

Discretization Error Estimation and Control for Farfield Acoustic Signatures

Sriram K. Rallabhandi*, Marian Nemec†, Michael J. Aftosmis‡
NASA Ames Research Center, Moffett Field, CA, 94035

We investigate the utility of adjoint-based error estimates for sonic boom farfield simulations governed by solutions of the augmented Burgers' equation. Solution of this nonlinear system uses operator splitting with a second-order finite volume discretization in space and second-order Runge-Kutta time marching, while the absorption and molecular relaxation are solved using second-order central differencing. The discretization error in selected ground sonic boom cost functionals is estimated using the method of adjoint-weighted residuals. Key elements of the implementation process are emphasized with details provided on the practical aspects as applied to the sonic boom farfield propagation. We establish the accuracy of the adjoint solutions using complex step and finite difference approaches, and examine the accuracy of the error estimates using analytical N-wave solutions. We then apply it to a pressure waveform corresponding to the X-59 research aircraft. The investigations demonstrate that the method of adjoint-weighted residuals accurately predicts the level of discretization error present in sonic boom farfield simulations while offering insight into which features of the nearfield signal are the primary drivers of ground noise metrics. The numerical results indicate that at sampling frequencies as low as 50 kHz, discretization error in the propagation is under 0.01 dB[A] for realistically complex examples.

Nomenclature

A-SEL	= A-weighted Sound Exposure Level (decibels)
B	= Blokhintzev invariant
B-SEL	= B-weighted Sound Exposure Level (decibels)
c_0	= Ambient speed of sound (m/s)
C-SEL	= C-weighted Sound Exposure Level (decibels)
H	= Pertaining to the current space-time mesh
h	= Pertaining to the refined space-time mesh
J	= Objective/cost function for adjoint calculation
J_G	= Ground pressure cost functional
L	= Number of uniformly spaced nodes used in propagation
m_v	= Dispersion parameter
n	= Propagation iteration counter
N	= Number of steps during propagation
p, P	= Pressure waveform during propagation
p'	= Pressure perturbation w.r.t. ambient pressures
R, R_x	= Residual vectors in numerical scheme
$R(U_h^H)$	= Residual vector reconstructed on the embedded mesh from the working mesh
t_v	= Relaxation time parameter
U	= Solution vector of the underlying model equation
U_h^H	= Solution vector reconstructed on the embedded mesh from the working mesh
v_g	= Magnitude of ray velocity (m/s)
β	= $1 + \frac{\gamma-1}{2}$

*Aerospace Engineer, Computational Aerosciences Branch, MS 258-5, sriram.rallabhandi@nasa.gov, Associate Fellow AIAA

†Aerospace Engineer, Computational Aerosciences Branch, MS 258-5; marian.nemec@nasa.gov, Senior Member AIAA

‡Aerospace Engineer, Computational Aerosciences Branch, MS 258-5, michael.aftosmis@nasa.gov, Associate Fellow AIAA

ρ_0	= Ambient density
σ	= Temporal variable along the ray path
τ	= Spatial variable along the pressure waveform
Γ, ψ	= Adjoint vectors
$\Gamma_h^H, \epsilon_h^H, \gamma_{i,h}^H, \delta_h^H, \beta_h^H, \lambda_{i,h}^H$	= Adjoint vectors reconstructed on the embedded mesh from the working mesh
$\overline{\Gamma}_h$	= Higher-order reconstruction of the adjoint vectors from the working mesh
δ	= Diffusivity of sound
η	= Total discretization error indicator
ϵ	= Relative error w.r.t. embedded mesh
Ω	= Doppler shift factor induced by prevailing winds

I. Introduction

SONIC boom prediction typically involves three key elements: a) determining the complex flow-field around the aircraft (termed as *nearfield*) using Computational Fluid Dynamics (CFD), b) propagating those pressure disturbances over long distances to reach the ground (termed as *farfield*), and c) computing the loudness metrics to quantify the impact of sonic booms. Due to the large computational domains with disparate scales and complex interfaces, it is crucial to control the discretization errors within these coupled nearfield-farfield simulations to ensure that the predicted results are reliable and accurate, especially as we move toward quieter en route noise levels. Previous work on coupled nearfield-farfield simulations [1] demonstrated an efficient procedure to estimate and control discretization errors when adaptively refining the nearfield meshes for predicting sonic boom metrics on the ground. However, in that study, the error estimates had contributions only from the nearfield CFD domain through adaptive mesh refinement. Related work [2] on farfield simulations used higher-order conservative schemes and enhanced the adjoint solver of the augmented Burgers' equation to obtain an asymptotically consistent adjoint discretization, enabling reliable estimation of adjoint variables and sensitivities that could be used for design optimization. However, no attempt was made to leverage those adjoint vectors to quantify and control the inherent discretization errors in the farfield simulations. The current paper focuses on the computation of discretization error estimates from the farfield solver building upon previous work [2] on the computation of robust adjoints. Discretization error is controlled through uniform refinement of both spatial and temporal resolution in the propagation simulation. Finally, the farfield discretization error control is demonstrated to effectively reduce numerical errors in farfield sonic boom simulations for complex configurations and flight conditions based on the X-59. Similar work [3, 4] utilizing a higher-order adaptive space-time finite element approach has been recently reported that seeks to adaptively control discretization errors in sonic boom simulations using non-uniform anisotropic meshes.

II. Problem Formulation and Governing Equations

The current work utilizes and builds upon sBOOM [5, 6], which uses a lossy augmented Burgers' equation model as given in Eq. (1) to numerically model the physics of sonic boom propagation through the atmosphere. Detailed derivations based on approximations to the Westervelt Equation using weakly nonlinear ray theory and the related underlying assumptions are provided in earlier publications [7–9].

$$\frac{\partial P}{\partial \sigma} = \frac{\frac{\partial B}{\partial \sigma}}{B} P + \frac{\beta \Omega}{2 \rho_0 c_0^3} \left(\frac{c_0}{v_g} \right) \frac{\partial P^2}{\partial \tau} + \frac{\delta \Omega^2}{2 \rho_0 c_0^3} \left(\frac{c_0}{v_g} \right) \frac{\partial^2 P}{\partial \tau^2} + \Sigma_v \frac{m_v}{2 c_0} \frac{t_v \Omega^2}{1 + t_v \Omega \frac{\partial}{\partial \tau}} \left(\frac{c_0}{v_g} \right) \frac{\partial^2 P}{\partial \tau^2} \quad (1)$$

As described in reference [2], an operator splitting scheme [10] is used to solve Eq. (1) as a sequential set under the assumption that if the time step is small, the error induced by splitting is small. Using this procedure, a total of seven equations corresponding to nonlinearity, thermo-viscous absorption, and molecular relaxation for nitrogen and oxygen are iteratively solved at each time step as the waveform propagates from the nearfield to the ground. The output quantities of interest are the ground pressures and the loudness metrics such as perceived level (PL) [11] or A/B/C/D/E-weighted sound exposure levels that are based on prescribed frequency weighting curves [12].

III. Adjoint-based Discretization Error Formulation

Discretization errors are inevitable in all numerical simulations. In the case of sonic boom prediction, particularly when attempting to analyze and design for low loudness levels needed for community acceptability, it is imperative

to quantify and control these errors to provide credible and verifiably accurate results. The classical approach for estimating the discretization error ϵ in a general cost functional $J(U(\tau))$ involves uniformly refining the underlying mesh, τ_H to τ_h , and computing the error relative to the embedded mesh as given in Eq. (2).

$$\epsilon = |J(U_h) - J(U_H)| \quad (2)$$

Asymptotic analysis [13] is used to estimate the total discretization error. The method of adjoint-weighted residuals [14–16] overcomes the progressively increasing cost of obtaining the cost functional values on the embedded mesh (τ_h) by using a Taylor series-based approximation as given in Eq. (3). Since we do not want to solve on the embedded fine mesh, the common procedure followed is to eliminate the dependence on U_h . We leverage solutions to the adjoint equations to obtain an approximation of the cost functional on the embedded mesh as shown in Eq. (4), where Ψ represents adjoint vectors, R the residual vectors, and the summation is carried over the number of time steps needed for the disturbance to march from the nearfield location to the ground.

$$J(U_h) \approx J(U_h^H) + \frac{\partial J(U_h^H)}{\partial U_h} (U_h - U_h^H) \quad (3)$$

$$J_{\text{Corr,Exact}}(U_h) \approx J(U_h^H) - \sum_{n=1}^N \Psi_{h,n}^T R(U_{h,n}^H) \quad (4)$$

The adjoint variables in Eq. (4) can become computationally expensive, especially on the final large meshes. To mitigate this, Eq. (4) can be further simplified to apply prolonged adjoint solutions from the current mesh to reconstruct on the embedded mesh. The resulting cost functional is given in Eq. (5). This approximation introduces an additional error, termed as the remaining error, as given in Eq. (6).

$$J_{\text{Corr}}(U_h) \approx J(U_h^H) - \sum_{n=1}^N \left(\Psi_{h,n}^H \right)^T R(U_{h,n}^H) \quad (5)$$

$$\eta \approx \sum_{n=1}^N (\Psi_{h,n} - \Psi_{h,n}^H)^T R(U_{h,n}^H) \quad (6)$$

The error relative to the embedded mesh is given by Eq. (7). When the cost functional on the embedded mesh is replaced by the exact solution, one obtains the error in the numerical simulations. Obtaining the remaining error term given in Eq. (6) may become impractical as it requires the computation of the adjoints on the finer, embedded mesh. To overcome this problem, a higher-order reconstruction of the adjoint vectors from the working mesh can be used to approximate the adjoint solutions on the embedded refined mesh. This allows approximation of the remaining error term as in Eq. (8), where $\overline{\Psi_{h,n}^H}$ is a higher-order reconstruction of the adjoint vectors solved on the current working mesh and projected onto the embedded refined mesh.

$$\epsilon \approx |J(U_h^H) - \sum_{n=1}^N (\Psi_{h,n}^H)^T R(U_{h,n}^H) - J(U_H)| \quad (7)$$

$$\eta \approx \sum_{n=1}^N (\overline{\Psi_{h,n}^H} - \Psi_{h,n}^H)^T R(U_{h,n}^H) \quad (8)$$

Several output functionals $J(U)$ are studied in this paper. These include noise metrics based on the Sound Exposure Level (SEL) to quantify the loudness of the ground signature. Specifically, A/B/C/D/E-SEL metrics correspond to various weightings of the frequencies in the ground signature expressed in decibels, and are used to both quantify sonic boom noise, study convergence of the outputs, and perform error analysis. In addition to the SEL-based loudness metrics, a simple integration of the ground pressure as indicated in Eq. (9) is used to study the convergence of the ground signatures. In this equation, $\Delta\tau$ is the cell size along the spatially uniform discretization.

$$J_G = \left[\sum_{i=1}^L P_i^2 \right] \Delta\tau \quad (9)$$

IV. Implementation

An operator splitting approach [10] is used to separate the terms on the right side of Eq. (1) into seven equations that handle the effects of nonlinearity, thermo-viscous absorption, and molecular relaxation. Ray-tube spreading and atmospheric stratification are implemented to impact pressure scaling. Discrete adjoint equations with multiple relevant sonic boom metrics have been implemented and reported in prior work [17]. More recently, significant updates [2] were

implemented to obtain adjoint consistent discretization by using conservative finite volume approach using Godunov's Flux function with explicit time marching. Each relaxation step was split into two steps involving internal species pressures [3] to ensure adjoint consistent discretization for the implementation of the relaxation mechanisms as well.

