Optimal Control using Composite Bernstein Approximants

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***Abstract*— In this work, we present composite Bernstein polynomials as a direct collocation method for approximating optimal control problems. An analysis of the convergence properties of composite Bernstein polynomials is provided, and beneficial properties of composite Bernstein polynomials for the solution of optimal control problems are discussed. The efficacy of the proposed approximation method is demonstrated through a bang-bang example. Lastly, we apply this method to a motion planning problem, offering a practical solution that emphasizes the ability of this method to solve complex optimal control problems.**

I. Introduction

Optimal control problems are often too complex to be solved analytically, thus requiring the application of numerical methods to find a solution. Numerical methods for solving optimal control problems are typically separated into two primary classifications: direct and indirect methods. Indirect methods involve the transformation of the optimal control problem into a boundary-value problem, resulting in a system of differential equations. These methods rely on the calculus of variations and Pontryagin's maximum principle, often resulting in solutions that can be harder to implement for complex problems. Alternatively, direct methods discretize the continuous optimal control problem into a nonlinear programming (NLP) problem, allowing the solution to be found via collocation and pseudospectral (PS) methods.

A common approach to solving optimal control problems using direct methods is to employ PS methods, which have been used to solve a wide range of optimization problems [1], [2], due in large part to their spectral (exponential) rate of convergence. Various collocation methods have been introduced, including Legendre-Gauss-Lobatto (LGL) PS for finite time-horizon problems [3], [4], as well as Legendre-Gauss (LG) PS [5] and Legendre-Gauss-Radau (LGR) PS [6], [7] for infinite time-horizon problems. Although PS methods have gained extensive adoption solving real-world optimal control problems, they are not without their own set of limitations.

The primary drawbacks of PS methods lie in the discretization and approximation of the optimal control problem [8]. When discretizing the state and input of the optimal control problem, the constraints are enforced only at the specific discretization points, with no guarantees on constraint satisfaction between each point. Additionally, PS methods struggle to approximate discontinuous solutions due to the Gibbs phenomenon, which can be observed as oscillations around a jump discontinuity. There are methods of reducing the effects of the Gibbs phenomenon, though the most powerful of which require that the location of the discontinuities are known *a priori* [9], a condition that is not feasible for many optimal control problems. Confronted with these challenges, researchers have been motivated to explore more robust alternatives that are not plagued by these limitations.

One alternative approach is Bernstein polynomial approximation, which has many properties that aid in the accurate solution of optimal control problems [10]. Due to geometric properties of Bernstein polynomials, constraint satisfaction can be guaranteed throughout the solution, not solely at the discretization points. Furthermore, Bernstein polynomials do not suffer from the Gibbs phenomenon, allowing for smooth approximations of discontinuous solutions. With these properties, Bernstein polynomials are able to approximate the solution for the whole time horizon, without inducing oscillations at jump discontinuities. Composite Bernstein polynomials, a series of Bernstein polynomials segmented together at various knotting locations along the time horizon, enjoy these same properties as Bernstein polynomials. However, unlike single Bernstein polynomials, composite Bernstein polynomials are able to accurately approximate both smooth and discontinuous solutions, and converge to the solution at a faster rate than single Bernstein polynomials, as presented in this work. Due to the introduction of these knotting locations, composite Bernstein polynomials are capable of accurately approximating discontinuous solutions, such as bang-bang control inputs, which are a common occurrence among optimal control problems. This paper extends our previous work [11]–[13] on Bernstein polynomial-based optimal control to composite Bernstein polynomials, and proposes a new direct method to solve optimal control problems using these approximants.

This paper is structured as follows: in Section II we present the Bolza-type optimal control problem. Section III introduces composite Bernstein polynomials and their convergence properties. Section IV presents the proposed direct method with the discretization of the Bolza-type optimal control problem. In Section V, a method to determine the quantity of knots is discussed. Numerical examples are provided in Section VI, including a collision avoidance motion planning problem. Finally, conclusions are presented in Section VII.

