



A Physics-Based Work-Energy Formulation for Real-Time Trajectory Guidance of a Lunar Lander

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02/02/2025





Introduction

The goal of the final approach maneuver is to get as close as possible to the intended landing site, but most importantly, to achieve full energy dissipation to attain stability and a close to zero velocity at a predetermined safe landing altitude in order not to crash.

The work-energy approach applies the energy conservation principle to determine instant guidance commands at any point during the approach maneuver. Instead of spending computational time optimizing and propagating a reference trajectory to achieve the desired final conditions. The approach seeks to instantly satisfy the local work-energy conditions that the laws of physics guarantee that the desired final landing will be achieved (The laws of Thermodynamics are convex in nature).

This approach simplifies the math, reduces the computational process time and eliminates the need of calculating reference trajectories and optimizations, thus providing astronauts with real-time vectors to follow to achieve manual control during different kind of maneuvers (including landing), as well as real-time calculation of the divert space.

This presentation only includes a summarized introduction to the concept and its applications in order not to make it mathematically heavy in its content.

Introduction

Problem with non-linear systems (Butterfly Effect)

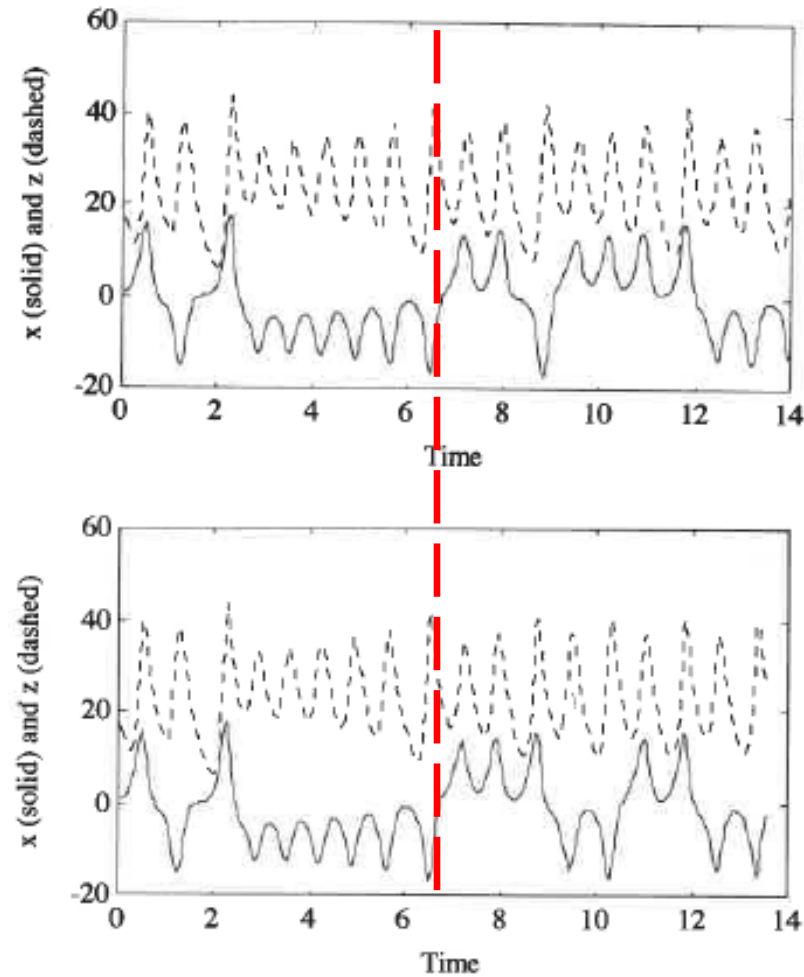


Figure 3.12 Time series for $x(t)$ (solid line) and $z(t)$ (dashed line) for the Lorenz model. The initial condition is $[x \ y \ z] = [1 \ 1 \ 20]$; the parameters are $\sigma = 10$, $b = 8/3$, and $r = 28$. Compare with Figure 3.13.

Figure 3.13 Time series for $x(t)$ (solid line) and $z(t)$ (dashed line) for the Lorenz model. The initial condition is $[x \ y \ z] = [1 \ 1 \ 20.01]$; the parameters are $\sigma = 10$, $b = 8/3$, and $r = 28$. Compare with Figure 3.12.

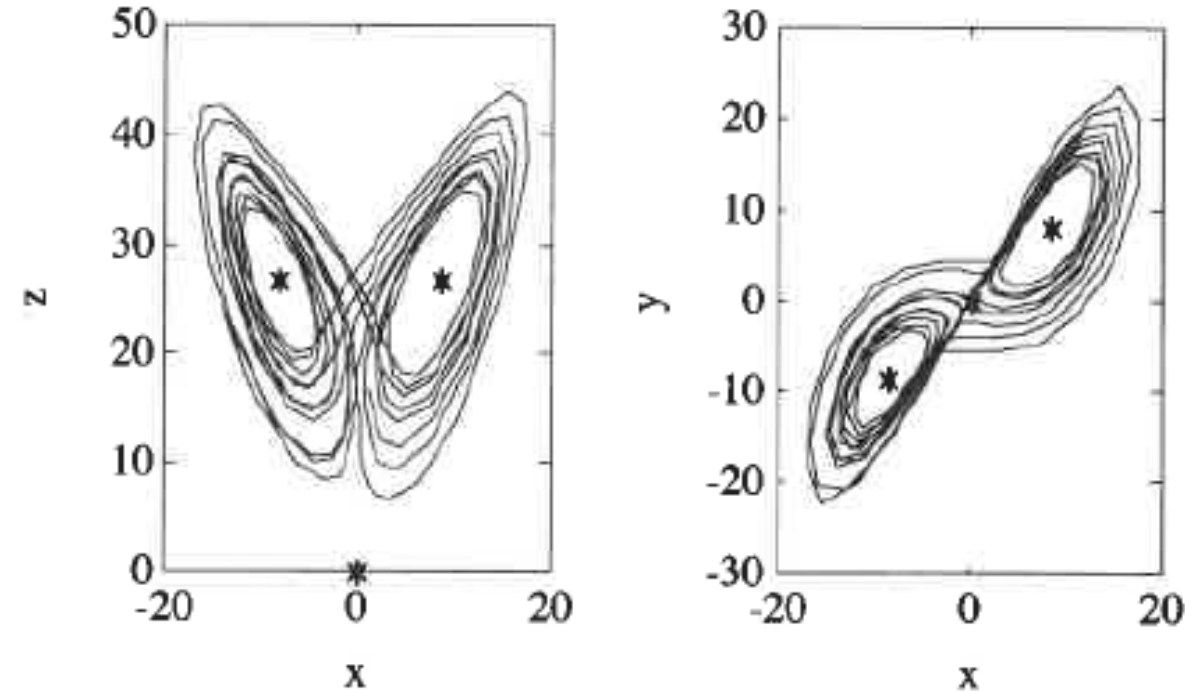


Figure 3.14 Phase space projections for the Lorenz model. The initial condition and parameters are the same as in Figure 3.12. Steady states are indicated by asterisks.

Ref. Numerical Methods for Physics, A. Garcia.

Summary of the Work-Energy Approach

For a lunar lander, there are only 2 energy dissipation mechanisms available:
Work done by the engine, and mass jettisoning.