When second-order spatial discretization is used to solve the nonlinearity, a van Leer limiter is used to reconstruct the solution to avoid oscillatory behavior near shocks. Differentiation of the limiter gave rise to errant behavior in the adjoint solutions. While adjoint sensitivities matched those obtained using complex variables for loudness metric cost functionals, they differed from those computed using a finite difference approach. To resolve this discrepancy, the limiter is assumed to be constant or independent of the pressure during the adjoint solve. After this change was incorporated, the adjoint, complex and finite difference methods all yielded the same sensitivities.

In the case of sonic boom modeling, we simulate propagation along each time step by splitting and sequentially solving nonlinearity, atmospheric absorption, and molecular relaxation mechanisms; therefore the summation term in Eq. (5) includes adjoint and residual vectors from each of those mechanisms in the full space-time discretization as shown in Eq. (10). In this equation, $\epsilon_{h,n}^H, \gamma_{0,h,n}^H, \delta_{h,n}^H, \gamma_{1,h,n}^H, \beta_{h,n}^H, \lambda_{n,1,h}^H$, and $\lambda_{n,2,h}^H$ are the intermediate adjoint solutions corresponding to each of the mechanisms in the augmented Burgers' equation reconstructed on the embedded mesh from the current mesh. The terms $R_\epsilon, R_{\gamma,0}, R_\delta, R_{\gamma,1}, R_\beta, R_{\lambda_1}$ and R_{λ_2} are the residuals reconstructed on the embedded mesh for each of those mechanisms. Additional details on what each term means, including details on the splitting mechanism, and the underlying manipulations are included in earlier work [2].

$$\begin{aligned} J_{\text{Corr}}(U_h) \approx & J(U_h^H) - \sum_{n=1}^N \left(\epsilon_{h,n}^H \right)^T R_\epsilon(P_{v,1,h,n}^H, P_{h,n}^H) - \sum_{n=1}^N \left(\gamma_{0,h,n}^H \right)^T R_{\gamma,0}(q_{h,n}^H, P_{v,1,h,n}^H, P_{h,n}^H) \\ & - \sum_{n=1}^N \left(\delta_{h,n}^H \right)^T R_\delta(P_{v,2,h,n}^H, q_{h,n}^H) - \sum_{n=1}^N \left(\gamma_{1,h,n}^H \right)^T R_{\gamma,1}(r_{h,n}^H, P_{v,2,h,n}^H, q_{h,n}^H) - \sum_{n=1}^N \left(\beta_{h,n}^H \right)^T R_\beta(t_{h,n}^H, r_{h,n}^H) \\ & - \sum_{n=1}^N \left(\lambda_{n,1,h}^H \right)^T R_{\lambda_1}(s_{h,n}^H, t_{h,n}^H) - \sum_{n=1}^N \left(\lambda_{n,2,h}^H \right)^T R_{\lambda_2}(p_{h,n+1}^H, s_{h,n}^H, t_{h,n}^H) \end{aligned} \quad (10)$$

In order to reconstruct the current space-time mesh residuals and adjoints onto the refined space-time discretization, we use both bilinear and biquadratic interpolations. For all the results shown in this paper, the adjoint vectors are reconstructed using biquadratic interpolation, while a bilinear approach was used for the residuals due to their high frequency content. To determine the remaining error term in Eq. (8), we compute the difference in the biquadratic and bilinear interpolants, resulting in a higher-order error term.

V. Results

A. Farfield Error Analysis: N-wave Propagation with an Exact Solution

To benchmark error analysis of sonic boom farfield propagation, we begin with an N-wave propagation problem for which an exact solution exists [18]. The initial ($\sigma = 0$) waveform is defined over an interval $[-\pi, \pi]$ with shocks present at the $\pm\tau_s$ location along the non-dimensional space coordinate (τ) axis as given in Eq. (11). The exact solution of this N-wave propagation with viscosity and molecular relaxation turned off under uniform atmospheric conditions, and cylindrical spreading at a general space-time coordinate of (τ, σ) is given by Eq. (12), where the location of the shock in the non-dimensional space is determined as in Eq. (13). For the current paper, we assume $\tau_s = 0.3\pi$, $\Delta P = 0.5$ and a Mach number of 1.4.

$$p'_{near}(\tau, 0) = -\Delta P \begin{cases} \frac{\tau}{\tau_s} & -\tau_s \leq \tau \leq \tau_s \\ 0.0 & \text{otherwise} \end{cases} \quad (11)$$

$$p'(\tau, \sigma) = -\Delta P \begin{cases} \frac{\tau}{\tau_s [1 + \frac{\sigma \Delta P}{\tau_s}]} & -\tau_f \leq \tau \leq \tau_f \\ 0.0 & \text{Otherwise} \end{cases} \quad (12)$$

$$\tau_f = \tau_s \sqrt{1 + \frac{\Delta P \sigma}{\tau_s}} \quad (13)$$

For this problem, we consider a series of uniform refinements along the spatial direction (τ), successively doubling from 201 to 25601 mesh nodes. To march along the direction of propagation (σ), we limit the step size based on a prescribed CFL – i.e., $\Delta\sigma = \frac{CFL\Delta\tau}{\max(|P_n|)}$. For results shown in this section, a CFL of 0.6 was chosen. The simulations are terminated at $\sigma = 1.8\pi$, ensuring that the duration of the final waveform is double that of the initial waveform. Figure 1(a) shows the nondimensional nearfield pressure along with the pressures obtained using the analytical solution (Eq. (12)) and numerical simulation with 25601 nodes. Second-order nonlinear implementation using a conservative Gudunov flux function with explicit Euler time marching was used in these comparisons. The numerical solution matches well with the analytical solution of the inviscid Burgers' equation. Figure 1(b) depicts the nondimensional nearfield waveform along with the progressively refined uniform spatial discretizations. As the underlying spatial discretization is refined, the sharpness of the shock discontinuity approaches the analytical solution of a jump discontinuity. This is depicted by the zoomed-in image of the front shock in Fig. 1(b).

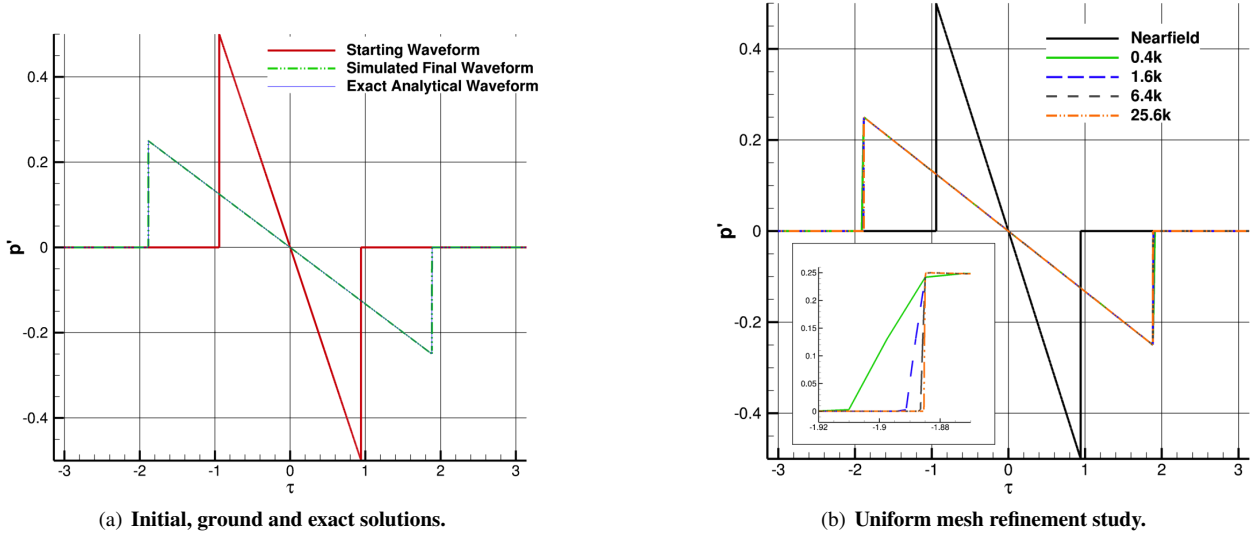


Fig. 1 N-wave propagation model problem.

To ensure that the primal solution is well behaved, we look at the L_1 norm of the final pressure perturbations against those generated using the analytical solution as the underlying space-time discretization is uniformly refined. Figure 2 shows that the solution error is $O(h)$. This is the expected rate of convergence in the presence of shocks.

To ensure that everything is working as expected, adjoint-weighted error analysis corresponding to the ground pressure cost functional (Eq. (9)) is presented and discussed in detail. The nearfield, ground signature (corresponding to $\sigma = \sigma_G = 1.8\pi$), the adjoint solution at the ground and nearfield corresponding to this cost functional are depicted in Fig. 3. The adjoint solution process begins at the ground level, and marches backward to the nearfield tracing the same space-time mesh that was used in the forward march of the primal solution. The adjoint solution at the ground level is an N-wave with twice the magnitude of the ground shock pressure owing to the form of the ground cost functional (See Eq. (9)) when scaled by the discretization spacing $\Delta\tau$. The adjoint solution at the nearfield is scaled by a factor of 450 to get to similar magnitudes for comparison against the other waveforms shown in the plot. During the primal march from the nearfield to the ground, the shock magnitude attenuates and the signature stretches according to the physics of the problem. The nearfield adjoint jumps to a constant value between the nearfield and ground shocks and remains continuous at that level through the shocks. This is consistent with the analytic adjoint solutions of the Burgers' equation [19, 20]. Additional details are

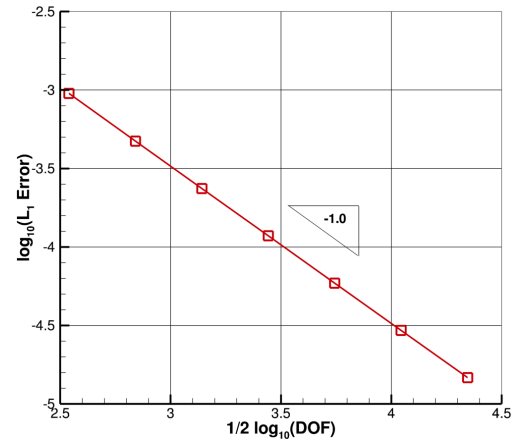


Fig. 2 L_1 norm error of the pressure perturbations with respect to the analytical solution.

also included and discussed in our earlier work [1].