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II. Problem Statement

Consider the Bolza-type optimal control problem:

***Problem** 1 (Problem P):*

$$\min_{\boldsymbol{x}(t),\boldsymbol{u}(t),t_f} J(\boldsymbol{x}(t),\boldsymbol{u}(t),t_f) = E(\boldsymbol{x}(0),\boldsymbol{x}(t_f),t_f) + \int_0^{t_f} F(\boldsymbol{x}(t),\boldsymbol{u}(t))dt \tag{1}$$

subject to

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{u}(t)), \quad \forall t \in [0,t_f], \tag{2}$$

$$\boldsymbol{e}(\boldsymbol{x}(0),\boldsymbol{x}(t_f),t_f) = \boldsymbol{0}, \tag{3}$$

$$\boldsymbol{h}(\boldsymbol{x}(t),\boldsymbol{u}(t)) \leq \boldsymbol{0}, \quad \forall t \in [0,t_f], \tag{4}$$

where $J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}$, $E : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, $\boldsymbol{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, $\boldsymbol{e} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}^{n_e}$, and $\boldsymbol{h} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_h}$. ∇

For Problem P, J represents the Bolza-type cost function, E is the initial point and end point cost, and F is the running cost; Equation (2) governs the dynamics of the system, Equation (3) imposes boundary conditions, and Equation (4) ensures compliance with the inequality constraints of the system.

III. Composite Bernstein Approximation

In this section, we describe the process of function approximation through composite Bernstein polynomials, and present key convergence results. The Bernstein basis for polynomials of degree N over the domain $I_k = t \in [t_{k-1}, t_k]$ is defined as

$$b_{j,N}^{[k]}(t) = \binom{N}{j} \frac{(t-t_{k-1})^j (t_k - t)^{N-j}}{(t_k - t_{k-1})^N}$$

for $j = 0, ..., N$, with

$$\binom{N}{j} = \frac{N!}{j!(N-j)!}$$

An Nth order Bernstein polynomial $x_N^{[k]}(t)$ defined over the domain I_k is a linear combination of $N+1$ Bernstein basis polynomials of order N, i.e.,

$$x_N^{[k]}(t) = \sum_{j=0}^{N} \bar{x}_{j,N}^{[k]} b_{j,N}^{[k]}(t), \qquad \forall t \in I_k, \forall k = 1, ..., K \tag{5}$$

where $\bar{x}_{j,N}^{[k]}$ for $j = 0, ..., N$ are referred to as Bernstein coefficients (or control points). Given time *knots* t_k, $k = 0, \ldots, K$, with $t_0 < t_1 < \ldots < t_K$, a *composite Bernstein polynomial* is defined as follows:

$$x_M(t) = x_N^{[k]}(t), \quad \forall t \in I_k, \forall k = 1, ..., K \tag{6}$$

The derivative of a composite Bernstein polynomial $x_M(t)$ is defined for the closed set $[0, t_f]$, and is computed as

$$\dot{x}_M(t) = \sum_{j=0}^{N-1} \sum_{i=0}^{N} \bar{x}_{i,N}^{[k]} D_{i,j}^{[k]} b_{j,N-1}^{[k]}(t), \quad t \in I_k \tag{7}$$

where $D_{i,j}^{[k]}$ is the (i,j)th entry of the Bernstein differentiation matrix

$$\boldsymbol{D}_{N-1}^{[k]} = \begin{bmatrix} -\frac{N}{t_k - t_{k-1}} & 0 & \ldots & 0 \\ \frac{N}{t_k - t_{k-1}} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\frac{N}{t_k - t_{k-1}} \\ 0 & \ldots & 0 & \frac{N}{t_k - t_{k-1}} \end{bmatrix} \in \mathbb{R}^{(N+1)\times N}, \ \forall k = 1, ..., K.$$

The definite integral of a composite Bernstein polynomial $x_M(t)$ is computed as

$$\int_0^{t_f} x_M(t)dt = \sum_{k=1}^{K} w^{[k]} \sum_{j=0}^{N} \bar{x}_{j,N}^{[k]} \tag{8}$$

where $w^{[k]} = \frac{t_k - t_{k-1}}{N+1}$, $\forall k = 1, \ldots, K$.

Composite Bernstein polynomials can be used to approximate functions.

