

A Comparison of Control Allocation Methods in the Presence of Parametric Model Uncertainty

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When allocating redundant effectors to virtual control commands, linear (generalized inverse) allocators have historically been used on aircraft and spacecraft. While simple to implement, generalized inverses are unable to realize a significant portion of the attainable moments. To address this drawback, the control allocation problem can also be formulated as a linear programming or quadratic programming problem and solved using convex optimization-based solvers. These approaches have been shown to access a larger set of attainable moments, however, little work has been done to understand the performance of convex optimization-based control allocation in the presence of parametric model uncertainty. This paper seeks to compare the performance of several control allocation approaches, including two forms of generalized inverse allocators, the pseudo inverse and minimum-variance, and the linear programming and quadratic programming approach in the presence of parametric model uncertainty. The performance of these four allocators were tested on an aircraft model in the presence of realistic parametric model uncertainty and the convex optimization approaches were shown to outperform the generalized inverses.

I. Nomenclature

B	=	control effectiveness matrix
d	=	tolerance bound
δ	=	control surface deflection, rad
ϵ	=	control allocation weighting term
J	=	cost function
L, M, N	=	aircraft body-axis moments, ft · lbf
p	=	number of virtual controls
r	=	number of control surfaces
R	=	ellipsoidal uncertainty set shape
X, Y, Z	=	aircraft body-axis forces, lbf
ρ	=	size of box uncertainty set
τ	=	generalized moments
θ	=	uncertain parameter vector
Θ	=	parametric model uncertainty set

Superscripts

$\hat{}$	=	estimate
^{n}	=	nominal value

Subscripts

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a	=	aileron
c	=	command
ds	=	differential stabilator
e	=	error
i	=	row index
j	=	effector index
p	=	preferred value
r	=	rudder
vc	=	vertical canard

II. Introduction

Optimal control allocation is a well-studied aspect of flight control design. Typically, control laws are designed to produce virtual moment commands to be imparted on a vehicle. These virtual moments commands encapsulate either forces, torques, or impulses. However, the vehicle must realize these commands and can do so through manipulation of its control effectors. Common vehicle control effectors include aerodynamic control surfaces and propulsors on aircraft. Reaction control systems, thrust vectoring control systems, control moment gyros are common control effectors on spacecraft. Control allocation is the algorithm which distributes the virtual moment commands to the individual effectors.

Generalized inverses are the most common control allocation approach which utilize a local linearization of control effectiveness. The subsequent mapping between virtual and physical commands is derived in a closed-form solution to a Lagrange optimization problem [1, 2]. The simplest generalized inverse is a pseudo-inverse (PI) allocator [3]. PI involves only a matrix multiplication, hence making it attractive for real-time implementation, especially on human-rated vehicles which safety is paramount. A major drawback to single matrix generalized inverses are that effector position constraints can only be satisfied by limiting the achievable moments.

The control allocation problem can also be formulated as a convex optimization problem and solved on-line using linear programming (LP) or quadratic programming (QP) techniques [4, 5]. The major benefit is that the QP and LP techniques can account for effector position constraints [6–9]. Furthermore, it has been demonstrated in Ref. [10] that additional convex objectives such as minimizing the l_∞ norm of the structural loads on the control surfaces can be incorporated into the optimization problem. Common solution techniques include interior-point methods, simplex algorithms, and more recently first-order methods such as Alternate Direction Method of Multiplier (ADMM) [11–13]. Historically, due to lack of computational power and solver maturity, the aerospace controls community has generally avoided on-line optimization as part of the flight controls architecture. Rigorous flight certification process requires convergence guarantees and fast solution times. The past 10–15 years has seen a sharp rise in solver technology [14–18]. The SpaceX Falcon F9 booster landing uses model predictive control to solve the real-time trajectory optimization problem with hundreds of decision variables in fractions of a second [19]. This shift in paradigm suggests that the formulation of the optimal control allocation problem as a convex program could supplant generalized inverses as the new industry standard.

