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REVISITING OPTIMAL GUIDANCE SOLUTION FOR VERTICAL LANDING AND TAKE-OFF

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The vertical landing and take-off are crucial phases of a rocket-powered spacecraft's powered descent and ascent trajectories, during which its motion is confined to a single (vertical) axis. It is well-known that the minimum-propellant guidance solutions for vertical landing and take-off in a vacuum, consist of only one coast and one full-thrust arc and no singular arc exists in these problems. However, an exact solution for the vertical descent problem is not found in the literature. In the present work, exact guidance solutions for the optimal vertical landing and take-off are presented that are valid for arbitrary boundary conditions. The time-based and switching function-based guidance law implementations are discussed and their performance is compared using lunar descent and ascent simulations.

INTRODUCTION

A rocket-powered spacecraft's powered descent and ascent trajectories are typically divided into multiple phases that include descent/ascent from/to orbit, pitch-up/down phases followed/preceded by vertical descent/ascent. During The vertical landing and take-off, the spacecraft or vehicle's motion is confined to a single (vertical) axis. For powered descent to a planet or moon, the vertical landing is the terminal part of the descent trajectory that enables soft-landing to the surface starting from an initial velocity and altitude. On the other hand, the vertical take-off is the first phase of the powered ascent trajectory to gain altitude quickly for staying clear of any surface features during subsequent pitch-over and ascent to orbit phases. Vertical take-off is also useful for launch vehicle ascent in dense atmosphere to reduce the aerodynamic stresses on the vehicle structure during the subsequent ascent to orbit phase.

The minimum-propellant guidance problem can be solved analytically for vertical landing and take-off in a vacuum. In both cases, it is well-known that the optimal solution consists of a single coast and full-thrust arc.[1] No singular arc exists in this problem and therefore, the optimal thrust for this problem is completely determined by Pontyragin's maximum principle.[5] For vertical landing, it has been shown that the minimum-propellant solution is equivalent to the minimum-time solution and consists of a coast arc (free-fall) followed by full thrust until touchdown. If impulsive thrusting is permitted, the optimal solution is free fall until touchdown at which time the velocity is reduced to zero using an impulsive burn instantaneously.

The classical analytical solution given by Meditch for the vertical landing is provided in the form of a switching function that determines the optimal timing of the transition from the free fall to the full-thrust phase.[2] The analytical expression for this switching function is derived by assuming

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that the propellant used for the vertical landing is small compared to the total mass of the vehicle. Additionally, the classical solution applies only to the special case of the zero final altitude and velocity. This approach is not desirable in the presence of sensor noise and biases as errors in the thrust ignition timing could compromise the soft touchdown requirement. A more recent treatment of the vertical landing problem is given in Reference [3].

In the present work, optimal guidance solutions for the vertical landing and take-off are derived that provide the exact analytical expression for the switching function without making any approximations. Additionally, explicit equations for direct computation of the burn and coast phase duration are derived that provides a time-based guidance law implementation as opposed to the switching-function based approach. These solutions work for completely arbitrary boundary conditions, i.e., the initial and final altitudes as well as velocities. The optimality of the free-fall and full-thrust control structure is also proved for completeness. The new solutions for the optimal vertical landing and take-off are validated using numerical simulations of a lunar landing and ascent vehicle. The performance of time-based and switching-function based guidance law implementations are also compared and contrasted.

DYNAMICAL MODEL

The equations of motion for the vertical descent or ascent of a rocket-powered vehicle assuming constant gravitational acceleration, are

$$\dot{h} = v, \tag{1}$$

$$\dot{v} = T/m - g, \tag{2}$$

$$\dot{m} = -\frac{T}{c}, \tag{3}$$

where c $(= g_0 I_{sp})$ is the effective exhaust velocity. The notation used in these equations are: h is the altitude, v is the velocity magnitude, m is the wet mass, T is applied thrust, g is the constant gravitational acceleration, I_{sp} is the constant specific impulse, g_0 is the standard acceleration due to gravity at Earth's surface. The propellant mass used by the rocket can be used as a metric for assessing the optimality of a trajectory solution. The propellant used by the rocket may be computed by first eliminating T from Eqs. (2) and (3) to obtain

$$\frac{\dot{m}}{m} = -\frac{\dot{v} + g}{c}, \tag{4}$$

which may be integrated from the initial to final time to obtain:

$$\ln \frac{m_f}{m_0} = -\frac{v_f - v_0 + g(t_f - t_0)}{c}. \tag{5}$$

Using this equation, the propellant mass used is simply

$$m_0 - m_f = m_0 \left[1 - \exp\left(\frac{v_0 - v_f - g(t_f - t_0)}{c}\right)\right]. \tag{6}$$

This result shows that propellant mass usage is a strictly increasing monotonic function of the time of flight ($t_f - t_0$) for a given value of the initial and final velocity difference. In case of vertical descent, for instance, the fuel-optimal solution for stopping the vehicle from an initial vertical velocity at a given altitude is also the solution that minimizes the time of flight. It is remarked that the same may not be true for a vertical ascent optimization problem if the vehicle's final velocity (at the targeted altitude) is not constrained.

OPTIMAL SOLUTION FOR VERTICAL DESCENT

The equations of motion described in the previous section are easily integrated analytically if either thrust or thrust acceleration during the entire time of flight is assumed constant. The optimal solution that minimizes the total propellant mass (or maximizes the final total mass of the vehicle) may consist of one or more coasting and/or thrusting arcs. In this section, an optimal thrust program is derived for the vertical descent problem that minimizes the propellant consumed. Unlike the existing solution,[2] the final altitude and velocity magnitude for landing are not assumed zero. This increases the applicability of this solution, especially for those cases that require a constant velocity phase just before the touchdown.