***Definition** 1:* Let $x(t)$ be a function defined over $[0, t_f]$. Let the time knots satisfy $t_0 = 0$, $t_K = t_f$, and $|t_k - t_{k-1}| \leq \frac{C_t}{K}$ for all $k = 1, \ldots, K$, where $C_t > 0$ is independent of K. Let

$$t_{k,j} = t_{k-1} + \frac{j}{N}(t_k - t_{k-1})$$

Then, the *composite Bernstein approximant* for $x(t)$ is a composite Bernstein polynomial $x_M(t)$ with coefficients $\bar{x}_{j,N}^{[k]} = x(t_{k,j})$.

Next, we introduce convergence results for composite Bernstein polynomials.

***Lemma** 1:* Let $x(t) \in \mathcal{C}^2([0, t_f])$. Let $x_M(t)$ be the composite Bernstein polynomial approximation of $x(t)$. The following bound holds:

$$\|x_M(t) - x(t)\| \leq \frac{A}{K^2 N}$$

for all $t \in [0, t_f]$, where

$$A = \frac{C_t^2}{8} \max_{\tau \in [t_{k-1}, t_k], k=1,\ldots,K} |\ddot{x}(\tau)|$$

Proof: The proof of Lemma 1 is presented in [14]. Additionally, an outline of the proof is given in Appendix I.

***Lemma** 2:* Let $x(t) \in \mathcal{C}^{r+2}(0, t_f)$ for some r $\in \mathbb{Z}^+$. Let $x_M(t)$ be the composite Bernstein polynomial approximation of $x(t)$. The following bound holds:

$$\|x_M^{(r)}(t) - x^{(r)}(t)\| \leq \frac{B_1}{K^2 N} + \frac{B_2}{KN} + \frac{B_3}{N}$$

where $x^{(r)}(t)$ denotes the rth derivative of $x(t)$, and B_1, B_2, and B_3 are constants independent of both K and N:

$$\begin{aligned} B_1 &= \frac{C_t^2(r^2+1)}{2} \max_{\tau \in [t_{k-1}, t_k], k=1,\ldots,K} \|x^{(r+2)}(\tau)\| \\ B_2 &= \frac{C_t r^2}{2} \max_{k=1,\ldots,K} \|x^{(r+1)}(t_{k-1})\| \\ B_3 &= \frac{r(r-1)}{2} \max_{k=1,\ldots,K} \|x^{(r)}(t_{k-1})\| \end{aligned} \tag{9}$$

Proof: The proof of Lemma 2 is presented in [14]. Additionally, an outline of the proof is given in Appendix II.

Lemma 3: Let $x_M(t)$ be the composite Bernstein polynomial approximation of $x(t) \in \mathcal{C}^2$. The following bound holds:

$$\left\| \int_0^{t_f} x_M(t)dt - \int_0^{t_f} x(t)dt \right\| \leq \frac{C}{K^2 N}$$

where

$$C = \frac{C_t^3}{8} \max_{\tau \in [t_{k-1}, t_k]} |\ddot{x}(\tau)|$$

Proof: The proof of Lemma 3 is presented in [14]. Additionally, an outline of the proof is given in Appendix III.

Notice that the convergence rate of composite Bernstein polynomials is quadratic w.r.t. K, in contrast to the linear convergence rate, w.r.t. K, observed with single Bernstein polynomials (where $K = 1$), see also [11]. Building upon this enhanced convergence rate, this paper extends our previous work [11]–[13] on Bernstein polynomial-based optimal control to composite Bernstein polynomials.

IV. Proposed Direct Method

Here we formulate a discretized version of Problem P, referred to as Problem P_M. Where $M + 1 = K(N + 1)$ is the number of control points of the composite Bernstein polynomial, with K denoting the number of polynomials in the approximant and N denoting the order of each polynomial.

We approximate the states and control inputs of Problem P with composite Bernstein polynomials, with each individual polynomial defined as:

$$\begin{aligned} \boldsymbol{x}_N^{[k]}(t) &= \sum_{j=0}^{N} \bar{\boldsymbol{x}}_{j,N}^{[k]} b_{j,N}(t), \quad t \in [t_{k-1}, t_k] \\ \boldsymbol{u}_N^{[k]}(t) &= \sum_{j=0}^{N} \bar{\boldsymbol{u}}_{j,N}^{[k]} b_{j,N}(t), \quad t \in [t_{k-1}, t_k] \end{aligned} \tag{10}$$

$\forall k = 1, ..., K$, where $\boldsymbol{x}_N^{[k]}$ represents the k-th polynomial of the composite Bernstein polynomial, $\bar{\boldsymbol{x}}_{j,N}^{[k]}$ is the j-th element of $\bar{\boldsymbol{x}}_N^{[k]}$, $\bar{\boldsymbol{u}}_{j,N}^{[k]}$ is the j-th element of $\bar{\boldsymbol{u}}_N^{[k]}$, and where $t_0, t_1, \dots, t_K$ are the time knots.