The aforementioned literature does not account for uncertainty in the elements of the controls matrix, $\mathbf{B}$. Robustness to modeling errors is evaluated by performing Monte Carlo simulations of the truth model. Grauer and Pei derived the minimum-variance (MV) control allocator, which is based on generalized inverse and considers the effects of parametric model uncertainty by minimizing the variance of the generalized moment error [20]. MV was shown to outperform PI in the presence of known parametric uncertainty in elements of $\mathbf{B}$. Robust control techniques have also gained attention to help minimize the effects of bounded disturbances in the allocation problem [21–24]. Proper compensation of the effects of parametric model uncertainty is critical, as a realized virtual control error will create unintended motion on the vehicle and additional burden on the pilot and/or feedback control loop [25].

To the knowledge of the authors, there appears to be little documentation that discusses the effects of convex optimization-based control allocation considering parametric model uncertainty. De Almeida applied robust convex optimization to offline control allocation design [26]. Duan, Zhang, Zhao, and Kong studied on-line robust control allocation under ellipsoidal uncertainty [27]. In this paper, an aircraft model was used to compare the PI, MV, LP, and QP allocators in the presence of parametric model uncertainty. The parametric model uncertainty was taken from previously derived examples [20], in practice, covariance matrix of aerodynamic effectors can be derived from system identification techniques [30] either off-line or in-real time [31, 32].

The remainder of this paper is organized as follows. Section III reviews the derivation of the linear programming

allocator. Section IV reviews the derivation of the quadratic programming allocator. Section V provides an example which compares the efficacy of the PI, MV, LP, and QP approaches. Lastly, section VI provides concluding remarks and recommendations for future research.

III. Control Allocation by Linear Programming

A common formulation of the control allocation problem is to minimize the 1-norm of the allocation error [4, 9] and then transform the objective into a standard form linear-programming problem. This allows the problem to be solved in real-time using linear programming solvers which are commonly available in commercial software. The problem is set up as a mixed optimization problem where the designer can satisfy multiple optimization objectives. To begin, consider the following 1-norm mixed optimization problem [4].

$$\min_{\mathbf{u}} \quad J(\mathbf{u}) = \|\mathbf{B}(\boldsymbol{\theta})\mathbf{u} - \boldsymbol{\tau}_c\|_1 + \epsilon\|\mathbf{u} - \mathbf{u}_p\|_1 \quad (1a)$$

$$\text{s.t.} \quad \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}} \quad (1b)$$

The parameter, ϵ is a weighting parameter which allows the control designer to choose between a solution which is close to preferred control surface deflections, or prioritizing minimizing moment error. In this paper $\epsilon = 1 \times 10^{-6}$. To handle rate constraints, an additional term can be included to ensure the difference between the current and previous solution does not exceed a certain threshold. The 1-norm constraints in (1) can be written as,

$$\|\mathbf{B}(\boldsymbol{\theta})\mathbf{u} - \boldsymbol{\tau}_c\|_1 = \sum_{i=1}^p |\mathbf{B}(\boldsymbol{\theta})_i \mathbf{u} - \tau_{c,i}| \quad (2a)$$

$$\|\mathbf{u} - \mathbf{u}_p\|_1 = \sum_{j=1}^r |u_j - u_{p,j}| \quad (2b)$$

the subscript, i denotes the i^{th} row. The subscript, j denotes the j^{th} effector. Setting an upper bound, s_i on the allocation error, $|\mathbf{B}(\boldsymbol{\theta})_i \mathbf{u} - \tau_{c,i}| \leq s_i$ and an upper bound, q_j on the preferred deflection, $|u_j - u_{p,j}| \leq q_j$.