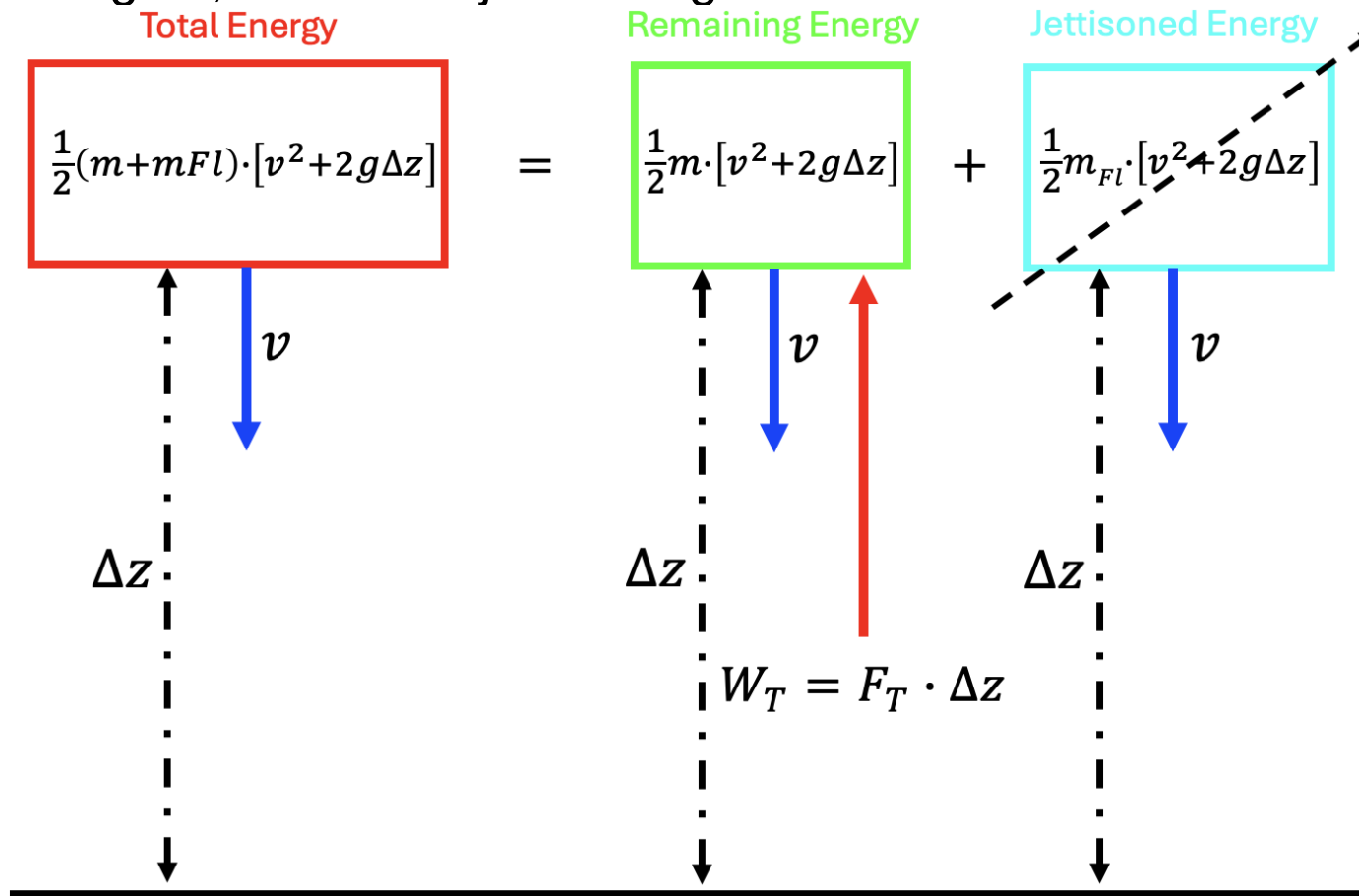


Fig. 1. Energy Dissipation Mechanisms

Local Landing-Site Coordinates Frame Definition

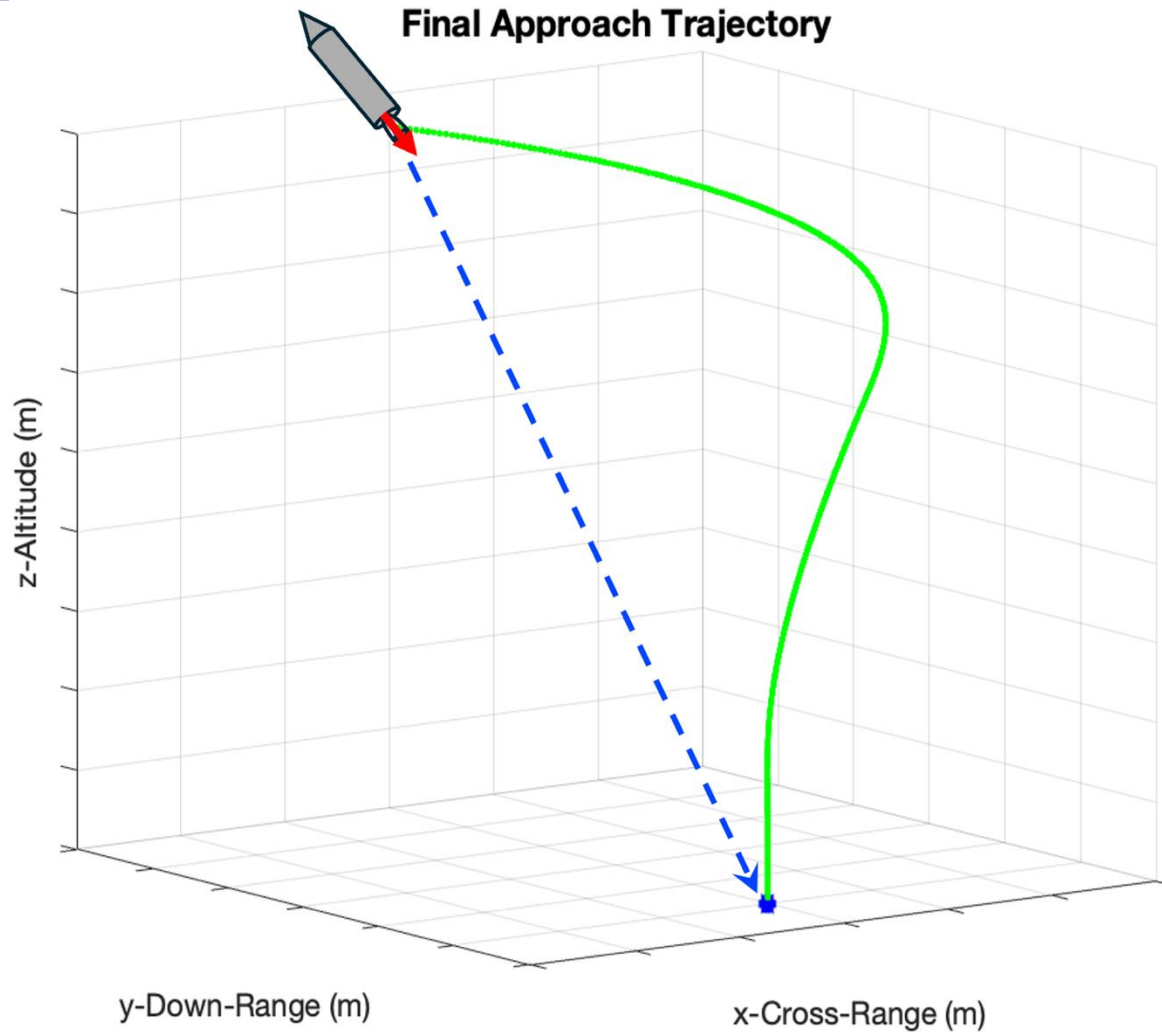


Fig. 2. Local LS-Coordinates

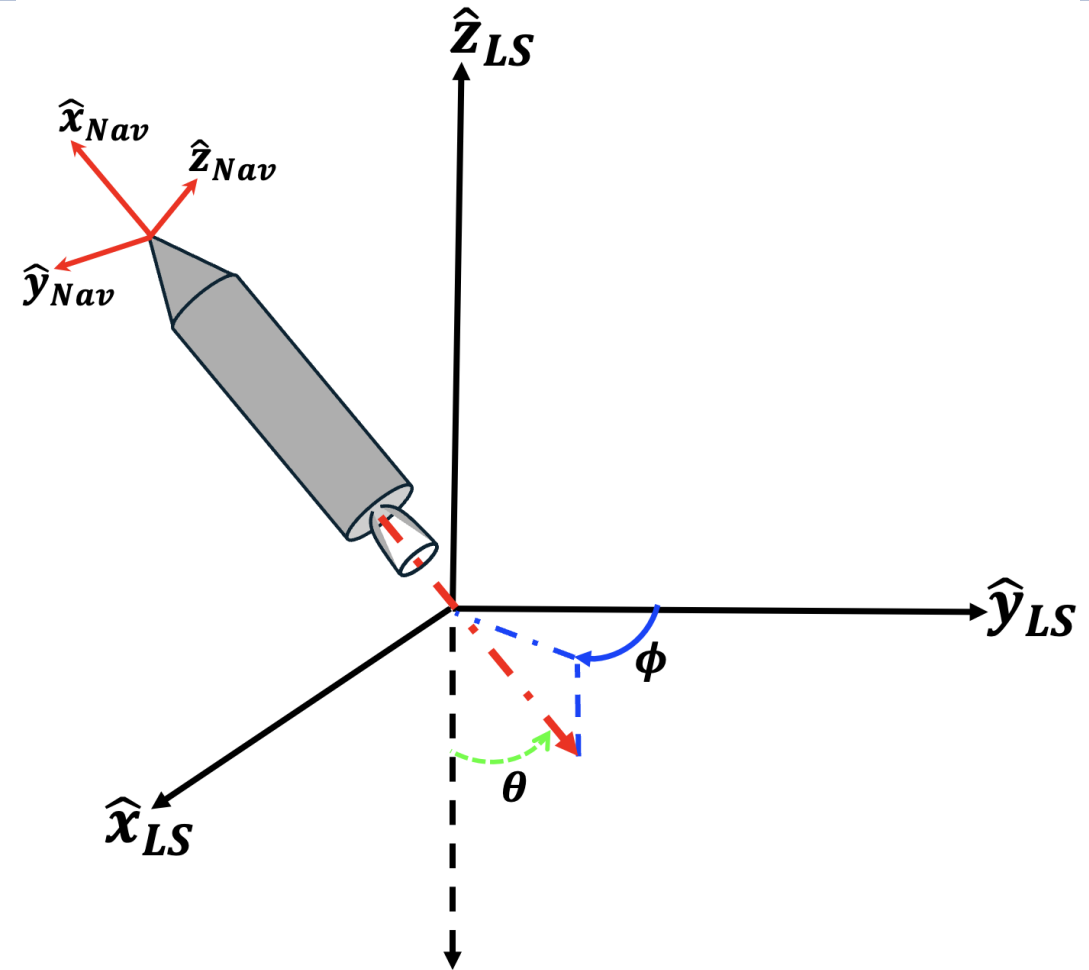


Fig. 3. LS-Attitude and Yaw Angles

*NOTICE $\Delta x = 0$



General Equations of Motion

The mass-flow rate expelled from the rocket engine is defined as

$$\frac{dm}{dt} = \dot{m} = T_{hrtl} \cdot \dot{m}_{max.} , \quad (1)$$

where $\dot{m} \leq 0$, and the engine throttle for the Raptor engine is defined as $0 \leq T_{hrtl} \leq 1$, and the magnitude of $\dot{m}_{max}$ represents the maximum magnitude mass-flow rate at full throttle.