The composite Bernstein polynomials $\boldsymbol{x}_M(t)$ and $\boldsymbol{u}_M(t)$, defined in the same form as Equation 6, approximate $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ with $\boldsymbol{x}_M : [t_0, t_K] \to \mathbb{R}^{n_x}$ and $\boldsymbol{u}_M : [t_0, t_K] \to \mathbb{R}^{n_u}$, where $t_0 = 0$ and t_K is an approximation of the final mission time t_f. Let

$$\bar{\boldsymbol{x}}_M = [\bar{\boldsymbol{x}}_{0,N}^{[1]}, \dots, \bar{\boldsymbol{x}}_{N,N}^{[1]}, \dots, \bar{\boldsymbol{x}}_{0,N}^{[K]}, \dots, \bar{\boldsymbol{x}}_{N,N}^{[K]}], \in \mathbb{R}^{n_x \times (M+1)}$$

$$\bar{\boldsymbol{u}}_M = [\bar{\boldsymbol{u}}_{0,N}^{[1]}, \dots, \bar{\boldsymbol{u}}_{N,N}^{[1]}, \dots, \bar{\boldsymbol{u}}_{0,N}^{[K]}, \dots, \bar{\boldsymbol{u}}_{N,N}^{[K]}], \in \mathbb{R}^{n_u \times (M+1)}$$

i.e., $\bar{\boldsymbol{x}}_M$ and $\bar{\boldsymbol{u}}_M$ are matrices that comprise all the control points from each Bernstein polynomial. Let $\bar{\boldsymbol{t}}_K$ be a vector of the time knots between each polynomial, i.e., $\bar{\boldsymbol{t}}_K = [t_0, t_1, ..., t_K] \in \mathbb{R}^{K+1}$. Then, Problem P_M can be defined as follows:

Problem 2 *(Problem P_M):*

$$\begin{aligned} \min_{\bar{\boldsymbol{x}}_M, \bar{\boldsymbol{u}}_M, t_K} J_M(\bar{\boldsymbol{x}}_M, \bar{\boldsymbol{u}}_M, \bar{\boldsymbol{t}}_K) &= E(\bar{\boldsymbol{x}}_{0,M}, \bar{\boldsymbol{x}}_{M,M}, t_K) \\ &+ \sum_{k=1}^{K} w^{[k]} \sum_{j=0}^{N} F(\bar{\boldsymbol{x}}_{j,N}^{[k]}, \bar{\boldsymbol{u}}_{j,N}^{[k]}) \end{aligned} \tag{11}$$

subject to

$$\left\| \sum_{i=0}^{M} \bar{\boldsymbol{x}}_{i,M} \boldsymbol{D}_{i,j}^{M} - \boldsymbol{f}(\bar{\boldsymbol{x}}_{j,M}, \bar{\boldsymbol{u}}_{j,M}) \right\| \leq \delta_P^M, \quad \forall j = 0, ..., M \tag{12}$$

$$\boldsymbol{e}(\bar{\boldsymbol{x}}_{0,M}, \bar{\boldsymbol{x}}_{M,M}, t_K) = 0, \tag{13}$$

$$\boldsymbol{h}(\bar{\boldsymbol{x}}_{j,M}, \bar{\boldsymbol{u}}_{j,M}) \leq 0, \ \forall j = 0, ..., M \tag{14}$$

$$\bar{\boldsymbol{x}}_{N,N}^{[k]} - \bar{\boldsymbol{x}}_{0,N}^{[k+1]} = 0, \ \forall k = 1, ..., K-1 \tag{15}$$

