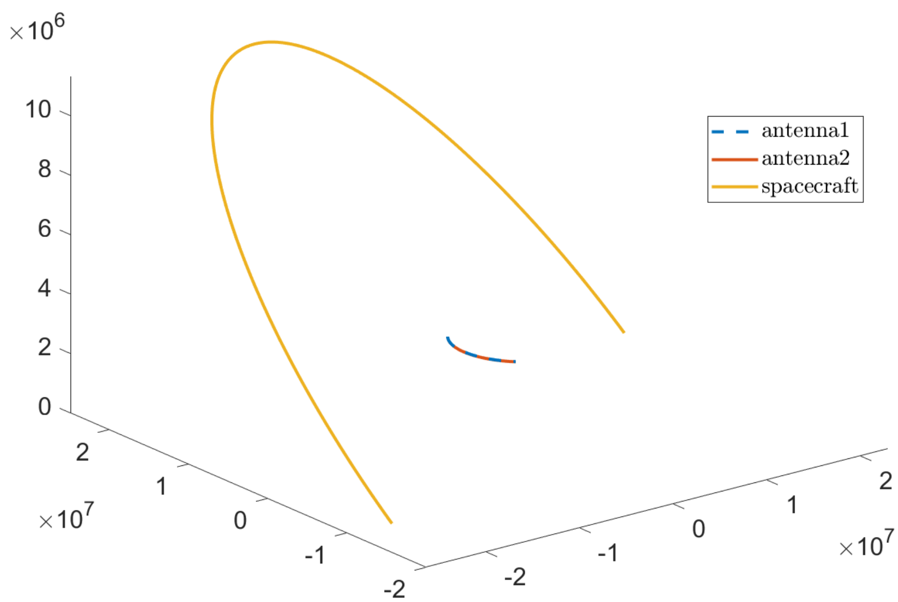

Figure 1: The ECI positions of the antennas and satellite.

5.4 Results

5.4.1 Delay

The VLBI delay as given in Equation 1 for the modeled satellite from antenna 1 to antenna 2 is shown in Figure 2.

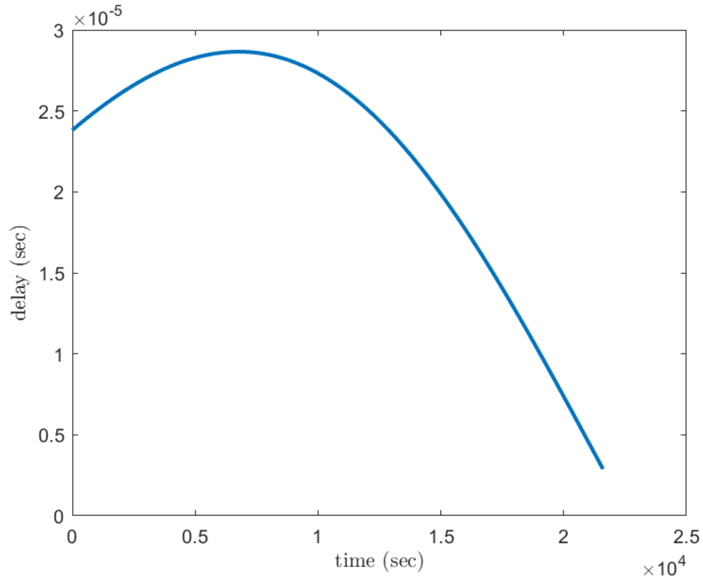

Figure 2: The modeled VLBI delay for the simulated Earth-orbiting satellite.

Figure 3 shows the difference in the VLBI delay between the full expressions for Δt_0 and Δt_2 in Jaron and Nothnagel (2019), and the simplified expressions in Equations 5 and 6. The difference is on the order of 10^{-16} seconds — undetectable in VLBI data.

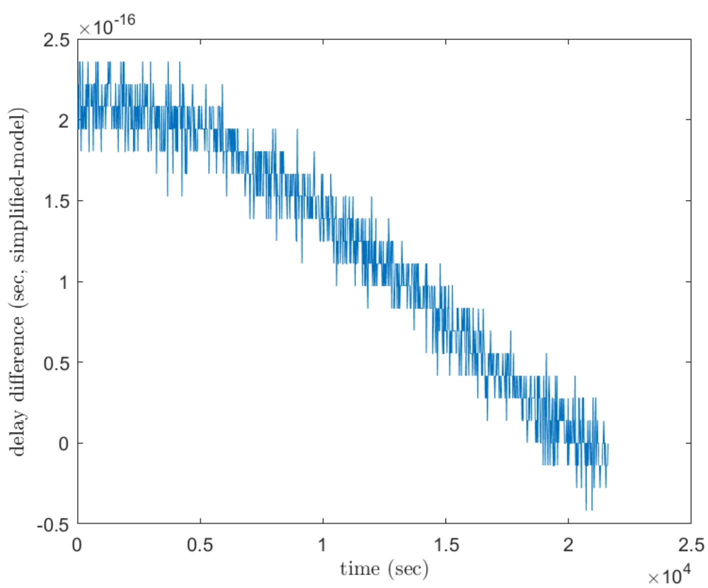

Figure 3: The difference between the simplified delay model and that of Jaron and Nothnagel (2019).

To test the delay partials with respect to antenna 1 and antenna 2 as given by Equations 23-32, we have taken the numerical partial derivatives of the VLBI delay by varying one component of one of the antenna positions at a time and recording the variation in the path delay,

$$\frac{\partial \tau}{\partial \mathbf{x}_{ij}} \approx \frac{\Delta \tau}{\Delta \mathbf{x}_{ij}}. \tag{46}$$

To maintain accuracy in the physical model, when we increase a component of the ECI position of the antennas as we calculate the numerical partial derivatives, we transform this position back to ECEF with a rotation matrix accounting for polar motion, nutation, spin, and precession (MATLAB's ECI2ECEF routine). We discard the velocity in ECEF, in effect constraining that the antenna co-rotates with the Earth and imparting a dependence of the ECI velocity on the ECI position, which we approximated in the analytical partial derivatives as $\mathbf{v}_{\mathrm{ECI}} \approx \omega \times \mathbf{r}_{\mathrm{ECI}}$. This difference appears only in the delay rate partial derivatives. The results of this investigation are shown in Figure 4, which demonstrates good agreement between the numerical and analytical partial derivatives.