For a **constant mass-flow rate** during a time interval t , the space-craft mass as a function of time is given by

$$m(t) = m_o + \dot{m} \cdot t , \quad (2)$$

with the initial mass at the beginning of the time interval m_o . The magnitude of the rocket engine force as derived by Konstantin E. Tsiolkovsky is

$$F_T = -v_e \cdot \dot{m} , \quad (3)$$

and the effective exhaust velocity $v_e = g_o Isp$, where g_o is the standard gravity, while Isp is the engine's specific impulse. The maximum thrust force possible is given by

$$F_{T_{max.}} = -v_e \cdot \dot{m}_{max.} . \quad (4)$$



General Equations of Motion

From the local landing-site coordinates frame perspective, the breaking force that the engine exerts on the lander is given by

$$\begin{aligned}\mathbf{F}_T &= F_{Tx} \cdot \hat{\mathbf{i}} + F_{Ty} \cdot \hat{\mathbf{j}} + F_{Tz} \cdot \hat{\mathbf{k}} \\ &= v_e \cdot \dot{m} \cdot \sin(\theta) \sin(\phi) \cdot \hat{\mathbf{i}} + v_e \cdot \dot{m} \cdot \sin(\theta) \cos(\phi) \cdot \hat{\mathbf{j}} - [m \cdot g + v_e \cdot \dot{m} \cdot \cos(\theta)] \cdot \hat{\mathbf{k}} .\end{aligned}\tag{5}$$

The components of the acceleration vector are obtained by dividing Eq. (5) by the mass from Eq. (2)

$$a_x(t) = v_e \cdot \frac{\dot{m}}{m} \cdot \sin(\theta) \sin(\phi) ,\tag{6a}$$

$$a_y(t) = v_e \cdot \frac{\dot{m}}{m} \cdot \sin(\theta) \cos(\phi) ,\tag{6b}$$

$$a_z(t) = -g - v_e \cdot \frac{\dot{m}}{m} \cdot \cos(\theta) .\tag{6c}$$

General Equations of Motion

Eqs. (6a) through (6c) are integrated with respect to time to obtain the velocities

$$v_x(t) = v_{x_o} + v_e \sin(\theta) \sin(\phi) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (7a)$$

$$v_y(t) = v_{y_o} + v_e \sin(\theta) \cos(\phi) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (7b)$$

$$v_z(t) = v_{z_o} - g \cdot t - v_e \cos(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (7c)$$

and integrating again to obtain the vector components of the position in the form

$$x(t) = x_o + \left[v_{x_o} - v_e \sin(\theta) \sin(\phi) \right] \cdot t + v_e \sin(\theta) \sin(\phi) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (8a)$$

$$y(t) = y_o + \left[v_{y_o} - v_e \sin(\theta) \cos(\phi) \right] \cdot t + v_e \sin(\theta) \cos(\phi) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (8b)$$

$$z(t) = z_o + \left[v_{z_o} + v_e \cos(\theta) \right] \cdot t - \frac{1}{2} g \cdot t^2 - v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] . \quad (8c)$$

General Equations of Motion

It can be shown that by combining Eqs. (7a) to (7c) with Eqs.(8a) to (8c) the time-intervals for each of the vector components are given by

$$\tau_x = \frac{\left\{ \frac{m_o}{\dot{m}} \cdot [v_{x_o} - v_x(\tau_x)] - [x_o - x(\tau_x)] \right\}}{[v_x(\tau_x) - v_e \sin(\theta) \sin(\phi)]}, \quad (9a)$$

$$\tau_y = \frac{\left\{ \frac{m_o}{\dot{m}} \cdot [v_{y_o} - v_y(\tau_y)] - [y_o - y(\tau_y)] \right\}}{[v_y(\tau_y) - v_e \sin(\theta) \cos(\phi)]}, \quad (9b)$$

$$\tau_z = - \left\{ \frac{1}{g} \cdot [v_z(\tau_z) + v_e \cos(\theta)] + \frac{m_o}{\dot{m}} \right\} + \sqrt{\left\{ \frac{1}{g} \cdot [v_z(\tau_z) + v_e \cos(\theta)] + \frac{m_o}{\dot{m}} \right\}^2 - \frac{2}{g} \cdot \left\{ [z_o - z(\tau_z)] - \frac{m_o}{\dot{m}} \cdot [v_{z_o} - v_z(\tau_z)] \right\}}. \quad (9c)$$



Work-Energy Formulation

The initial and final energy states of the lunar lander during the approach maneuver can be expressed as

$$E_o = \frac{1}{2}m_o \cdot [\mathbf{v}_o^2 + 2g \cdot z_o] , \quad (10a)$$

$$E_f = \frac{1}{2}(m_o - m_{Fl}) \cdot [\mathbf{v}_f^2 + 2g \cdot z_f] , \quad (10b)$$

where the horizontal velocities are set to zero on the landing, and the vertical velocity is set to a small negative value v_{zf} at a final altitude z_f , therefore

$$\mathbf{v}_o^2 = v_{x_o}^2 + v_{y_o}^2 + v_{z_o}^2 , \quad (11a)$$

$$\mathbf{v}_f^2 = v_{z_f}^2 . \quad (11b)$$



Work-Energy Formulation

To achieve the desired final energy conditions at landing, most of the energy is dissipated by the work done by the engine along the displacement vector $\Delta \mathbf{r}$ in the form

$$W_T = \mathbf{F}_T \cdot \Delta \mathbf{r} . \quad (12)$$

The second energy dissipation mechanism is due to the jettisoning of mass in the form of fuel during time intervals of constant $\dot{m}$ in the form

$$E_{Jet} = \frac{1}{2} m_{Fl} \cdot [\bar{\mathbf{v}}^2 + 2g \cdot \bar{z}] , \quad (13)$$

where the jettisoned fuel mass during the interval τ with constant $\dot{m}$ is given by

$$m_{Fl} = -\tau \cdot \dot{m} . \quad (14)$$



Work-Energy Formulation

The average square of the velocity $\overline{v^2}$ and average altitude displacement $\bar{z}$ in Eq. (13) are calculated using the average integral functions

$$\bar{v^2} = \frac{1}{\tau} \int_0^{\tau} dt \cdot v^2(t) , \quad (15a)$$

$$\bar{z} = \frac{1}{\tau} \int_0^{\tau} dt \cdot z(t) . \quad (15b)$$

$$E_{Jet} = \frac{1}{2} m_{Fl} \cdot [\bar{v^2} + 2g \cdot \bar{z}] , \quad (13)$$



Work-Energy Formulation

$$\begin{aligned}
 \bar{v}_x^2 &= v_{x_o}^2 + 2 \cdot v_e^2 \sin^2(\theta) \sin^2(\phi) - 2 \cdot v_{x_o} v_e \sin(\theta) \sin(\phi) \\
 &+ \frac{1}{\tau} \cdot v_e \sin(\theta) \sin(\phi) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\
 &\times \left\{ 2 \cdot [v_{x_o} - v_e \sin(\theta) \sin(\phi)] + v_e \sin(\theta) \sin(\phi) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\
 &\times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] ,
 \end{aligned} \tag{16a}$$

By plugging in the velocities from Eqs. (7a) through (7c) into Eq. (15a) and integrating along the interval τ , the average square of the velocity components are obtained

$$\begin{aligned}
 \bar{v}_y^2 &= v_{y_o}^2 + 2 \cdot v_e^2 \sin^2(\theta) \cos^2(\phi) - 2 \cdot v_{y_o} v_e \sin(\theta) \cos(\phi) \\
 &+ \frac{1}{\tau} \cdot v_e \sin(\theta) \cos(\phi) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\
 &\times \left\{ 2 \cdot [v_{y_o} - v_e \sin(\theta) \cos(\phi)] + v_e \sin(\theta) \cos(\phi) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\
 &\times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] ,
 \end{aligned} \tag{16b}$$

$$\begin{aligned}
 \bar{v}_z^2 &= v_{z_o}^2 + 2 \cdot v_e^2 \cos^2(\theta) + 2 \cdot v_{z_o} v_e \cos(\theta) \\
 &+ g \tau \cdot \left\{ \frac{1}{3} \cdot g \tau - [v_{z_o} + \frac{1}{2} \cdot v_e \cos(\theta)] + \frac{1}{\tau} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \right\} \\
 &+ \frac{1}{\tau} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\
 &\times \left\{ g \cdot \left[\tau - \frac{m_o}{\dot{m}} \right] - 2 \cdot [v_{z_o} + v_e \cos(\theta)] + v_e \cos(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\
 &\times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] .
 \end{aligned} \tag{16c}$$



Work-Energy Formulation

By also plugging in the altitude from Eq. (8c) into Eq. (15b) and integrating along the interval τ , the average altitude displacement is obtained in the form

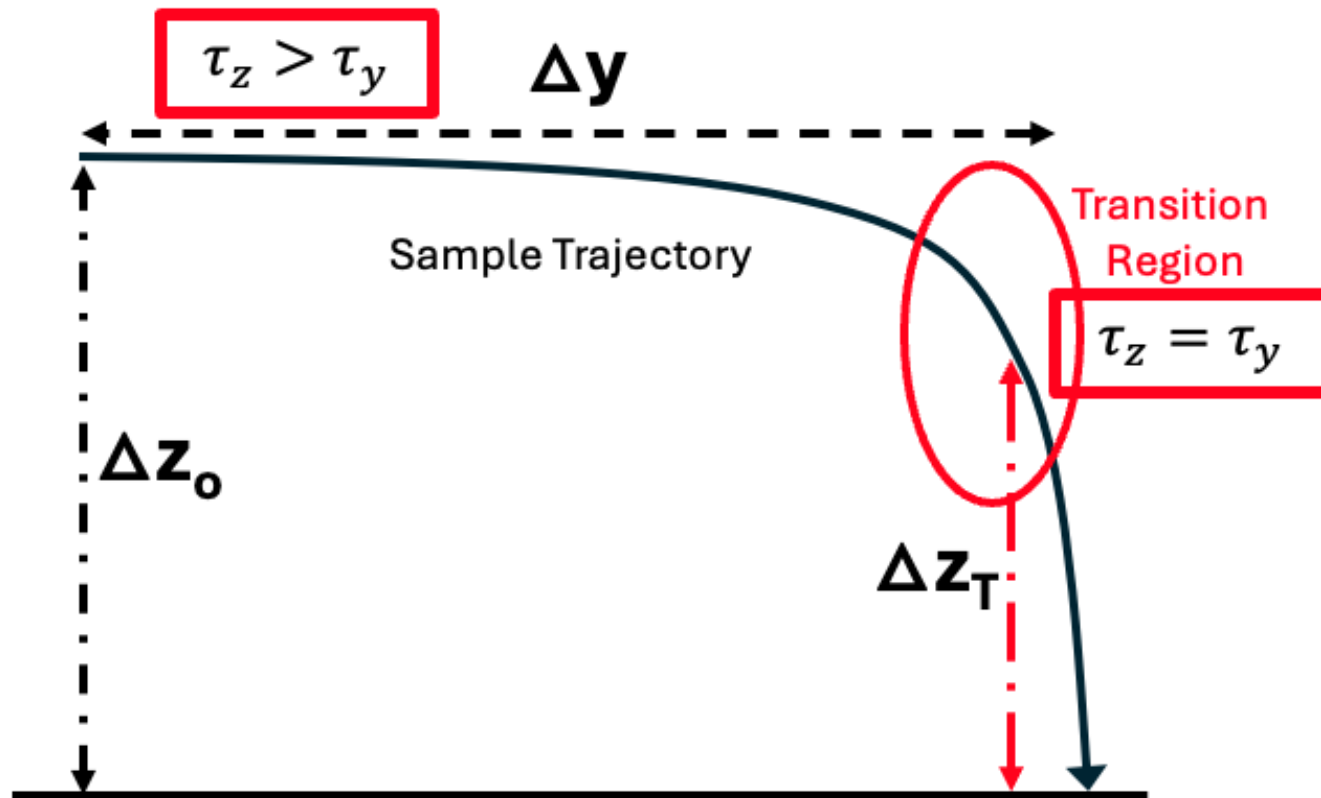
$$\begin{aligned}\bar{z} = z_o + \frac{1}{2}\tau \cdot \left[v_{z_o} + \frac{3}{2} \cdot v_e \cos(\theta) \right] - \frac{1}{6}g \cdot \tau^2 \\ + \frac{1}{2} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left\{ 1 - \frac{1}{\tau} \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right]^2 \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} .\end{aligned}\quad (17a)$$