$$\min_{\mathbf{u}, s, q} \quad J = \begin{bmatrix} 0 \dots 0 & 1 \dots 1 & \epsilon \dots \epsilon \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ s \\ q \end{bmatrix} \quad (3a)$$

$$\text{s.t.} \quad \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}} \quad (3b)$$

$$|\mathbf{B}(\boldsymbol{\theta})_i \mathbf{u} - \tau_{c,i}| \leq s_i \quad \forall i \quad (3c)$$

$$-s \leq \mathbf{0} \quad (3d)$$

$$|u_j - u_{p,j}| \leq q_j \quad \forall j \quad (3e)$$

The constraints in (3b), (3c), and (3e) do not follow the standard form for linear programming solvers. The constraints are rewritten in the following form,

$$\min_{\mathbf{u}, s, q} \quad J = \begin{bmatrix} 0 \dots 0 & 1 \dots 1 & \epsilon \dots \epsilon \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ s \\ q \end{bmatrix} \quad (4a)$$

$$\text{s.t.} \quad -\mathbf{u} \leq -\underline{\mathbf{u}} \quad (4b)$$

$$\mathbf{u} \leq \bar{\mathbf{u}} \quad (4c)$$

$$\mathbf{B}(\boldsymbol{\theta})_i \mathbf{u} - \tau_{c,i} \leq s_i \quad \forall i \quad (4d)$$

$$\mathbf{B}(\boldsymbol{\theta})_i \mathbf{u} - \tau_{c,i} \geq -s_i \quad \forall i \quad (4e)$$

$$-s \leq \mathbf{0} \quad (4f)$$

$$u_j - u_{p,j} \leq q_j \quad \forall j \quad (4g)$$

$$u_j - u_{p,j} \geq -q_j \quad \forall j \quad (4h)$$

and further refining the optimization to yield the final form of the linear programming problem:

$$\min_{\mathbf{u}, \mathbf{s}, \mathbf{q}} \quad \mathbf{J} = \begin{bmatrix} 0 \dots 0 & 1 \dots 1 & \epsilon \dots \epsilon \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \\ \mathbf{q} \end{bmatrix} \quad (5a)$$

$$\text{s.t.} \quad -\mathbf{u} \leq -\underline{\mathbf{u}} \quad (5b)$$

$$\mathbf{u} \leq \bar{\mathbf{u}} \quad (5c)$$

$$\mathbf{B}(\boldsymbol{\theta})_i \mathbf{u} - s_i \leq \tau_{c,i} \quad \forall i \quad (5d)$$

$$-\mathbf{B}(\boldsymbol{\theta})_i \mathbf{u} - s_i \leq -\tau_{c,i} \quad \forall i \quad (5e)$$

$$-\mathbf{s} \leq \mathbf{0} \quad (5f)$$

$$u_j - q_j \leq u_{p,j} \quad \forall j \quad (5g)$$

$$-u_j - q_j \leq -u_{p,j} \quad \forall j \quad (5h)$$

In this paper, (5) is solved considering a nominal value, $\boldsymbol{\theta}^n$ using the LINPROG MATLAB command. The vector of uncertain parameters, $\boldsymbol{\theta}$ belongs to a given set $\boldsymbol{\theta} \in \Theta$. In general, (5d) and (5e) have infinitely many constraints corresponding to all possible realizations of the uncertainty, $\boldsymbol{\theta}$. The problem can be solved in real-time using gradient-based, active-set algorithms or interior-point algorithms[27].

IV. Control Allocation by Quadratic Programming

Another common formulation of the control allocation problem is to formulate the objective function as a mixed-objective 2-norm optimization problem [33]. The resultant optimization problem will transform into a quadratic programming problem, which similar to the previous section, could be solved with readily-available commercial software. To review control allocation by quadric programming begin by considering the following mixed-objective 2-norm optimization problem [33].

$$\min_{\mathbf{u}} \quad \mathbf{J}(\mathbf{u}) = \|\mathbf{B}(\boldsymbol{\theta})\mathbf{u} - \boldsymbol{\tau}_c\|_2^2 + \epsilon \|\mathbf{u} - \mathbf{u}_p\|_2^2 \quad (6a)$$

$$\text{s.t.} \quad \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}} \quad (6b)$$

The cost function in (6) can be expanded to

$$\min_{\mathbf{u}} \quad \mathbf{J}(\mathbf{u}) = (\mathbf{B}(\boldsymbol{\theta})\mathbf{u} - \boldsymbol{\tau}_c)^T (\mathbf{B}(\boldsymbol{\theta})\mathbf{u} - \boldsymbol{\tau}_c) + \epsilon (\mathbf{u} - \mathbf{u}_p)^T (\mathbf{u} - \mathbf{u}_p) \quad (7a)$$

$$\text{s.t.} \quad \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}} \quad (7b)$$

