

Generating Dominating Sets Using Locally Defined Centrality Measures

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Abstract—The dominating set problem has many practical applications but is well-known to be NP-hard. Therefore, there is a need for efficient heuristic algorithms, especially in applications such as ad hoc wireless networks. Most distributed algorithms proposed in the literature assume that each node has knowledge of the network structure. We propose a distributed heuristic algorithm that uses two rounds of communication, and where each node has only local information, both in terms of network structure and dominating set assignment. First, each node calculates a local centrality measure to determine whether it is part of the dominating set D . The second round guarantees D is a dominating set by adding any non-dominated nodes. We compare several centrality measures and show that the Shapley centrality, derived from the Shapley value in game theory, is theoretically motivated and performs well in practice on several synthetic and real-world networks.

Index Terms—Network, dominating, Shapley.

I. INTRODUCTION

A dominating set in a network is a subset of nodes so that every node is either in the dominating set or has a neighbor in the set. Dominating sets and their variations have many applications [1]–[3]. Often, the objective is to find a dominating set of minimum size.

Deciding whether or not a graph has a dominating set of size k is NP-complete [4]. The current fastest exact algorithm for identifying a minimum dominating set was introduced in 2011 by Van Rooij and Bodlaender [5] and runs in $O(1.4969^n)$ time. Certain classes of graphs admit polynomial-time approximation schemes for the minimum dominating set problem. One example of such a class is unit disk graphs. This class includes random geometric graphs, as well as many types of ad hoc wireless networks [6]–[8]. Instead of identifying a dominating set globally, there are also distributed algorithms where decisions are made at the level of individual nodes with limited rounds of communications [9]–[14]. In most of these distributed algorithms, it is assumed that each node has full global information regarding the structure of the network, but only local information regarding the dominating set assignment. In some applications, nodes may have limited or no global information. A pertinent example of such an application is ad-hoc and wireless sensor networks [15]–[18].

The Space Communication and Navigation (SCaN) program office manages NASA’s space communication activities. One of their goals is to understand satellite communication, including routing, radio technology, and network modeling. In

the future of space science, autonomous robotic missions to other planets will be a primary means for humans to explore our solar system. In particular, satellite swarm missions akin to HelioSwarm have the opportunity to offer us great insight into the nature of other planets [19]. The power of satellite swarm missions comes from the wide variety of different roles that small satellites can serve. However, to enable interplanetary satellite swarm missions, it will be essential that satellites can autonomously organize and optimize the variety of roles available to the swarm.

Our motivation for this paper comes from this consideration. A satellite swarm orbiting a distant planet, such as Neptune, must be able to organize autonomously independently of the number of satellites involved. Two general roles that satellites fall into are communications satellites and data collection satellites. Future swarms may have the flexibility to switch between these roles as the needs of the swarm demand. However, there may not be a centralized node that can witness the full swarm at any given time. To enable the most efficient distribution of communications and data collection satellites, each satellite needs to be able to select its own role without knowing the full distribution of roles within the swarm. Moreover, any such selection algorithm must be adaptable to changes within the swarm network as satellite mobility can affect the overall network topology.

To solve these problems, we propose that the communications satellites form a dominating set over the swarm so that each data collector has the capacity to collect and transmit their data. However, we also want to minimize the number of communications satellites so as to maximize the amount of data the swarm can collect. So, we want to find a minimum dominating set among the satellites using a process that functions independently from the size of the swarm, that does not require centralized computations or for each satellite to be aware of the entire swarm, and that can be re-computed quickly among satellites upon detecting changes to the network topology.

Finding a minimum dominating set in these types of applications is often not feasible, because there may not be a central processor with access to the full network structure. Instead, the objective is to find a near minimum dominating set in a distributed manner where nodes can access only local information in the network.

Let $G(V, E)$ be a simple, undirected graph on n nodes, and let $\gamma(G)$ be the domination number of G ; the cardinality of a

minimum dominating set. One of the most fundamental results in the theory of dominating sets is an upper bound in terms of the minimum degree $\delta(G)$ of G :

$$\gamma(G) \leq \frac{1 + \ln(1 + \delta(G))}{1 + \delta(G)} n. \quad (1)$$

This result was proved independently by [20]–[23]. The proof given in [20] has become a standard example of the power of the probabilistic method. It gives a simple, two-step probabilistic algorithm that yields a dominating set whose cardinality is bounded above in expectation by Inequality 1. The algorithm works as follows. In the first round, each vertex is assigned to a set X with a fixed probability p , independently of other vertices. In the second round, each vertex not dominated by X is added to a set Y . The output is a dominating set $X \cup Y$. This algorithm is both fast, as it is linear in the size of the graph, and highly local, as decisions are made at the vertex level with communication only to immediate neighbors.