$$t_K > t_{K-1} > \ldots > t_0 = 0, \tag{16}$$

where $w^{[k]} = \frac{t_k - t_{k-1}}{M+1}$, $\forall k = 1, ..., K$, $\bar{\boldsymbol{x}}_{j,M}$ is the jth element of $\bar{\boldsymbol{x}}_M$, $\bar{\boldsymbol{u}}_{j,M}$ is the jth element of $\bar{\boldsymbol{u}}_M$, δ_P^M is a relaxation bound equal to a small positive number that depends on M and converges uniformly to 0, i.e., $\lim_{M\to\infty} \delta_P^M = 0$. Finally, $\boldsymbol{D}_{i,j}^M$ is the (i, j)th entry of the differentiation matrix $\boldsymbol{D}_M \in \mathbb{R}^{(M+1)\times(M+1)}$,

$$\boldsymbol{D}_M = \text{blkdiag}\left(\boldsymbol{D}_{N-1}^{[1]} \boldsymbol{E}_{N-1}^{N}, \dots, \boldsymbol{D}_{N-1}^{[K]} \boldsymbol{E}_{N-1}^{N}\right)$$

where blkdiag() is defined as the block diagonal operator, and $\boldsymbol{E}_{N-1}^N$ is the degree elevation matrix of order $N - 1$ to N, see [10]. Differentiation of Bernstein polynomials results in a polynomial of $N - 1$ order, degree elevation returns D_{N-1} to a square matrix.

The following results can be stated regarding the feasibility and consistency of the proposed method, by establishing that solutions to the approximated Problem 2 exist and converge to the optimal solution of the original Problem 1. Due to limitations in space, detailed proofs are omitted here. However, the proofs follow steps similar to the proofs in [13]. We highlight that, while reference [13] is primarily concerned with single Bernstein polynomials, our investigation extends this framework to the application of composite Bernstein polynomials.

Theorem 1: Assume that Problem 1 is feasible, and the solution satisfies $\boldsymbol{x} \in \mathcal{C}^2$, $\boldsymbol{u} \in \mathcal{C}^2$. Then, there exist order of approximations K^* and N^*, and for any approximation orders $K \geq K^*$ and $N \geq N^*$ there exist Bernstein coefficients $\bar{\boldsymbol{x}}_M$, $\bar{\boldsymbol{u}}_M$ and final time t_K that constitute a feasible solution to Problem 2.

Theorem 2: Assuming the functions J, $\boldsymbol{f}$, and $\boldsymbol{h}$ in Problem 2 are Lipschitz with respect to their arguments, and under the same assumptions outlined in Theorem 1, it follows that as $K \to \infty$ or $N \to \infty$, the solution to Problem 1 i.e., $\bar{\boldsymbol{x}}_M^*$, $\bar{\boldsymbol{u}}_M^*$ and final time t_K^* gives composite Bernstein polynomials $\boldsymbol{x}_M^*(t)$ and $\boldsymbol{u}_M^*(t)$ that converge to the optimal solution to Problem 1.

The above theorems are based on the convergence principles detailed in Lemmas 1-3, which form the basis for the assumptions in Theorems 1 and 2, where it is assumed that Problem 1 possesses solutions in $\mathcal{C}^2$. These results could be extended by integrating insights from research on the composite Bernstein approximation of $\mathcal{C}^0$ functions [15], as well as functions with Lipschitz derivatives and Hölder continuous functions [15]–[17].

Finally, one distinct advantage of composite Bernstein polynomials lies in their ability to approximate discontinuous functions. This feature is critically relevant to optimal control, as many optimal control problems manifest bang-bang control functions. Traditional polynomial approximation methods, such as Fourier series, Lagrange interpolation, Hermite and Laguerre polynomial approximations, and Chebyshev or Legendre polynomials, often struggle when approximating step functions [18]. This typically manifests as oscillatory behavior near points of discontinuity, a phenomenon often referred to as the Gibbs phenomenon. However, Bernstein polynomials are notable for their immunity to this phenomenon [19] as well as their convergence when approximating discontinuous functions with bounded variations [15].

V. Knotting Method

The collocation method presented in this paper can be used by simply predefining K and optimizing the location of the knots. However, it might be more efficient to directly estimate the number of discontinuities and their locations to deduce an optimal K. When the number of discontinuities is known *a priori*, it is apparent how many segments should be used to approximate the solution. However, in cases where the number of discontinuities is unknown, the number of segments required to approximate the solution is not clear. One such method used to identify these discontinuities is to evaluate the derivative of the control against some derivative threshold [20], chosen as a design variable by the user. Intuitively, this method identifies the location of sudden changes, i.e., the discontinuities, through the following steps: (1) Solve the NLP with $K = 1$; (2) Calculate the derivative of the control input, i.e., $\dot{u}_N$; (3) Evaluate the elements of $\dot{u}_N$ against the threshold D_{th}; (4) For each element of $\dot{u}_N$ in violation of D_{th}, $K = K + 1$.