For brevity, define the following variables

$$\mathbf{H}(\boldsymbol{\theta}) = 2(\mathbf{B}(\boldsymbol{\theta})^T \mathbf{B}(\boldsymbol{\theta}) + \epsilon \mathbf{I}_{r \times r}) \quad (8a)$$

$$\mathbf{c}^T(\boldsymbol{\theta}) = -2(\boldsymbol{\tau}_c^T \mathbf{B}(\boldsymbol{\theta}) + \epsilon \mathbf{u}_p^T) \quad (8b)$$

$$k = \boldsymbol{\tau}_c^T \boldsymbol{\tau}_c + \epsilon \mathbf{u}_p^T \mathbf{u}_p \quad (8c)$$

Since k is constant in $\mathbf{u}$ it can be dropped from the objective function. The optimization problem in (7) now reduces to,

$$\min_{\mathbf{u}} \quad \mathbf{J}(\mathbf{u}) = \frac{1}{2} \mathbf{u}^T \mathbf{H}(\boldsymbol{\theta}) \mathbf{u} + \mathbf{c}^T(\boldsymbol{\theta}) \mathbf{u} \quad (9a)$$

$$\text{s.t.} \quad \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}} \quad (9b)$$

Similar to Equation (9) is also solved considering a nominal value, $\boldsymbol{\theta}^n$ and solved using the QUADPROG MATLAB command. Equation (9a) also has infinitely many constraints corresponding to all possible realizations of the uncertainty, $\boldsymbol{\theta}$. The problem can also be solved in real-time using gradient-based, active-set algorithms or interior-point algorithms[27].

V. Example

For a fixed-wing aircraft, several reasons for significant uncertainty in the control effectiveness matrix include: high angle-of-attack flight conditions where unsteady flow and vortex shedding can occur, control effectiveness degradation due to icing, control surface interaction, damaged or failed control surfaces, or in real-time modeling research where the initial control effectiveness is unknown, but learned in-flight [34]. The example in this paper considers the F-16 Advanced Fighter Technology Integration (AFTI) model which is depicted in Fig. 1. The control surfaces used to control the lateral-directional dynamics are: anti-symmetric ailerons δ_a , differential stabilator δ_{ds} , rudder δ_r , and vertical canards δ_{vc} mounted underneath the engine intake. The virtual controls are body-axis roll acceleration $\dot{p}$ and body-axis yaw acceleration $\dot{r}$. The model for the lateral-directional dynamics was evaluated in straight and level flight at Mach 0.6 and 30,000ft altitude [35].



Fig. 1 F-16 AFTI aircraft in flight (credit: NASA).

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -0.154 & 0.0824 & -0.998 & 0.0521 \\ -19.2 & -0.894 & 0.319 & 0 \\ 2.30 & -0.0009 & -0.279 & 0 \\ 0 & 1 & 0.0823 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} 0.0004 & 0.0144 & 0.0212 & 0.0074 \\ -17.5 & -13.6 & 3.92 & 0.415 \\ -0.268 & -1.51 & -1.97 & 1.51 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_{ds} \\ \delta_r \\ \delta_{vc} \end{bmatrix} \quad (10)$$

The control deflection limits and control surface rate limits in this example are also taken from [35] and are given in Table 1.

Table 1 F-16 AFTI aircraft control surface position and rate limits.

Control surface	Position limit, deg	Rate limit, deg/s
Ailerons	± 20	52
Stabilators	± 25	60
Rudder	± 30	120
Vertical canards	± 27	100

The uncertainty in the B matrix is given in Table 2 and is taken from Ref. [20]. This uncertainty is typical for that of fixed-wing aircraft [20] and expressed in percentages of control derivatives. The PI, MV, LP, and QP allocators are designed based on the information in (10) and Table 1.

Table 2 Standard errors used in example.