Wu and Li [14], proposed a similar algorithm with a deterministic first step. It uses the local clustering coefficient to select a dominating set [24]. This algorithm assigns a node v to the dominating set if it has a clustering coefficient less than 1. In this case, a connected dominating set is guaranteed, and there is no need for a second round.

Meghanathan investigates a similar algorithm with other centrality measures such as closeness and betweenness centrality [25]. Others have proposed a variety of greedy algorithms that are similar [26]–[30], but none that provide more analysis on other locally defined centrality measures. As far as the authors are aware, [14] and [31] are the only papers which perform a formal threshold analysis on a dominating set algorithm that uses two or less rounds of communication and local degree information.

Inspired by this style of algorithm, we propose an algorithm for generating dominating sets with the help of local centrality measures. Our algorithm is not guaranteed to generate a minimum dominating set, but we show that it performs well in practice in finding small dominating sets. In the first step, the algorithm selects vertices based on a centrality measure meeting a certain threshold. In the second step, more vertices are added as needed to guarantee a dominating set.

We compare several different centrality measures on synthetic and real-world networks. The best-performing measure in our analysis is the Shapley centrality, which is derived from the Shapley value. The Shapley value was introduced in game theory [32]. It is used in various contexts to measure the importance of an actor in game theory [33]–[36]. There are various definitions of the Shapley value in the literature. Our definition for Shapley centrality derives from the definition in [34], which defines the Shapley value of a node as the sum of the reciprocals of the degrees of its neighbors. (These authors then subtract 1 from each value, which we will not do for ease of computations.)

In Section II, we define the two-step algorithm and a few centrality measures that appear to effectively select small dominating sets. We compare their performance in the two-step algorithm on various synthetic and real-world networks in Section III. Section IV provides counterexamples to a few

conjectures about the two-step algorithm that we developed from the simulations. Lastly, in Section V, we discuss a few natural future directions.

II. DOMINATING SET ALGORITHMS USING THRESHOLDS

In Section II-A we outline the two-step algorithm explicitly. In Section II-B, we define the centrality measures considered in this study with a few basic observations that motivate their use in this context.

A. The two-step algorithm

Let $c(v)$ be a locally computed node centrality measure on the node set of a network $V(G)$, and $\tau \geq 0$ a threshold constant. The two-step algorithm has two rounds: the first round selects all nodes v such that $c(v) > \tau$, and the second round selects all nodes v such that neither v nor any of its neighbors were selected in the first round. The second step ensures that the final set of selected nodes forms a dominating set in G . Let $N(v) = \{u \in V \mid (u, v) \in E(G)\}$.

Two-step Algorithm

- 1) Let X be the set of nodes v with $c(v) > \tau$.
- 2) Let Y be the set of nodes v such that $(N(v) \cup \{v\}) \cap X = \emptyset$.
- 3) Let $D = X \cup Y$.

This algorithm can be executed in a distributed manner. To perform the first step, each node sends the necessary information to its neighbors so that each of them can calculate $c(v)$. Then, each node communicates to its neighbors whether they are in the set and carries out the second step. All of the measures we consider in Section II-B can be computed using one round of communication with the immediate neighborhood of each node, supposing each node knows its degree. However, local or not, any centrality measure can be used in the algorithm. If the algorithm is implemented in a distributed manner and a local centrality measure is used, it runs in time $O(\Delta(G))$, where $\Delta(G)$ is the maximum degree of the network. In the application of ad-hoc wireless networks, this often implies sublinear or even constant run time.

This algorithm performs best when X is close to a dominating set so that Y is small. Thus, the goal of the measure $c(v)$ is to capture nodes in X likely to appear in a minimum dominating set. In Section II-B, we define a few centrality measures of interest. We compare their performance in the two-step algorithm on various synthetic and real-world networks in Section III.

B. Centrality measures

Since we want our two-step algorithm to be local, we focus on centrality measures that can be determined locally. Specifically, we consider centrality measures that depend only on the immediate neighborhood of a node. We chose four centrality measures to compare in this study: two that have been analyzed in the past for dominating set constructions (URM and ICC), and two that have not (RND and SC). Networks may or may not be connected, but we consider only networks without isolated vertices.

Uniform random measure (URM): The uniform random measure assigns each node a randomly selected value between 0 and 1. Alon investigates which threshold minimizes the expected size of the set constructed in the two-step algorithm and gives an upper bound on the domination number [31]. Refer to Section I for more information.