The optimization problem for vertical descent considered in this work, is defined as: find the fuel-optimal (more strictly, propellant-optimal) thrust profile for vertical landing of a rocket that can apply thrust only in a direction anti-parallel to the gravity vector, given the initial altitude and vertical velocity as well as targeted final altitude and vertical velocity. That is,

$$\begin{aligned} \textbf{Minimize} \quad & J(T) = (m_0 - m_f) \\ \textbf{Subject to} \quad & 0 \leq T \leq \mathcal{T},\ m_f < m_0 \\ \textbf{Boundary Conditions} \quad & h_0, v_0, m_0, h_f, v_f \end{aligned}$$

where $\mathcal{T}$ represents the maximum thrust. The equations of motion for this optimization problem are the same as in Eqs. (1)-(3). The targeted final velocity is specified in this optimization problem to ensure safe landing of the vehicle on the surface and it is, typically, set to a small number depending on the landing hardware characteristics. As a result, for this optimization problem, the minimum-propellant solution is equivalent to a minimum-time solution as per the discussion in the previous section. Therefore, the optimization problem may be reformulated as

$$\begin{aligned} \textbf{Minimize} \quad & J(T) = (t_f - t_0) \\ \textbf{Subject to} \quad & 0 \leq T \leq \mathcal{T},\ m_f < m_0,\ t_f > t_0 \\ \textbf{Boundary Conditions} \quad & h_0, v_0, m_0, h_f, v_f \end{aligned}$$

The cost function for this optimization problem is in Mayer form and therefore, the Hamiltonian is

$$H = \lambda_h v + \lambda_v (T/m - g) + \lambda_m (-T/c), \tag{7}$$

where λ_h, λ_v, and λ_m are the costates corresponding to the states h, v, and m, respectively. The costate differential equations are

$$\dot{\lambda}_h = -\frac{\partial H}{\partial h} = 0, \tag{8}$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_h, \tag{9}$$

$$\dot{\lambda}_m = -\frac{\partial H}{\partial m} = \lambda_v T/m^2. \tag{10}$$

The final time and mass of the vehicle are free, so the transversality condition requires

$$\lambda_m(t_f) = 0, \tag{11}$$

$$H(t_f) = -1. \tag{12}$$

The last condition gives $H = -1$ on the optimal solution for any time because H is not an explicit function of time. The thrust is constrained in this problem, therefore, Pontryagin's minimum principle may be applied to compute the control. First, Eq. (55) may be written as

$$H = \lambda_h v - g\lambda_v + ST, \tag{13}$$

where

$$S = \lambda_v/m - \lambda_m/c \tag{14}$$

is referred to as the switching function. For minimizing H, thrust is determined using

$$T = \begin{cases} \mathcal{T} & S < 0, \\ 0 & S > 0, \\ 0 < T < \mathcal{T} & S = 0. \end{cases} \tag{15}$$

The $S = 0$ condition represents the singular control arc during which S assumes zero value for a finite amount of time. It is known that this condition can not occur along an optimal solution irrespective of the boundary condition values.[2] This can be proven easily by considering that during a singular arc, the following two conditions must occur: $S = 0$ and $\dot{S} = 0$. It is easy to verify by substituting Eq. (63) in the two singular arc conditions that

$$\dot{\lambda_v} = 0 \quad \text{if } S = \dot{S} = 0.$$

This shows that λ_v must be a constant for singular arcs, which gives rise to three cases: $\lambda_v = 0$, $\lambda_v > 0$, $\lambda_v < 0$. All the three cases gives rise to the following contradictions in each case:

$$\lambda_v \begin{cases} = 0 & \Rightarrow \lambda_m = 0 \Rightarrow H = 0, \\ < 0 & \Rightarrow \lambda_m > 0 \Rightarrow S < 0, \\ > 0 & \Rightarrow \lambda_m < 0 \Rightarrow S > 0, \end{cases} \tag{16}$$

which proves that the singular arcs cannot exist for this problem. Next, it is proven that the switching between zero and max thrust can occur only once during $[t_0, t_f]$. This is accomplished by showing that S is a monotonic function of time. Differentiating Eq. (63) multiple times and using Eqs. (56)-(58) gives

$$\dot{S} = -\frac{1}{m}\lambda_h, \tag{17}$$
$$\ddot{S} = -\frac{T}{m^2 c}\lambda_h, \tag{18}$$
$$\dddot{S} = -\frac{2T^2}{m^3 c^2}\lambda_h. \tag{19}$$

The same pattern continues for the higher derivatives. These equations show that all the derivatives of the switching function are always of the same sign and therefore, S is a monotonic increasing or decreasing function of time. If the constant λ_h is zero, then all the derivatives are zero. These observations show that the optimal solution may be of four types: 1) a coast arc with no thrust from the initial to final time, 2) a full thrust burn arc from the initial to final time, 3) a full thrust burn arc followed by a coast arc, 4) a coast arc followed by a full thrust burn arc. The first case of zero thrust cannot happen in case of a landing because $h_f < h_0$ and $v_f < v_0$. The third case is also not valid for landing because the velocity at the end of the burn arc will be lower than v_f, resulting in a non-optimal solution. The other two cases are valid optimal thrust profiles for this problem.

