

Design Under Uncertainty with Design-Dependent Uncertain Variables

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Uncertainty quantification (UQ) can provide a more robust understanding of a system, leading to better informed decisions earlier in the design process. The additional information that UQ provides can be leveraged during a design optimization process known as design under uncertainty that, when incorporated with multidisciplinary design and optimization, can become computationally infeasible due to the large number of responses required for meaningful results. Previous work addressed reducing the computational expense in design under uncertainty by incorporating analytic derivatives throughout polynomial chaos expansion. Although this allows design under uncertainty to be feasible for more systems, some multidisciplinary systems have design-dependent uncertain variables. This paper details an implementation of design-dependent uncertain variables in a manner than preserves derivatives required for efficient gradient-based optimization throughout the process. Two analytic examples of design-dependent uncertain variables are given: the first transforms a uniform uncertain variable with one design variable and the second transforms a normal uncertain variable with two design variables. The polynomial chaos expansion (PCE) results are comparable to both the Monte Carlo (MC) results and the analytic results for the two examples. A case study that maximizes the lift-to-drag ratio with a design-dependence between the wing leading edge sweep angle and uncertain parameter percentage of laminar flow is compared to a MC and alternative optimization formulations. This paper demonstrates that design-dependent uncertain variables are valid and hold throughout PCE.

Nomenclature

AR	Wing Aspect Ratio	c_r	Wing Root Chord
a	Significance Level	D	Drag
a	Uniform Distribution Lower Bound	g	Gravity Constant
b	Span	$\vec{J}$	Design Variables
b	Uniform Distribution Upper Bound	$\vec{K}$	Constraint Functions
C	Camber	L	Lift
C_{l_α}	Wing Section Lift Curve Slope	M	Mach Number
$C_{l_{\alpha 0}}$	Wing Section C_l at $\alpha = 0$	N_z	Ultimate Load Factor
C_{L_α}	Wing Lift Curve Slope	P_{minx}	Chordwise Location of Minimum Pressure
C_L	Wing Lift Coefficient	Re_L	Reynolds Number
C_{D_0}	Wing Zero Lift Drag Coefficient	S_w	Wing Area
C_{D_i}	Wing Induced Drag Coefficient	t/c	Wing Thickness-to-Chord Ratio
C_{D_w}	Wing Wave Drag Coefficient	$\vec{U}$	Uncertainty Space
C_D	Wing Total Drag Coefficient	$\mathbf{u}_0$	Initial Uncertain Samples
C_{f_L}	Wing Laminar Flow Skin Friction Coefficient	$\mathbf{u}_t$	Transformed Uncertain Samples
C_{f_T}	Wing Turbulent Flow Skin Friction Coefficient		

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W_{wing}	Wing Weight	λ	Wing Taper Ratio
x_i	Design Variable for Parameter i	Λ	Wing Leading Edge Sweep Angle
$\%c_L$	Percentage of Laminar Flow	μ	Mean
α	Angle of Attack	σ	Standard Deviation
β	Mach Correction	$\sigma_{C_{D_0}}$	Drag Multiplication Factor
η	Airfoil Efficiency	σ_{C_L}	Stall Loss of Lift Factor
κ_α	Korn Factor		

I. Introduction

Accounting for uncertainty in the preliminary design process of aircraft is of critical concern to NASA's Aeronautics Research Mission Directorate (ARMD). Uncertainty quantification (UQ) studies can give designers insight into the system level performance based on a statistical representation of uncertainty. Furthermore, coupling uncertainty quantification with state-of-the-art gradient-based optimization programs such as OpenMDAO [1] reduces the overhead necessary to perform design under uncertainty. Reducing computational expense is essential to eventually incorporating high-fidelity analysis in optimizations with uncertainty, enabling future researchers to explore an aircraft design space in detail under uncertain conditions. Recent work by Philips et al. [2] explored design under uncertainty leveraging analytic gradients throughout polynomial chaos expansion (PCE), including a novel approach for calculating a differentiable confidence interval (CI). This enabled researchers to start placing objectives and constraints on upper confidence intervals, lower confidence intervals, and central values. However, additional challenges can arise when an uncertain distribution changes as a function of design variables. This work addresses this additional layer of uncertainty variation in a manner consistent with gradient-based optimization methods. The method detailed in this paper is generalized such that it is easily tractable to any gradient-based optimization.

Previous work by Kriegesmann [3] showed that it is possible to account for the dependency of design variables on uncertain variables in design optimization. This concept was demonstrated with both Monte Carlo (MC) simulation and Taylor series approaches of representing system uncertainty. Other approaches have modeled design-dependent uncertain variables with analytic derivatives; however, these methods applied directly to derivatives for the probability density functions of the uncertain variables [4]. Approaching some Multi-Disciplinary Analysis and Optimization (MDAO) problems with MC simulations can be infeasible due to the number of samples needed for the appropriate accuracy. Similarly, it can be impractical to use the analytical probability density function (PDF) to represent some probabilistic systems, such as those encountered in aircraft preliminary design. Philips et al. [2] demonstrated an efficient approach for calculating a differentiable CI but did not explore design-dependent uncertainties.