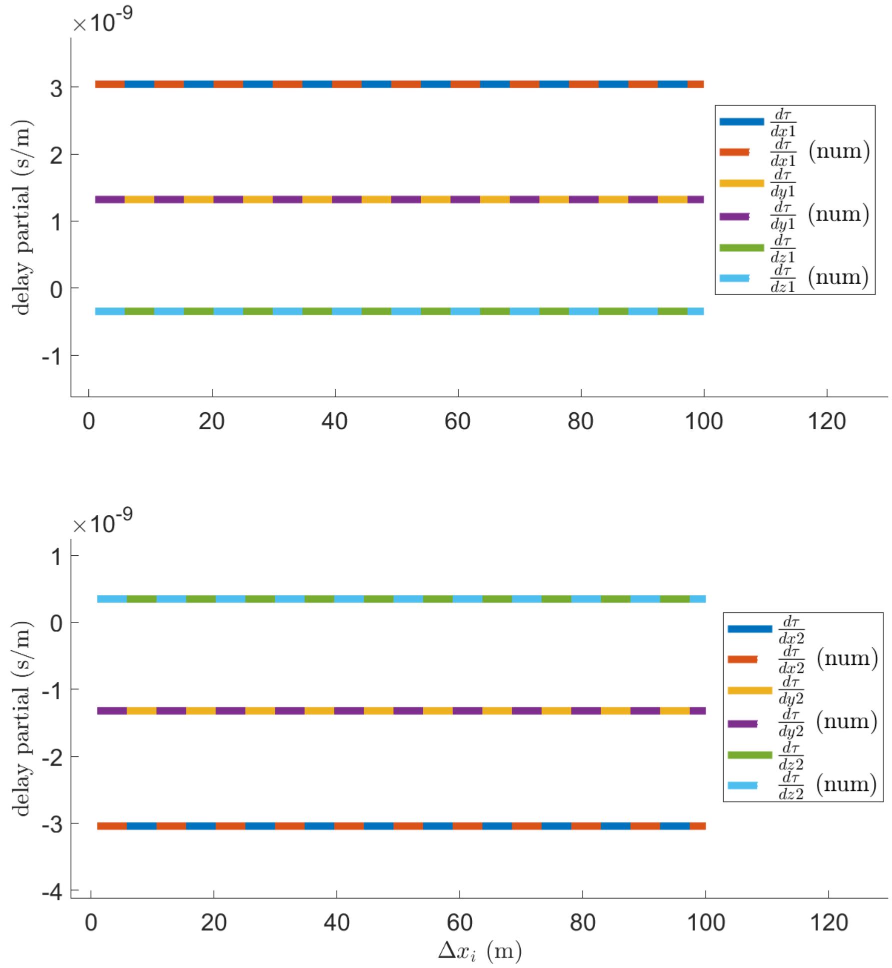

Figure 4: The numerical and analytical delay partial derivatives with respect to the first antenna position (top) and second antenna position (bottom).

5.4.2 Delay rate

Similar to Equation 46, we have verified the delay rate expression by comparing it to a numerical derivative, $\dfrac{\partial \tau}{\partial t} \approx \dfrac{\Delta \tau}{\Delta t}$. Figure 5 shows again that there is good agreement between the numerical and analytical derivatives.

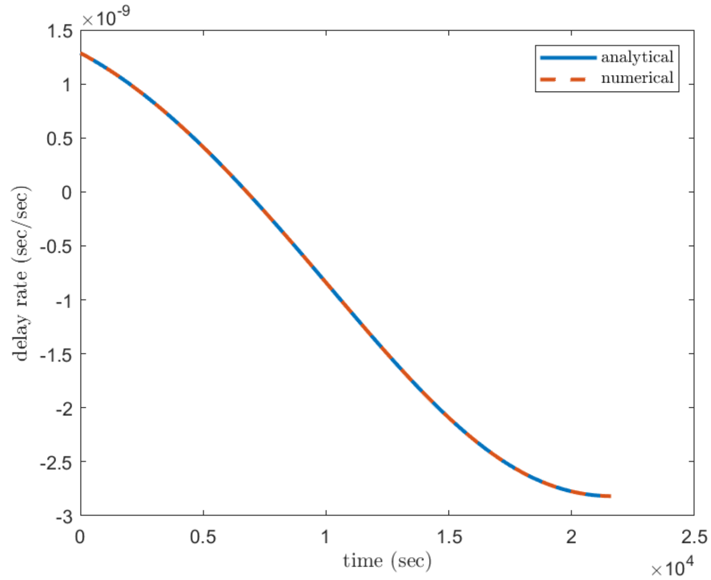

Figure 5: The noiseless numerically differentiated delay rate and the derived analytical delay rate.

Finally, as with the delay partials, we verified the delay rate partials by varying one component of the antenna positions and recording the variation in the analytical delay rate,

$$\frac{\partial \dot{\tau}}{\partial \mathbf{x}_{ij}} \approx \frac{\Delta \dot{\tau}}{\Delta \mathbf{x}_{ij}}. \tag{47}$$

Figure 6 shows tight agreement between the numerical and analytical partial derivatives of the VLBI delay rate.