Once the number of discontinuities is known, knots are placed near each jump. Subsequently, the locations of the knots can be defined as a decision variable in the optimal control problem formulation, enabling the optimization algorithm to find their optimal placement.

VI. Numerical Results

In this section, two numerical examples are presented, highlighting the efficacy of using composite Bernstein polynomials to solve optimal control problems. These solutions were found using a constrained nonlinear optimization algorithm.

Bang-Bang Problem [21]

Determine $y : [0, 2] \to \mathbb{R}$ and $u : [0, 2] \to \mathbb{R}$ that minimize

$$J(y(t), u(t)) = \int_0^2 (3u(t) - 2y(t))dt,$$

Order	$K=1$	$K=2$
$N=10$	$J \approx -59.35$	$J \approx -59.83$
$N=15$	$J \approx -59.50$	-
$N=30$	$J \approx -59.67$	-
$N=55$	$J \approx -59.74$	-

(I) Bang-Bang Problem Results: Evaluated cost function for each scenario.

subject to

$$\dot{y}(t) = y(t) + u(t), \forall t \in [0, 2],$$
$$y(0) = 4,\ y(2) = 39.392,$$
$$0 \le u(t) \le 2, \forall t \in [0, 2].$$

This problem is initially solved using the approximation method with one segment, i.e., the Bernstein approximation method. From Figure 1(a), it is obvious that even though the approximation improves as N increases, it remains inadequate for discontinuous solutions. The algorithm then evaluates the derivative of the control and compares it against some threshold to determine whether additional segments are required. A discontinuity is detected at $t = 1.2s$, shown in Figure 1(b), where the threshold is exceeded by one control point, indicating that two segments should be used to approximate the solution. The example is then solved with $K = 2$, i.e., the composite Bernstein approximation method, with the results shown in Figure 1(c).

The composite Bernstein approximant, used with the presented knotting method, is able to detect the exact value of the discontinuity at $t = 1.096s$, from the initial guess of $t = 1.2s$. The evaluated cost functions of each scenario are shown in Table I. The cost of the analytical solution shown as a dotted line in Figures 1(a) and 1(c), is $J \approx -59.83$, the exact solution obtained with this method.

Multi-vehicle Motion Planning Problem

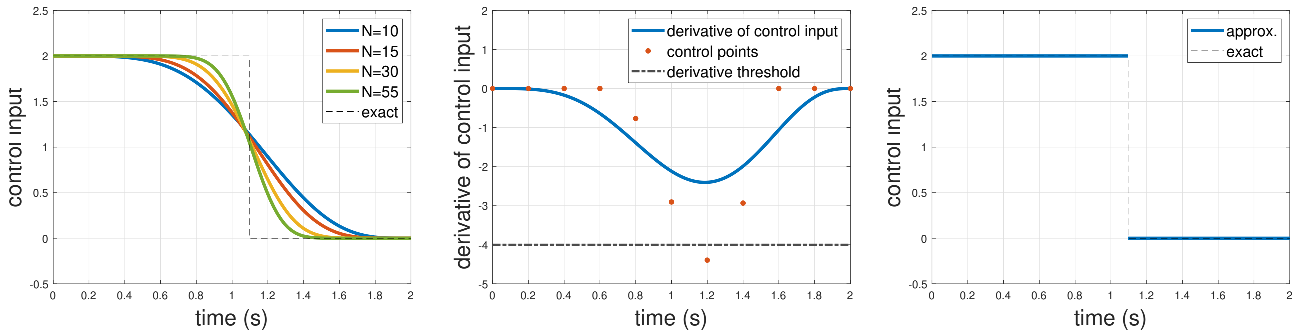
For this example, a multi-vehicle motion planning problem is formalized as an optimal control problem. The scenario described below is that of an aircraft conducting a standard 45-degree entry into the traffic pattern, which is subsequently alerted to a potential collision with an intruding aircraft, requiring replanning of the remaining trajectory to reach the runway at a free final time.