Extending previous work, this research will demonstrate the validity of design-dependent uncertain variables through PCE with analytic gradients. Section II will discuss the uncertainty modeling methodology using polynomial chaos expansion and an overview of the analytical gradients through PCE. Next, Section III will detail the case study in which design-dependent uncertain variables are implemented for a wing design that maximized lift-to-drag ratio (L/D). Results of the study will be presented in Section IV. Finally, a summary is presented in Section V.

II. Methodology

A. Uncertainty Quantification Methodology

Uncertainty quantification encompasses the study of how the input parameter uncertainty and modeling simplifications affect the responses of a process or simulation. UQ can vary in scope by including only a single model or multiple models of varying fidelity levels as well as experimental data. The overarching objective of UQ is to create a more robust design or evaluation process by identifying sensitivities and mitigating the potential impact of uncertainties through informed, targeted resource investments.

1. Uncertainty Quantification with Polynomial Chaos Expansion (UQPCE)

All of the uncertainty modeling and analysis contained in this research was performed with one of NASA's in-house uncertainty codes, Uncertainty Quantification with Polynomial Chaos Expansion, UQPCE [5]. UQPCE* is an open source, Python-based research code for use in parametric, non-deterministic computational analysis and design. UQPCE

*<https://github.com/nasa/UQPCE>

utilizes a non-intrusive polynomial chaos expansion surrogate modeling technique to efficiently estimate uncertainties for computational analyses. Further details on polynomial chaos theory are given by Refs. [6–9]. UQPCE uses the second-order probability approach outlined by Eldred and Swiler [10] for the treatment of mixed aleatory and epistemic uncertainties. The software enables the user to perform an automated uncertainty analysis for any given computational code without requiring modification to the source. UQPCE estimates sensitivities, CIs, and other model statistics which can be useful in the conceptual design and analysis of flight vehicles. This software was originally developed with funding from the Commercial Supersonic Technology (CST) Project to study the potential impacts of uncertainties on the prediction of ground noise generated from commercial supersonic aircraft concepts [11–14]. The code development is currently supported by the Transformational Tools and Technologies (TTT) Project and has been leveraged in uncertainty analysis for electrified aircraft propulsion studies [15].

2. Analytic Derivatives

Ensuring that accurate gradients are available throughout every part of a system is essential to support efficient gradient-based optimization methods. Previous work incorporated analytic derivatives through PCE, including a novel approach for calculating differentiable CIs, to reduce the expense for design under uncertainty [2]. The CI calculation is shown in Fig. 1. A histogram of a standard normal distribution is shown in the top subfigure, with the cumulative distribution function (CDF) of the histogram in the middle subfigure. The lowest portion shows the hyperbolic tangent activation function used to give a differentiable counting function, with the location of the solved upper CI shown by the black dashed line in all subfigures.

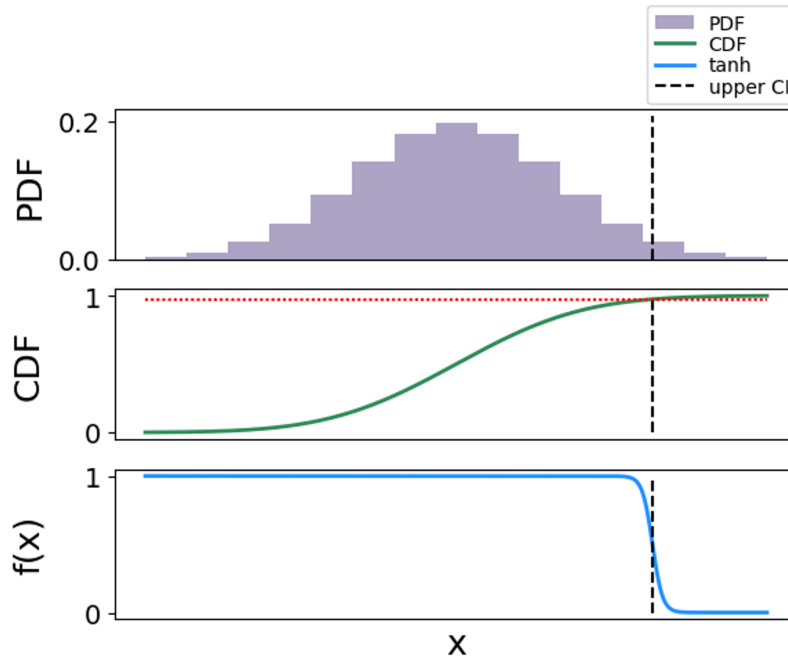


Fig. 1 The hyperbolic tangent activation function.

B. Design-Dependent Uncertain Variables

There are cases in which an input uncertainty is not adequately represented by an unchanging distribution. For example, an epistemic uncertain variable could have a larger width in one area of the design space than in others. A varying uncertain space within a gradient-based MDAO problem presents particular challenges; modeling these dependencies requires establishing a functional relationship between uncertain and design variables. The underlying methodology must adjust the uncertain space at each step of the optimization while still providing analytical derivatives. This work proposes a transformation of the uncertain variable samples in the initial uncertain space as the optimizer explores the design space. This treats the transformation as a response, no different than any other analytic equation,

and leaves the initial uncertain space unchanged for the PCE. Since these transformations are analytic equations used to change an uncertain variable, the analytical gradient capability from the previous work is maintained.

1. Uniform Transform Example

The simplest case considers the transformation of a uniform uncertain variable shown in Table 1. The design variable, x_b , is given in Table 2; x_b transforms the original uniformly distributed samples from being on interval $[0, 1]$ to being on interval $[0, x_b]$.