	δ_a	δ_{ds}	δ_r	δ_{vc}
L'	5%	10%	20%	25%
N'	50%	50%	5%	10%

A. Static Allocator Performance

This section compares the static performance of all four allocators with and without uncertainty in the B matrix. Begin by defining the theoretical attainable moment set (TAMS) as the set of moments due to all possible combinations of control effector positions. Similarly, define the particular attainable moment set (PAMS) is the subset of the TAMS which can be realized by the allocator. One performance metric of an allocation scheme is allocation efficiency and is defined as the ratio of the size of the PAMS to the TAMS, η . If the allocator can realize the entire space in the TAMS then $\eta = 1$ and the allocator is perfectly efficient. Perfect knowledge of the control effectiveness matrix is necessary to characterize the nominal TAMS, however, the control effectiveness matrix is subject to parametric model uncertainty which will change the TAMS. Fig. 2 shows the TAMS of the F-16 AFTI aircraft based on a nominal control effectiveness matrix (TAMS B_{des}), the TAMS computed with a dispersed control effectiveness matrix (TAMS B_{disR}), and PAMS for each allocator. There are several regions where the TAMS B_{des} and TAMS B_{disR} do not overlap. The regions occupied by TAMS B_{des} but not TAMS B_{disR} mean that moments predicted to be attainable by the model are unattainable in practice.

Optimization-based approaches in LP and QP provide superior allocation efficiency as compared to the generalized inverses. The efficiencies are provided in Fig. 2 in terms of percentages. Due to its conservative nature, MV has the smallest value of η , as expected.

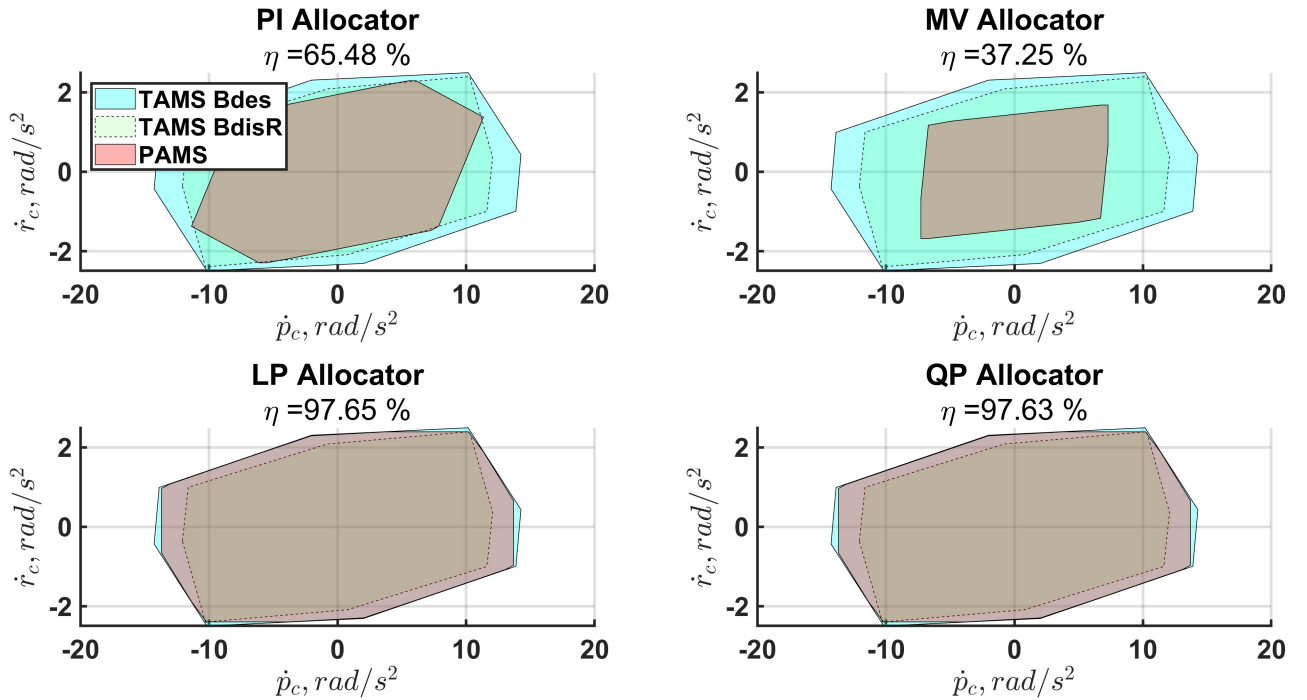


Fig. 2 Allocation efficiency.