The complete solution of the landing optimization problem includes expressions for the switching function S, the switching time (t_s) separating the coast arc and the burn arc, and the final time t_f in terms of the initial conditions and known quantities. These are derived in the sequel. During the coast arc from t_0 to t_s, the thrust is zero and the state equations (Eqs.(1)-(3)) are easily integrated to obtain

$$h_s = h_0 + v_0(t_s - t_0) - \frac{g}{2}(t_s - t_0)^2, \tag{20}$$
$$v_s = v_0 - g(t_s - t_0), \tag{21}$$
$$m_s = m_0, \tag{22}$$

where the subscript s represents quantities at the switching time (t_s). During the interval $[t_s, t_f]$, the thrust is set at $\mathcal{T}$ and the resulting expressions after integrating the state equations are

$$h_f = h_s + v_s(t_f - t_s) - \frac{g}{2}(t_f - t_s)^2 + c(t_f - t_s) + \frac{m_f c^2}{\mathcal{T}} \ln\left(\frac{m_f}{m_0}\right), \tag{23}$$
$$v_f = v_s - g(t_f - t_s) - c \ln\left(\frac{m_f}{m_0}\right), \tag{24}$$
$$m_f = m_0\left(1 - \frac{\mathcal{T}}{m_0 c}(t_f - t_s)\right). \tag{25}$$

Meditch's Approximate Solution

Meditch derived an approximate solution for vertical descent by assuming that the mass used during vertical descent is small compared to the total wet mass of the vehicle and he simplified the integrated equations of motion by using zero value for the final altitude and velocity of the vehicle.[2] In this solution, the natural log term in the integrated equations of motion is also replaced by its series expansion with terms retained up to second order only. Using these simplifications in Eqs. (23) and (24) along with using the expression of m_f from Eq. (25) yields

$$\begin{aligned} h_s &= -c(t_f - t_s) - v_s(t_f - t_s) + \frac{g}{2}(t_f - t_s)^2 \\ &\quad - \left(\frac{m_0\, c^2}{\mathcal{T}} - c(t_f - t_s)\right)\left(-\frac{\mathcal{T}}{m_0 c}(t_f - t_s) - \frac{\mathcal{T}^2}{m_0^2 c^2}(t_f - t_s)^2/2\right), \quad (26) \\ v_s &= g(t_f - t_s) - \frac{\mathcal{T}}{m_0}(t_f - t_s) - \frac{\mathcal{T}^2}{m_0^2 c}(t_f - t_s)^2/2, \quad (27) \end{aligned}$$

in which $\ln(1-x)$ term for any arbitrary x is replaced by its second-order series expansion $(-x - x^2/2)$. Substituting the second equation for v_s into the first results in

$$h_s = \frac{g}{2}\left(N_0 - 1\right)\left(t_f - t_s\right)^2, \quad (28)$$

where

$$N_0 = \frac{\mathcal{T}}{m_0 g} \quad (29)$$

is the thrust-to-weight ratio (note g may not be the standard acceleration due to gravity) at the initial time. It is clear that N_0 must be greater than unity to have a positive value for the thrust arc duration. Meditch does not provide the explicit expression for the thrust and coast arc durations, however, he uses the switching function to determine the time of initiation of the max thrust phase.[2] In his solution, the expression for the thrust arc duration from the last result is substituted in Eq. (27) to find an expression for the switching function:

$$S(h_s, v_s) = v_s + \sqrt{2\left(N_0 - 1\right) g h_s} + \frac{N_0^2}{\left(N_0 - 1\right)} \frac{g h_s}{c} = 0, \quad (30)$$

that is satisfied by all states represented by (h_s, v_s) and m_0 from which it is possible to reach the origin for a soft landing using the maximum thrust $\mathcal{T}$. It should be noted that the switching function gives a family of feasible solutions corresponding to given N_0 and c; and it is guaranteed that if a descent trajectory reaches any point on the switching curve defined by Eq. (30) in the phase plane then it will achieve soft landing if the constant thrust associated with that switching function (through N_0) is maintained during the thrusting phase. For a pure descent trajectory, h_s is non-negative and v_s is non-positive (the positive direction is pointing upwards from the surface). It is clear that on the switching curve defined by Eq. (30), v_s increases (or decreases in magnitude) as h_s decreases for $N_0 > 1$. It is guaranteed that once a descent vehicle with the initial thrust-to-weight ratio N_0 achieves a state (h_s, v_s) that satisfies the switching function equation $S(h_s, v_s) = 0$, it will reach the origin at the end of the burn phase. If the initial altitude and velocity of the vehicle does not satisfy the switching function equation then the descent starts with a free-fall phase for which the trajectory in the phase plane is given by

$$F(h_s, v_s) = v_s^2 - v_0^2 - 2g(h_0 - h_s) = 0, \tag{31}$$

which is obtained from Eqs. (20) and (21) after eliminating $(t_s - t_0)$. This equation shows that along the free-fall trajectory v_s decreases (or magnitude increases) as h_s decreases. For a viable solution, the two curves in the phase plane given by Eqs. (30) and (31) must intersect at (h_s, v_s).

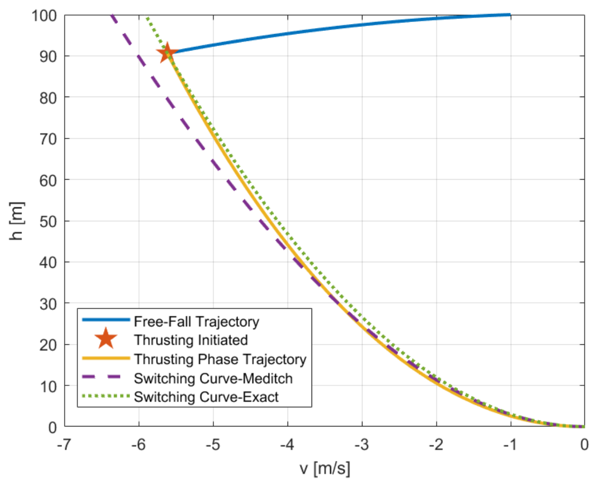

Figure 1. Switching and free-fall curves for a lunar descent case

Figure 1 shows the free-fall trajectory and Meditch's switching curve for a lunar descent vehicle with $N_0 = 1.1$, $c = 2942.1$ m/s and the vehicle is starting descent from 100 m altitude with 1 m/s downward velocity. For the chosen vertical descent boundary conditions and the vehicle's propulsion system specifications, it is seen that the switching function as computed using Meditch's approach does not intersect the free-fall trajectory curve at the correct point and the errors are due to the second-order approximations used in deriving Eq. (30). It is clear from Fig. 1 that the condition for a feasible solution is $v_0 \geq v_s$ for v_s satisfying $S(h_0, v_s) = 0$. That is, for the feasible descent solution