Table 1 Uncertain variable

Input	Range
uniform	$[0, 1]$

Table 2 Design variable

Input	Range
x_b	$[0.5, 2]$

The equation for the normalization of a uniform variable can be used to transform the original samples to represent a distribution with bounds as shown in Eq. (1).

$$\mathbf{u}_t = (x_b - x_a) \cdot \left(\frac{\mathbf{u}_0 - a_0}{b_0 - a_0} \right) + x_a \quad (1)$$

Since $a_0 = x_a = 0$ was chosen for this example, Eq. (1) becomes:

$$\mathbf{u}_t = x_b \cdot \left(\frac{\mathbf{u}_0}{b_0} \right) \quad (2)$$

The relationship between the uncertain distribution and the design variable, x_b , is shown in Fig. 2. The initial uniform samples have been transformed to have a new upper bound. As a uniform distribution, the PDF has an equal probability at all points, which leads to a constant relative density. The initial and transformed lower bound on the samples, a_0 and x_a , respectively, both remain at 0 for simplicity. As shown in Fig. 2, the transformed uncertain samples, whose values are given on the y-axis, change as x_b changes.

This example is represented in the eXtended Design Structure Matrix (XDSTM) in Fig. 3. In this work, we adopt the convention of the [†] superscript for input or outputs containing uncertain parameters in the XDSTM. Subscript _{CI} indicates a CI value estimated from a UQPCE analysis, and subscript _μ indicates a mean or central value estimated from UQPCE analysis. Superscript ⁽⁰⁾ indicates the initial guess and superscript * indicates the optimized value.

To confirm that these transformations are valid through PCE with optimization, two different optimization cases are considered. One of these cases is minimizing the upper CI of the transformed uniform distribution. In this case, the optimizer should choose the lower bound of the design variable, $x_b = 0.5$, since this reduces the upper bound on the distribution. Conversely, if maximizing the upper CI, the optimizer should give the upper bound of the design variable, $x_b = 2$, since this widens the distribution, resulting in a higher upper CI. For an analytic comparison of the CIs, the UQPCE calculation is replaced with the analytical calculation of the CI for a uniform distribution. A MC analysis is performed using $N = 1 \times 10^6$ samples from the original uniform distribution transformed according to Eq. (2) during optimization. The comparisons between UQPCE, the analytical case, and the MC case for the transformed distribution are shown in Tables 3 and 4. Figure 2 visualizes that the CI of $\mathbf{u}_t$ is minimized at $x_b = 0.5$ and maximized at $x_b = 2.0$.

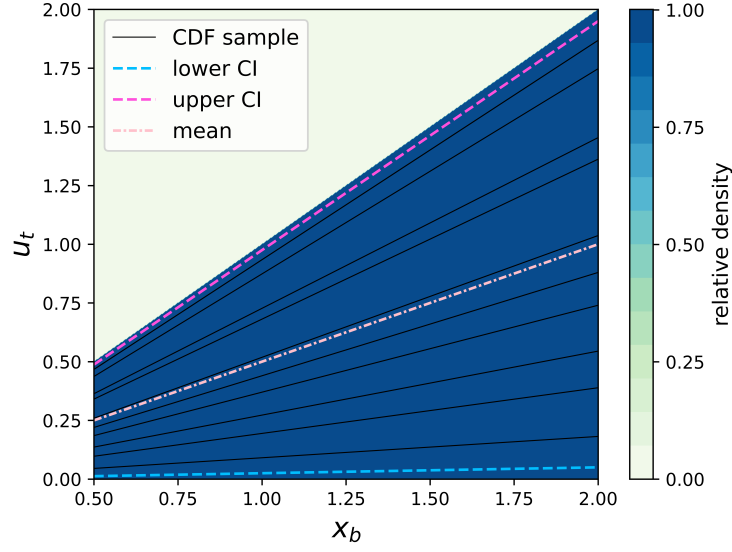


Fig. 2 A uniform uncertain variable with design-dependent upper bound, x_b , and a constant lower bound $a = 0$.

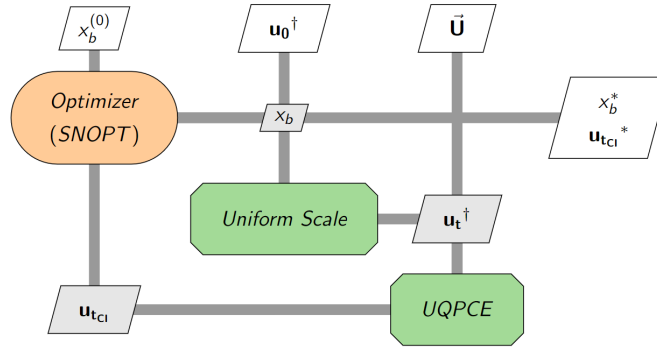


Fig. 3 XDSM for the uniform transform case.