Therefore, the total energy dissipated by both the work done by the engine and mass jettisoning is expressed as

$$E_o - E_f = W_T + E_{Jet} . \quad (18)$$

Work-Energy Formulation

To employ the work-energy approach to calculate an optimal trajectory, the problem is first formulated in 2D because $\Delta x = 0$. Fig. 4 illustrates the formulation between the horizontal and vertical directions



* The cross-range calculation is performed by simply solving for the yaw angle ϕ in Eqs. (7a), (8a), (9a), and (9b).

Fig. 4. 2D Approach Trajectory



Work-Energy Formulation

Therefore, Eq. (18) is broken into the horizontal and vertical components in the form

$$E_{o_y} - \cancel{E_{f_y}}^0 = F_y \cdot \Delta y + E_{Jet_y} , \quad (19a)$$

$$E_{o_z} - E_{f_z} = F_z \cdot \Delta z + E_{Jet_z} , \quad (19b)$$

where

$$E_{Jet_y} = \frac{1}{2} m_{Fl_y} \cdot \bar{v}_y^2 , \quad (20a)$$

$$E_{Jet_z} = \frac{1}{2} m_{Fl_z} \cdot [\bar{v}_z^2 + 2g \cdot \bar{z}] , \quad (20b)$$

with

$$m_{Fl_y} = -\tau_y \cdot \dot{m} , \quad (21a)$$

$$m_{Fl_z} = -\tau_z \cdot \dot{m} . \quad (21b)$$



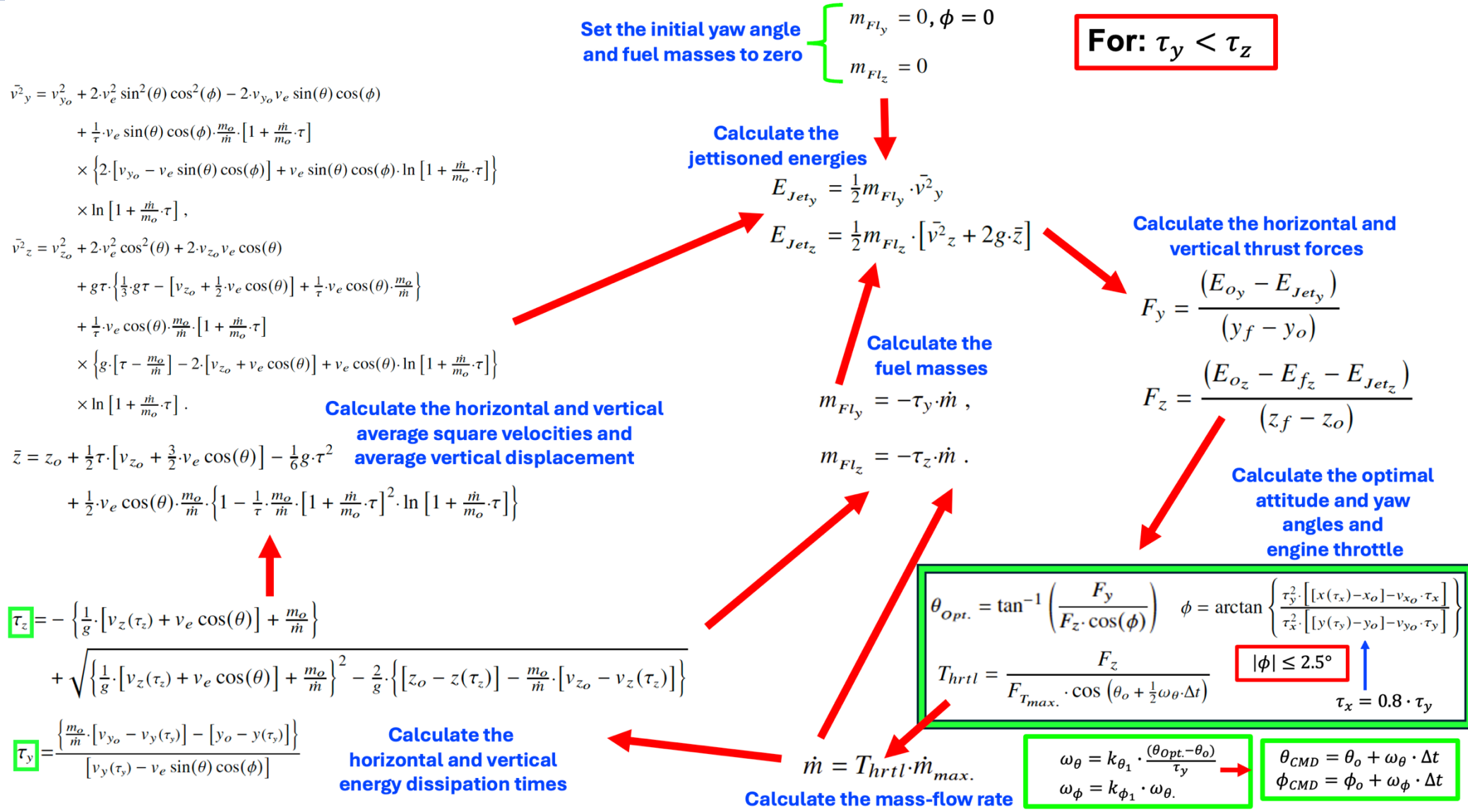
Work-Energy Formulation

By combining and solving Eqs. (7a) and (7b) together with Eqs. (8a) and (8b), it can be show that the solutions for the yaw angle ϕ along the trajectory to correct for the cross-range are simply given by

$$\begin{aligned}\phi &= \arctan \left\{ \frac{[x(\tau_x) - x_o] - v_{x_o} \cdot \tau_x}{[y(\tau_y) - y_o] - v_{y_o} \cdot \tau_y} \cdot \frac{\left[\tau_y - \frac{m_o}{\dot{m}} \cdot \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right) \cdot \ln \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right) \right]}{\left[\tau_x - \frac{m_o}{\dot{m}} \cdot \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right) \cdot \ln \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right) \right]} \right\} \\ &\approx \arctan \left\{ \frac{\tau_y^2 \cdot [x(\tau) - x_o] - v_{x_o} \cdot \tau_x}{\tau_x^2 \cdot [y(\tau) - y_o] - v_{y_o} \cdot \tau_y} \right\},\end{aligned}\tag{22a}$$

$$\phi = \arctan \left\{ \frac{\left[(x_f - x_o) - \frac{m_o}{\dot{m}} \cdot (v_{x_f} - v_{x_o}) \right]}{\left[(y_f - y_o) - \frac{m_o}{\dot{m}} \cdot (v_{y_f} - v_{y_o}) \right]} \right\}.\tag{22b}$$

Work-Energy Formulation



Work-Energy Formulation

Set the initial fuel masses to zero and enters fixed time

$$\begin{cases} m_{Fl_y} = 0 \\ m_{Fl_z} = 0 \\ \tau_y = \tau_z = \tau \end{cases}$$

For: $\tau \geq t \geq 0$

Calculate the jettisoned energies

$$E_{Jet_y} = \frac{1}{2} m_{Fl_y} \cdot \bar{v}_y^2$$

$$E_{Jet_z} = \frac{1}{2} m_{Fl_z} \cdot [\bar{v}_z^2 + 2g \cdot \bar{z}]$$

Calculate the horizontal and vertical thrust forces

$$F_y = \frac{(E_{Oy} - E_{Jet_y})}{(y_f - y_o)}$$

$$F_z = \frac{(E_{Oz} - E_{fz} - E_{Jet_z})}{(z_f - z_o)}$$

Calculate the fuel masses

$$m_{Fl_y} = -\tau_y \cdot \dot{m}$$

$$m_{Fl_z} = -\tau_z \cdot \dot{m}$$

Calculate the optimal attitude and yaw angles and engine throttle

$$\begin{aligned} \theta_{Opt.} &= \tan^{-1} \left(\frac{F_y}{F_z \cdot \cos(\phi)} \right) & \phi &= \arctan \left\{ \frac{[(x_f - x_o) - \frac{m_o}{\dot{m}} \cdot (v_{xf} - v_{xo})]}{[(y_f - y_o) - \frac{m_o}{\dot{m}} \cdot (v_{yf} - v_{yo})]} \right\} \\ T_{hrtl} &= \frac{F_z}{F_{Tmax.} \cdot \cos(\theta_o + \frac{1}{2} \omega_\theta \cdot \Delta t)} & |\phi| &\leq 10^\circ \end{aligned}$$