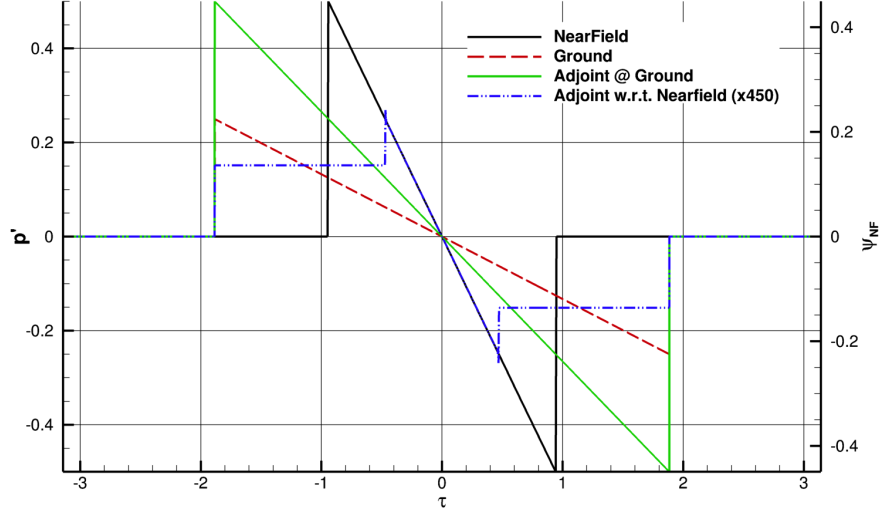
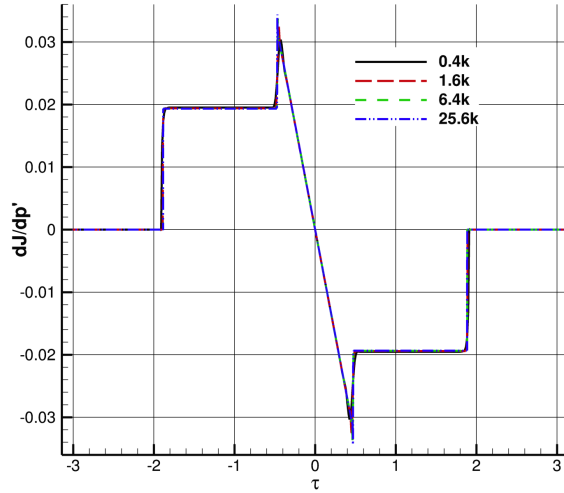
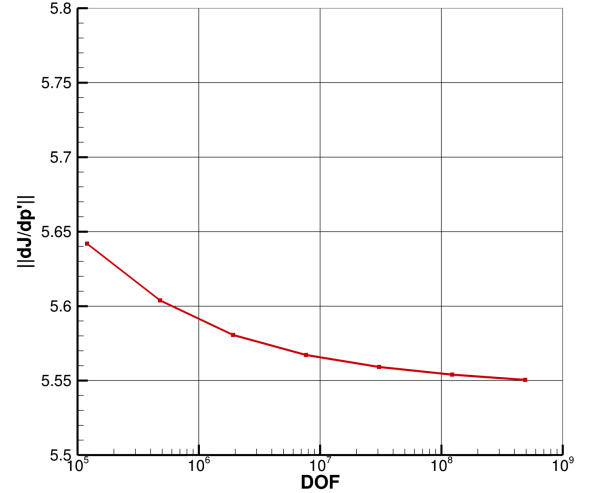


Fig. 3 Nearfield, ground signature and nearfield adjoint corresponding to the ground signature output.

Figure 4(a) depicts the adjoint solutions corresponding to the ground functional (Eq. (9)) along with the L_2 norm of the same with increasing uniform refinement as shown in Fig. 4(b). The adjoint solutions and their norms are converging without any spurious oscillations near features and at the boundaries, indicating that the discretization and boundary conditions (BCs) are all adjoint consistent.



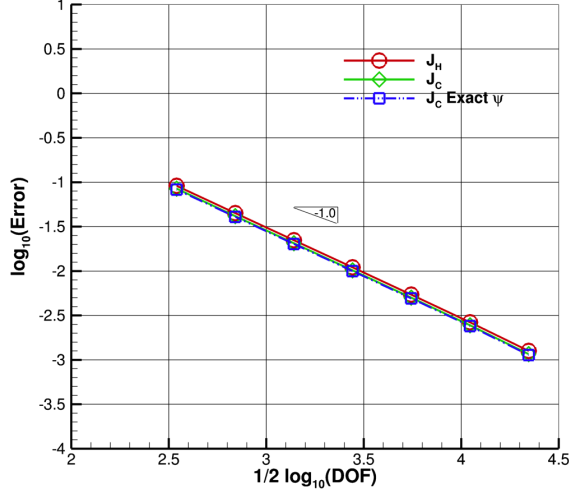
(a) Normalized nearfield adjoint sensitivities with increasing spatial discretizations.



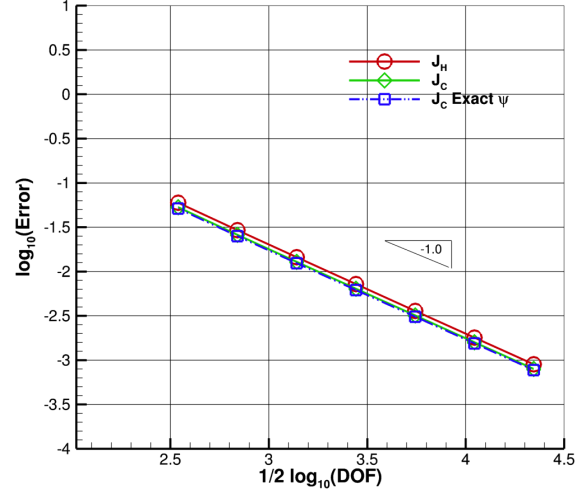
(b) Convergence of the normalized L_2 norm.

Fig. 4 Convergence of the nearfield adjoint sensitivities for the ground cost functional with uniform refinement.

Error analysis as described in section III is carried out to analyze the behavior of the underlying discretization errors for this problem with successive uniform mesh refinement. At the highest mesh refinement level, the signal has 25601 nodes, and the corresponding degrees of freedom in the entire space-time domain is $\approx 0.5 \times 10^9$. At this level, the sampling frequency used is ≈ 174 kHz. This is much more than ≈ 20 kHz, the threshold below which we cannot expect asymptotic convergence for the loudness metrics. Figures 5(a) and 5(b) show the error in predicting Eq. (9) with respect to the true analytical solution and the exact solution evaluated on the current mesh, respectively. While the error is slightly higher when the exact solution is computed analytically (Fig. 5(a)), both cases show that the error is $O(h)$, which is consistent with the L_1 norm of the ground signature pressures.



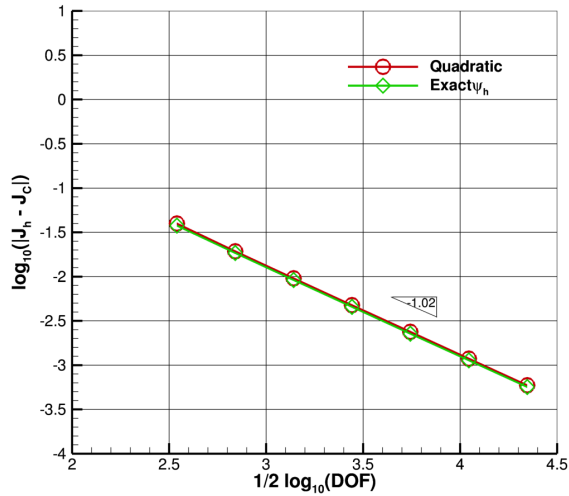
(a) Error w.r.t. analytical solution.



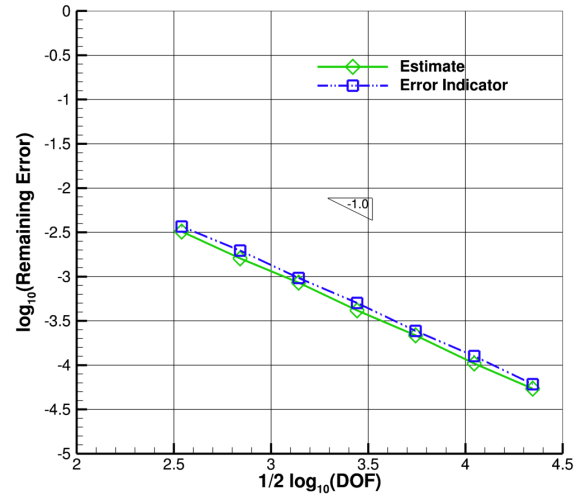
(b) Error w.r.t. exact solution evaluated on the current mesh.

Fig. 5 Error convergence of the functional J_H and its adjoint correction J_C for the ground pressure cost functional.

Figure 6(a) depicts the error behavior relative to the ground pressure cost functional (Eq. (9)) evaluated on the embedded mesh. Figure 6(b) depicts the remaining error in the simulations leveraging Eq. (8) and using biquadratic and bilinear reconstruction of the adjoint solutions on the coarse mesh onto the embedded mesh. In addition to the estimate of the remaining error, the absolute values of the errors at each step along the direction of propagation are summed, to result in what is termed as the *error indicator*. The absolute value operator prevents the error cancellation at each stage, and therefore the error indicator values are expected to be larger than the error estimate. In finite volume implementations, the error indicator provides an estimate of the error bound. For both cases, the error is $O(h)$, though the starting error is offset and lower by an order of magnitude for the remaining errors compared to the relative error with respect to the embedded mesh cost functional.



(a) Relative error w.r.t. embedded mesh cost functional.

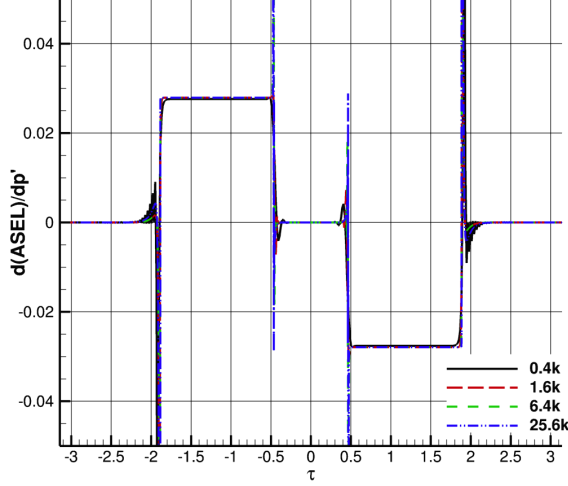


(b) Remaining error and bound estimate.