Clustering coefficient (ICC): In [14], Li and Wu show that the collection of all nodes with two non-adjacent neighbors is a connected dominating set. This is quantified by finding the proportion of non-adjacent neighbors, known as the *clustering coefficient* [24] of a node v is

$$CC(v) = \frac{t_v}{\binom{k_v}{2}} = \frac{2t_v}{k_v(k_v - 1)}.$$

Here, t_v is the number of triangles that node v is a part of, and k_v is the degree of node v . Note that if $k_v = 0$ or 1, then $CC(v) = 0$.

Based on simulated data, we suspect that the nodes representative of stronger dominating set candidates tend to have lower values of $CC(v)$. For this reason, we let

$$ICC(v) = 1 - CC(v) = \frac{\binom{k_v}{2} - t_v}{\binom{k_v}{2}}.$$

Similarly to $CC(v)$, if $k_v = 0$ or 1, then $ICC(v) = 1$. It is worth noting that ICC is not a new centrality measure but rather our adjustment of the clustering coefficient that allows for more convenient comparisons with the other centrality measures, as they all tend to correlate positively with the likelihood of appearing in minimum dominating sets.

Using the language of our paper, Li and Wu show that the first step of two-step algorithms constructs a connected dominating set with measure ICC and a threshold of 0.

Relative neighbor degree (RND): Ai, Li, Su, Jiang, and Xiong define the neighbor-degree centrality of v as $ND(v) = \frac{\sum_{u \in N(v)} k_u}{k_v}$ [37]. We propose the *relative neighbor-degree centrality* of v as

$$RND(v) = \frac{k_v}{ND(v)} = \frac{k_v^2}{\sum_{u \in N(v)} k_u}$$

which measures how the average degree of nodes in $N(v)$ compares to the degree of v itself. If a node v has a relative neighbor degree of 1, then the average degree of the neighbors of v equals the degree of v . If v has a relative neighbor degree larger than 1, then v makes for a good dominating set candidate as its degree is larger than its neighbor's degree on average.

Shapley centrality (SC): The Shapley value was originally proposed in [32], but was not introduced in network theory until later. Bozzo, Franceschet, and Rinaldi describe the Shapley value of a node v in a network as the sum of the reciprocals of the neighbors' degrees, and we use this as our definition for Shapley centrality:

$$SC(v) = \sum_{u \in N(v)} \frac{1}{\deg(u)} \quad [32], [34].$$

A node with a high Shapley centrality is likely adjacent to many low-degree nodes.

For the remainder of the paper, we compare all four measures and specifically focus on how SC and RND, as they seem to perform best in this setting. From some simple analysis, these two measures are related. For instance, Proposition II.1 shows that the number of nodes selected in the first round of the two-step algorithm with SC is always greater than that with RND.

Proposition II.1. *For node v in a network N , if $RND(v) \geq \tau$, then $SC(v) \geq \tau$.*

Proof. Suppose that $RND(v) \geq \tau$. Then the average degree of a node in N_v is less than k_v/τ . By Jensen's inequality, the average of the set $\{1/k_u | u \in N_v\}$ is at least τ/k_v . Hence, $SC(v) = \sum_{u \in N_v} 1/k_u \geq k_v \frac{\tau}{k_v} = \tau$. $\square$

We give the expected value of SC in Proposition II.2 below. This proposition shows that even though SC is a locally computed measure, we know how each node will compare to the average without knowing anything about the network's structure. The expected value of RND is not as easily described.

Proposition II.2. *Let $G = (V, E)$ graph. The mean Shapley centrality is 1.*

Proof. Let A be the adjacency matrix of a network $N = (V, E)$ with $n = |V|$. Let M be the column stochastic matrix with $M_{ij} = \frac{A_{ij}}{k_j}$ and $\mathbf{1}$ be the n -dimensional vector with 1 in each entry. Notice that $(M \cdot \mathbf{1})_i = SC_i$ for each $i \in V$. Since the stochastic matrix preserves the average of vectors, then the expected Shapley centrality value is 1. $\square$

From Proposition II.2, $\tau = 1$ is a natural choice of threshold if we use SC in our two-step algorithm. To round out this section, we give additional evidence suggesting that $\tau = 1$ is a reasonable choice of threshold in a vacuum.

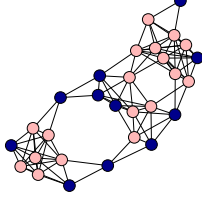
Observation II.3. *If $\deg(v) = 1$, then v is adjacent to a node u with $SC(u) \geq 1$.*

Observation II.4. *If $\deg(v) \geq \deg(u)$ for all $u \in N(v)$, then $RND(v) \geq 1$.*

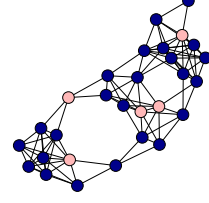
According to Observation II.3, a threshold of $\tau = 1$ will, among others, select all nodes adjacent to degree 1 nodes in the first round of the algorithm. We will see in Section III and IV, other thresholds better find minimal dominating sets using the two-step algorithm.