Determine x_1, x_2, x_3, x_4, x_5, u_1, u_2, and u_3 that minimize

$$J(x(t), u(t)) = t_f,$$

subject to

$$\dot{x}_1(t) = u_1(t)\cos(x_4(t))\sin(x_5(t)), \forall t \in [0, t_f]$$
$$\dot{x}_2(t) = u_1(t)\sin(x_4(t))\cos(x_5(t)), \forall t \in [0, t_f]$$
$$\dot{x}_3(t) = u_1(t)\sin(x_5(t)), \forall t \in [0, t_f]$$
$$\dot{x}_4(t) = u_2(t),\ \dot{x}_5(t) = u_3(t), \forall t \in [0, t_f]$$
$$x_1(0) = 0,\ x_2(0) = -4000,$$
$$x_3(0) = 1000,\ x_4(0) = 180,\ x_5(0) = 0,$$
$$x_1(t_f) = -3110.9,\ x_2(t_f) = 0,$$



(a) (b) (c)

(1) Solution to Example 1: (a) Approximation of a bang-bang input using a single Bernstein polynomial for orders $N = 10, 15, 30, 55$. (b) Derivative of the control input for the single Bernstein polynomial solution for $N = 10$. (c) Approximation of a bang-bang input using a composite Bernstein polynomial consisting of two polynomials of order $N = 10$.

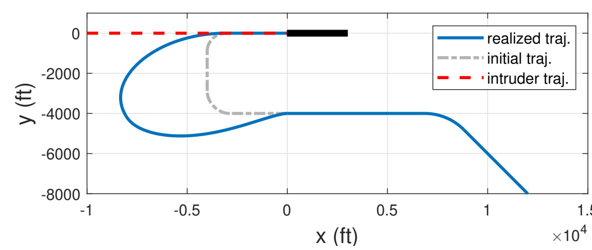
$$x_3(t_f) = 259.2395,\ x_4(t_f) = 0,\ x_5(t_f) = 0,$$
$$0 \le x_3(t) \le 25000,\ -3 \le x_5(t) \le 3.5, \forall t \in [0, t_f]$$
$$100 \le u_1(t) \le 295,\ -3 \le u_2(t) \le 3, \forall t \in [0, t_f]$$

where x_1, x_2, and x_3 are the respective $[x, y, z]$ position of the aircraft in ft, x_4 is the heading in $\deg$, x_5 is the angle of attack in $\deg$, u_1 is the aircraft's velocity in $\frac{ft}{s}$, u_2 is the yaw rate in $\frac{\deg}{s}$, and u_3 is the pitch rate in $\frac{\deg}{s}$. A separation variable of 500 ft is defined to specify the minimum safe distance between the aircraft and the intruding vehicle to avoid collision.

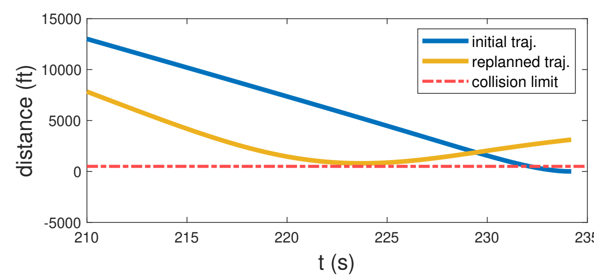
The initial, replanned, and intruder trajectories are shown in Figure 2(a). A potential collision is detected when the vehicle is at roughly $[0, -4000, 1000]$ ft, thus replanning occurs. With the replanned trajectory, the vehicle evades the intruder, avoiding collision. This maneuver is shown in Figure 2(b). Flight trajectories, like the initial trajectory shown in Figure 2(a), exhibit explicit geometric patterns contingent upon vehicle position and velocity. This example highlights the ability of composite Bernstein polynomials to precisely capture specific flight patterns, which a single Bernstein polynomial would struggle to represent at low orders.