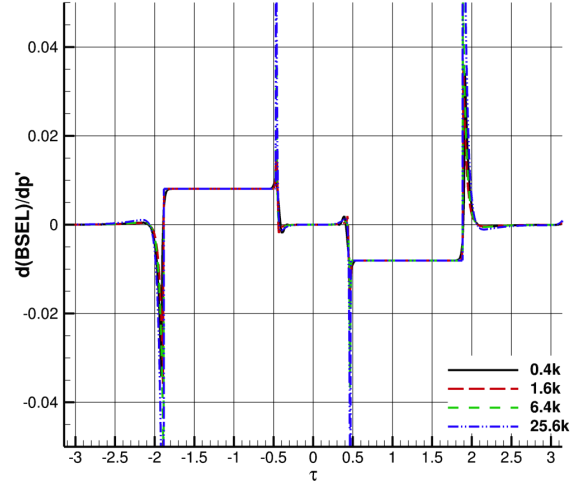
Fig. 6 Relative error w.r.t. embedded mesh and remaining error for ground pressure cost functional.

With behavior of the ground pressure cost functional well understood, we turn our attention to the A/B/C-weighted Sound Exposure Level (SEL) outputs. Figures 7(a), 7(b), and 7(c) show the nearfield adjoint solutions corresponding to A-SEL, B-SEL and C-SEL respectively with uniform refinement levels discussed earlier for the ground pressure

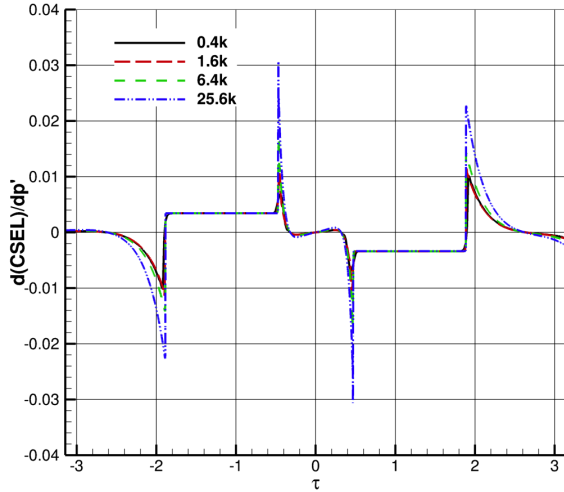
cost functional, Eq. (9). The adjoint solutions are converging without any spurious oscillations near shocks or at the boundaries of the domain, indicating that the discretization and BCs are all adjoint consistent for the loudness metrics as in the case of ground pressures. The behavior of the nearfield adjoint sensitivities corresponding to the loudness metrics is very similar to those computed for the ground functional (See Fig. 4(a)) except at the location of the shocks. In this region, a small perturbation between the nearfield shock and the ground shock locations is consumed by the moving shock during propagation and leads to a constant sensitivity value as depicted by the plateau in each of these plots. This behavior is similar to that of the adjoint solutions using the ground pressure functional (J_G). However, for the loudness metrics, the maximum sensitivity corresponds to the foot of the shock system and this gets progressively steeper from C-SEL to A-SEL. Figure 7(d) shows the comparison of the nearfield sensitivities corresponding to A/B/C-SEL loudness metrics and their magnitudes for the largest mesh run for this problem.



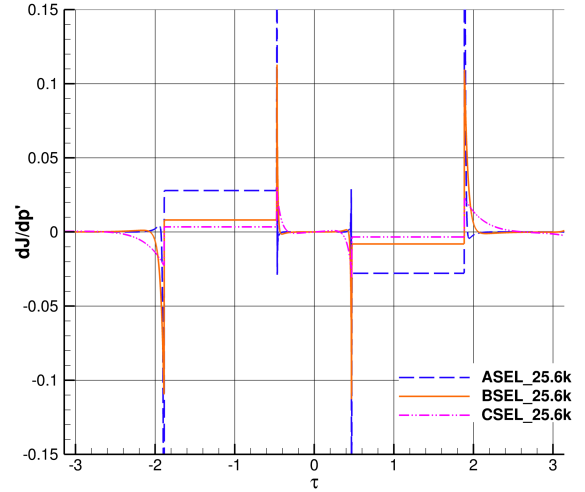
(a) Normalized nearfield adjoint sensitivities with increasing spatial discretization for A-SEL.



(b) Normalized nearfield adjoint sensitivities with increasing spatial discretization for B-SEL.



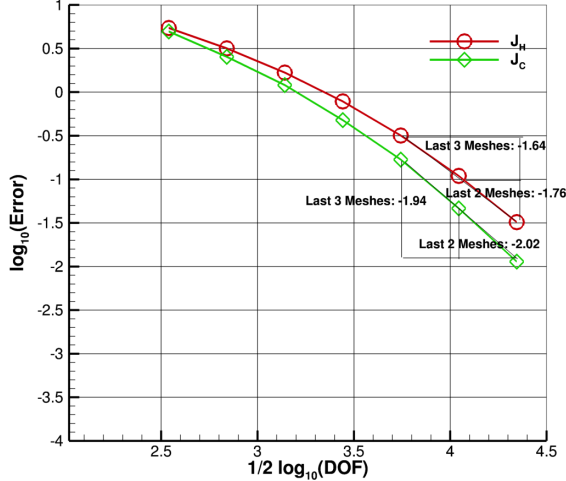
(c) Normalized nearfield adjoint sensitivities with increasing spatial discretization for C-SEL.



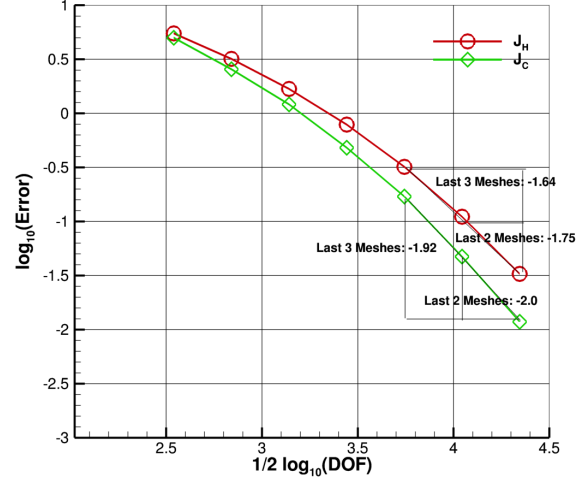
(d) Normalized nearfield adjoint sensitivities for loudness metrics at the highest spatial discretization of 25.6k nodes.

Fig. 7 Normalized nearfield adjoint sensitivities for the loudness metrics.

Figures 8(a) and 8(b) show the errors in decibels with respect to the analytical solution and exact solution evaluated on the current mesh, respectively. Unlike the ground pressures, these loudness metrics cannot be computed for discontinuous data; instead we substitute a fine-mesh simulation with 100k nodes in place of the exact analytic solution. As the mesh is uniformly refined, the error in A-SEL drops monotonically and the convergence is super-linear. The error convergence rate over the last three meshes and the final two meshes are shown, and both curves are nearing $O(h^2)$.



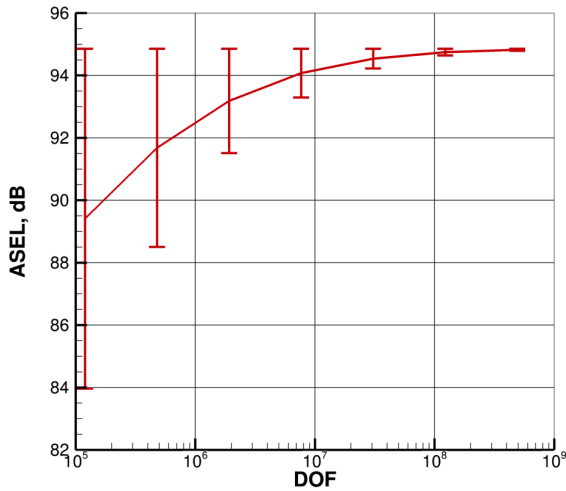
(a) Error w.r.t. analytical solution.



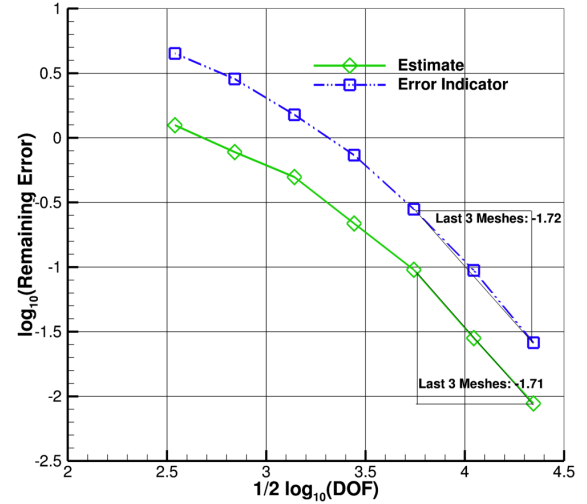
(b) Error w.r.t. exact solution evaluated on the current mesh.

Fig. 8 Error convergence of the functional J_H and its adjoint correction J_C for A-SEL.

The convergence of A-SEL is shown in Fig. 9(a) with error bars, where it is seen that the errors computed on the current mesh are within 0.05 decibels on the most refined mesh investigated for this problem. The remaining error is computed as an inner product of the residuals and difference between the higher- and lower-order reconstruction of the adjoint solutions along the entire propagation distance (see Eq. (8)). It is shown for A-SEL in Fig. 9(b). In addition to the error estimate, the error indicator comprising of the sum of absolute values of the remaining error at each level are plotted. As expected, the magnitude of the error indicator is larger (i.e., more conservative) relative to the estimate of the remaining error in the numerical simulations.



(a) A-SEL convergence with increasing refinement.



(b) Remaining error and bound estimate.

Fig. 9 Cost functional convergence with error bars and remaining error for A-SEL.

Figures 10(a) and 10(b) show the error in decibels corresponding to B-SEL as a function of the analytical and embedded cost functional respectively. The rate of convergence is faster than for A-SEL, and is shown to reach $O(h^{5/2})$ for the error when using reconstructed adjoint correction, and $\approx O(h^2)$ when directly using the cost functional on the current grid. The convergence behavior for B-SEL along with the error bars is shown in Fig. 11(a). Compared to A-SEL convergence, the initial errors are approximately half those corresponding to that of A-SEL, roughly 4 dB as opposed to slightly more than 10 dB for A-SEL. This is expected because A-SEL loudness metric computation weights the lower

frequencies more than B-SEL, and the underlying mesh needs to be more refined to achieve comparable discretization error levels. The error prediction on the final mesh investigated for this problem shows the errors to be less than 0.01 decibels, and even lower when using adjoint error correction. Figure 11(b) depicts the remaining error estimate as well as the estimate of the bound error, referred to as the error indicator. The rate of convergence on the final three and last two meshes are also overlaid on the plot, showing that the errors are nearing $O(h^{3/2})$ and $O(h^2)$ for the error estimate and the conservative bound estimate, respectively.