Table 3 Results from minimizing the 95% upper CI

Method	x_b	Objective (Upper CI)
UQPCE	0.5	0.49
Analytic	0.5	0.49
Monte Carlo	0.5	0.49

Table 4 Results from maximizing the 95% upper CI

Method	x_b	Objective (Upper CI)
UQPCE	2	1.95
Analytic	2	1.95
Monte Carlo	2	1.95

2. Normal Transform Example

A more interesting extension of this is that of an uncertain normal variable with initial mean, μ_0 , and initial standard deviation, σ_0 . Parameters x_μ and x_σ are design variables that affect the mean and standard deviation, respectively, and the initial normally distributed samples can be transformed to represent a normal distribution with the new parameters. This transformation is given by Eq. (3). Figure 4 shows design variable x_μ on the x-axis, and the y-axis shows the scaled normal distribution whose mean value equals x_μ . Figure 5 shows design variable x_σ on the x-axis, and the y-axis shows the scaled normal distribution whose standard deviation equals x_σ . The solid black lines in Figs. 4 and 5 represent the initial distribution samples used for UQPCE, and the lines show how they each change as the design parameters change; these can be thought of as percentile locations in the CDFs.

$$\mathbf{u}_t = x_\sigma \left(\frac{\mathbf{u}_0 - \mu_0}{\sigma_0} \right) + x_\mu \quad (3)$$

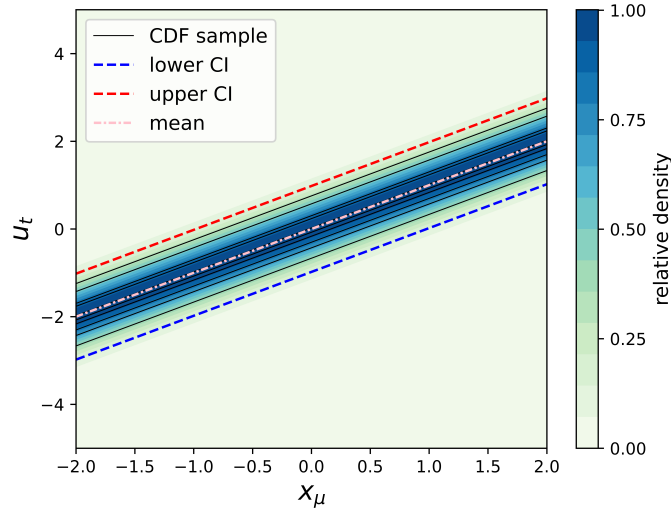


Fig. 4 A normal uncertain variable with design-dependent mean, x_μ , and constant $\sigma = 0.5$.

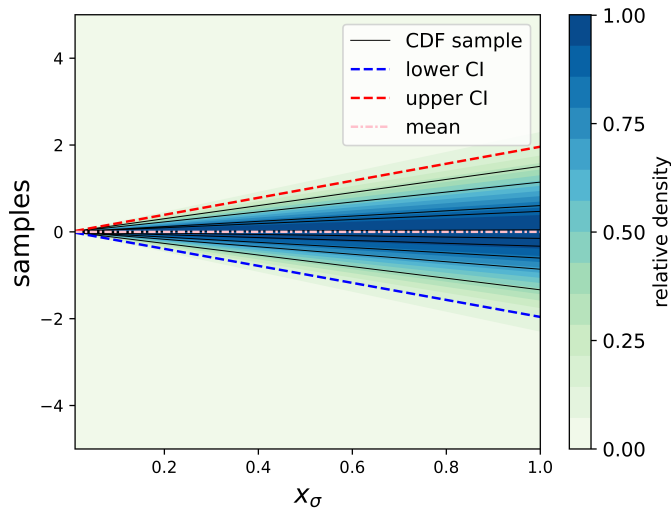


Fig. 5 A normal uncertain variable with design-dependent standard deviation, x_σ , and constant $\mu = 0$.

A simple optimization problem is described in Tables 5 and 6 to demonstrate the validity of this transform with

robust design optimization and PCE. This example is shown in the XDSM in Fig. 6.

Table 5 Uncertain variable

Input	μ_0	σ_0
normal	0	1

Table 6 Design variables

Input	Range
x_μ	[-2, 2]
x_σ	[0.01, 1]

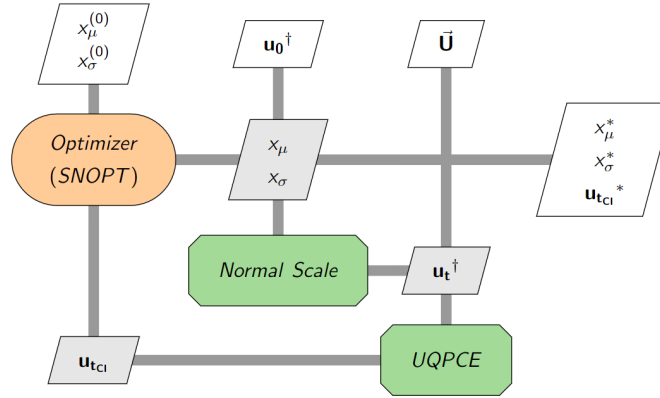


Fig. 6 XDSM for the normal transform case.

The same problem is optimized for two different objectives: the first case minimizes the upper CI of the new distribution and the second case maximizes the upper CI of the new distribution. Minimizing the upper CI should result in the lower bounds of design variables x_μ and x_σ ; this is because the lowest x_μ value intuitively lowers the upper CI and the lowest x_σ will narrow the distribution, also lowering the upper CI. Similarly, for the case of maximizing the upper CI, the upper bounds of both x_μ and x_σ are optimal. For an analytic comparison of the CIs, the UQPCE calculation is replaced with the analytical calculation of the CI for a normal distribution with given parameters. A MC analysis is performed using $N = 1 \times 10^6$ samples from the original uniform distribution transformed according to Eq. (3) during optimization. Comparisons between UQPCE, the analytical case, and the MC case for the transformed distribution are shown in Tables 7 and 8.