Calculate the horizontal and vertical average square velocities and average vertical displacement

$$\begin{aligned} \bar{v}_y^2 &= v_{y_o}^2 + 2 \cdot v_e^2 \sin^2(\theta) \cos^2(\phi) - 2 \cdot v_{y_o} v_e \sin(\theta) \cos(\phi) \\ &+ \frac{1}{\tau} \cdot v_e \sin(\theta) \cos(\phi) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\ &\times \left\{ 2 \cdot [v_{y_o} - v_e \sin(\theta) \cos(\phi)] + v_e \sin(\theta) \cos(\phi) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\ &\times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right], \end{aligned}$$

$$\begin{aligned} \bar{v}_z^2 &= v_{z_o}^2 + 2 \cdot v_e^2 \cos^2(\theta) + 2 \cdot v_{z_o} v_e \cos(\theta) \\ &+ g \tau \cdot \left\{ \frac{1}{3} g \tau - [v_{z_o} + \frac{1}{2} \cdot v_e \cos(\theta)] + \frac{1}{\tau} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \right\} \\ &+ \frac{1}{\tau} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\ &\times \left\{ g \cdot \left[\tau - \frac{m_o}{\dot{m}} \right] - 2 \cdot [v_{z_o} + v_e \cos(\theta)] + v_e \cos(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\ &\times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right]. \end{aligned}$$

$$\begin{aligned} \bar{z} &= z_o + \frac{1}{2} \tau \cdot [v_{z_o} + \frac{3}{2} \cdot v_e \cos(\theta)] - \frac{1}{6} g \cdot \tau^2 \\ &+ \frac{1}{2} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left\{ 1 - \frac{1}{\tau} \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right]^2 \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \end{aligned}$$

$$\dot{m} = T_{hrtl} \cdot \dot{m}_{max.}$$

Calculate the mass-flow rate

$$\begin{aligned} \omega_\theta &= k_{\theta_2} \cdot \frac{(\theta_{Opt.} - \theta_o)}{\tau} \\ \omega_\phi &= k_{\phi_2} \cdot \omega_\theta \end{aligned}$$

$$\begin{aligned} \theta_{CMD} &= \theta_o + \omega_\theta \cdot \Delta t \\ \phi_{CMD} &= \phi_o + \omega_\phi \cdot \Delta t \end{aligned}$$

GLASS Monte-Carlo (Apollo vs Work-Energy)

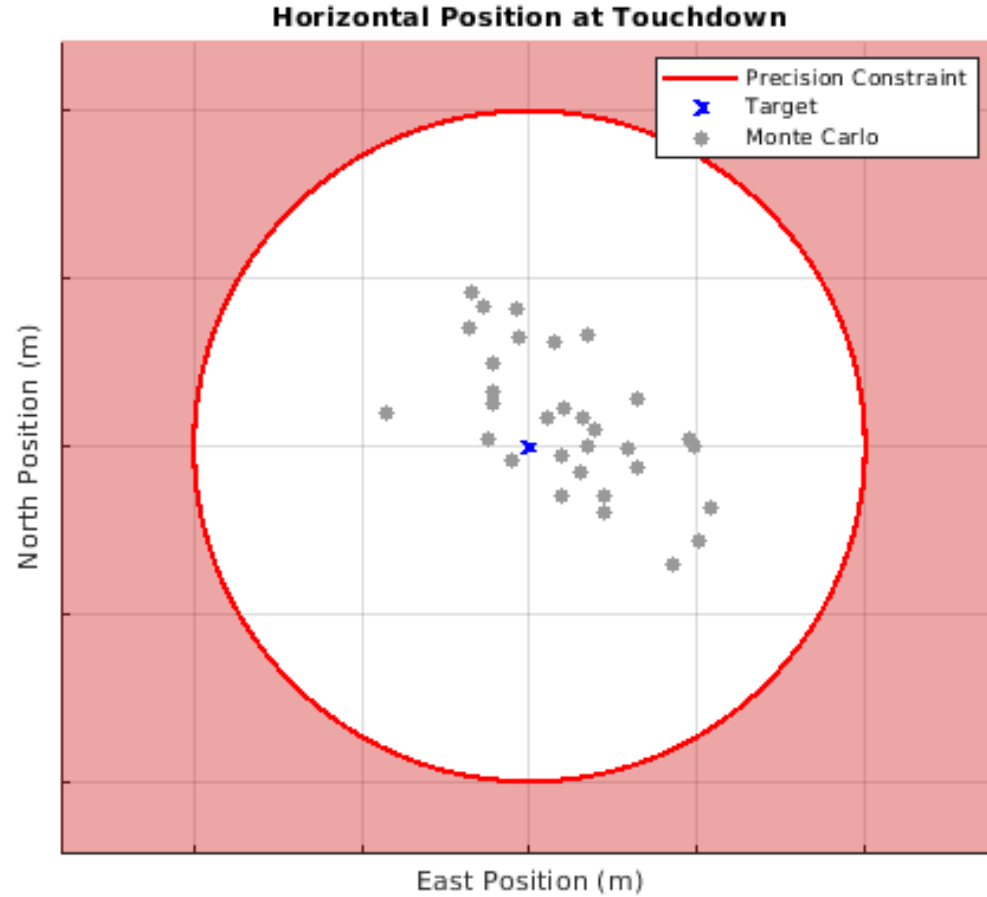


**Fig. 11a. Apollo Guidance
(Fractional-Polynomial)**

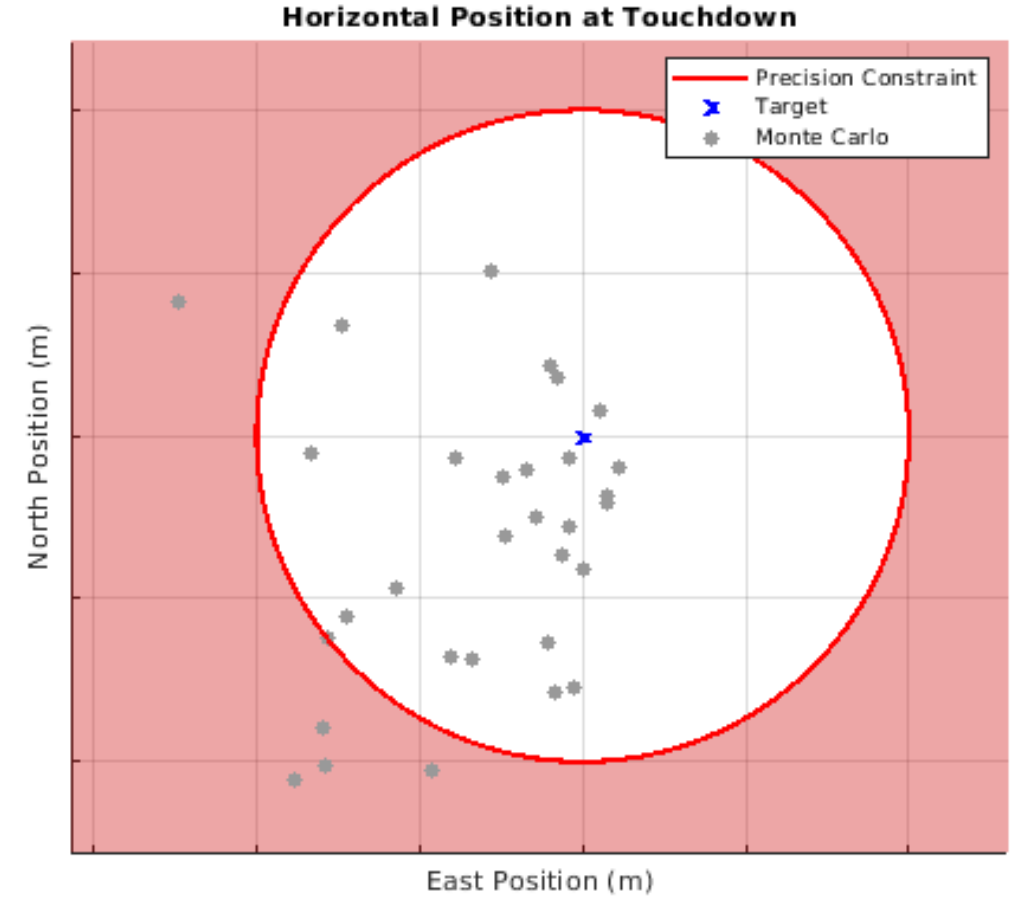


**Fig. 11b. Work-Energy
Guidance**

GLASS Monte-Carlo (Apollo vs Work-Energy)