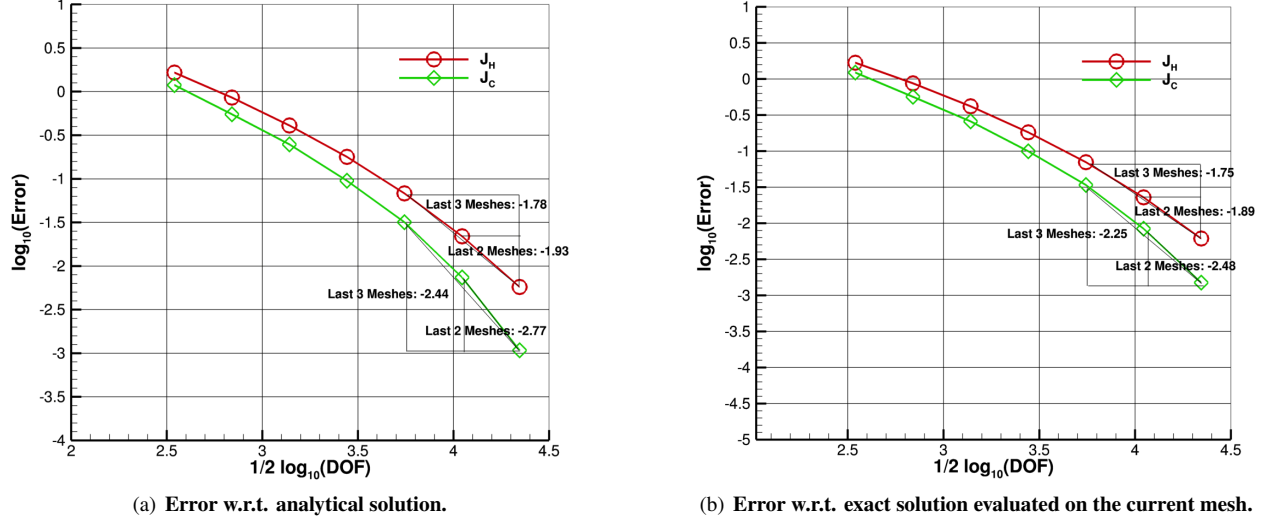


Fig. 10 Convergence of the functional J_H and its adjoint correction J_C for B-SEL.

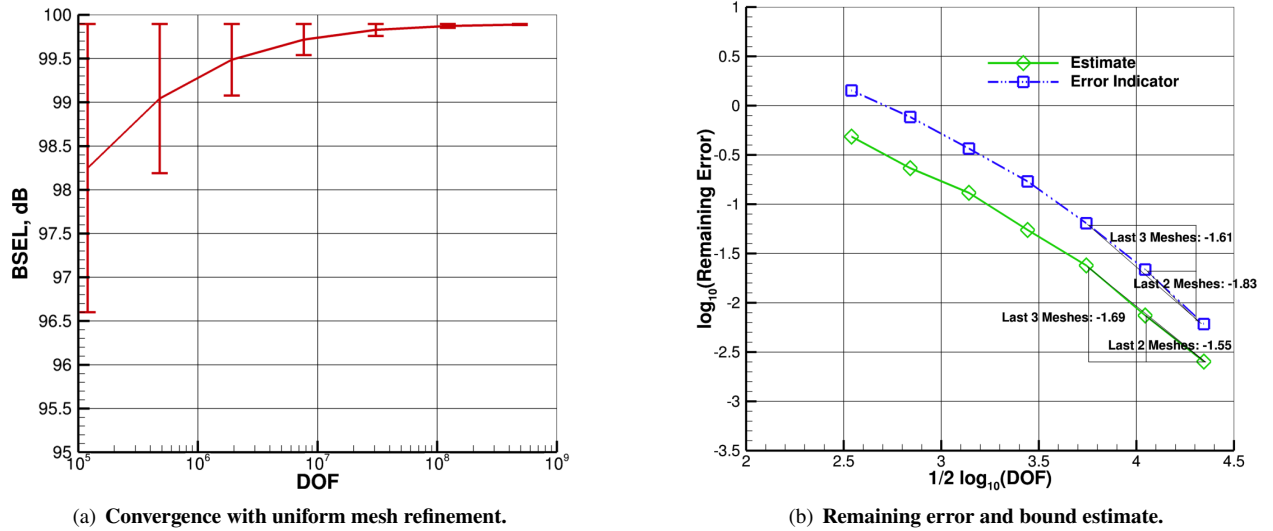
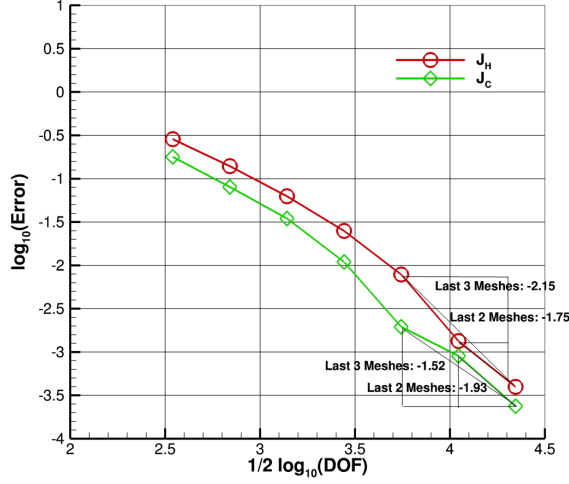


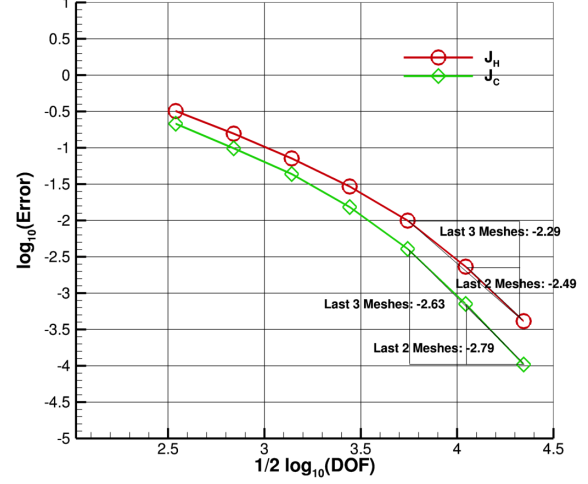
Fig. 11 Cost functional convergence with error bars and remaining error for B-SEL.

Similar error analysis was also conducted on C-SEL, and the results are shown in Figs. 12(a) and 12(b). Compared to the analytical cost functional, the errors reach within 0.0003 decibels while the errors are much smaller ($\approx 1 \times 10^{-4}$ decibels) with respect to the embedded mesh cost functional. The rates of convergence on the final three and last two meshes are also listed on the plots. These are approaching $O(h^2)$ and $O(h^{5/2})$ when evaluated with respect to the analytical and embedded solutions respectively.

The errors are much tighter for C-SEL, compared to other loudness metrics, even for smaller initial meshes as seen in Fig. 13(a). This is because C-SEL loudness computation does not weight lower frequencies of the sonic boom



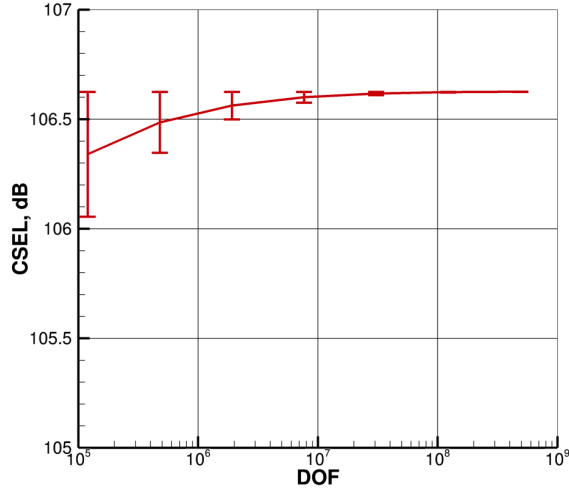
(a) Error w.r.t. analytical solution.



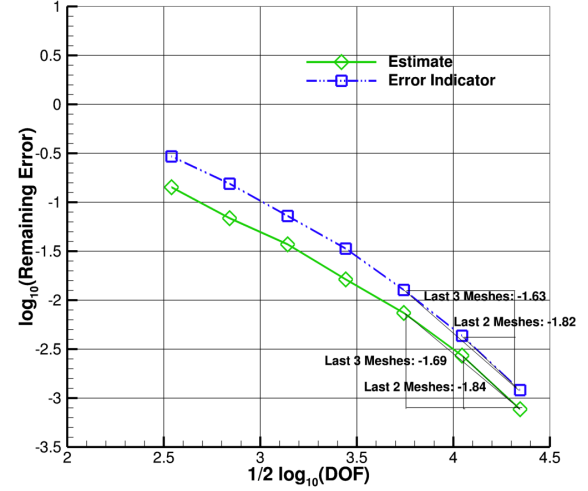
(b) Error w.r.t. exact solution evaluated on the current mesh.

Fig. 12 Convergence of the functional J_H and its adjoint correction J_C for C-SEL.

waveform and therefore the mesh refinement requirements are much less stringent to reduce discretization errors and achieve cost functional convergence. The remaining errors and estimates of the bound error are shown in Fig. 13(b), and the behavior is similar to the other loudness metrics.



(a) Convergence with uniform mesh refinement.



(b) Remaining error and bound estimate.

Fig. 13 Cost functional convergence with error bars and remaining error for C-SEL.

Figure 14 shows the errors in cost functionals corresponding to the ground pressures (9) and multiple loudness metrics (A/B/C-SEL) as a function of the degrees of freedom in the simulations. The error when using the ground pressure functional is the smallest to begin with and continues to diminish at a linear rate. The loudness metrics start with notably larger errors, but the rate of convergence is similar, and accelerates with successive mesh refinement. Among the loudness metrics, A-SEL has the highest error and C-SEL has the lowest. At the highest refinement level shown, the corrected cost functional values using the adjoint-weighted residual error estimates are approximately within 0.01, 0.001 and 0.0001 decibels compared to the analytical value for A-, B- and C-SEL cost functionals respectively. This is not surprising since the A/B/C-SEL metrics increasingly weigh lower frequencies which are progressively better resolved as the mesh is refined. Thus, the error magnitudes for these metrics follow intuition and shows well-behaved error convergence. Figure 15 depicts the remaining error and bound estimate for each of the four cost functionals shown.

The relative positions of each of the loudness metrics is the same as in Fig. 14 and shows the remaining errors and error indicators to be decreasing with uniform refinement of the space-time discretization.