VII. Conclusion

In this paper, we introduced composite Bernstein polynomials as a means to solve optimal control problems. This method builds off of the favorable properties of Bernstein polynomials for motion planning, and extends this work to composite Bernstein polynomials, allowing for faster convergence and more accurate approximations of discontinuous solutions, due to their enhanced convergence rate and the added flexibility of additional segments. Two numerical examples and their solutions via this method were exhibited, including a motion planning problem, demonstrating the efficacy of using composite Bernstein polynomials to solve complex optimal control problems. Additionally, rigorous analyses of the convergence properties of composite Bernstein polynomials were presented in the context of smooth functions. In future work, we plan to extend this work by studying the convergence rate of composite Bernstein polynomials to discontinuous functions.



(a)



(b)

(2) (a) Aerial map of runway with aircraft following 45° entry until an intruding aircraft is detected and the initial trajectory is replanned to avoid collision. (b) Distance between the initial trajectory of the aircraft and the replanned trajectory of the aircraft, and the trajectory of the intruder for the motion planning example, with a separation variable of 500 ft.

Appendix I
Outline of Proof of Lemma 1

First, notice that

$$\|x_M(t) - x(t)\| \le \max_{k=1,\dots,K} \max_{\tau \in [t_k, t_{k+1}]} \|x_N^{[k]}(\tau) - x(\tau)\|, \tag{17}$$

for all $k \in 1, \ldots, K$ and for all $\tau \in [t_{k-1}, t_k]$. Let $[t_0, \ldots, t_k; x]$ denote the k-th order divided difference of $x(\tau)$ at the points $t_0, \ldots, t_k$. Then, the *generalized* Stancu's remainder formula is derived following steps outlined in [12], resulting in:

$$
\begin{aligned}
&x_N^{[k]}(\tau) - x(\tau) \\
&\quad = \frac{(\tau - t_{k-1})(t_k - \tau)}{N} \sum_{j=0}^{N-1} \left[t_{k-1} + j\frac{t_k - t_{k-1}}{N}, t_{k-1} \right. \\
&\qquad \left. + (j+1)\frac{t_k - t_{k-1}}{N}, \tau; x \right] b_{j,N-1}^{[k]}(\tau)\,,
\end{aligned} \tag{18}
$$

that is

$$
|x_N^{[k]}(\tau) - x(\tau)| \leq \frac{(\tau - t_{k-1})(t_k - \tau)}{N} \frac{1}{2} \max_{\tau \in [t_{k-1}\,,\, t_k]} |\ddot{x}(\tau)|,
$$

Thus, the following result holds:

$$
|x_N^{[k]}(\tau) - x(\tau)| \leq \frac{C_t^2}{8K^2N} \max_{\tau \in [t_{k-1}\,,\, t_k]} |\ddot{x}(\tau)|\,,
$$

which proves Lemma 1.

Appendix II
Outline of Proof of Lemma 2

Consider the operator $B_{n,s,m}x(\tau)$, defined in [22], for all $k \in 1, \ldots, K$ and for all $\tau \in [t_{k-1}, t_k]$. Then, Equation (18) can be rewritten as follows:

$$
x_N^{[k]}(\tau) - x(\tau) = \frac{(\tau - t_{k-1})(t_k - \tau)}{N} B_{N,1,1}x(\tau),
$$

The above equation is differentiated using the Leibniz rule, and with the derivatives of $B_{N,1,1}x(\tau)$, from [22], we get

$$
\begin{aligned}
|x_N^{[k](r)}(\tau) - x^{(r)}(\tau)| \leq \frac{1}{2N}((\tau - t_{k-1})(t_k - \tau)\|x^{(r+2)}(\tau)\| \\
+ r(t_k - 2\tau + t_{k-1})\|x^{(r+1)}(\tau)\| + r(r-1)\|x^{(r)}(\tau)\|),
\end{aligned} \tag{19}
$$

Then, Lemma 2 follows with Equation 9.

Appendix III
Outline of Proof of Lemma 3

Using Lemma 1, we have

$$
\begin{aligned}
&\left\| \int_{t_{k-1}}^{t_k} x_N^{[k]}(\tau) d\tau - \int_{t_{k-1}}^{t_k} x(\tau) d\tau \right\| \\
&\leq \int_{t_{k-1}}^{t_k} \frac{C_t^2}{8K^2N} \max_{\tau \in [t_{k-1}, t_k]} |\ddot{x}(\tau)| d\tau,
\end{aligned}
$$

which gives the result in Lemma 3.

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