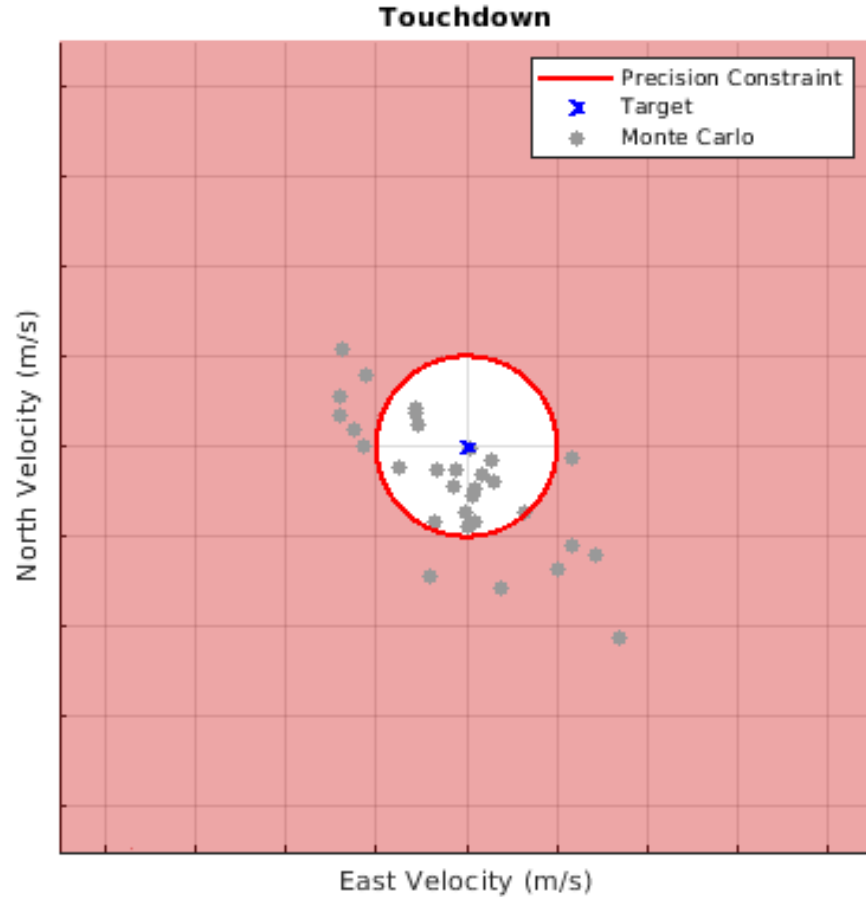


**Fig. 12a. Apollo Guidance
(Fractional-Polynomial)**

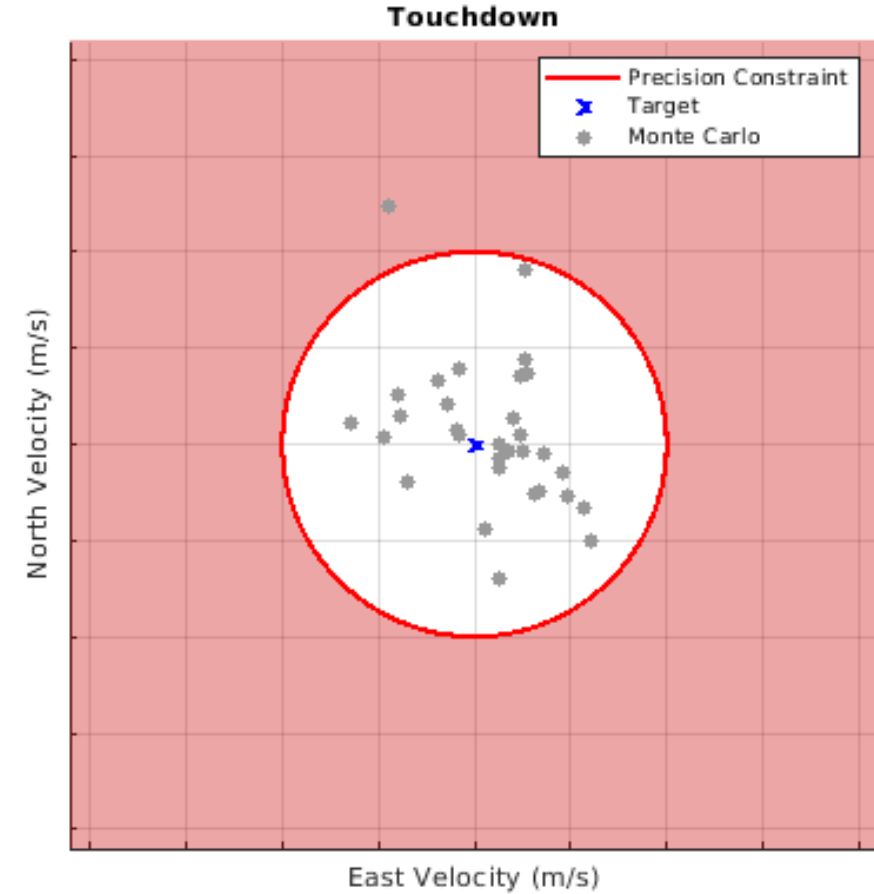


**Fig. 12b. Work-Energy
Guidance**

GLASS Monte-Carlo (Apollo vs Work-Energy)