$$v_0 \geq -\sqrt{2\,(N_0 - 1)\,gh_0} - \frac{N_0^2}{(N_0 - 1)}\frac{gh_0}{c}. \tag{32}$$

Substituting v_0 from this result for v_s in Eq. (30), it is seen that the equivalent condition for a feasible solution is $S(h_0, v_0) \geq 0$. Based on these insights, the optimum thrust program may be devised that uses the switching function to determine the instant the maximum thrusting is initiated. In this program, the current altitude and velocity estimates of the vehicle are substituted in the switching function $S(h, v)$ and its value is constantly checked along the descent trajectory. At the initial condition, $S(h, v) > 0$ and the vehicle is on a free-fall trajectory until $S(h, v)$ becomes 0 and at that instant, the maximum thrusting is initiated. If $S(h, v) < 0$ at the initial condition then a feasible solution for soft landing $(h = v = 0)$ is not possible for the given initial conditions and thrust-to-weight ratio. It is noted from Eq. (30) that the switching function also becomes zero for $h_s = v_s = 0$.

The accuracy of Meditch's solution is affected by the magnitude of the ratio $(m_0 - m_f)/m_0$ and it is valid for final boundary conditions of zero altitude and velocity.[2] However, it may be advantageous to target a point slightly above the surface with a slightly negative velocity followed by a constant velocity descent phase until the altitude becomes zero and engines are shut down. This reduces touchdown state sensitivity to dispersions caused by navigation and control errors. The next subsection generalizes Meditch's solution for arbitrary boundary conditions and provides an exact guidance solution for the vertical descent problem without using any approximations.

Exact Guidance Solution for Arbitrary Boundary Conditions

The exact solution for the optimal descent problem is derived by first substituting Eqs. (20)-(21) into Eq. (23)-(24) to eliminate h_s and v_s:

$$h_f = h_0 + v_0(t_f - t_0) - \frac{g}{2}(t_f - t_0)^2 + c(t_f - t_s) + \frac{m_f c^2}{\mathcal{T}} \ln\left(\frac{m_f}{m_0}\right), \tag{33}$$

$$v_f = v_0 - g(t_f - t_0) - c \ln\left(\frac{m_f}{m_0}\right). \tag{34}$$

If $(t_f - t_0)$ is eliminated from the first equation using the second one, a nonlinear equation for the burn arc duration is obtained:

$$h_f = h_0 + \frac{v_0^2 - v_f^2}{2g} + c(t_f - t_s) - \frac{c\,v_f}{g} \ln\left(\frac{m_f}{m_0}\right) - \frac{c^2}{2g}\left(\ln\left(\frac{m_f}{m_0}\right)\right)^2 + \frac{m_f c^2}{\mathcal{T}} \ln\left(\frac{m_f}{m_0}\right), \tag{35}$$

which may be written as

$$(1 - x)\exp(x) + \frac{N_0}{2}x^2 + N_0\frac{v_f}{c}\,x - 1 - \frac{N_0}{2c^2}\left(v_0^2 - v_f^2 + 2g(h_0 - h_f)\right) = 0. \tag{36}$$

where

$$x \equiv \ln\left(\frac{m_f}{m_0}\right). \tag{37}$$

This equation may also be expressed in terms of a different solve-for variable y that is a linear function of the ratio m_f/m_0 as

$$\ln(1 - y)\left(\ln(1 - y) - \frac{2(1 - y)}{N_0} + \frac{2\,v_f}{c}\right) - \frac{2y}{N_0} - a = 0, \tag{38}$$

where

$$y \equiv 1 - \frac{m_f}{m_0}, \tag{39}$$

$$a \equiv \frac{v_0^2 - v_f^2 + 2g(h_0 - h_f)}{c^2}. \tag{40}$$

This final result is a nonlinear equation with a single unknown y and it may be easily solved using nonlinear root solving methods with bounds on y as $0 \leq y \leq 1$. If the natural log term is approximated by its series expansion up to second order in y and terms up to $\mathcal{O}(y^2)$ are retained, a quadratic equation is found for y:

$$4\left(1 - \frac{1}{N_0} - \frac{v_f}{c}\right) y^2 - \frac{8v_f}{c} y - 4a = 0, \tag{41}$$

and its positive root is given by

$$y = \frac{\frac{2v_f}{c} + \sqrt{\left(\frac{2v_f}{c}\right)^2 + a(1 - \frac{1}{N_0} - \frac{v_f}{c})}}{(1 - \frac{1}{N_0} - \frac{v_f}{c})}. \tag{42}$$