III. EXPERIMENTS

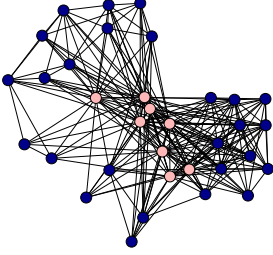
We test the performance of the two-step algorithm on four satellite constellations: Globalstar [38], Iridium [39], Orbcomm [40], and SpaceBEE [41]. Orbit information was collected from CelesTrak and processed using Satellite Orbital Analysis Program [42] to generate a sequence of graphs based on line-of-sight connections. Figure 1 contains an instance of each satellite constellation. Each node in Figure 1 is colored pink (light) if it was selected as part of the dominating set by the two-step algorithm using the Shapley centrality, and blue (dark) otherwise. We test the algorithm with two threshold values, namely, $\tau = 1$ and $\tau = 1.2$. There were fewer nodes



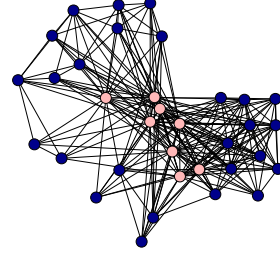
(a) Dominating set in the Iridium satellite network identified by the two-step algorithm with $\tau = 1$.



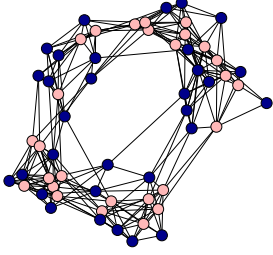
(b) Dominating set in the Iridium satellite network identified by the two-step algorithm with $\tau = 1.2$.



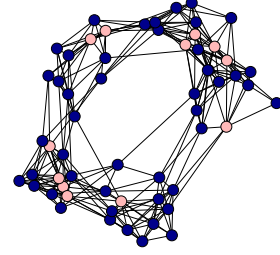
(c) Dominating set in the SpaceBEE satellite network identified by the two-step algorithm with $\tau = 1$.



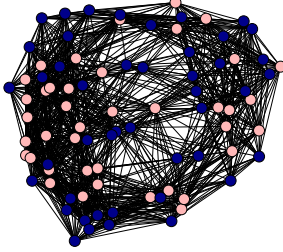
(d) Dominating set in the SpaceBEE satellite network identified by the two-step algorithm with $\tau = 1.2$.



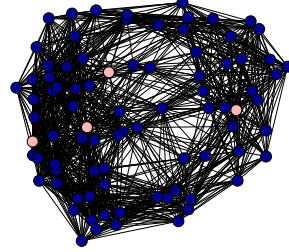
(e) Dominating set in the Orbcomm satellite network identified by the two-step algorithm with $\tau = 1$.



(f) Dominating set in the Orbcomm satellite network identified by the two-step algorithm with $\tau = 1.2$.



(g) Dominating set in the Globalstar satellite network identified by the two-step algorithm with $\tau = 1$.



(h) Dominating set in the Globalstar satellite network identified by the two-step algorithm with $\tau = 1.2$.

Fig. 1: Composite figure showcasing the dominating sets (pink vertices) across four satellite networks (Iridium, SpaceBEE, Orbcomm, Globalstar) for two distinct values of τ . Each network's response to varying τ values highlights the adaptability and efficiency of the two-step algorithm in identifying critical nodes.

selected by the two step algorithm with threshold $\tau = 1.2$ in the Iridium, Orbcomm, and Globalstar constellations, and in the SpaceBEE constellation the number of nodes selected did not change from $\tau = 1$ to $\tau = 1.2$.

To investigate the differences in different thresholds more fully, we then simulated additional graphs based on the parameters witnessed in these constellation graph sequences. The average degree observed in the Iridium, SpaceBEE, Orbcomm, and Globalstar constellations is around 6, 14, 9, and 29, respectively. We tested the algorithm on three random graph

models, namely, the Chung Lu, Erdős-Rényi, and Random Geometric graph models. We chose the parameters such that the expected degree of each model matched the average degree in each constellation and tested the two step algorithm with the uniform random measure, clustering coefficient, relative neighbor degree, and Shapley centrality with a variety of thresholds τ between 0 and 2.5. A description of each model is below:

Chung-Lu Model [43]–[45] The Chung-Lu model takes as input the order the graph n and a list of n integers

$(w_1, w_2, \dots, w_n)$ where $w_i \in [0, n]$. Each edge (i, j) is included in the model with probability

$$p_{i,j} = \frac{w_i w_j}{\sum_k w_k}$$

We use the Chung-Lu model to sample graphs with a longer-tailed degree distribution and create the degree sequence using a negative binomial degree distribution. The parameters we use for the negative binomial distribution are described in the description of Figures 2a, 2d, 2g, and 2j.