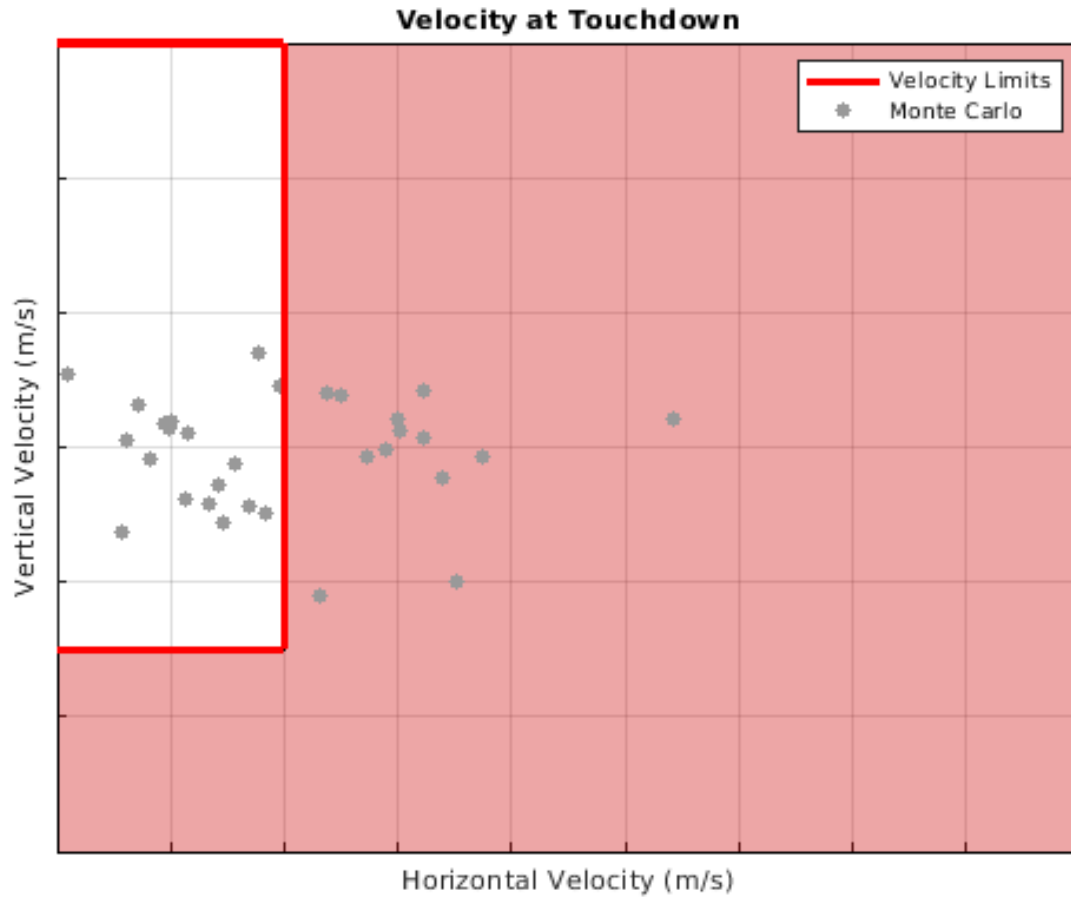


**Fig. 13a. Apollo Guidance
(Fractional-Polynomial)**

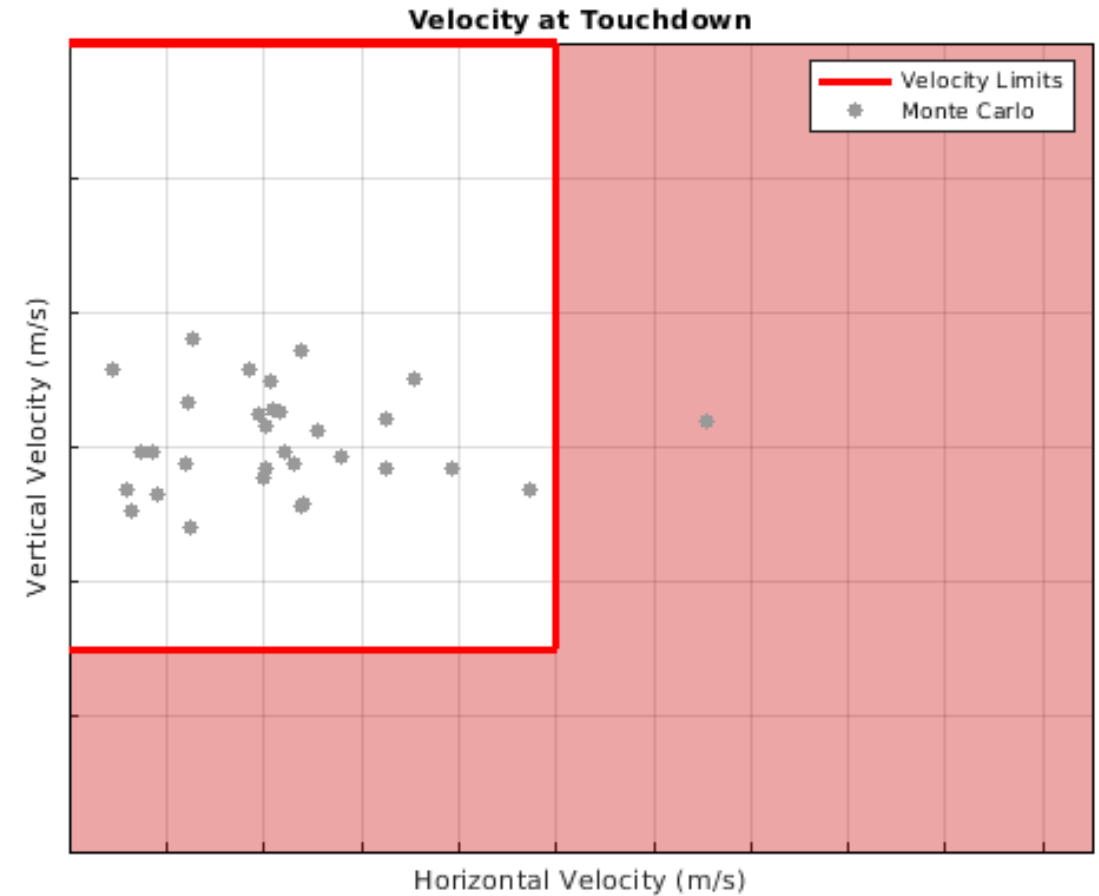


**Fig. 13b. Work-Energy
Guidance**

Work-Energy Fast Divert Space Determination

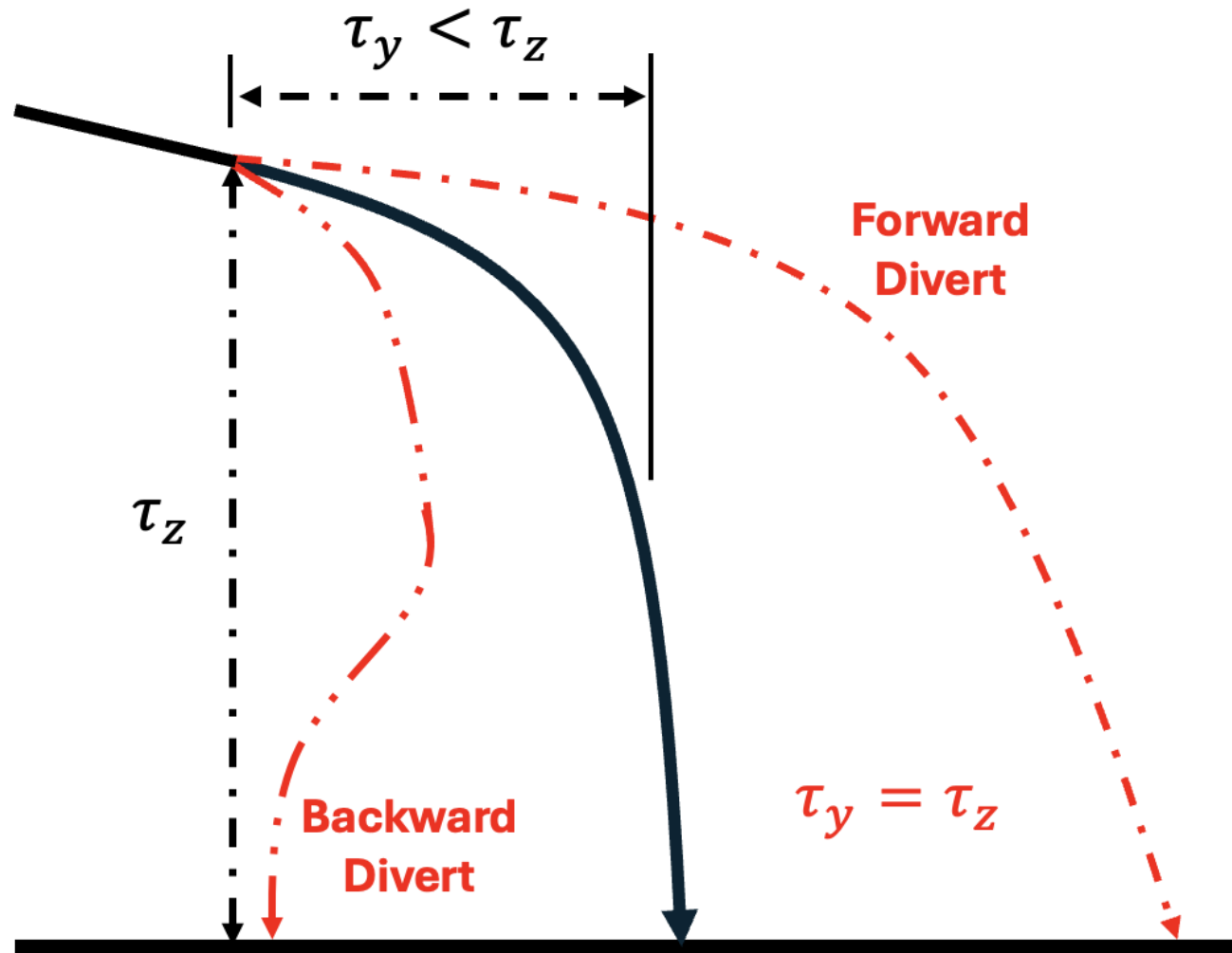


Fig. 19. Divert Space Determination

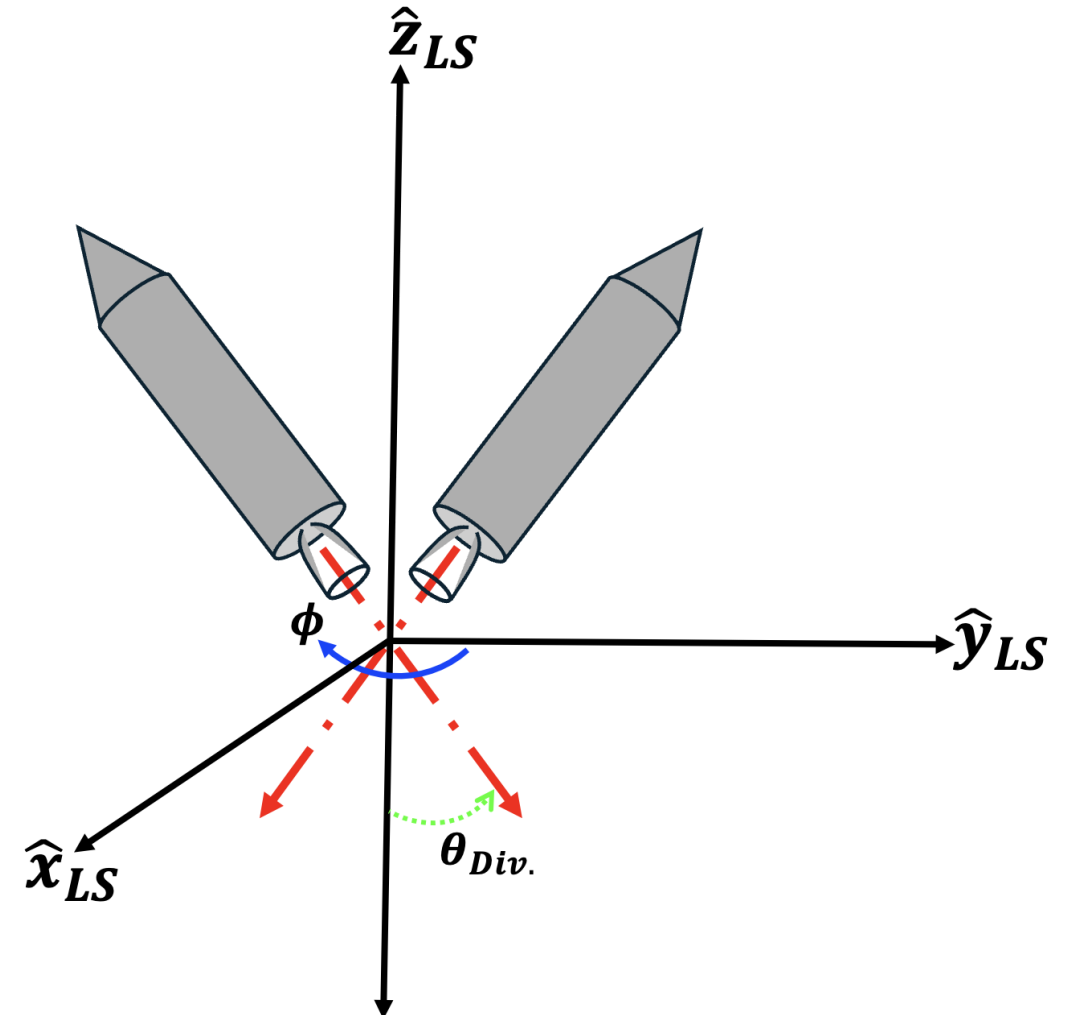
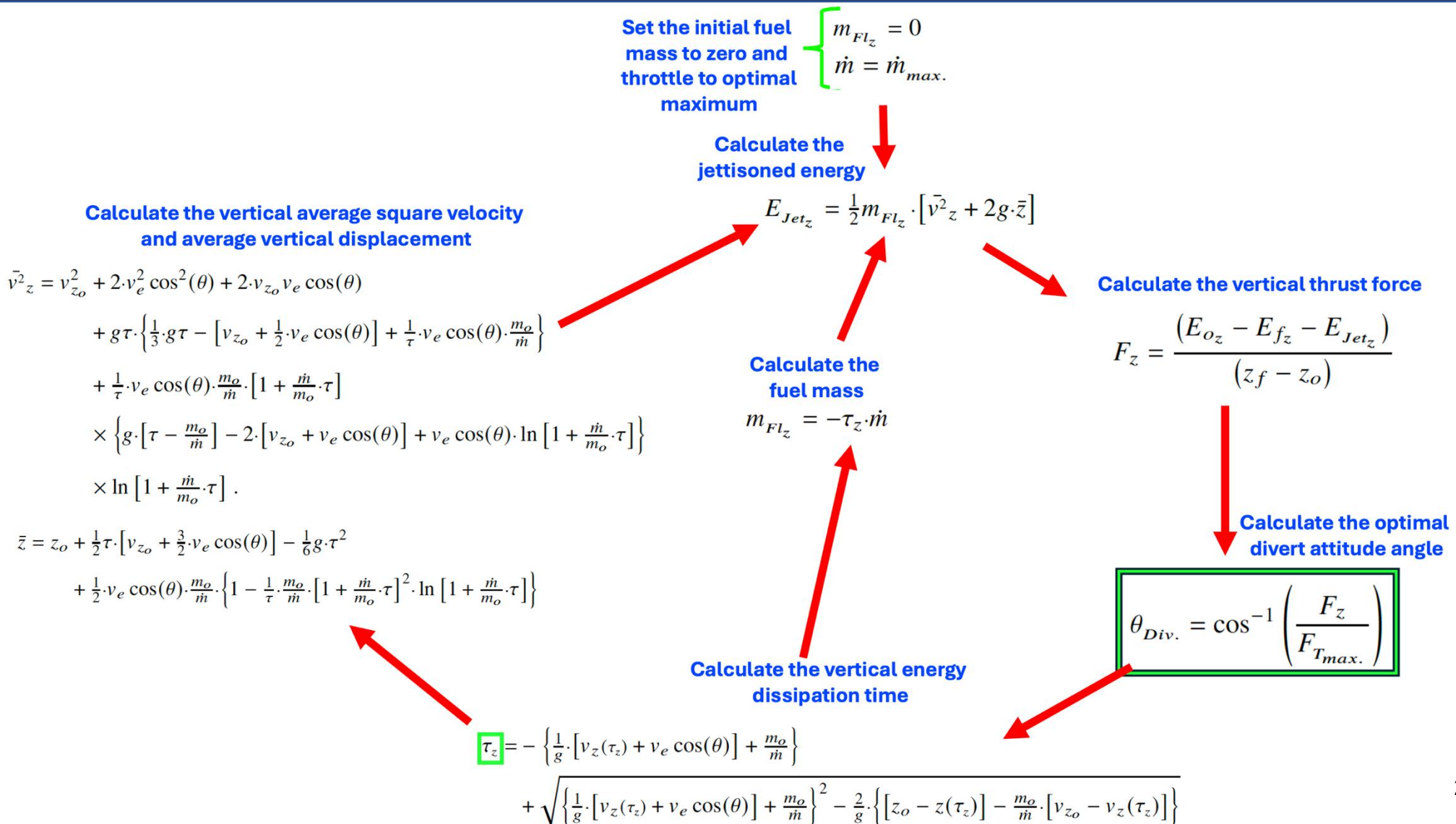