Table 7 Results from minimizing the 95% upper CI

Method	x_μ	x_σ	Objective (Upper CI)
UQPCE	-2	0.01	-1.98
Analytic	-2	0.01	-1.98
Monte Carlo	-2	0.01	-1.98

Table 8 Results from maximizing the 95% upper CI

Method	$\mathbf{x}_\mu$	$\mathbf{x}_\sigma$	Objective (Upper CI)
UQPCE	2	1	3.96
Analytic	2	1	3.96
Monte Carlo	2	1	3.96

The results from the analytical solution, UQPCE, and MC show that this transformation during design under uncertainty is valid through the PCE.

III. Case Study

This case study is an example from analytic equations and is intended to demonstrate the design-dependent uncertain variables; it is not meant to be a rigorous aeronautics research example but rather to illustrate the validity of the use of the UQ tools and methods for aerospace multidisciplinary design. For this case study, a simple model representing a linear relationship between the bounds of the percentage of laminar flow and the sweep of the wing will demonstrate a design-dependent uncertain variable. The objective function is chosen to maximize the L/D . The design and uncertain variables for this case study are shown in Tables 9, 10, and 11.

This example case is from previous work [2] with updates including additional analytic derivatives, minor changes to the calculation of the differentiable CI, and a change to one of the uncertain distributions. This example focuses only on impacts from the wing planform, and it does not consider vehicle-level trim. The following sections will discuss the disciplines and analyses incorporated in this case study. Section III.A covers aerodynamics, Section III.B discusses weights, and Section III.C explains the optimization problem formulation.

Table 9 Uncertain parameters, Epistemic

Input	Name	Interval
$\%_L^*$	Percentage of Laminar Flow	[0, 1]
σ_{CD_0}	Drag Multiplication Factor	[4, 6]
κ_α	Korn Factor	[0.8, 0.95]

Table 10 Uncertain parameters, Aleatory

Input	Name	Distribution	Mean	Std. Dev.
σ_{CL}	Stall Loss of Lift Factor	Gaussian	2×10^{-5}	3×10^{-6}
M	Mach Number	Gaussian	0.72	0.01

This case models a relationship in which $\%_L$ is an epistemic uncertain variable with a dependency on the design variable for wing leading edge sweep angle, Λ . Epistemic variable $\%_L^*$ is the untransformed parameter that transforms to $\%_L$. The lower bound of $\%_L$, $\%_{La}$, has a relation with Λ shown in Eq. (4), and the upper bound has the relation with Λ shown in Eq. (5). Since $\%_L$ has lower and upper bounds that are transformed like the previous uniform example, Eq. (1) can be used to scale the $\%_L^*$ samples from their original interval of [0, 1] to a design-dependent interval of $[\%_{La}, \%_{Lb}]$. Equation (6) shows the transform applied with these terms. The samples transformed to lie within this new bound results in the $\%_L$ values used for the aerodynamics calculations.

$$\%_{La} = -0.01\Lambda + 0.4 \quad (4)$$

Table 11 Design variables

Input	Name	Range	Units
t/c	Wing Thickness-to-Chord Ratio	[0.1, 0.15]	
P_{minx}	Chordwise Location of Minimum Pressure	[0.3, 0.5]	
C	Camber	[0.01, 0.043]	
α	Angle of Attack	[-8, 14]	deg
λ	Wing Taper Ratio	[0.1, 1]	
Λ	Wing Leading Edge Sweep Angle	[0, 40]	deg
b	Span	[20, 50]	m
c_r	Wing Root Chord	[5, 7]	m

$$c_{l_{Lb}} = -0.015\Lambda + 1.0 \quad (5)$$

$$c_{l_L} = (c_{l_{Lb}} - c_{l_{La}}) \cdot (c_{l_L}^*) + c_{l_{La}} \quad (6)$$

A visual representation of the relationship between c_{l_L} and Λ is shown in Fig. 7, where the lower bound on the c_{l_L} variable is shown in blue and the upper bound on the c_{l_L} is shown in red. This relationship is defined to create a significant relationship between Λ and c_{l_L} , and the relationship is not intended to be a representation of the true relationship between these variables.

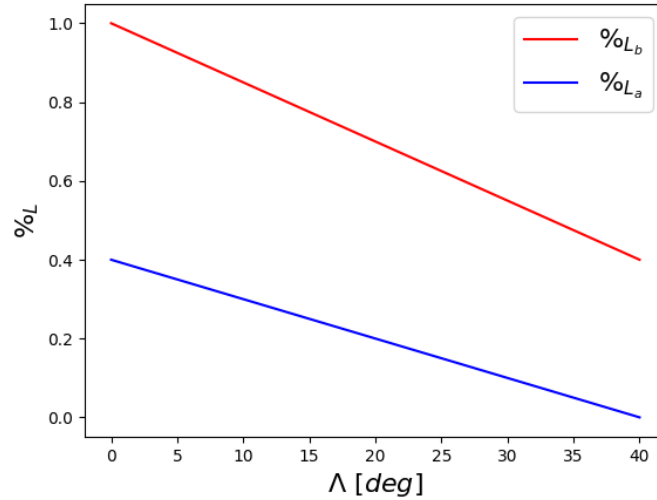


Fig. 7 The relationship between the bounds of the uncertain variable for percentage of laminar flow, c_{l_L} , and the design variable sweep, Λ .