Erdős-Rényi Model [46] The Erdős-Rényi model has two input values: the number of nodes n and the probability of an edge p . We list the parameters used in our analysis to model each constellation in Figures 2b, 2e, 2h, and 2k.

Random Geometric Graphs [47] The random geometric graph model takes as input the number of nodes n and a radius r . To construct the network, n points are sampled randomly from the unit square $[0, 1]^2$ as the nodes, and two nodes v_1 and v_2 are connected by an edge if $d(v_1, v_2) < r$, where d is the Euclidean distance. We list the parameters used in our analysis to model each constellation in Figures 2c, 2f, 2i, and 2l.

In Figure 2, for each family and each set of parameters, we generated 1000 networks to form the basis of our experiment. SC and RND construct the smallest dominating sets across all thresholds, with SC slightly outperforming the rest in the random geometric graph model (i.e. the construction most similar to that of satellite networks). Notice that the minimal dominating set in each family typically occurs at a threshold greater than 1 rather than precisely at 1. The most notable behavior is that of the Chung-Lu model due to its heavier tailed degree distribution. The thresholds that provide the smallest dominating set across the 1000 graphs in this model are higher on average compared to the Erdős-Rényi and random geometric graph models.

IV. ODDITIES

This section presents a set of counterexamples to natural conjectures regarding the two-step algorithm with the SC centrality measure. These counterexamples are enlightening, revealing exciting relationships between the measures we examined here and some general oddities surrounding the two-step algorithm.

With a threshold in the first filtering for the two-step algorithm, much of our questioning centered on choosing the best threshold. Inspired by Wu and Li [14], we initially sought a universal threshold for constructing a dominating set. If there were such a threshold for SC or RND, we could reduce our two-step algorithm to just one step: compute the measure and determine whether or not you are in the dominating set. If the threshold is 0, only one step is necessary, but perhaps a small non-zero threshold will also guarantee a dominating set in one step.

It turns out that there is no such non-zero threshold, but our construction to show this is unwieldy and unlikely to appear in graphs with relatively small numbers of nodes. We present this construction in the proof of the following proposition.

Proposition IV.1. *Given a threshold $\tau > 0$, let S_τ be the set of nodes such that $SC(v) > \tau$. There exists a graph G such that S_τ is not a dominating set of G .*

Proof. We construct a graph with node a that satisfies the following properties:

- 1) For node a , $SC(a) < \tau$.
- 2) For all nodes $b \in N(a)$, $SC(b) < \tau$.

Given $\tau > 0$, there exists some $N_1 \in \mathbb{N}$ such that $\frac{1}{N_1} < \tau$. So, we can construct a set of N_1 nodes $\{b_i\}_{i=1}^{N_1}$ such that each b_i is adjacent to a . Thus, $\deg(a) = N_1$, and $N(a) = \{b_i\}_{i=1}^{N_1}$.

Since $N_1, \tau > 0$, there exists $N_2 \in \mathbb{N}$ such that $N_2 > \frac{N_1}{\tau}$. For each b_i , construct $N_2 - 1$ nodes $\{c_{i,j}\}_{j=1}^{N_2-1}$ such that each is adjacent to b_i . Thus, $\deg(b_i) = N_2$, and $N(b_i) = \{a\} \cup \{c_{i,j}\}_{j=1}^{N_2-1}$ for each i . Then, by construction, $SC(a) = \frac{N_1}{N_2} < \tau$.

Since $\frac{1}{N_1} < \tau$, $\tau - \frac{1}{N_1} > 0$, and since $N_2 \in \mathbb{N}$, $N_2 - 1 \geq 0$. Thus, there exists $N_3 \in \mathbb{N}$ such that $N_3 > \frac{N_2-1}{\tau - \frac{1}{N_1}}$. So, for each $c_{i,j}$, construct $N_3 - 1$ nodes $\{d_{i,j,k}\}_{k=1}^{N_3-1}$ such that each is adjacent to $c_{i,j}$. Thus, $\deg(c_{i,j}) = N_3$ for each j . So, by construction, for each b_i ,

$$SC(b_i) = \frac{1}{\deg(a)} + \sum_{j=1}^{N_2-1} \frac{1}{\deg(c_{i,j})} = \frac{1}{N_1} + \frac{N_2-1}{N_3} < \tau.$$

Thus, any graph containing this construction as a subgraph would satisfy the conditions above. $\square$

From the construction in Proposition IV.1, and the relationship between RND and SC given in Proposition II.1, we get the following result.