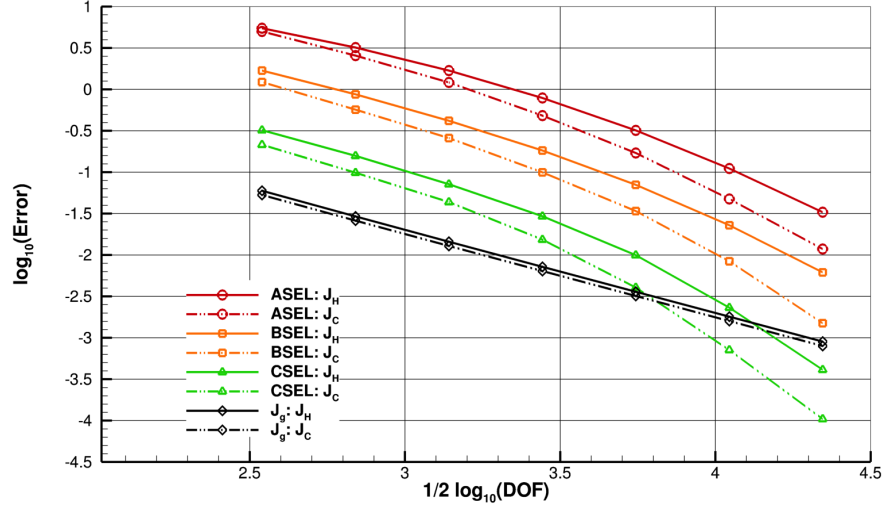


Fig. 14 Error w.r.t. exact solution evaluated on the current mesh corresponding to Eq. 9 and A/B/C-SEL.

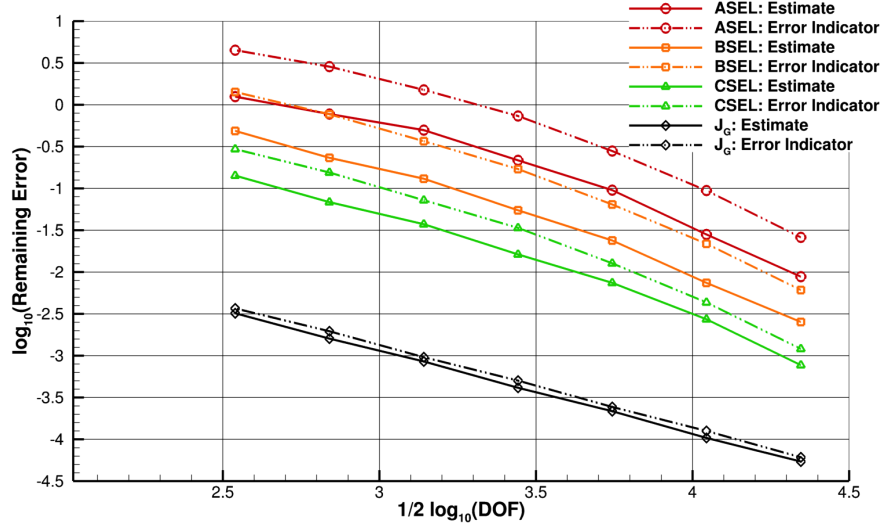


Fig. 15 Remaining error and estimate of the error bound corresponding to Eq. 9 and A/B/C-SEL.

B. Farfield Error Analysis: NACA 0003 Airfoil

For the second case, we consider flow simulation over a NACA 0003 airfoil with a 100 ft chord at $M_\infty = 1.3$, $\alpha = 0^\circ$, and altitude of 40000 ft. The nearfield pressures are extracted at 50 ft below the airfoil and sonic boom analysis is performed to test for error in farfield simulations. All features of the augmented Burgers' equation including viscosity and molecular relaxation are turned on for error analysis in this case while maintaining the assumption of uniform atmosphere. The initial and final pressure waveforms, along with the adjoint solutions at the ground ($\sigma = \sigma_G = 4.311$) and nearfield location are shown in Fig. 16 for the case of the ground pressure functional (Eq. (9)). Viscosity and relaxation have the effect of smoothing the adjoint solutions by providing a gradual rise near the shock features (See Fig. 3) as the mesh is uniformly refined; otherwise the behavior of the adjoint solutions is similar to farfield analysis using only nonlinearity.

The convergence of the ground cost functional (J_G from Eq. (9)) is plotted in Fig. 17(a) as a function of the degrees

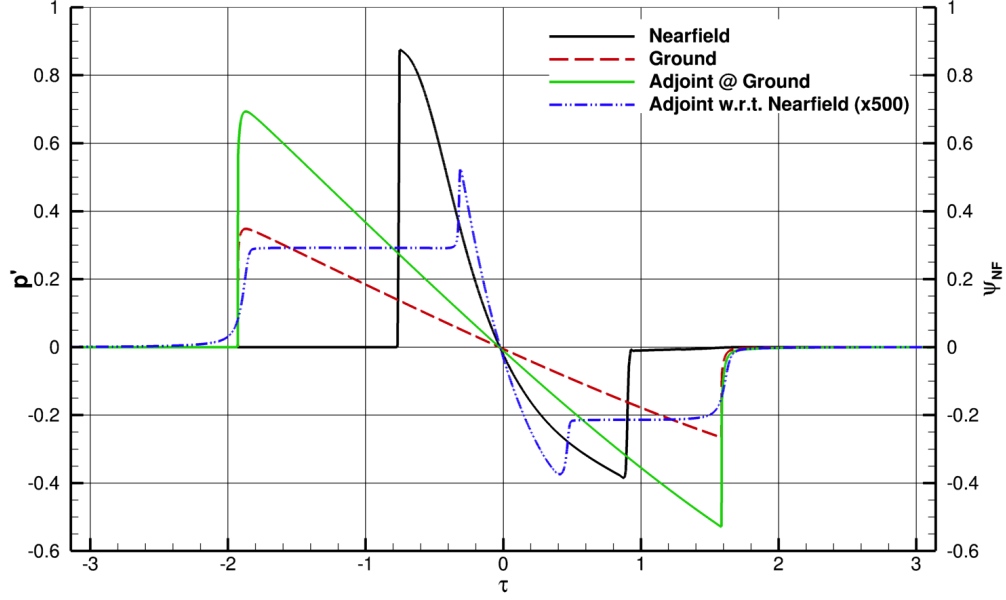


Fig. 16 Nearfield, ground signature and adjoint solutions corresponding to the ground signature output for the NACA 0003 airfoil.

of freedom in the space-time grid. For the most refined mesh shown, the spatial discretization has 16001 nodes, and corresponds to a sampling frequency of approximately 54 kHz. The performance of the error prediction from the adjoint-weighted residuals is shown in Fig. 17(b), where the errors are with respect to the cost functional computed over a refined spatial mesh of 64001 nodes. Error levels between the adjoint-weighted and current mesh estimates are initially comparable; however, by the final mesh the former is diminishing the error at an increased rate compared to the latter. The rates of convergence on the final three and last two meshes are also listed on the plots. These are approaching $O(h^3)$ and $O(h^{5/2})$ for the correction based on adjoint-weighted residuals and the cost functional computed on the current mesh, respectively.

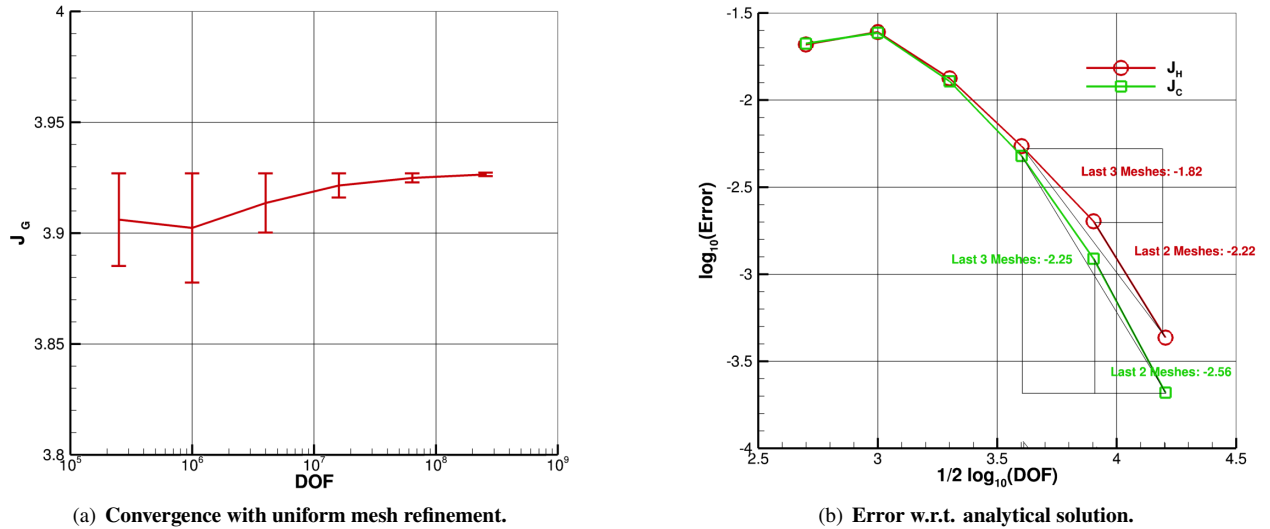


Fig. 17 Cost functional convergence with error bars and error behavior for J_G corresponding to NACA 0003 airfoil.

C. Farfield Error Analysis: X-59

For the final test case in this paper, we examine a complete case featuring the X-59 low-boom flight demonstrator aircraft [21] to assess the impact of discretization error in a realistically complex example. This aircraft is being purpose built as a technology demonstrator by integrating airframe, engine inlet, nozzle and control surface deflections to generate a nearfield flow solution that results in a benign sonic thump on the ground. The sonic boom propagation error analysis reported in this paper relies heavily on detailed nearfield CFD simulations computed using Cart3D [1]. The CFD simulations are performed at $M = 1.4$, $\alpha = 2.09^\circ$ and an altitude of 52026 ft. High fidelity sonic boom analysis requires computation of differential pressures on an imaginary cylindrical surface a certain distance away from the aircraft. This off-body distance is configuration dependent to sufficiently resolve the three dimensional effects near the aircraft, before augmented Burgers' equation-based farfield waveform propagation analysis is carried out from this cylindrical surface to the ground. For all results shown here, we use an off-body distance of 3 body lengths or roughly 270 ft from the aircraft to start the propagations, and include all terms of the Burgers' equation under standard atmospheric conditions (Ref. [22] for temperature/pressure, and Ref. [23] for relative humidity profiles) without winds.