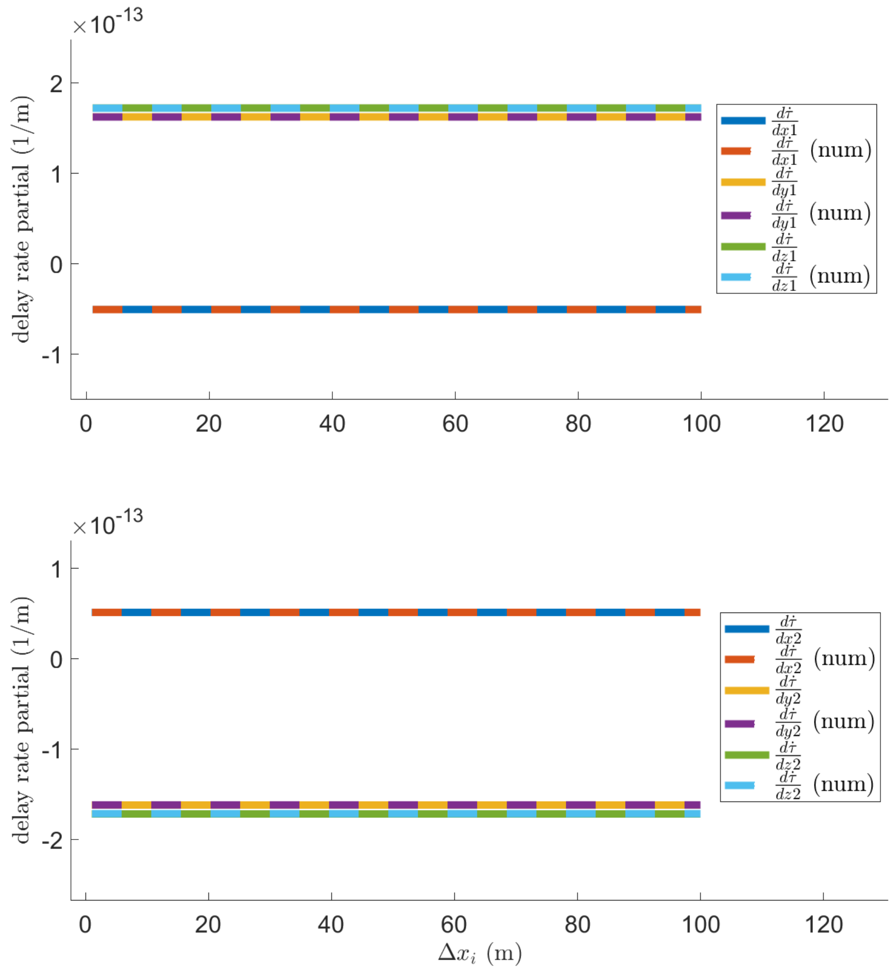

Figure 6: The numerical and analytical delay rate partial derivatives with respect to the first antenna position (top) and second antenna position (bottom).

6 Deriving and testing the satellite position and velocity partial derivatives

To estimate satellite position and velocity, $\mathbf{x}_0$ and $\mathbf{v}_0$, from VLBI data, we also need the partial derivatives of the delay and delay rate with respect to these quantities.

$$\frac{\partial \tau}{\partial \mathbf{x}_0} = \left(\frac{\partial \Delta t_0}{\partial \mathbf{x}_0} + \frac{\partial \Delta t_2}{\partial \mathbf{x}_0} \right) (1 - L_G) \tag{48}$$

$$\frac{\partial \dot{\tau}}{\partial \mathbf{x}_0} = \left(\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{x}_0} + \frac{\partial \Delta \dot{t}_2}{\partial \mathbf{x}_0} \right) (1 - L_G) \tag{49}$$

$$\frac{\partial \tau}{\partial \mathbf{v}_0} = \left(\frac{\partial \Delta t_0}{\partial \mathbf{v}_0} + \frac{\partial \Delta t_2}{\partial \mathbf{v}_0} \right) (1 - L_G) \tag{50}$$

$$\frac{\partial \dot{\tau}}{\partial \mathbf{v}_0} = \left(\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{v}_0} + \frac{\partial \Delta \dot{t}_2}{\partial \mathbf{v}_0} \right) (1 - L_G) \tag{51}$$

6.1 Delay partial derivatives

The partial derivative of the time delay from the satellite to antenna 1 with respect to the position of the satellite is given by,

$$\frac{\partial \Delta t_0}{\partial \mathbf{x}_0} = \frac{1}{c^2} \frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0} \cdot \mathbf{v}_0 - \frac{\gamma_0}{c} \frac{\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0} \cdot \mathbf{x}_{01} + \frac{\mathbf{x}_{01} \cdot \mathbf{v}_0}{c^2} \left(\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0} \cdot \mathbf{v}_0 \right)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01} \cdot \mathbf{v}_0)^2}{c^2}}}, \tag{52}$$

where

$$\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0} = \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{x}_0(t_1 - \tilde{\delta}_1)}{\partial \mathbf{x}_0} \approx \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} + \mathbf{I}, \tag{53}$$

and

$$\frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} = \frac{-(\mathbf{x}_1 - \mathbf{x}_0)}{c\|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{54}$$

Similarly, the partial derivative of the time delay from the satellite to antenna 2 with respect to the position of the satellite is,

$$\frac{\partial \Delta t_2}{\partial \mathbf{x}_0} = -\frac{1}{c^2} \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0} \cdot \mathbf{v}_2 + \frac{\gamma_2}{c} \frac{\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0} \cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02} \cdot \mathbf{v}_2}{c^2} \left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0} \cdot \mathbf{v}_2 \right)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02} \cdot \mathbf{v}_2)^2}{c^2}}}, \tag{55}$$

where

$$\begin{aligned} \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0} &= \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{x}_0(t_1 - \tilde{\delta}_1)}{\partial \mathbf{x}_0} \\ &\quad + \mathbf{v}_2(t_1 + \tilde{\tau}) \otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_0} + (\mathbf{v}_0 - \mathbf{v}_2) \otimes \frac{\partial \Delta t_0}{\partial \mathbf{x}_0} \\ &\approx \mathbf{v}_0(t_1 - \tilde{\delta}_1) \otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} + \mathbf{I} \\ &\quad + \mathbf{v}_2(t_1 + \tilde{\tau}) \otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_0} + (\mathbf{v}_0 - \mathbf{v}_2) \otimes \frac{\partial \Delta t_0}{\partial \mathbf{x}_0}, \end{aligned} \tag{56}$$

and

$$\frac{\partial \tau}{\partial \mathbf{x}_0} = \frac{\partial \tilde{\delta}_2}{\partial \mathbf{x}_0} - \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} = \frac{-(\mathbf{x}_2 - \mathbf{x}_0)}{c\|\mathbf{x}_2 - \mathbf{x}_0\|} + \frac{\mathbf{x}_1 - \mathbf{x}_0}{c\|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{57}$$