It is clear that for $N_0 > 1$ and $a > 0$, a positive value of y exist, which can be used to start the iterations for solving Eq. (38). The burn phase duration and total descent time are then immediately determined from y by using Eqs. (25) and (34), respectively, as shown next:

$$t_f - t_s = \frac{c}{g}\frac{y}{N_0}, \tag{43}$$

$$t_f - t_0 = \frac{v_0 - v_f - c\ln(1-y)}{g}. \tag{44}$$

These results reveal that the burn phase duration decreases with increasing N_0 and as a result, the propellant cost also decreases. The absolute minimum propellant cost occurs for infinite N_0 or implusive thrusting in which case the vehicle falls freely under the effect of gravity for the entire duration and at the instant of touchdown, an impulsive burn reduces the velocity to zero instantaneously. The velocity impulse required for soft landing for a vehicle under free fall may be derived from Eq. (31):

$$\Delta v = \sqrt{v_0^2 + 2g(h_0 - h_f)}, \tag{45}$$

which gives the lower bound on the propellant required for soft landing assuming that thrust is only used to slow down the vehicle.[2]

The expression for the switching function is useful to directly trigger the maximum thrust arc without any need for accurate timekeeping. The first step is to express burn phase duration in terms of h_s and v_s. Eliminating the natural log term from Eq. (23) using Eq. (24) yields

$$(t_f - t_s)^2 + 2\left(\frac{N_0(v_f + c) - c}{gN_0}\right)(t_f - t_s) + 2\left(\frac{cN_0(v_s - v_f) + gN_0^2(h_s - h_f)}{g^2N_0^2}\right) = 0, \quad (46)$$

which is a quadratic equation in $(t_f - t_s)$ and its positive root is given by

$$t_f - t_s = \frac{c\,y(h_s, v_s)}{gN_0}, \quad (47)$$

where

$$\begin{aligned} y(h_s, v_s) &= b_1 + \sqrt{b_1^2 - b_2}, & (48)\\ b_1 &= (1 - N_0) - N_0 v_f/c, & (49)\\ b_2 &= -2N_0((v_s - v_f)/c + gN_0(h_s - h_f)/c^2). & (50)\end{aligned}$$

It is noted that $y(h_s, v_s) \geq 0$. The switching function without any approximations may be obtained by substituting this result for $(t_f - t_s)$ in Eq. (24) to get:

$$S(h_s, v_s) = v_f - v_s + \frac{c\,y(h_s, v_s)}{N_0} + c\ln\left(1 - y(h_s, v_s)\right) = 0, \quad (51)$$

It is noted that this expression of the exact switching function is valid for any arbitrary boundary conditions. The condition for a feasible solution is $v_0 \geq v_s$ for v_s satisfying $S(h_0, v_s) = 0$. In order to have real values for $y(h_s, v_s)$ and $\ln(1 - y(h_s, v_s))$, we must also have $b_1^2 - b_2 \geq 0$ and $y(h_s, v_s) < 1$. It can be experimentally checked that $S(h_s, v_s) > 0$ at the initial condition (similar to Meditch's switching function). Therefore, the optimal guidance scheme is same as that was for the Meditch's solution. The free-fall phase is started if the initial conditions satisfy $S(h_0, v_0) > 0$. During the free fall, the altitude and velocity estimates are used to check the value of the switching function and when $S(h_s, v_s) = 0$ is satisfied, the maximum thrusting (corresponding to N_0) is initiated that guarantees the achievement of the desired soft landing targets assuming no navigation and control errors. Similar to the case of Meditch's solution, the switching function is parameterized by m_0 (through N_0), which is difficult to measure during the flight. Cherry[4] defined a parameter τ that along with c can be inferred onboard the flight from the accelerometer measurements using the least-square approach. The relation between the thrust acceleration a_T and τ at time t is given by:

$$\begin{aligned} a_T(t) &= \frac{\mathcal{T}}{m_0 - |\dot{m}|(t - t_0)},\\ &= \frac{c}{\tau - (t - t_0)}, \end{aligned} \quad (52)$$

where

$$\tau = \frac{m_0}{|\dot{m}|}. \quad (53)$$

It is clear that τ has a unit of time and its value is equal to the time it takes for a rocket-propelled vehicle which is all propellant and no structure, would completely consume itself.[4] In the switching function expression given by Eq. (51), N_0 may be replaced by τ using

$$N_0 = \frac{c}{\tau g}. \tag{54}$$

Figure 1 shows the exact switching function curve computed using Eq. (51) in the phase plane for a lunar lander with $N_0 = 1.1$, $c = 2942.1$ m/s and initial altitude and downward velocity as 100 m and 1 m/s, respectively. The results show that for this lunar vertical descent case, for which Meditch's expression for the switching function did not reproduce the results accurately, the exact expression for the switching function correctly computed the state at which the thrusting is required to be initiated for the desired soft landing. In other words, the switching function curve computed using the exact expression intersects the free-fall trajectory curve at the correct location (computed using numerical optimization) in the phase plane. Figures 2 and 3 show the performance of Meditch's solution and the exact switching-function based guidance solution for a lunar landing vehicle descending from 100 m altitude with 1 m/s initial downward velocity and final soft landing targets as zero for both final altitude and velocity. The mass of the vehicle was fixed at 10000 kg with Isp as 300 s but the thrust-to-weight ratio was varied from 1.0001 to 2. The results show Meditch's approximated switching function has severely degraded performance for lower thrust-to-weight ratios.