A. Aerodynamics

Aerodynamic calculations for lift and drag are implemented from analytic, low-fidelity methods. Only lift and drag are considered in this case study. First, 72 NACA 6-series airfoils of varying thickness and camber were analyzed with a 2-D panel code [16] to estimate their sectional lift curve slope and lift at $\alpha = 0$ values. Then, a regression model was assembled from the data to fit characteristics from the NACA 6-series (location of minimum pressure, P_{minx} ,

thickness-to-chord ratio, t/c , and camber, C) to the sectional lift values. This process provided the optimizer with continuous design variables that influenced the sectional lift properties.

$$C_{l_\alpha} = f(P_{minx}, t/c, C) \quad (7)$$

$$C_{l_{\alpha_0}} = f(P_{minx}, t/c, C) \quad (8)$$

A knockdown factor of 0.5, which was chosen to match aerodynamic performance data for Boeing 737 class vehicles, was applied to $C_{l_{\alpha_0}}$ to take into account 3-D wing effects.

$$C_{L_{\alpha_0}} = 0.5C_{l_{\alpha_0}} \quad (9)$$

The wing lift curve slope, C_{L_α} , from Ref. [17] is given in Eq. (10). This formula considers compressibility, sweep, and finite span effects.

$$C_{L_\alpha} = \frac{2\pi AR}{2 + \sqrt{4 + \frac{AR^2\beta^2}{\eta^2} \left(1 + \tan^2(\Lambda)/\beta^2\right)}} \quad (10)$$

where β and η are given by:

$$\beta = \sqrt{1 - M^2} \quad (11)$$

$$\eta = \frac{C_{l_\alpha}}{2\pi/\beta} \quad (12)$$

The lift coefficient then becomes:

$$C_L = C_{L_{\alpha_0}} + C_{L_\alpha}\alpha - \sigma_{C_L}\alpha^4 \quad (13)$$

The last term of Eq. (13) contains an uncertain parameter, σ_{C_L} , that models the stall loss of lift at higher angles of attack. To calculate the skin friction drag or zero lift drag, flat plate equations [18] were utilized:

$$C_{f_L} = \frac{1.328}{\sqrt{Re_L}} \quad (14)$$

$$C_{f_T} = \frac{0.455}{\log(Re_L)^{2.58} (1 + 0.144M^2)^{0.65}} \quad (15)$$

The zero lift drag coefficient is then estimated by:

$$C_{D_0} = 2 \left(\%_L C_{f_L} + (1 - \%_L) C_{f_T} \right) \sigma_{C_{D_0}} \quad (16)$$

where the uncertain parameter, $\%_L$, denotes the percentage of laminar flow over the wing. The factor of two represents drag on both "sides" of the flat plate (upper and lower wing surface). Here, $\sigma_{C_{D_0}}$ is an uncertain parameter that multiplies the zero lift drag of the wing to estimate a value for the theoretical full vehicle. The range of $\sigma_{C_{D_0}}$ is [4, 6], which translates to the wing accounting for approximately 17% to 25% of the overall theoretical full vehicle zero lift drag. Equations to determine the wing efficiency, which is an input into the induced drag calculation, are based on historical data, and regression models are given by Pamadi [19] in Eqs. (17), (18), and (19). These equations account for wing aspect ratio, leading edge sweep, taper ratio, and section lift coefficient.

$$\kappa = \frac{AR\lambda}{\cos \Lambda} \quad (17)$$

$$r = 0.0004\kappa^3 - 0.008\kappa^2 + 0.05\kappa + 0.86 \quad (18)$$

$$e = \frac{1.1C_{L_\alpha}}{rC_{L_\alpha} + (1 - r)\pi AR} \quad (19)$$

The induced drag, C_{Di} , is then calculated by:

$$C_{Di} = \frac{C_L^2}{\pi e AR} \quad (20)$$

The wave drag, C_{Dw} , is given by:

$$C_{Dw} = 20(M - M_{cr})^4 \quad (21)$$

where the critical Mach number, M_{cr} , is:

$$M_{cr} = M_{DD} - \left(\frac{0.1}{80} \right)^{1/3} \quad (22)$$

and the drag divergence Mach number, M_{DD} , is given by:

$$M_{DD} = \frac{\kappa_\alpha}{\cos \Lambda} - \frac{t/c}{\cos \Lambda^2} - \frac{C_L}{10 \cos \Lambda^3} \quad (23)$$

The Korn factor, κ_α , [20] was included as an uncertain parameter (see Table 9). The total drag is given by the summation of the zero lift drag, induced drag, and wave drag.

$$C_D = C_{D0} + C_{Di} + C_{Dw} \quad (24)$$

B. Weights

An empirical method based on historical data and regression estimations given by Raymer [17] (imperial units) was employed for wing weight calculations. This equation takes into account all the wing planform characteristics and estimates an empty wing mass based on historical data. A control surface area, S_{csw} , of 10% was assumed.

$$W_{wing} = 0.0051 (W_{dg} N_z)^{0.557} S_w^{0.649} AR^{0.5} (t/c)^{-0.4} (1 + \lambda)^{0.1} \cos(\Lambda)^{-1} S_{csw}^{0.1} \quad (25)$$

C. Optimization Problem Formulation

Three optimization cases are constructed to examine the effects of coupled uncertainty. Each case is slightly different based on a different question posed by the practitioner. In Case 1, the objective is to make the worst-case wing L/D values as large as possible. In Case 2, the objective is to maximize the mean L/D value while constraining the worst-case L/D . In Case 3, the objective is to maximize deterministic L/D and then perform a post-optimality uncertainty analysis.