The partial derivative of the delay Δt_0 with respect to the satellite velocity $\mathbf{v}_0$ is given by,

$$\frac{\partial \Delta t_0}{\partial \mathbf{v}_0} = \frac{1}{c^2}\left(\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\right) - \frac{1}{c}\frac{\partial \gamma_0}{\partial \mathbf{v}_0}\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}}$$
$$- \frac{\gamma_0}{c}\frac{\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{x}_{01} + \frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}\left(\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\right)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}}}, \tag{58}$$

where

$$\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{v}_0} = \frac{\partial \mathbf{v}(t_1 - \tilde{\delta}_1)}{\partial \mathbf{v}_0}\tilde{\delta}_1 + \frac{\partial \mathbf{x}_0(t_1 - \tilde{\delta}_1)}{\partial \mathbf{v}_0} \approx \tilde{\delta}_1\mathbf{I} - \tilde{\delta}_1\mathbf{I} = \mathbf{0}. \tag{59}$$

A proof that $\frac{\partial \mathbf{x}_0(t_1 - \tilde{\delta}_1)}{\partial \mathbf{v}_0} \approx -\tilde{\delta}_1\mathbf{I}$ is shown in Appendix B using the Keplerian state transition matrix. The partial derivatives of position, velocity, and acceleration with respect to each other evaluated at different epochs can all be improved in principle by using this state transition matrix rather than assuming either unity or zero, but in the simulations we conducted for this document, the additional complexity was unnecessary to get accurate partial derivatives.

The derivative of the Lorentz factor γ_0 with respect to the satellite velocity $\mathbf{v}_0$, which is *non-negligible*, is given by,

$$\frac{\partial \gamma_0}{\partial \mathbf{v}_0} = \frac{\mathbf{v}_0}{c^2(1 - \frac{v_0^2}{c^2})^{3/2}}. \tag{60}$$

The partial derivative of the delay Δt_2 with respect to the satellite velocity is,

$$\frac{\partial \Delta t_2}{\partial \mathbf{v}_0} = -\frac{1}{c^2}\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_2 + \frac{\gamma_2}{c}\frac{\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_2\right)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}}}, \tag{61}$$

where

$$\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0} = \Delta t_0\mathbf{I} + (\mathbf{v}_0 - \mathbf{v}_2)\frac{\partial \Delta t_0}{\partial \mathbf{v}_0}. \tag{62}$$

6.2 Delay rate partial derivatives

The partial derivative of the time derivative of the delay Δt_0 with respect to the satellite position is given by,

$$
\begin{aligned}
\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{x}_0} = & \frac{1}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_0 + \frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{a}_0 + \mathbf{x}_{01}\cdot \frac{\partial \mathbf{a}_0}{\partial \mathbf{x}_0}\right) \\
& + \frac{\gamma_0}{c}\frac{\left(\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{x}_{01} + \frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_0\right)\left(\dot{\mathbf{x}}_{01}\cdot \mathbf{x}_{01} + \frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}(\dot{\mathbf{x}}_{01}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\cdot \mathbf{a}_0)\right)}{\left(x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}\right)^{\frac{3}{2}}} \\
& - \frac{\gamma_0}{c}\frac{\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{x}_{01} + \dot{\mathbf{x}}_{01}\cdot \frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0} + \frac{1}{c^2}\left(\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_0\right)(\dot{\mathbf{x}}_{01}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\cdot \mathbf{a}_0)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}}} \\
& - \frac{\gamma_0}{c}\frac{\frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_0 + \frac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_0}\cdot \mathbf{a}_0 + \mathbf{x}_{01}\cdot \frac{\partial \mathbf{a}_0}{\partial \mathbf{x}_0}\right)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}}},
\end{aligned}
\tag{63}
$$

where

$$
\begin{aligned}
\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{x}_0} &= \frac{\partial \mathbf{a}_0(t_1 - \tilde{\delta}_1)}{\partial \mathbf{x}_0}\tilde{\delta}_1 + \mathbf{a}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} + \mathbf{v}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_0} \\
&\approx \frac{\partial \mathbf{a}_0}{\partial \mathbf{x}_0}\tilde{\delta}_1 + \mathbf{a}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} + \mathbf{v}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_0}
\end{aligned}
\tag{64}
$$

and

$$
\frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_0} = -\frac{\mathbf{v}_1 - \mathbf{v}_0}{c\|\mathbf{x}_1 - \mathbf{x}_0\|} + \frac{(\mathbf{v}_1 - \mathbf{v}_0)\cdot(\mathbf{x}_1 - \mathbf{x}_0)}{c\|\mathbf{x}_1 - \mathbf{x}_0\|^3}(\mathbf{x}_1 - \mathbf{x}_0), \tag{65}
$$

$$
\frac{\partial \mathbf{a}_0}{\partial \mathbf{x}_0} = -\frac{\mu}{x_0^3}\mathbf{I} + \frac{3\mu}{x_0^5}(\mathbf{x}_0 \otimes \mathbf{x}_0). \tag{66}
$$

Similarly,

$$\begin{aligned}
\frac{\partial \Delta \dot{t}_2}{\partial \mathbf{x}_0} &= -\frac{1}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_2 + \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{a}_2\right) \\
&\quad - \frac{\gamma_2}{c}\frac{\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_2\right)\right)\left(\dot{\mathbf{x}}_{02}\cdot \mathbf{x}_{02} + \frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}(\dot{\mathbf{x}}_{02}\cdot \mathbf{v}_2 + \mathbf{x}_{02}\cdot \mathbf{a}_2)\right)}{\left(x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}\right)^{\frac{3}{2}}} \\
&\quad + \frac{\gamma_2}{c}\frac{\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{x}_{02} + \dot{\mathbf{x}}_{02}\cdot \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0} + \frac{1}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_2\right)(\dot{\mathbf{x}}_{02}\cdot \mathbf{v}_2 + \mathbf{x}_{02}\cdot \mathbf{a}_2)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}}} \\
&\quad + \frac{\gamma_2}{c}\frac{\frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{v}_2 + \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{x}_0}\cdot \mathbf{a}_2\right)}{\sqrt{x_{02}^2 + \frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}}},
\end{aligned} \tag{67}$$