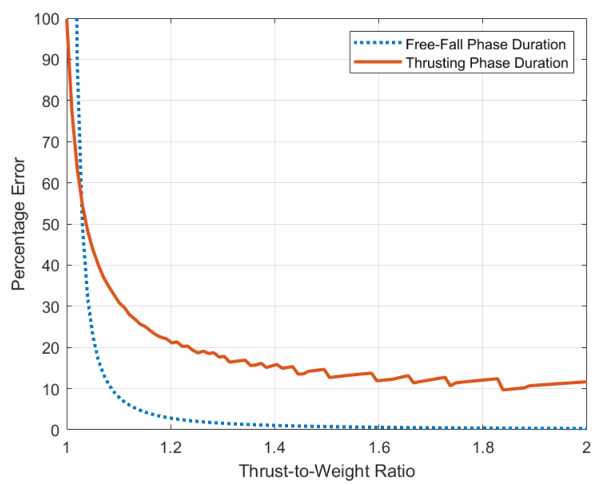

Figure 2. Accuracy of Meditch's solution for determining coast and burn arc durations compared to the exact solution for a vertical lunar landing case.

In this subsection, the exact guidance equations for an optimal vertical landing are derived. There are two equivalent ways to implement the guidance scheme using this solution given the terminal conditions and the vehicle specifications: either compute explicitly the coast and thrusting phase time duration by solving the nonlinear equation from Eq. (38) or evaluate the switching function expression from Eq. (51) along the free-fall trajectory and initiate thrusting when its value becomes zero. Both schemes produce the same results in the absence of navigation and control errors, however, the latter method has advantages owing to the fact that in contrast to the former, the switching function does not depend on the initial conditions except for the initial thrust-to-weight ratio. As a

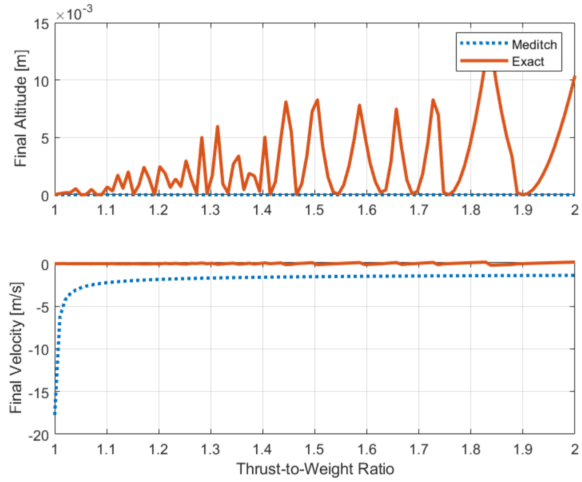

Figure 3. Guidance performance of Meditch's solution and the exact solution for a range of thrust-to-weight ratios of an example lunar lander.

result, the switching-function based guidance implementation is agnostic to dispersion in the trajectory states at the end of the free-fall phase and thus, its accuracy is not affected in case the lander is not exactly following the free-fall trajectory. Figure 4 shows the performance of time-based and switching function-based guidance implementation for a lunar lander whose trajectory is deviated from the pure free-fall trajectory. This simulation uses a lunar landing case for a vehicle with $N_0 = 1.1$ and $Isp = 300$ s. The optimal guidance is used to soft-land the vehicle from an altitude of 100 m with 1 m/s downward velocity with guidance targets as 1 m and -1 m/s for soft landing. Random noise with samples taken from a normal distribution with zero bias and standard deviations of 0.1 m and 0.01 m/s for altitude and velocity states, respectively, is added in the vehicle's states at the end of the free-fall phase. A total of 500 simulation runs were carried out and the results in Fig. 4 show that the switching-function based guidance is able to achieve the desired soft-landing targets exactly whereas the time-based implementation has a degraded performance. The switching function method is akin to a closed-loop guidance method because it inherently provides a solution starting from the current state (except for the initial mass) of the lander. Since the switching function is associated with a particular thrust-to-weight ratio, multiple switching curves can be generated offline and used during the flight to compensate for the navigation and control errors. Alternatively, the switching function approach can also be used to solve for thrust magnitude to find a descent solution starting from the current vehicle state to the desired soft landing targets.

OPTIMAL SOLUTION FOR VERTICAL ASCENT

An optimal thrust program for vertical ascent is derived in this section that minimizes the propellant consumed to achieve a target altitude. The derivation follows closely to that in the previous section for vertical descent. It is remarked that the considered vertical ascent problem is slightly different from the well-known Goddard's problem in which the final altitude of the rocket is maximized given the fixed quantity of propellant.[5]

The optimization problem for vertical ascent considered in this work, is defined as: find the fuel-optimal (more strictly, propellant-optimal) thrust profile for vertical ascent of a rocket that can apply thrust only in a direction anti-parallel to the gravity vector, given the initial mass, altitude,

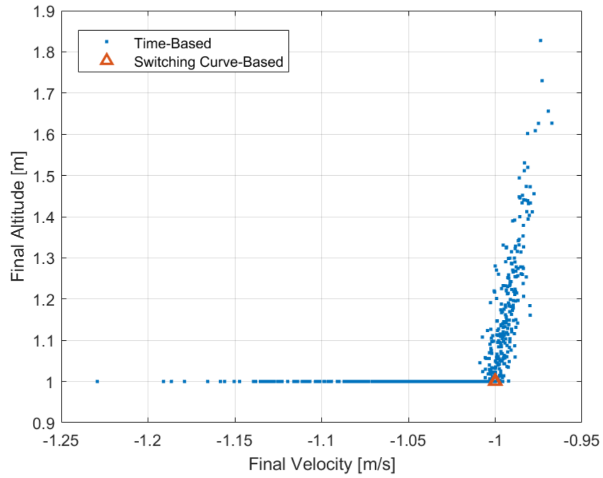

Figure 4. Performance comparison of time-based and switching function-based exact guidance law in a 500-sample monte carlo run for achieving desired lunar soft-landing targets in the presence of noise during the free-fall phase.