The detailed setup for Case 1 is provided below; the other cases are similar with minor modifications to the objective and constraint functions. An example XDSM incorporating the calculations in the prior sections is shown in Fig. 8.

In this optimization, the different metrics of L/D were optimized subject to a maximum wing weight constraint. The arbitrary wing weight constraint was enforced to prevent the wing from sizing unrealistically heavy. The general formulation of the Case 1 optimization problem is shown below.

$$\begin{array}{ll} \text{Maximize} & L/D_{CIL} \\ \text{Subject to Constraints} & \vec{K} \text{ and } W_{wing} < 6000 \text{ kg} \\ \text{And Fixed Uncertain Inputs} & \vec{U} \text{ (Tables 9 and 10)} \\ \text{With Respect to Design Variables} & \vec{J} \text{ (Table 11)} \end{array}$$

As shown in Fig. 8, the $\%L^*$ value is transformed according to its relation with Λ . (Note that Λ is part of the design values shown as $\vec{J}$). The new, transformed $\%L$ values are then input into the aerodynamics calculations.

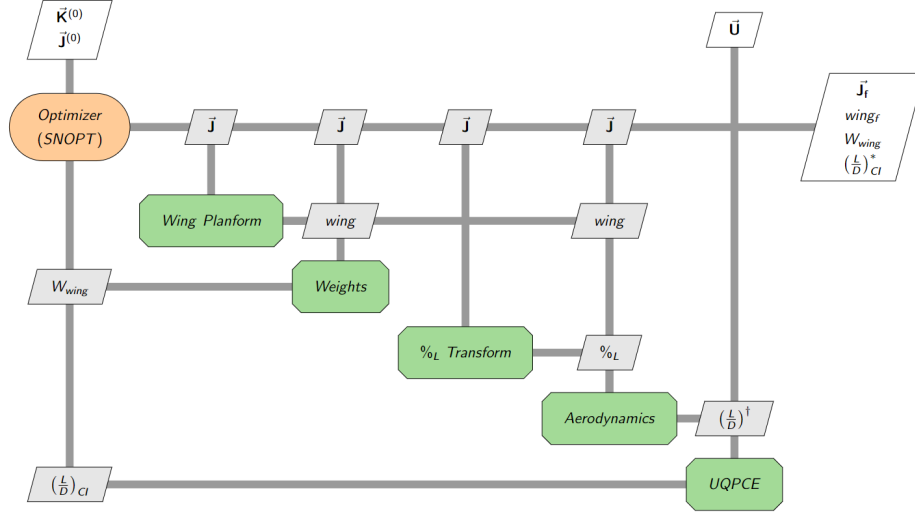


Fig. 8 The XDSM for the L/D optimization.

Table 12 Definitions of the different optimization cases considered for the case study

Case Number	Objective	Constraints	Design-Dependence	Type
1	L/D_{CIL}	W_{wing}	yes	uncertain
2	L/D_{μ}	$W_{wing}, L/D_{CIL} \geq 10$	yes	uncertain
3	L/D	W_{wing}	yes	deterministic

Table 13 Comparison of UQPCE and MC results from maximizing the lower 95% L/D_{CI} with design-dependent $\%L$

Method	UQPCE	Monte Carlo
t/c	0.1	0.1
P_{minx}	0.312	0.312
C	0.043	0.043
α , deg	3.197	3.270
λ	0.246	0.252
Λ , deg	37.920	39.761
b , m	30.019	29.366
c_r , m	5.0	5.0

IV. Results

Both the chosen optimal design and corresponding uncertain L/D values from optimization with UQPCE are compared to results from optimization with MC. Three optimization cases are compared: Case 1 maximizes L/D_{CIL} with design-dependent $\%L$, Case 2 maximizes L/D_{μ} with a constraint on L/D_{CIL} and design-dependent $\%L$, and Case 3 maximizes a deterministic L/D . These cases are outlined in Table 12.

To demonstrate the validity of this approach with PCE, the UQPCE results are compared to MC results for maximizing the lower CI of L/D , L/D_{CIL} , with the design-dependent $\%L$ parameter. The optimal designs for UQPCE and MC are shown in Table 13. UQPCE used 42 evaluations of the aeronautics codes, whereas the MC used 5 million

evaluations.

Table 14 Comparison of optimizations using UQPCE and MC

Variable	L/D_{CIL} (UQPCE)	L/D_{CIL} (Monte Carlo)
Central Value	16.700	16.325
CI	[12.999, 21.152]	[12.056, 20.488]

As shown in Table 14, the central value, which is the middle value of the response distribution, and CI for UQPCE and MC are comparable. The differences between the UQPCE and MC results can be attributed to approximating the total derivatives in the MC case in order to reduce the computational time required, leading this optimization to no longer be an exact truth value but a proxy for the truth value.