where

$$\begin{aligned}
\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{x}_0} &= \frac{\partial \mathbf{a}_0}{\partial \mathbf{x}_0}\tilde{\delta}_1 + \mathbf{a}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \tilde{\delta}_1}{\partial \mathbf{x}_0} + \mathbf{v}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_0} \\
&\quad + \mathbf{a}_2(t_1 + \tilde{\tau})\otimes \frac{\partial \tilde{\tau}}{\partial \mathbf{x}_0} + \mathbf{v}_2(t_1 + \tilde{\tau})\otimes \frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{a}_0}{\partial \mathbf{x}_0}\Delta t_0 \\
&\quad + (\mathbf{a}_0 - \mathbf{a}_2)\frac{\partial \Delta t_0}{\partial \mathbf{x}_0} + (\mathbf{v}_0 - \mathbf{v}_2)\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{x}_0},
\end{aligned} \tag{68}$$

$$\frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{x}_0} = \frac{\partial \dot{\tilde{\delta}}_2}{\partial \mathbf{x}_0} - \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{x}_0}, \tag{69}$$

and

$$\frac{\partial \dot{\tilde{\delta}}_2}{\partial \mathbf{x}_0} = -\frac{\mathbf{v}_2 - \mathbf{v}_0}{c\|\mathbf{x}_2 - \mathbf{x}_0\|} + \frac{(\mathbf{v}_2 - \mathbf{v}_0)\cdot(\mathbf{x}_2 - \mathbf{x}_0)}{\|\mathbf{x}_2 - \mathbf{x}_0\|^3}(\mathbf{x}_2 - \mathbf{x}_0). \tag{70}$$

The partial derivative of the time derivative of the delay Δt_0 with respect to the satellite velocity is given by,

$$
\begin{aligned}
\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{v}_0} &= \frac{1}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_0 + \dot{\mathbf{x}}_{01}\right) \\
&- \frac{1}{c}\frac{\partial \gamma_0}{\partial \mathbf{v}_0}\frac{\dot{\mathbf{x}}_{01}\cdot \mathbf{x}_{01} + \frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}(\dot{\mathbf{x}}_{01}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\cdot \mathbf{a}_0)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}}} \\
&+ \frac{\gamma_0}{c}\frac{\left(\frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}\left(\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\right)\right)\left(\dot{\mathbf{x}}_{01}\cdot \mathbf{x}_{01} + \frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}(\dot{\mathbf{x}}_{01}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\cdot \mathbf{a}_0)\right)}{\left(x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}\right)^{\frac{3}{2}}} \\
&- \frac{\gamma_0}{c}\frac{\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{x}_{01} + \dot{\mathbf{x}}_{01}\cdot \frac{\partial \mathbf{x}_{01}}{\partial \mathbf{v}_0} + \frac{1}{c^2}\left(\frac{\partial \mathbf{x}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\right)(\dot{\mathbf{x}}_{01}\cdot \mathbf{v}_0 + \mathbf{x}_{01}\cdot \mathbf{a}_0)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}}} \\
&- \frac{\gamma_0}{c}\frac{\frac{\mathbf{x}_{01}\cdot \mathbf{v}_0}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_0 + \dot{\mathbf{x}}_{01}\right)}{\sqrt{x_{01}^2 + \frac{(\mathbf{x}_{01}\cdot \mathbf{v}_0)^2}{c^2}}},
\end{aligned}
\tag{71}
$$

where

$$
\begin{aligned}
\frac{\partial \dot{\mathbf{x}}_{01}}{\partial \mathbf{v}_0} &= \frac{\partial \mathbf{v}_0(t_1 - \tilde{\delta}_1)}{\partial \mathbf{v}_0}\dot{\tilde{\delta}}_1 + \mathbf{v}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{v}_0} + \frac{\partial \mathbf{v}_0(t_1 - \tilde{\delta}_1)}{\partial \mathbf{v}_0} \\
&\approx (1 + \dot{\delta}_1)\mathbf{I} + \mathbf{v}_0(t_1 - \tilde{\delta}_1)\otimes \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{v}_0},
\end{aligned}
\tag{72}
$$

and

$$
\frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{v}_0} = \frac{-(\mathbf{x}_1 - \mathbf{x}_0)}{c\|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{73}
$$

Finally, the partial derivative of the time derivative of the delay Δt_2 with respect to the satellite velocity is given by,

$$
\begin{aligned}
\frac{\partial \Delta \dot{t}_2}{\partial \mathbf{v}_0} = &-\frac{1}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_2\right) \\
&-\frac{\gamma_2}{c}\frac{\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{x}_{02}+\frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_2\right)\right)\left(\dot{\mathbf{x}}_{02}\cdot \mathbf{x}_{02}+\frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}(\dot{\mathbf{x}}_{02}\cdot \mathbf{v}_2+\mathbf{x}_{02}\cdot \mathbf{a}_2)\right)}{\left(x_{02}^2+\frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}\right)^{\frac{3}{2}}} \\
&+\frac{\gamma_2}{c}\frac{\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{x}_{02}+\dot{\mathbf{x}}_{02}\cdot \frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}+\frac{1}{c^2}\left(\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_2\right)(\dot{\mathbf{x}}_{02}\cdot \mathbf{v}_2+\mathbf{x}_{02}\cdot \mathbf{a}_2)}{\sqrt{x_{02}^2+\frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}}} \\
&+\frac{\gamma_2}{c}\frac{\frac{\mathbf{x}_{02}\cdot \mathbf{v}_2}{c^2}\left(\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{v}_2+\frac{\partial \mathbf{x}_{02}}{\partial \mathbf{v}_0}\cdot \mathbf{a}_2\right)}{\sqrt{x_{02}^2+\frac{(\mathbf{x}_{02}\cdot \mathbf{v}_2)^2}{c^2}}},
\end{aligned} \tag{74}
$$

where

$$
\begin{aligned}
\frac{\partial \dot{\mathbf{x}}_{02}}{\partial \mathbf{v}_0} = &(1+\dot{\tilde{\delta}}_1+\Delta \dot{t}_0)\mathbf{I}+\mathbf{v}_0(t_1-\tilde{\delta}_1)\otimes \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{v}_0}+\mathbf{v}_2(t_1+\tilde{\tau})\otimes \frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{v}_0} \\
&+(\mathbf{a}_0-\mathbf{a}_2)\frac{\partial \Delta t_0}{\partial \mathbf{v}_0}+(\mathbf{v}_0-\mathbf{v}_2)\frac{\partial \Delta \dot{t}_0}{\partial \mathbf{v}_0},
\end{aligned} \tag{75}
$$

$$
\frac{\partial \dot{\tilde{\tau}}}{\partial \mathbf{v}_0} = \frac{\partial \dot{\tilde{\delta}}_2}{\partial \mathbf{v}_0} - \frac{\partial \dot{\tilde{\delta}}_1}{\partial \mathbf{v}_0}, \tag{76}
$$

and

$$
\frac{\partial \dot{\tilde{\delta}}_2}{\partial \mathbf{v}_0} = \frac{-(\mathbf{x}_2-\mathbf{x}_0)}{c\|\mathbf{x}_2-\mathbf{x}_0\|}. \tag{77}
$$