Fig. 20. Divert Vehicle Orientation

Work-Energy Fast Divert Space Determination



Work-Energy Fast Divert Space Determination



Divert Space Trajectory

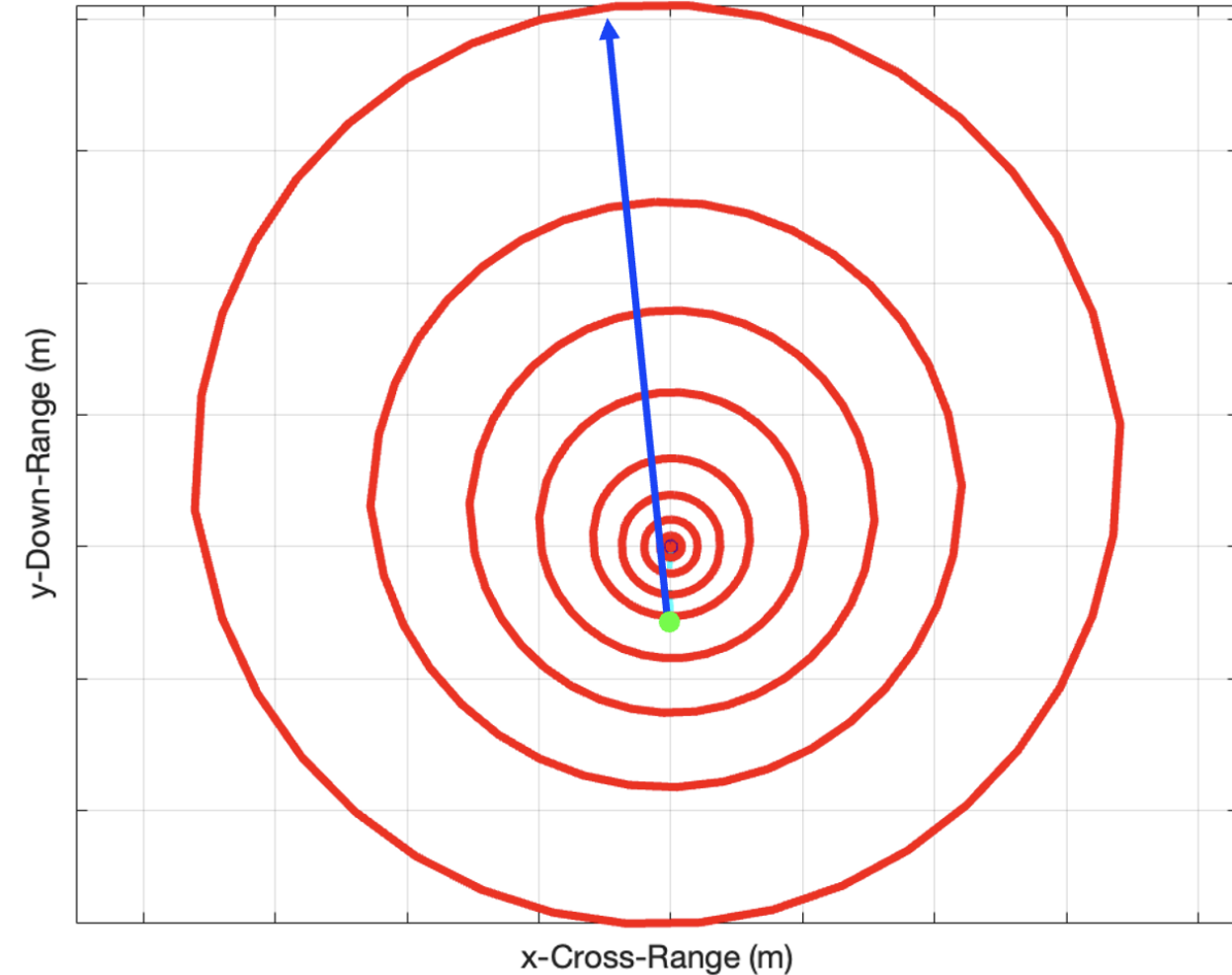
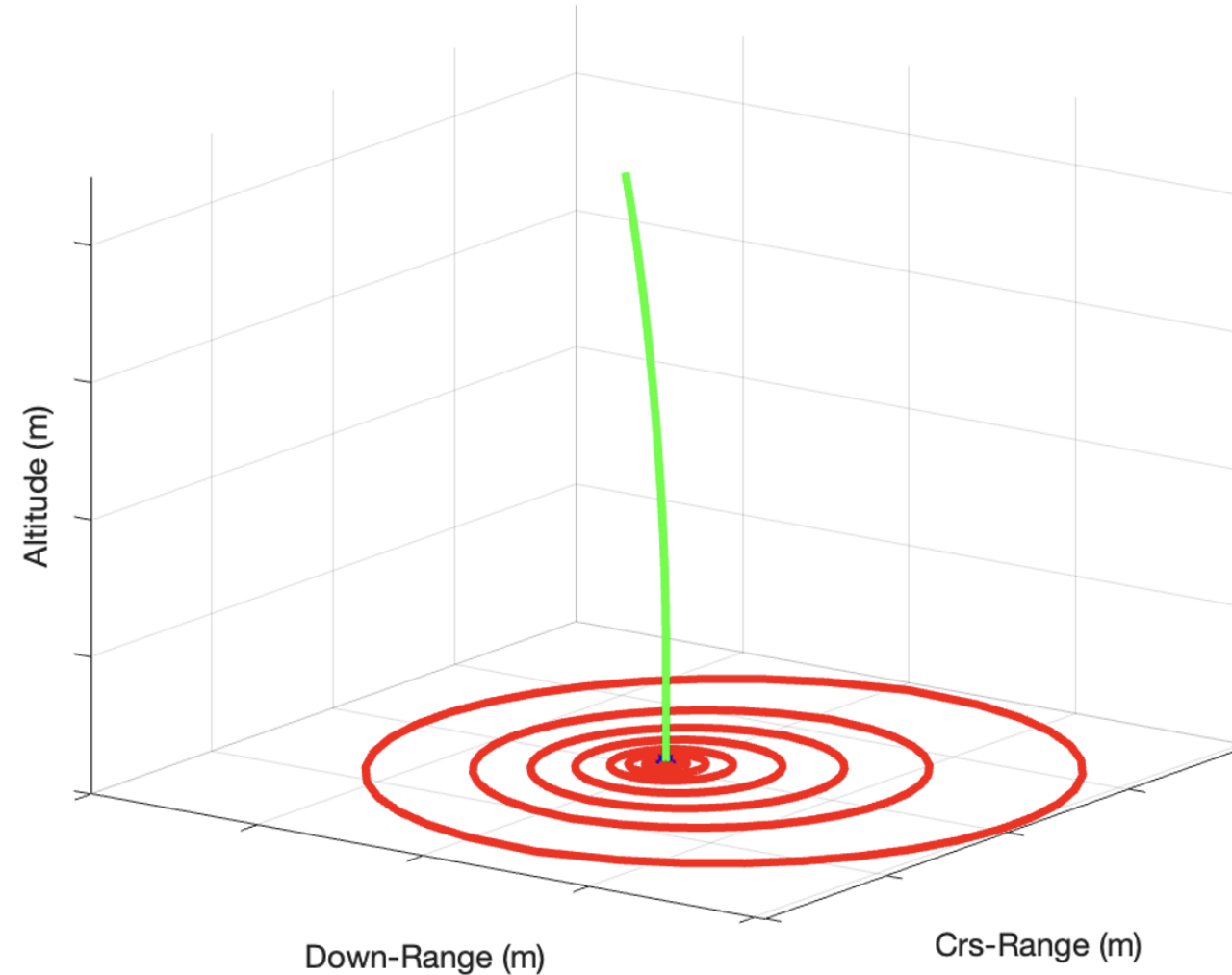


Fig. 21. Divert Space Calculation



Conclusions

Employing the Work-Energy formulation:

- Allows a more intuitive physical understanding of the trajectory and the consequences of changes due to perturbations and manual control.
- Reduces the need of constant calculation of reference trajectories and optimization (trajectory is optimized by the physics).
- Allows fast real-time calculation and recalculation of guidance commands at any point along the flight to achieve successfully the determined final state of the vehicle for landing.
- Allows fast real-time determination of the physically possible divert space for alternate landing.