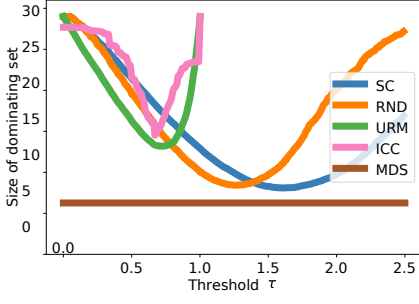
Corollary IV.2. *Given a threshold $\tau > 0$, let R_τ be the set of nodes such that $RND(v) > \tau$. There exists a graph G such that R_τ is not a dominating set of G .*

The choice of threshold then depends on the graph structure we are examining. However, in the plots we generated, the graphs appeared to have a consistent shape, including a single local minimum. For the minimum to be unique, one condition would be that for any three thresholds $\tau < \sigma < \rho$, one could never see $f(\tau) < f(\sigma) > f(\rho)$.

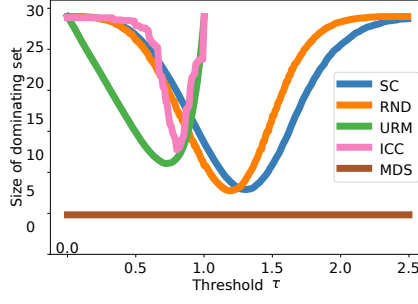
Unfortunately, this is not true in general. Figure 3 depicts one counterexample of this idea. In particular, when we use SC with the two-step algorithm at three different thresholds (0.8, 0.9, and 1), the size of the dominating set generated goes from 19 to 20 and back to 19.

We conclude our catalog of oddities by focusing on the threshold of 1. Thanks to Proposition II.2, we know that 1 holds a special place concerning the SC and RND values.

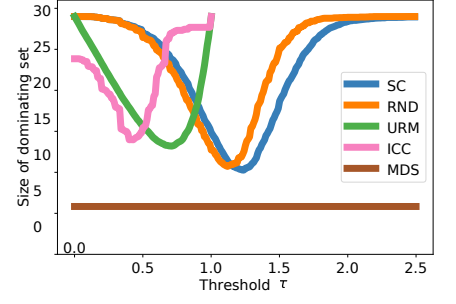
However, returning to the data represented in Figure 2, there was one further counterexample to consider. When we set the threshold at 1, the dominating set generated with RND was generally larger than the dominating set developed with SC. However, Figure 4 shows an example where the number of nodes selected in the two-step algorithm is smaller in SC (Figure 4a) than in RND (Figure 4b) using the threshold $\tau = 1$. Here, the black (dark) nodes are chosen in the first round and the pink (light) nodes in the second.



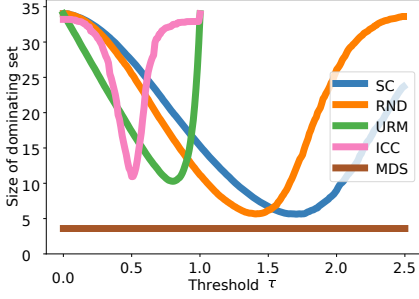
(a) Chung-Lu model of the Iridium constellation with $NB(k = 5, p = 0.574)$, showing an average degree of 6 across 29 nodes.



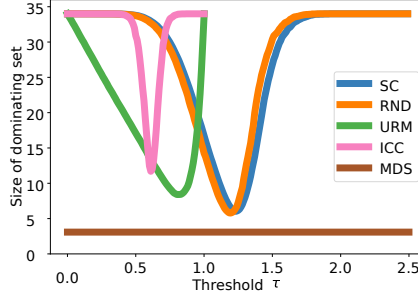
(b) Erdős-Rényi model representation of the Iridium satellite constellation, with $n = 29$ nodes and edge probability $p = 0.241$, achieving an average degree of 6.



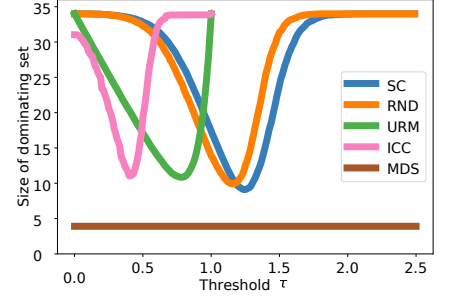
(c) Random geometric graph model representation of the Iridium satellite constellation, with parameters $n = 29$ and $r = 0.277$, achieving an average degree of 6.



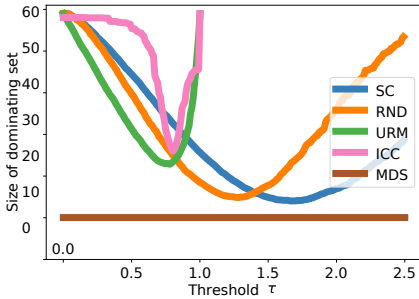
(d) Chung-Lu model of the SpaceBEE constellation with $NB(k = 5, p = 0.263)$, showing an average degree of 14 across 34 nodes.