6.3 Results

As with the antenna position partial derivatives, we have verified these expressions for the partial derivative of the delay and delay rate with respect to the satellite position and velocity by comparing them to numerically computed derivatives for a satellite with the given Keplerian elements and the two given antenna positions while changing the satellite's position and velocity.

6.3.1 Delay

As shown in Figure 7, the numerically and analytically evaluated partial derivatives of the VLBI delay with respect to the position and velocity are closely matched.

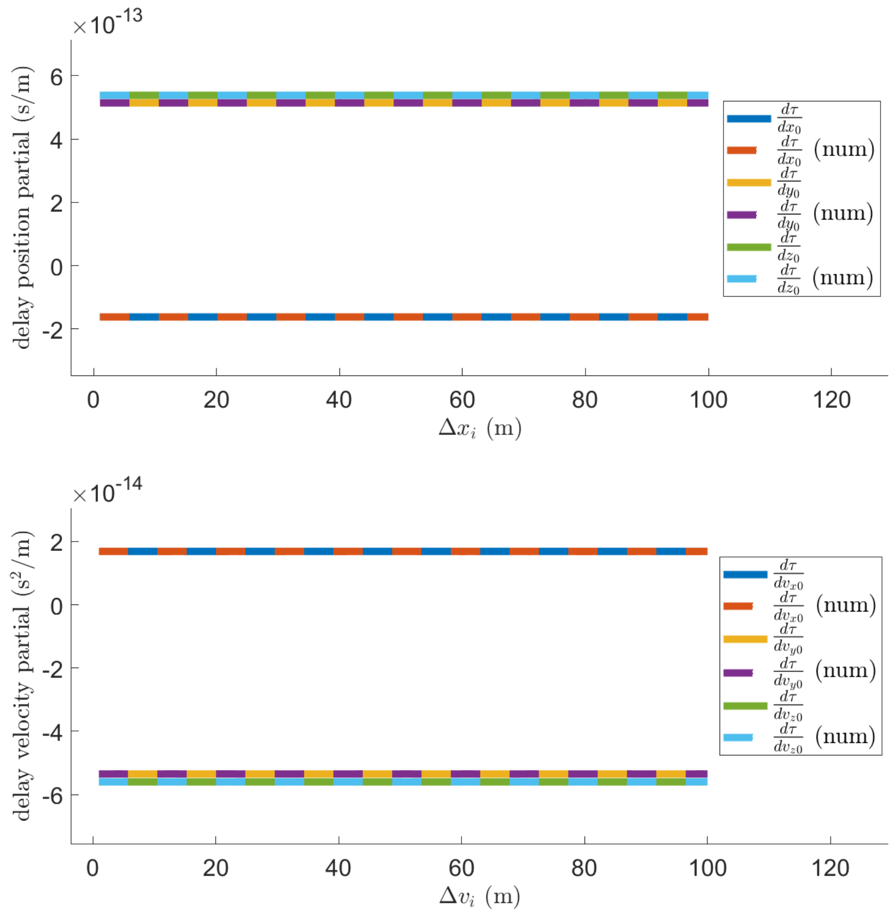

Figure 7: The numerical and analytical delay partial derivatives with respect to the satellite position (top) and satellite velocity (bottom).

6.3.2 Delay rate

The partial derivatives of the VLBI delay rate also closely match their numerically evaluated counterparts (Figure 8).

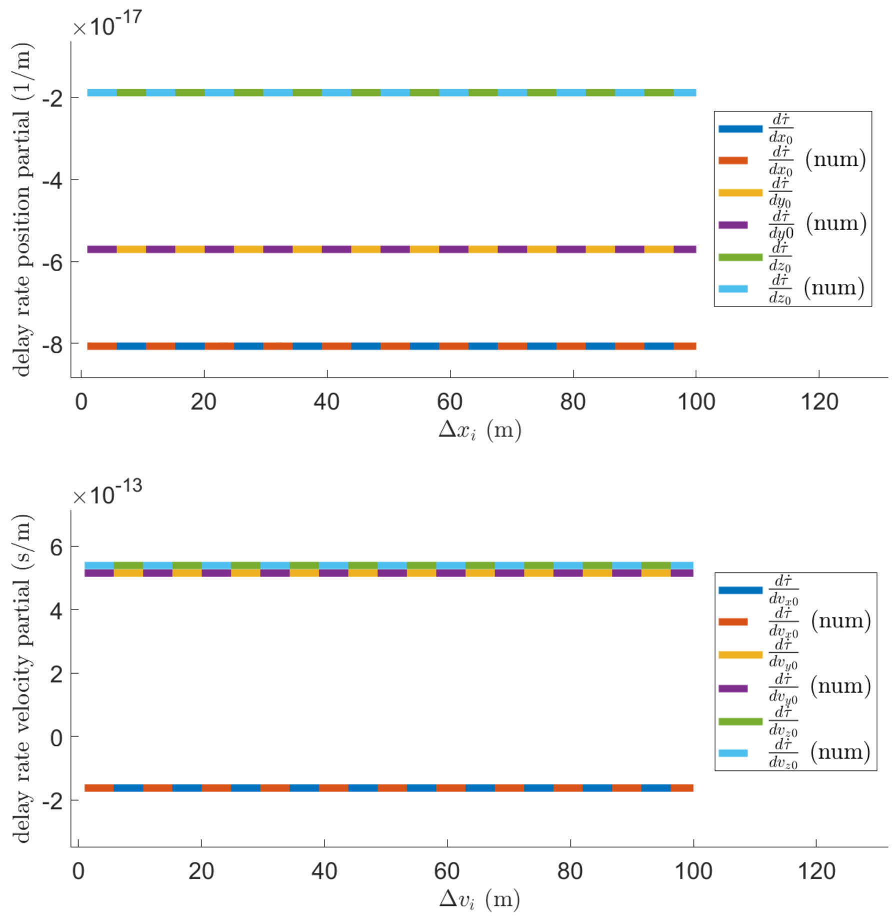

Figure 8: The numerical and analytical delay rate partial derivatives with respect to the satellite position (top) and satellite velocity (bottom).