While it is necessary to characterize the allocation efficiency when comparing allocator static performance, it is

equally important to quantify the allocation error. The allocation error is defined as the error between the realized and commanded moment. The sources of this error might be due to aforementioned dispersions in the control effectiveness matrix or a moment command outside the PAMS of the respective allocator. Fig. 3 compares the static control allocation performance of all four approaches for a single moment command that is outside of the TAMS and considers dispersions in the control effectiveness matrix. . The commanded moment is $\tau_c = \begin{bmatrix} 13 & 2 \end{bmatrix}^T$ rad/s² and is represented by the blue arrow. Known for direction-preserving in the case of control saturation, if no uncertainties are considered, the expected result is that the PI and MV allocators will be co-linear with τ_c and reach the edge of their respective PAMS. With optimization-based approaches the resultant moment is not guaranteed to be co-linear with τ_c . The LP and QP allocators will also reach the edges of their respective PAMS, with a non-zero direction error. The direction error of the LP and QP allocators is attributed to the formulations of their respective objective functions which are to minimize the 1-norm and the 2-norm of the moment error without regard to moment direction.

In the presence of uncertainty, it is expected that neither the direction nor the magnitude will be preserved. In this example the MV allocator shows the smallest direction error and the largest magnitude error. Conversely, LP and QP allocators show the largest direction errors, but the least magnitude errors.

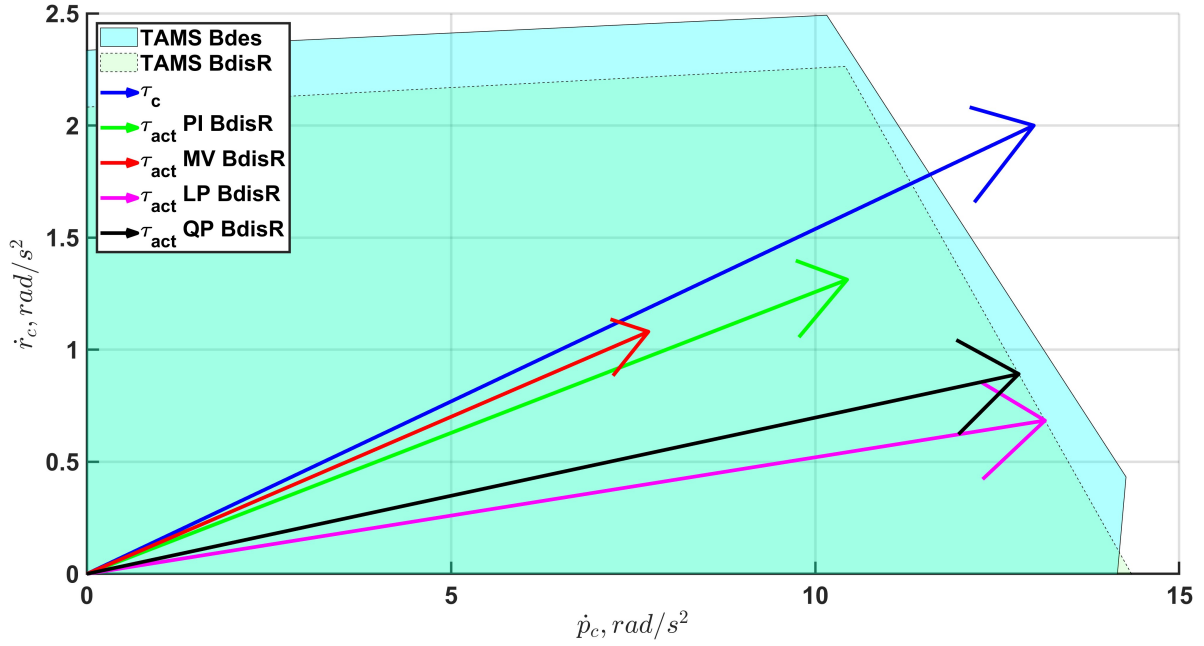


Fig. 3 Control allocation error comparison for a single moment command.