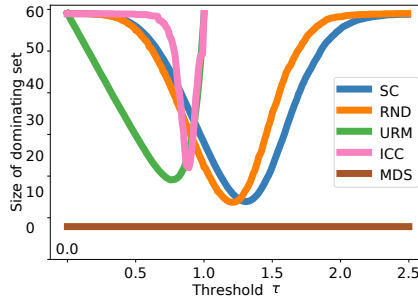
(e) Erdős-Rényi model representation of the SpaceBEE satellite constellation, with $n = 34$ nodes and edge probability $p = 0.4242$, achieving an average degree of 14.



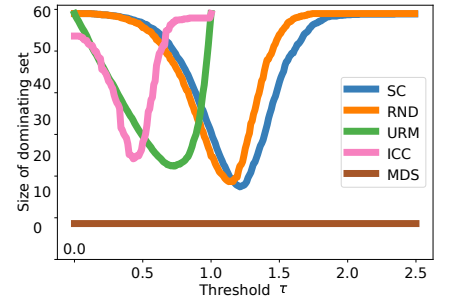
(f) Random geometric graph model representation of the SpaceBEE satellite constellation, with parameters $n = 34$ and $r = 0.367$, achieving an average degree of 14.



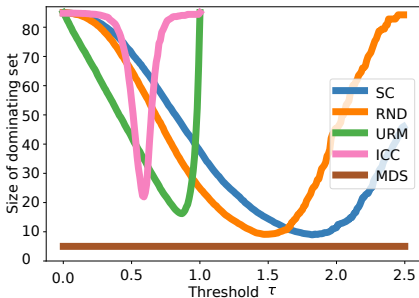
(g) Chung-Lu model of the Orbcomm constellation with $NB(k = 5, p = 0.357)$, showing an average degree of 9 across 59 nodes.



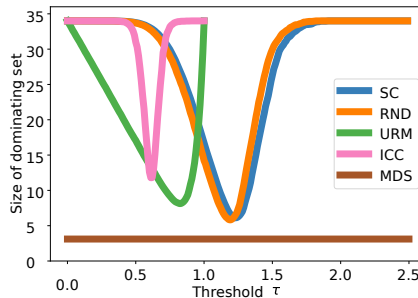
(h) Erdős-Rényi model representation of the Orbcomm satellite constellation, with $n = 59$ nodes and edge probability $p = 0.155$, achieving an average degree of 9.



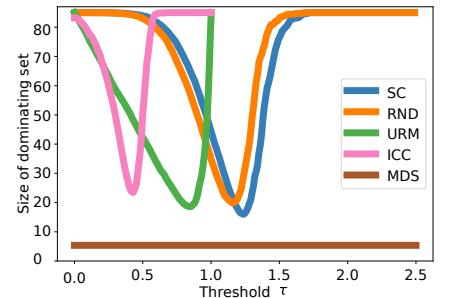
(i) Random geometric graph model representation of the Orbcomm satellite constellation, with parameters $n = 59$ and $r = 0.222$, achieving an average degree of 9.



(j) Chung-Lu model of the Globalstar constellation with $NB(k = 5, p = 0.147)$, showing an average degree of 29 across 85 nodes.



(k) Erdős-Rényi model representation of the Globalstar satellite constellation, with $n = 85$ nodes and edge probability $p = 0.345$, achieving an average degree of 29.



(l) Random geometric graph model representation of the Globalstar satellite constellation, with parameters $n = 85$ and $r = 0.333$, achieving an average degree of 29.

Fig. 2: Comparative visualization of satellite constellations (Iridium, SpaceBEE, Orbcomm, Globalstar) via Chung-Lu, Erdős-Rényi, and Random Geometric models, highlighting network diversity. Each figure is created by averaging over 1000 graphs.