References

1. Jaron, F., Nothnagel, A. Modeling the VLBI delay for Earth satellites. *J Geod* 93, 953–961 (2019). https://doi.org/10.1007/s00190-018-1217-0

2. Der, G.J. An Elegant State Transition Matrix. *J of Astronaut Sci* 45, 371–390 (1997). https://doi.org/10.1007/BF03546398

Appendix A

Gravitational delay

The gravitational delay has two components that have approximately non-negligible contribution to the VLBI delay — the delay due to the Sun and the delay due to the Earth:

$$t_{g\ 01} = t_{g\ 01,\ \mathrm{Sun}} + t_{g\ 01,\ \mathrm{Earth}} = \frac{2\mu}{c^3} \ln \frac{R_0 + R_1 + R_{01}}{R_0 + R_1 - R_{01}} + \frac{2\mu}{c^3} \ln \frac{x_0 + x_1 + x_{01}}{x_0 + x_1 - x_{01}}. \quad \text{(A78)}$$

In Equation A78, R_0, R_1, and R_{01} are x_0, x_1, and x_{01} in the Solar System Barycentric (SSB) frame. Interestingly, the Sun's gravitational delay to antenna 1 is larger than that of the Earth (Figure A9), but the *differential* delay from antenna 1 to antenna 2 is much larger for the Earth from the simulated satellite (Figure A10).

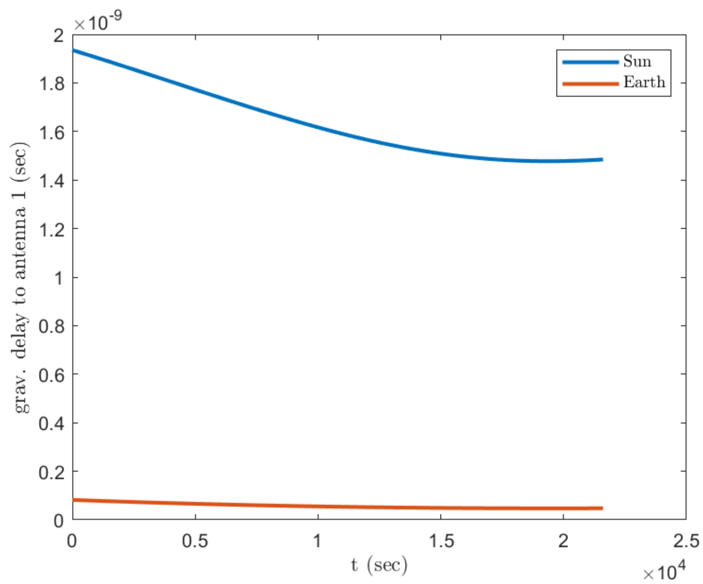

Figure A9: The gravitational delay to antenna 1 due to the Earth and Sun.

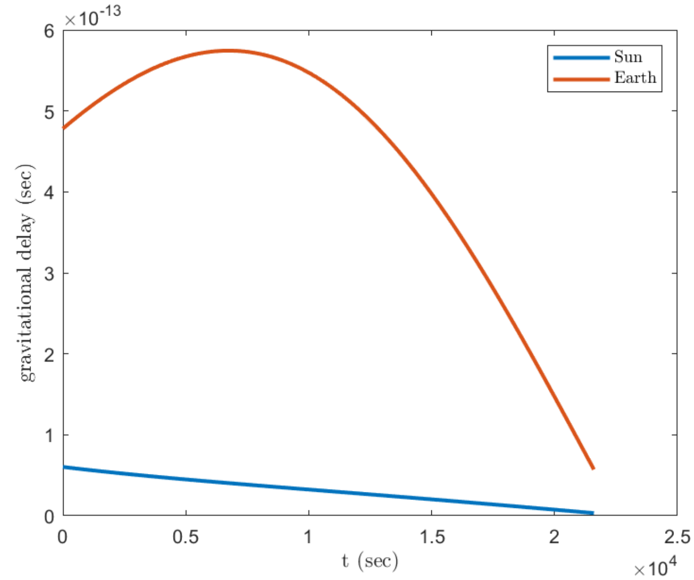

Figure A10: The VLBI delay due to the Earth and Sun.

Neglecting the gravitational delay from the Sun, we have a weak contribution to the partial derivatives of the VLBI delays from the Earth's gravitational delay:

$$\frac{\partial t_{g\ 01}}{\partial \mathbf{x}_1} = \frac{2\mu}{c^3}\left(\frac{\dfrac{\mathbf{x}_1}{x_1} - \dfrac{\mathbf{x}_{01}}{x_{01}}}{x_0 + x_1 + x_{01}} - \frac{\dfrac{\mathbf{x}_1}{x_1} + \dfrac{\mathbf{x}_{01}}{x_{01}}}{x_0 + x_1 - x_{01}}\right), \tag{A79}$$

$$\frac{\partial t_{g\ 02}}{\partial \mathbf{x}_2} = \frac{2\mu}{c^3}\left(\frac{\dfrac{\mathbf{x}_2}{x_2} - \dfrac{\mathbf{x}_{02}}{x_{02}}}{x_0 + x_2 + x_{02}} - \frac{\dfrac{\mathbf{x}_2}{x_2} + \dfrac{\mathbf{x}_{02}}{x_{02}}}{x_0 + x_2 - x_{02}}\right). \tag{A80}$$

These contributions are on the order of $\dfrac{\mu}{c^3(x_0 + x_i - x_{0i})} \approx \dfrac{4 \cdot 10^{14}}{(3 \cdot 10^8)^3 (10^6)} \approx 10^{-17}\ \dfrac{\mathrm{s}}{\mathrm{m}}$.