Figure 18 depicts the nearfield, ground signature and the adjoint solutions corresponding to the ground signature output (Eq. (9)) along the undertrack (pressures directly below the aircraft) of the flight path. The nearfield pressure is made up of multiple shocks and expansions emanating from each component on the aircraft in such a way that the ground signature is a shaped pressure profile designed to have low noise at supersonic cruise. Also shown in the figure are the adjoint solutions at the ground and at the nearfield corresponding to J_G (See Eq. (9)). The behavior of the adjoint solution at the nearfield is similar to the N-wave solution except now there are multiple adjoint solution "plateaus" while being continuous across shocks. This is in line with expectation and is a natural extension of the analytical solution with a two-shock N-wave pressure profile. All plots shown here are with respect to the nondimensional spatial coordinate (τ) for easy comparison of all waveforms. The actual nearfield signal in dimensional terms is approximately 100 ft long, and the ground signature has a duration of about 150 ms from the start of the initial pressure rise till the pressure disturbances reach ambient conditions in the wake of the aircraft.

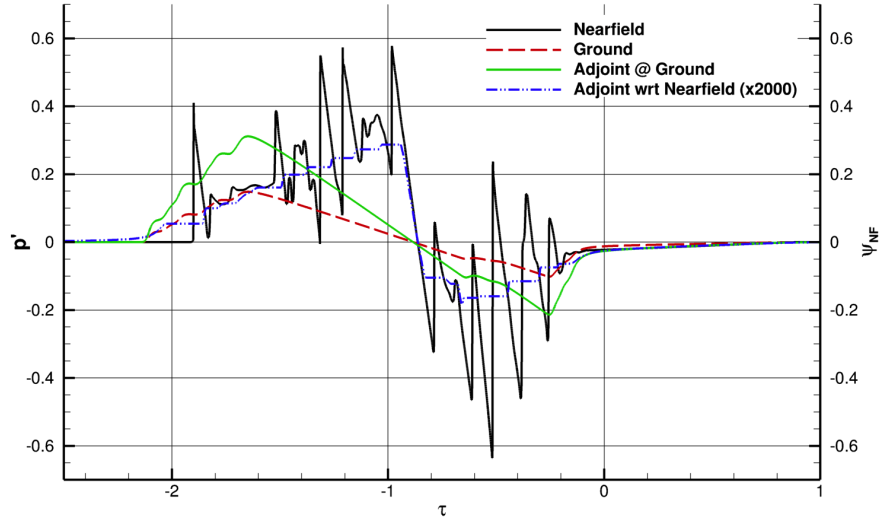
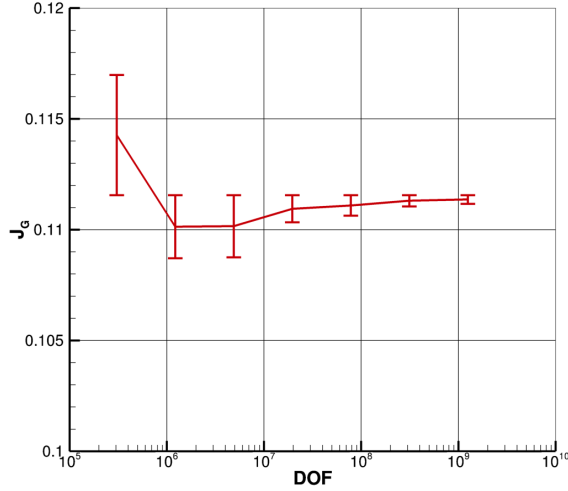


Fig. 18 Nearfield, ground signature and nearfield adjoint corresponding to the ground signature output.

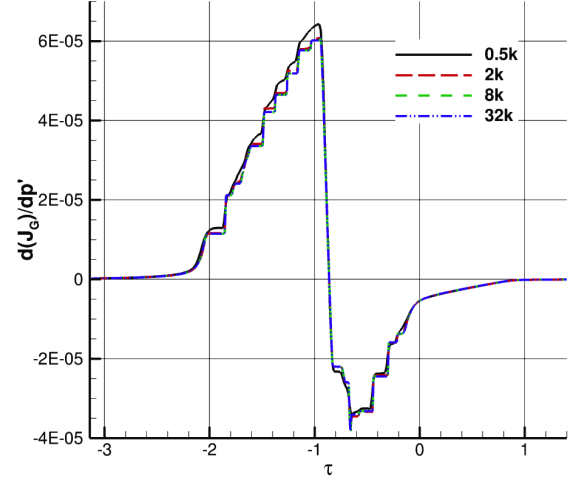
The convergence of the ground functional (J_G) with uniform refinement in the space-time discretization is depicted in Fig. 19(a). As the functional value converges, the error bounds also progressively tighten as expected. Figure 19(b) shows that the adjoint sensitivities are also well behaved for this cost functional at the spatial discretizations shown ranging from 501 to 32001 nodes.

Figure 20 compares the error estimate computed in the ground output on the current mesh (labeled J_H) with the error after the adjoint correction is applied (J_C). On the coarsest mesh, the adjoint-based correction term has large errors; however with increasing refinement, the error from adjoint-weighted residual correction gradually outperforms the error compared to the cost functional from the current mesh. On the last three meshes, both errors are $\approx O(h^{2/3})$.

The adjoint-weighted residual error analysis is carried out for A-SEL loudness metric and the convergence is depicted in Fig. 21(a) along with error bars at each mesh refinement level. The coarsest mesh has 501 nodes with



(a) Convergence with uniform mesh refinement.



(b) Normalized nearfield adjoint sensitivities with increasing spatial discretization for J_G .

Fig. 19 Convergence of J_G and its nearfield adjoint corresponding to X-59 pressure waveform.

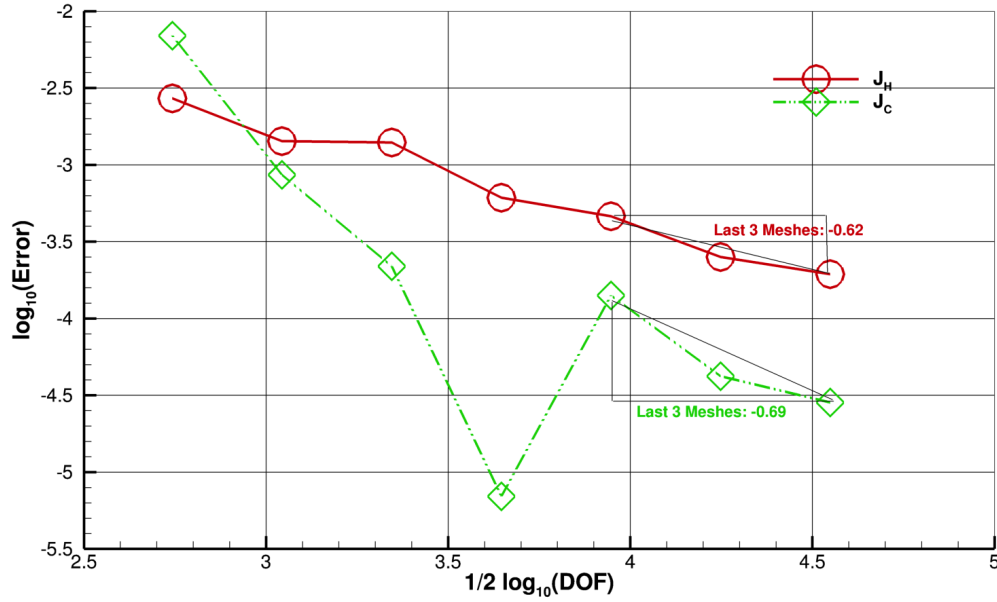
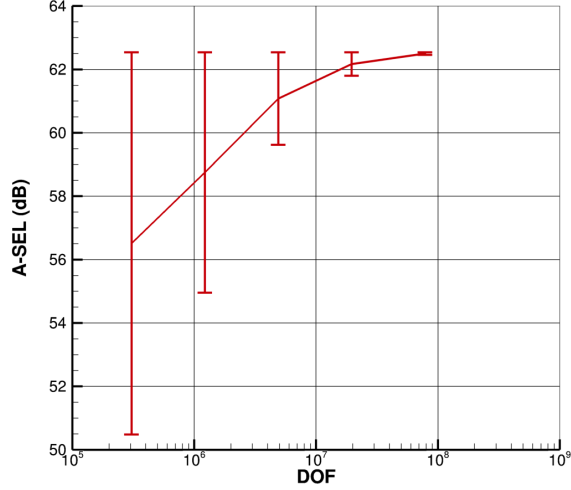


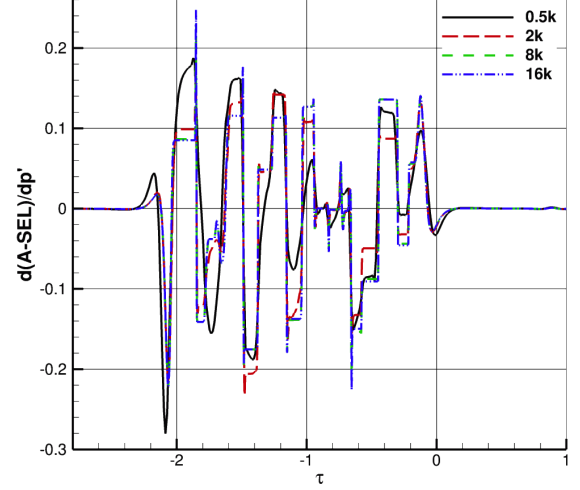
Fig. 20 Error w.r.t. exact solution evaluated on the current mesh for the X-59 J_G metric.

a sampling frequency of 1.5 kHz, while the finest mesh has 16001 corresponding to a sampling frequency of ≈ 53 kHz. At this refinement level, the predicted errors are within 0.01 decibels of the predicted levels at 64001 nodes and ≈ 210 kHz sampling frequency. Figure 21(b) shows the nearfield adjoint solutions corresponding to A-SEL for the X-59 at some selected spatial discretizations. When compared to A-SEL nearfield adjoint sensitivities for the N-wave case (See Fig. 7(a)), the magnitudes are an order of magnitude larger. Additionally, the multiple plateau regions in the nearfield adjoints is a reflection of the complex shock-expansion features in the nearfield and their movement as the pressure waveform propagates from the aircraft altitude to the ground. Most importantly, as the underlying discretization is refined, the nearfield adjoint waveforms converge to the same solution and the behavior is without any spurious oscillations within the domain and at the boundaries. The convergence of the ground signatures at these mesh refinements is shown in Fig. 22, where it is seen that the ground signatures converge sooner than the adjoint solutions.

The plot in Fig. 23 shows the predicted errors in A-SEL with respect A-SEL predicted on a refined mesh for the



(a) Convergence with uniform mesh refinement.



(b) Normalized nearfield adjoint sensitivities with increasing spatial discretization for A-SEL.

Fig. 21 Convergence of A-SEL and its nearfield adjoint corresponding to X-59 pressure waveform.