Table 15 Optimal designs for the three L/D optimization cases

Parameter	Case 1	Case 2	Case 3
t/c	0.1	0.1	0.1
P_{minx}	0.312	0.314	0.314
C	0.043	0.043	0.0308
α , deg	3.197	2.321	2.000
λ	0.246	0.225	0.211
Λ , deg	37.920	21.803	0.000
b , m	30.019	34.368	36.586
c_r , m	5.0	5.0	5.0

Table 15 shows the optimal design for each of the three cases using UQPCE. Case 1 is maximizing the lower L/D_{CIL} and has a design-dependent relationship between Λ and $\%L$. This leads to a trade-off of the Λ value of 37.920 deg and a lower CI of 12.999. Case 2 is maximizing the L/D_μ with a constraint of $L/D_{CIL} \geq 10.0$. This leads to a lower CI of 10.000 and a central value of 19.657. Case 3 is the deterministic case which has all uncertain inputs replaced with their mean values. Case 3 maximizes the deterministic L/D value with Λ dependence of $\%L$. The Λ value chosen in this case is 0.000 deg.

Table 16 Results of the L/D uncertain space from the three optimal designs

Variable	Case 1	Case 2	Case 3
Central Value	16.700	19.657	24.099
CI	[12.999, 21.152]	[10.000, 28.025]	[6.630, 43.372]

Table 16 shows the central values and CIs from the three cases. Case 1 has a lower CI value of 12.999. Case 2 has a lower CI value of 10.000, which is due to the constraint on the CI in this case. Although the lower CI has a smaller value than in case 1, the central value is larger as that was maximized. Case 3 is deterministic, where the deterministic L/D was maximized. Because of this, it makes sense that the central value is much larger than the other cases. Case 3 has a CI width of 36.742[†], whereas Case 1 has a CI width of 8.163 and Case 2 has a CI width of 18.025.

Figure 9 shows the L/D distribution across Λ with the other design variables at their optimal values for Case 1. Figure 9 shows that a $\Lambda = 0$ would not be chosen for this case as the uncertainty in L/D is widest at this value.

[†]The confidence interval for the deterministic case is calculated post-optimization at the deterministic optimal design.

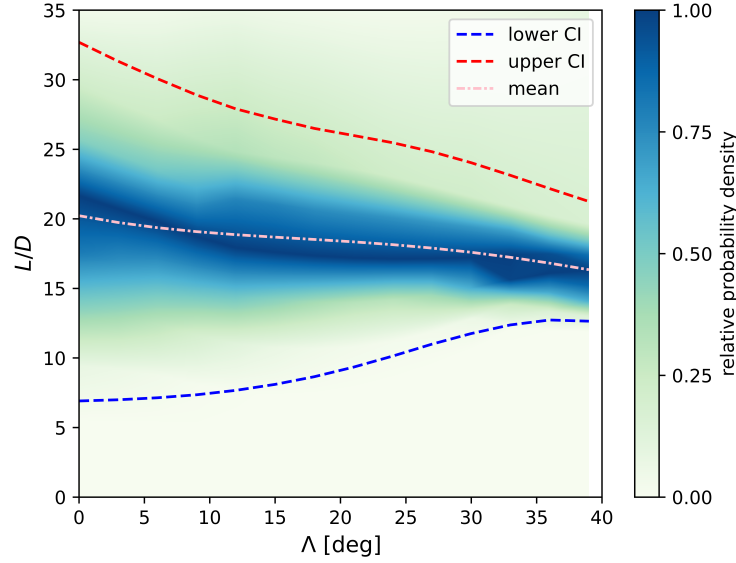


Fig. 9 L/D versus design variable Λ at the optimal design space for maximizing L/D_{CIL} .

V. Summary

Table 15 shows that the notable differences between Cases 1 and 2 are b and Λ . Case 1 leads to a higher Λ and lower b because maximizing the L/D_{CIL} penalizes a higher sweep value less than maximizing L/D_μ does. Table 16 shows that for Case 2 $L/D_{CIL} = 10.000$, which is at the limit of the constraint, whereas the lower CI for Case 1 is higher as expected due to its objective. Cases 1 and 3 have very different Λ values. This is because Case 3 is deterministic, with the uncertain distributions replaced with the mean values of each distribution. When considering the deterministic values in optimization, Λ is driven to 0 in order to maximize the $\%L$ value, driving C_D to its lowest value in relation to Λ by increasing the C_{D_i} and C_{D_w} values slightly at the benefit of decreasing C_{D_0} significantly. Consequently, L/D has its highest value with relation to Λ at $\Lambda = 0$ for the deterministic case. Case 1 does not drive Λ to zero because the uncertain values of M lead C_{D_w} to have variable behavior, widening the uncertainty of L/D at lower Λ as shown in Fig. 9. Note that the CI for Case 3 is the widest of the three cases, which is because it was the only one of the three designs that did not optimize or constrain with uncertainty in the design, and the C_{D_w} parameter leads to a wider uncertainty on C_D and thus L/D at the chosen value of $\Lambda = 0$.

This paper demonstrates the validity of design-dependent uncertain variables in robust design optimization with PCE. Table 13 shows that design-dependent uncertainty can be appropriately represented in PCE and the effect of an uncertain variable that changes during optimization can be achieved without affecting the validity of the underlying polynomial basis functions.

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