Appendix B

Partial derivatives from the Keplerian state transition matrix

A simple representation for the state transition matrix of a Keplerian orbit can be found in Der (1997):

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} f\mathbf{I} & g\mathbf{I} \\ \dot{f}\mathbf{I} & \dot{g}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{bmatrix}, \tag{B81}$$

where

$$f = 1 - \frac{x^2 C}{r_0}, \tag{B82}$$

$$g = (t - t_0) - \frac{x^3 S}{\mu}, \tag{B83}$$

$$\dot{f} = \frac{-\sqrt{\mu}}{r r_0} x(1 - \alpha x^2 S), \tag{B84}$$

and

$$\dot{g} = 1 - \frac{x^2 C}{r}. \tag{B85}$$

x is a so-called universal variable, or universal anomaly, $\mathbf{r}$, $\mathbf{r_0}$, $\mathbf{v}$, $\mathbf{v_0}$ are the current and initial positions and velocities of the orbiting object, α is the inverse of the semi-major axis, $\alpha = \frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}$, and C and S are functions that depend on the conic section represented by the satellite's orbit. For a satellite in a stable, elliptical orbit, $x = \frac{\theta}{\sqrt{\alpha}} = \theta\sqrt{a}$, where θ is the angle between $\mathbf{r}$ and $\mathbf{r}_0$. The functions C and S then simplify to,

$$C = \frac{1 - \cos\theta}{\theta^2}, \tag{B86}$$

and

$$S = \frac{\theta - \sin\theta}{\theta^3}. \tag{B87}$$

The state transition matrix can then be simplified to,

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} (1 - \frac{a}{r_0}(1 - \cos\theta)\mathbf{I} & (t - t_0 - \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\theta - \sin\theta))\mathbf{I} \\ \frac{-\sqrt{\mu a}}{r r_0}\sin\theta\mathbf{I} & (1 - \frac{a}{r}(1 - \cos\theta))\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{bmatrix}, \tag{B88}$$

Using the original formalism, relating satellite position at time t_0 to satellite position and velocity at time $t_0 - \tilde{\delta}_1$, we have,

$$\mathbf{x}_0 = \left(1 - \frac{a}{x_0}(1 - \cos\theta)\right)\mathbf{x}_0(t_0 - \tilde{\delta}_1) + \left(\tilde{\delta}_1 - \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\theta - \sin\theta)\right)\mathbf{v}_0(t_0 - \tilde{\delta}_1). \tag{B89}$$

The velocity at time t_0 is similarly given by,

$$\mathbf{v}_0 = \frac{-\sqrt{\mu a}\sin\theta}{x_0 x_0(t_0 - \tilde{\delta}_1)}\mathbf{x}_0(t_0 - \tilde{\delta}_1) + \left(1 - \frac{a}{x_0}(1 - \cos\theta)\right)\mathbf{v}_0(t_0 - \tilde{\delta}_1). \tag{B90}$$

Rearranging Equation B89 and taking the partial derivative with respect to $\mathbf{v}_0$,

$$\frac{\partial\mathbf{x}_0(t_1 - \tilde{\delta}_1)}{\partial\mathbf{v}_0} = -\frac{\tilde{\delta}_1 - \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\theta - \sin\theta)}{1 - \frac{a}{x_0}(1 - \cos\theta)}\frac{\partial\mathbf{v}_0(t_1 - \tilde{\delta}_1)}{\partial\mathbf{v}_0}. \tag{B91}$$

From Equation B90, $\frac{\partial\mathbf{v}_0(t_1 - \tilde{\delta}_1)}{\partial\mathbf{v}_0}$ is given by,

$$\frac{\partial \mathbf{v}_0(t_1-\tilde{\delta}_1)}{\partial \mathbf{v}_0} = \frac{1}{1-\dfrac{a}{x_0}(1-\cos\theta)}\left(\mathbf{I}+\frac{\sqrt{\mu a}}{x_0 x_0(t_1-\tilde{\delta}_1)}\sin\theta\frac{\partial \mathbf{x}_0(t_1-\tilde{\delta}_1)}{\mathbf{v}_0}\right), \quad \text{(B92)}$$

thus,

$$\frac{\partial \mathbf{x}_0(t_1-\tilde{\delta}_1)}{\partial \mathbf{v}_0} =$$

$$-\frac{\tilde{\delta}_1-\dfrac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\theta-\sin\theta)}{\left(1-\dfrac{a}{x_0(t_1-\tilde{\delta}_1)}(1-\cos\theta)\right)\left(1-\dfrac{a}{x_0}(1-\cos\theta)\right)}\left(\mathbf{I}+\frac{\sqrt{\mu a}}{x_0 x_0(t_1-\tilde{\delta}_1)}\sin\theta\frac{\partial \mathbf{x}_0(t_1-\tilde{\delta}_1)}{\mathbf{v}_0}\right). \quad \text{(B93)}$$

Solving Equation B93 for $\dfrac{\partial \mathbf{x}_0(t_1-\tilde{\delta}_1)}{\partial \mathbf{v}_0}$, we get,

$$\frac{\partial \mathbf{x}_0(t_1-\tilde{\delta}_1)}{\partial \mathbf{v}_0} =$$

$$\frac{-\left(\tilde{\delta}_1-\dfrac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\theta-\sin\theta)\right)\mathbf{I}}{\left(\tilde{\delta}_1-\dfrac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\theta-\sin\theta)\right)\dfrac{\sqrt{\mu a}}{x_0 x_0(t_1-\tilde{\delta}_1)}\sin\theta+\left(1-\dfrac{a}{x_0(t_1-\tilde{\delta}_1)}(1-\cos\theta)\right)\left(1-\dfrac{a}{x_0}(1-\cos\theta)\right)}. \quad \text{(B94)}$$

As $\theta \to 0$,

$$\frac{\partial \mathbf{x}_0(t_1-\tilde{\delta}_1)}{\partial \mathbf{v}_0} \approx -\tilde{\delta}_1\mathbf{I}. \quad \text{(B95)}$$