and velocity (typically the initial altitude and velocity are zero) as well as targeted final altitude with the final vertical velocity unconstrained. This problem was solved previously by using Miele's optimization of linear integrals method by Hull and it was shown that the optimal thrust profile is a maximum thrust phase followed by a coast phase.[1] The optimization problem may be succinctly described as

$$\begin{aligned} \textbf{Minimize} \quad & J(T) = (m_0 - m_f) \\ \textbf{Subject to} \quad & 0 \leq T \leq \mathcal{T},\, m_f < m_0 \\ \textbf{Boundary Conditions} \quad & h_0, v_0, m_0, h_f \end{aligned}$$

where $\mathcal{T}$ represents the maximum thrust. The consequence of the final velocity not being constrained is that unlike vertical descent, the minimum-time problem is not equivalent to minimum-propellant problem for vertical ascent as seen from Eq. (6). This conclusion makes intuitive sense because the fuel-optimal profile minimizes the gravity losses by using maximum thrust to gain enough velocity as quickly as possible so that during the subsequent coast phase, the vehicle reaches the desired altitude with zero velocity. It is also remarked that if the final velocity is specified, then the minimum-time and minimum-fuel problems for the ascent case are equivalent as in the case of vertical descent.

The equations of motion for this optimization problem are given in Eqs. (1)-(3). The Hamiltonian and the costate equations are same as that for the vertical descent case (see Eqs. (55) and (56)-(58)).

$$H = \lambda_h v + \lambda_v (T/m - g) + \lambda_m(-T/c), \tag{55}$$

$$\dot{\lambda}_h = -\frac{\partial H}{\partial h} = 0, \tag{56}$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_h, \tag{57}$$

$$\dot{\lambda}_m = -\frac{\partial H}{\partial m} = \lambda_v T/m^2. \tag{58}$$

The final time and mass of the vehicle are free and in addition, for ascent the final velocity is also free. The cost function is selected to be the propellant mass usage. Therefore, the transversality condition requires

$$\lambda_v(t_f) = 0, \tag{59}$$

$$\lambda_m(t_f) = -1, \tag{60}$$

$$H(t_f) = 0. \tag{61}$$

The last condition gives $H = 0$ on the optimal solution for any time because H is not an explicit function of time. The thrust is constrained in this problem, therefore, Pontryagin's minimum principle may be applied to compute the control. First, Eq. (55) may be written as

$$H = \lambda_h v - g\lambda_v + ST, \tag{62}$$

where

$$S = \lambda_v/m - \lambda_m/c \tag{63}$$

is the switching function. For optimal solution, H may be minimized in all cases by choosing thrust as

$$T = \begin{cases} \mathcal{T} & S < 0, \\ 0 & S > 0, \\ 0 < T < \mathcal{T} & S = 0. \end{cases} \tag{64}$$

The $S = 0$ condition represents the singular control arc and to determine singular control, the higher derivatives of S are computed:

$$S = \lambda_v/m - \lambda_m/c = 0, \tag{65}$$

$$\dot{S} = -\lambda_h/m = 0, \tag{66}$$

$$\ddot{S} = 0. \tag{67}$$

The second equation is only satisfied when $\lambda_h = 0$ and from Eq. (57) and (59), it is implied that $\lambda_v = 0$, which in turn means $\lambda_m = 0$ as well. This shows that once the system is on a singular arc, the control will have no effect and the system will stay on the singular arc till the final time. However, from Eq. (60), we have seen that λ_m must be non-zero at the final time, which is a contradiction and thus, the singular arcs cannot exist for this problem. Additionally, the preceding equations show that S is a monotonic function of time because λ_h is a constant and thus, this problem has at most one switching between the zero and maximum thrust. As a consequence, the only choices for the optimal solutions for the ascent problem are either maximum thrust from the initial to final time or the maximum thrust phase followed by a coast phase. The former case cannot be optimal because the vehicle will reach the targeted altitude with non-zero velocity and thus, more propellant is spent than was necessary to reach the desired altitude. It is remarked that the coast phase without a thrust phase and a coast phase followed by a thrust phase are not valid solutions for obvious reasons. If the ascent vehicle already has a positive initial velocity, the coast phase followed by the maximum thrust phase is still not optimal because in this case also the vehicle will reach the desired altitude with non-zero final velocity. It is easily proven from the condition $H(t_f) = 0$ that the final velocity for the optimal solution is zero. It is interesting to note that if the coast phase is not allowed then a definite minimum-propellant solution for the ascent problem exists with a unique optimal control (or thrust) value that is not necessarily the maximum thrust available. However, this minimum-propellant solution (without a coast phase) can be improved further by adding a coast phase after the thrust phase and using maximum available thrust as clear from the discussion above.

The complete solution of the ascent optimization problem includes expressions for the switching function S, the switching time (t_s) separating the burn and the coast arcs, and the final time t_f in terms of the initial conditions and known quantities. During the burn phase from t_0 to t_s, the thrust is at maximum and the state equations (Eqs.(1)-(3)) are easily integrated to obtain

$$h_s = h_0 + v_0(t_s - t_0) - \frac{g}{2}(t_s - t_0)^2 + c(t_s - t_0) + \frac{m_f c^2}{\mathcal{T}} \ln\left(\frac{m_f}{m_0}\right), \tag{68}$$

$$v_s = v_0 - g(t_s - t_0) - c \ln\left(\frac{m_f}{m_0}\right), \tag{69}$$

$$m_f = m_0 \left(1 - \frac{\mathcal{T}}{m_0 c}(t_s - t_0)\right), \tag{70}$$

where the subscript s represents quantities at the switching time (t_s). During the interval $[t_s, t_f]$, the thrust is set at zero and the resulting expressions after integrating the state equations are