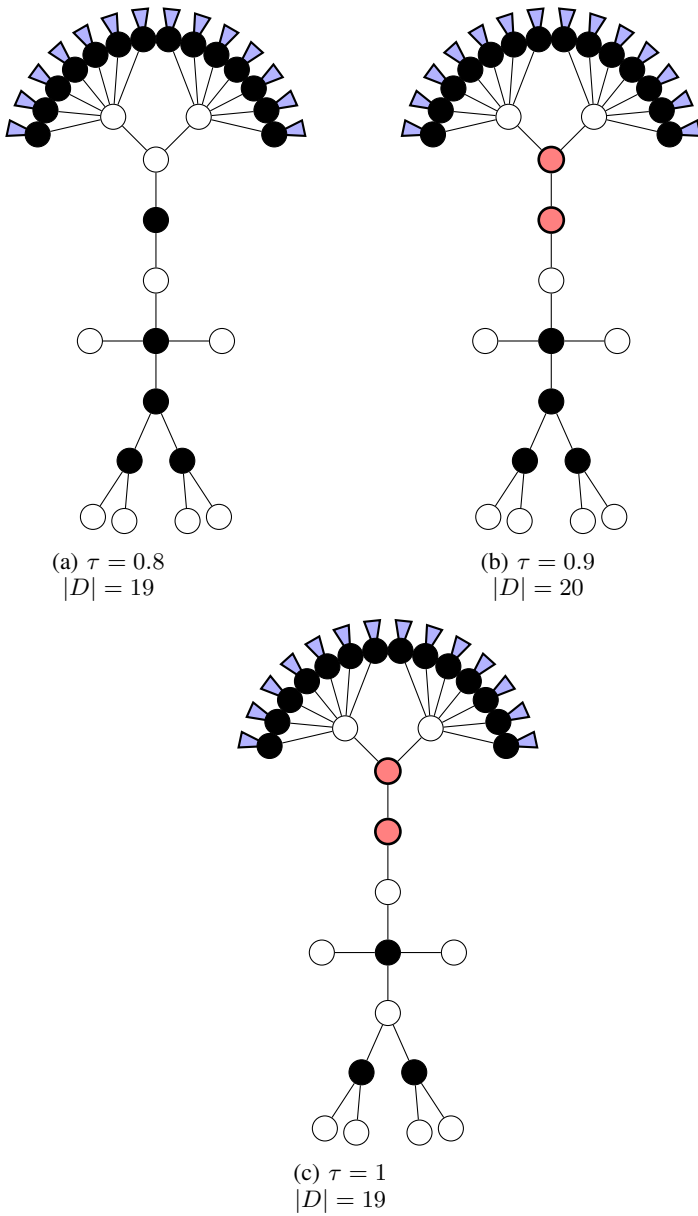


Fig. 3: The size of the dominating set formed by the two-step algorithm with the Shapley centrality does not achieve a unique local minimum. The purple triangles represent 100 leaf nodes. A black node suggests that it was chosen in the first round of the two-step algorithm, and red suggests it was chosen in the second round.

There is undoubtedly more exploration in characterizing the function described in the plots. Both experimentally and theoretically, further investigation is warranted.

V. FUTURE DIRECTIONS

Various algorithms aim to approximate a minimum dominating set for graphs [48]. Our approach of using a two-step algorithm allows for a larger variety of node measures than we propose for generating dominating sets. However, we focused on four local measures to highlight some methods

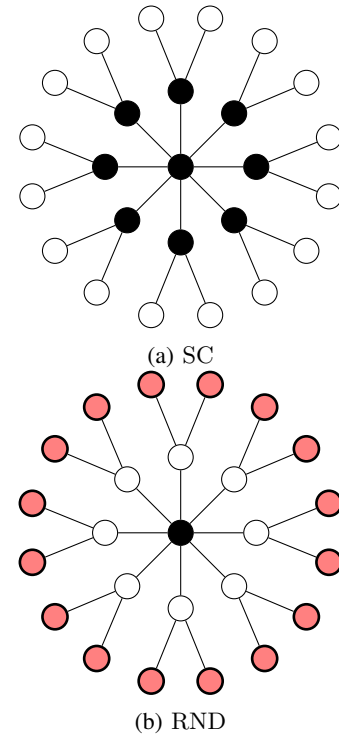


Fig. 4: An example where the dominating set generated using RND is greater than the dominating set generated using SC.

that could be practical in robot swarm and autonomous system communications.

We see several directions to take this work into the future. First, an analysis of different measures and their properties when plugged into the two-step algorithm may be an excellent method for comparing the effects of other measures. Included with this, a more in-depth theoretical analysis of the two-step algorithm is warranted to explore its effectiveness further.

One concept we have toyed with is constructing bounds for the thresholds in various situations. For example, one might wonder whether there are certain classes of networks that yield natural bounds. For instance, with regular graphs, the Shapley centrality of each node is necessarily 1. In graphs that are regular or close to regular, it may be effective to combine for example the Shapley centrality with a random measure. Predicting optimal thresholds based on network structure alone or time-varying network structure alone is a powerful tool in robotic mission design. Moreover, if the ability to find bounds analytically proves intractable, passing this information into a neural network may be an exciting problem to pursue. A Graph Neural Network that intakes graph structure and outputs a substantial threshold for achieving a dominating set could enable more optimizations, especially if this could be computed locally in some way.

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