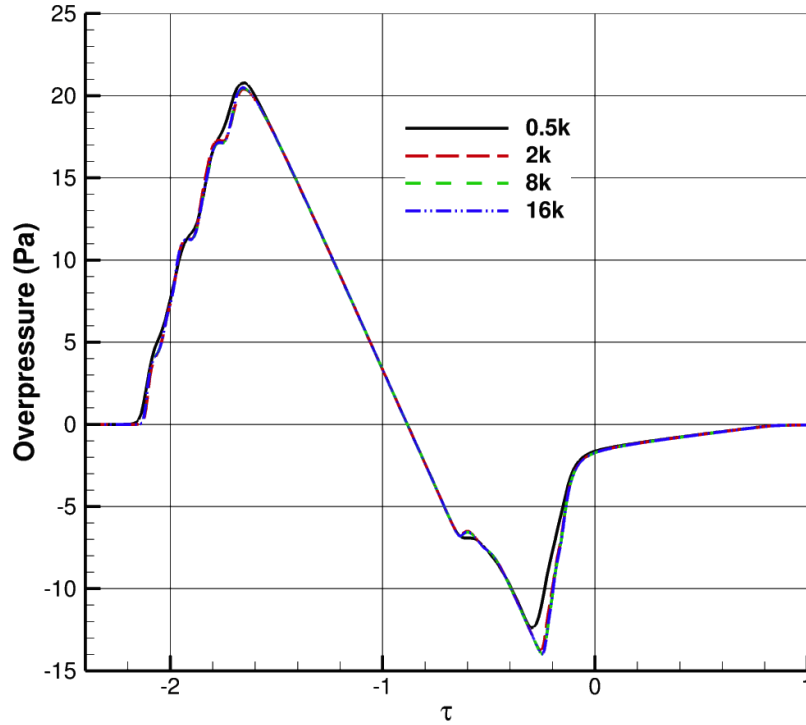


Fig. 22 Convergence of ground signatures with uniform refinement.

undertrack sonic boom signature corresponding to the X-59. At the lowest mesh refinement levels, the behavior of the adjoint-weighted residual error correction is insignificant; however with uniform mesh refinement, the significance and impact of the approach is clear. On the last three meshes looked at, the errors are $\approx O(h^3)$ and $\approx O(h^{5/2})$ with and without adjoint-weighted residual error correction. The behavior of the other loudness metrics is not shown but was observed to be similar. This example clearly demonstrates that at a sampling frequency of only ≈ 53 kHz, the propagation is already producing results accurate to within one hundredth of a decibel.

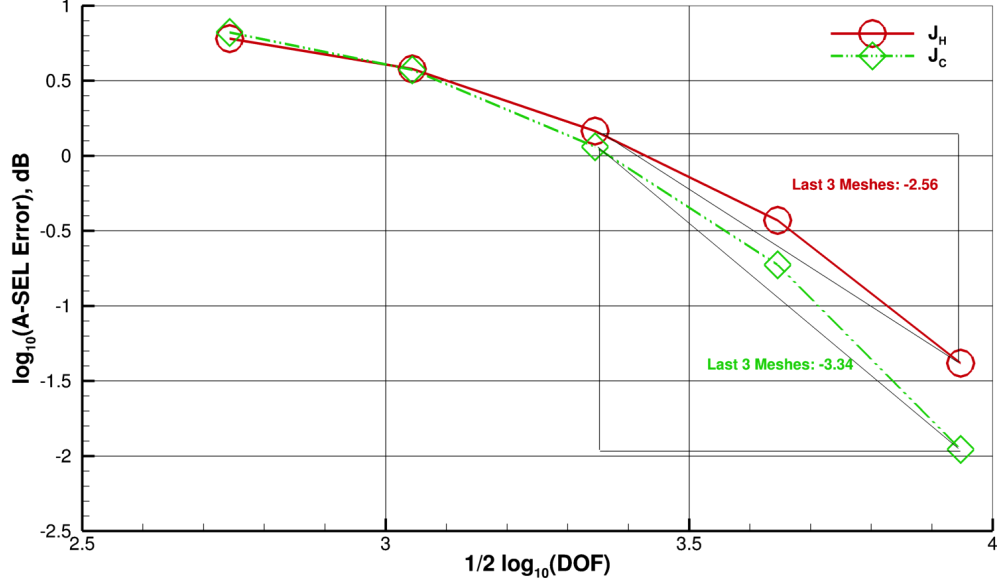


Fig. 23 Error w.r.t. exact solution evaluated on the current mesh for the X-59 A-SEL metric.

VI. Conclusions

The method of adjoint-weighted residuals has been applied to the farfield predictions of sonic booms to estimate the discretization errors in the simulations. The approach leverages updates to the asymptotically-consistent adjoint discretization for all the mechanisms involved in the underlying governing equations and provides an effective approach to estimate and control discretization errors in sonic boom farfield simulations. Correct estimation of errors by the numerical simulations is demonstrated by comparison against the exact, analytic N-wave solutions. For this problem, we show that the farfield pressure functionals, gradients and errors all converge at a first-order rate, which is the expected order of accuracy. When loudness metrics are considered, the error convergence accelerates as the underlying mesh is refined, both on the current mesh as well as using the adjoint-weighted residual correction. The other test cases demonstrate the amount of discretization error inherent at the current mesh level and informs the designer on whether additional refinement is needed to address and control those errors.

Near term plans include a detailed investigation of the errors from adjoint-weighted residuals for multiple loudness metrics and exploring opportunities for reducing the memory footprint and improving computational efficiency of both primal and adjoint solves in the farfield propagation. We also plan to leverage the updated asymptotically-consistent farfield adjoints and couple them with nearfield adjoints to allow reliable and robust aerodynamic shape optimization of low-boom aircraft.

Acknowledgments

The authors gratefully acknowledge support from NASA Aeronautics Mission Directorate (ARMD) Commercial Supersonic Technology (CST) project. The authors thank Dave Darmofal and Renato Trono Figueras, of Massachusetts Institute of Technology (MIT) for several insights and discussions related to the adjoint-consistent discretization of the molecular relaxation schemes as well as the notes pertaining to the analytical solution of an N-wave.

References

- [1] Nemec, M., Aftosmis, M. J., Rallabhandi, S. K., and Rodriguez, D. L., "Goal-Oriented Discretization Error Control in Coupled Nearfield-Farfield Low-Boom Simulations," AIAA Paper 2022-4085, 2022.
- [2] Rallabhandi, S. K., Nemec, M., and Aftosmis, M. J., "Recent Enhancements to Modeling Sonic Boom Propagation Using Augmented Burgers' Equation," AIAA Paper 2023-3727, 2023.
- [3] Darmofal, D. L., Allmaras, S. R., and Galbraith, M., "Sonic Boom Propagation using an Output-based Adaptive, Higher-order Finite Element Method," AIAA Paper 2023-1514, 2023.

- [4] Figueiras, R. T., Darmofal, D. L., Allmaras, S. R., and Galbraith, M. C., “Improvements to an Output-based Adaptive, Higher-order Finite Element Method for Sonic Boom Propagation,” AIAA Paper 2024-xxxx, 2024.
- [5] Rallabhandi, S. K., “Advanced Sonic Boom Prediction Using Augmented Burger’s Equation,” *Journal of Aircraft*, Vol. 48, No. 4, 2011, pp. 1245–1253.
- [6] Rallabhandi, S. K., “Propagation Analysis Using sBOOM for the Third AIAA Sonic Boom Prediction Workshop,” *AIAA Journal of Aircraft*, Vol. 59, No. 3, 2022.
- [7] Hamilton, M., and Blackstock, D., *Nonlinear Acoustics*, Academic Press, 1998. URL <https://books.google.com/books?id=k-kWozCE0jIC>.
- [8] Robinson, L. D., “Sonic Boom Propagation Through an Inhomogeneous Windy Atmosphere,” Ph.D. thesis, University of Texas at Austin, 1991.
- [9] Cleveland, R. O., “Propagation of Sonic Booms Through a Real, Stratified Atmosphere,” Ph.D. thesis, University of Texas at Austin, 1995.
- [10] Lee, Y.-S., and Hamilton, M. F., “Time-domain Modeling of Pulsed Finite-amplitude Sound Beams,” *The Journal of the Acoustical Society of America*, Vol. 97, No. 2, 1995, pp. 906–917.
- [11] Stevens, S. S., “Perceived Level of Noise by Mark VII and Decibels (E),” *Journal of the Acoustical Society of America*, Vol. 51, No. 2B, 1972, pp. 575–601.
- [12] American National Standards on Acoustics, “Design Response of Weighting Networks for Acoustical Measurements,” ANSI Standard S1.42-2020, American National Standards Institute, New York, NY, September 2020.
- [13] Roache, P. J., “Verification of Codes and Calculations,” *AIAA Journal*, Vol. 36, No. 5, 1998, pp. 696–702.
- [14] Becker, R., and Rannacher, R., “An Optimal Control Approach to *a posteriori* Error Estimation in Finite Element Methods,” *Acta Numerica*, Vol. 10, 2001, pp. 1–102. doi:10.1017/S0962492901000010.
- [15] Venditti, D. A., and Darmofal, D. L., “Grid Adaptation for Functional Outputs: Application to Two-Dimensional Inviscid Flows,” *Journal of Computational Physics*, Vol. 176, No. 1, 2002, pp. 40–69. doi:10.1006/jcph.2001.6967.
- [16] Nemec, M., and Aftosmis, M. J., “Toward Automatic Verification of Goal-Oriented Flow Simulations,” NASA TM-2014-218386, Ames Research Center, Aug. 2014. doi:2060/20150000864.
- [17] Rallabhandi, S. K., Nielsen, E. J., and Diskin, B., “Sonic-Boom Mitigation Through Aircraft Design and Adjoint Methodology,” *Journal of Aircraft*, Vol. 51, No. 2, 2014, pp. 502–510.
- [18] Pierce, A. D., “Acoustics: An Introduction to Its Physical Principles and Applications,” McGraw-Hill, Melville, NY, 1989, Chap. 11, pp. 587–588.
- [19] Alauzet, F., and Pironneau, O., “Continuous and discrete adjoints to the Euler equations for fluids,” *International Journal for Numerical Methods in Fluids*, Vol. 70, No. 2, 2012, pp. 135–157.
- [20] Jayasinghe, S., Darmofal, D. L., Galbraith, M. C., Burgess, N. K., and Allmaras, S. R., “Adjoint Analysis of Buckley-Leverett and Two-phase Flow Equations,” *Computational Geosciences*, Vol. 22, 2018, pp. 527–542. doi:10.1007/s10596-017-9708-2.
- [21] NASA, “Low-Boom Flight Demonstrator Project,” <https://www.nasa.gov/aeroresearch/programs/iasp/lbfd>, 2020. Online; accessed 20 May 2020.
- [22] “U.S. Standard Atmosphere,” *U.S. Government Printing Office, Washington, D.C.*, 1976.
- [23] Committee S1, Acoustics, “Method for Calculation of the Absorption of Sound by the Atmosphere, Annex C,” ANSI Standard S1.26-1995, American National Standards Institute, New York, NY, September 1995.