$$h_f = h_s + v_s(t_f - t_s) - \frac{g}{2}(t_f - t_s)^2, \tag{71}$$

$$v_f = v_s - g(t_f - t_s). \tag{72}$$

Substituting these relations in Eqs. (68) and (69) to eliminate h_s and v_s, we get

$$h_f = h_0 + v_0(t_f - t_0) - \frac{g}{2}(t_f - t_0)^2 + c(t_s - t_0) + \left(\frac{m_f c^2}{\mathcal{T}} - c(t_f - t_s)\right) \ln\left(\frac{m_f}{m_0}\right), \tag{73}$$

$$v_f = v_0 - g(t_f - t_0) - c \ln\left(\frac{m_f}{m_0}\right), \tag{74}$$

from which $(t_f - t_0)$ may be eliminated by substituting the second equation into the first one to get

$$\ln(1-y)\left(\ln(1-y) + \frac{2}{N_0} - \frac{2v_0}{c}\right) + \frac{2y}{N_0} + a = 0, \tag{75}$$

where

$$y \equiv 1 - \frac{m_f}{m_0}, \tag{76}$$

$$a \equiv \frac{v_0^2 - v_f^2 + 2g(h_0 - h_f)}{c^2}. \tag{77}$$

This nonlinear equation in terms of the single unknown y is analogous to the similar equation derived earlier (see Eq. (38)) for the descent problem. The burn and coast phase durations are immediately determined from y using

$$t_s - t_0 = \frac{c}{gN_0} y, \tag{78}$$

$$t_f - t_0 = \frac{v_0 - c\ln(1-y)}{g}. \tag{79}$$

The expression for the switching function can also be found as was obtained for the descent case. The usefulness of the switching-curve based guidance implementation is evident from the previous section. The steps for deriving the switching function for the ascent problem follow closely to that of the descent problem, however, the expressions are much simpler in the ascent case. First, Eq. (72) is substituted in Eq. (71) to eliminate $(t_f - t_s)$:

$$S(h_s, v_s) = h_f - h_s - \frac{v_s^2 - v_f^2}{2g} = 0. \tag{80}$$

The optimal guidance scheme using the switching function is: the thrust phase is initiated at the initial time with maximum thrust and the switching function value is checked using the current altitude and velocity estimates of the vehicle. At the initial condition ($h_s = v_s = 0$), $S(h_s, v_s) = h_f > 0$. At the exact instant when $S(h_s, v_s)$ evaluates to zero, the engines are shutdown and the vehicle enters the coast phase and reaches the targeted altitude when its final velocity reaches zero. In the phase plane, the switching curve intersects the burn phase trajectory exactly at the point where the thrust must be cut off for the optimal solution. The switching curve in the phase plane is coincident with the coast phase trajectory for the ascent problem. It is noted that for the ascent problem the switching curve is not parameterized by the initial thrust-to-weight ratio of the vehicle. Figure 5 shows the switching curve computed using Eq. (80) along with the burn and coast phase trajectories in the phase plane for an example lunar vertical ascent. The results show that the switching curve intersects the burn-phase trajectory once and that provides the time at which the ascent vehicle shuts down the engine and enters the coast phase till it reaches the final targeted altitude of 100 m with zero vertical velocity.

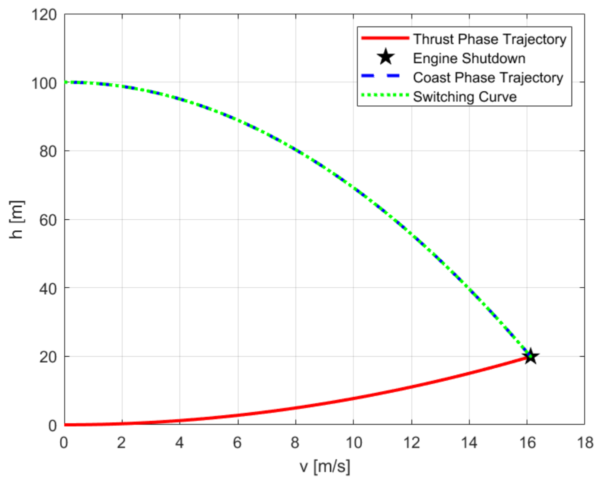

Figure 5. Burn and coast phase trajectories along with switching curves in the phase plane for a lunar ascent example.

CONCLUSION

The fuel-optimal solutions for the vertical descent and ascent of a rocket-powered vehicle save significant propellant over non-optimal solutions because the gravity losses are maximum during these phases of flight. The classical solution for optimal vertical descent suffers from performance degradation when the propellant mass during vertical descent is not negligible compared to the total wet mass of the landing vehicle as shown by simulations of a sample lunar lander performing vertical descent from 100 m altitude. It was shown that the accuracy of the classical solution deteriorates with decreasing thrust-to-weight ratio of the landing vehicle. In this work, the complete exact solutions for the vertical descent and ascent of a rocket-powered vehicle in vacuum are presented. The exact solutions can be used to implement the optimal guidance law for vertical landing/take-off by either computing the duration of the coast and burn arcs or by evaluating the switching function along the descent/ascent trajectory to determine the switching time. The former method involves root-solving of a single nonlinear equation whereas the latter method involves only the evaluation of a nonlinear function and thus, is more computationally efficient for onboard implementation. The switching-function based guidance is also agnostic to noise during the free-fall phase of the vertical descent unlike the time-based guidance implementation. Unlike the descent case, the switching function for optimal ascent is not dependent on the initial thrust-to-weight ratio of the vehicle and therefore, is completely independent of the vehicle's hardware characteristics.

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