While it is important compare allocator performances for a single τ_c , it is equally important to characterize performances across the entire moment space and dispersions in the $\mathbf{B}$ matrix. Figure 4 shows the magnitude error normalized by the moment command magnitude alongside the direction error. Figure 5 shows the standard magnitude error expressed as a percentage of moment command alongside the standard error in direction. To generate Fig. 4 and 5 a range of moment commands which spans and encompasses the nominal TAMS. At each moment command 1,000 Monte Carlo simulations were run where the control effectiveness matrix was varied according to the parameters provided in Table 2. The outline of the nominal TAMS is shown by the solid black line and the outline of the respective PAMS is shown by the dotted black line. The mean magnitude error for the k -th dispersion, $(\bar{M}_{e,k})$, the mean direction error for the k -th dispersion, $(\bar{D}_{e,k})$, the magnitude standard error, (SE_M) , and the direction standard error, (SE_D) were computed by (11)-(14), respectively. Since there are 1,000 Monte Carlo simulations, $N = 1,000$.

$$\bar{M}_{e,k} = \frac{\sum_{k=1}^N \|\mathbf{B}_{dis} \mathbf{u}_{c,k} - \tau_{c,k}\|_2}{N \cdot \|\tau_{c,k}\|_2} \quad (11)$$

$$\bar{D}_{e,k} = \frac{\sum_{k=1}^N \text{acos} \left(\frac{\mathbf{B}_{dis} \mathbf{u}_{c,k}}{\|\mathbf{B}_{dis} \mathbf{u}_{c,k}\|_2} \cdot \frac{\boldsymbol{\tau}_{c,k}}{\|\boldsymbol{\tau}_{c,k}\|_2} \right)}{N} \quad (12)$$

$$SE_M = \sqrt{\frac{\sum_{k=1}^N (\|\mathbf{B}_{dis} \mathbf{u}_{c,k} - \boldsymbol{\tau}_{c,k}\|_2 - \bar{M}_{e,k})^2}{N^2 \cdot \|\boldsymbol{\tau}_{c,k}\|_2}} \quad (13)$$

$$SE_D = \sqrt{\frac{\sum_{k=1}^N \left(\text{acos} \left(\frac{\mathbf{B}_{dis} \mathbf{u}_{c,k}}{\|\mathbf{B}_{dis} \mathbf{u}_{c,k}\|_2} \cdot \frac{\boldsymbol{\tau}_{c,k}}{\|\boldsymbol{\tau}_{c,k}\|_2} \right) - \bar{D}_{e,k} \right)^2}{N^2}} \quad (14)$$

In Fig. 4 all allocators show small percentage of allocation error for commands within the respective PAMS of the allocator. For all allocators the percentage of moment error rapidly increases near the origin, this is because the magnitude of the moment command is small near the origin and exacerbates the percent of magnitude error. Outside the PAMS the mean moment error magnitude increases radially outward, the error increases most with the MV allocator, and the least with the LP and QP allocators.

Mean direction errors are also shown in Fig. 4 for all allocators. It is apparent that all allocators exhibit significantly larger direction errors when $\boldsymbol{\tau}_c$ is a pure yaw command. These areas of higher mean direction error are an artifact of the higher control effectiveness in the roll channel. Consequently, a small magnitude error in the yaw channel will result in a large direction error as opposed to the roll channel. The PI, LP, and QP allocators show a mean direction error of about 10deg at the two peaks. The MV allocator shows a higher mean direction error, about 14deg. The larger mean direction error for the MV allocator is attributed to the reduced PAMS, as a large τ_c will lead to a greater direction error. Additionally, large mean magnitude in the control allocation errors could lead to degraded tracking performance or handling qualities.

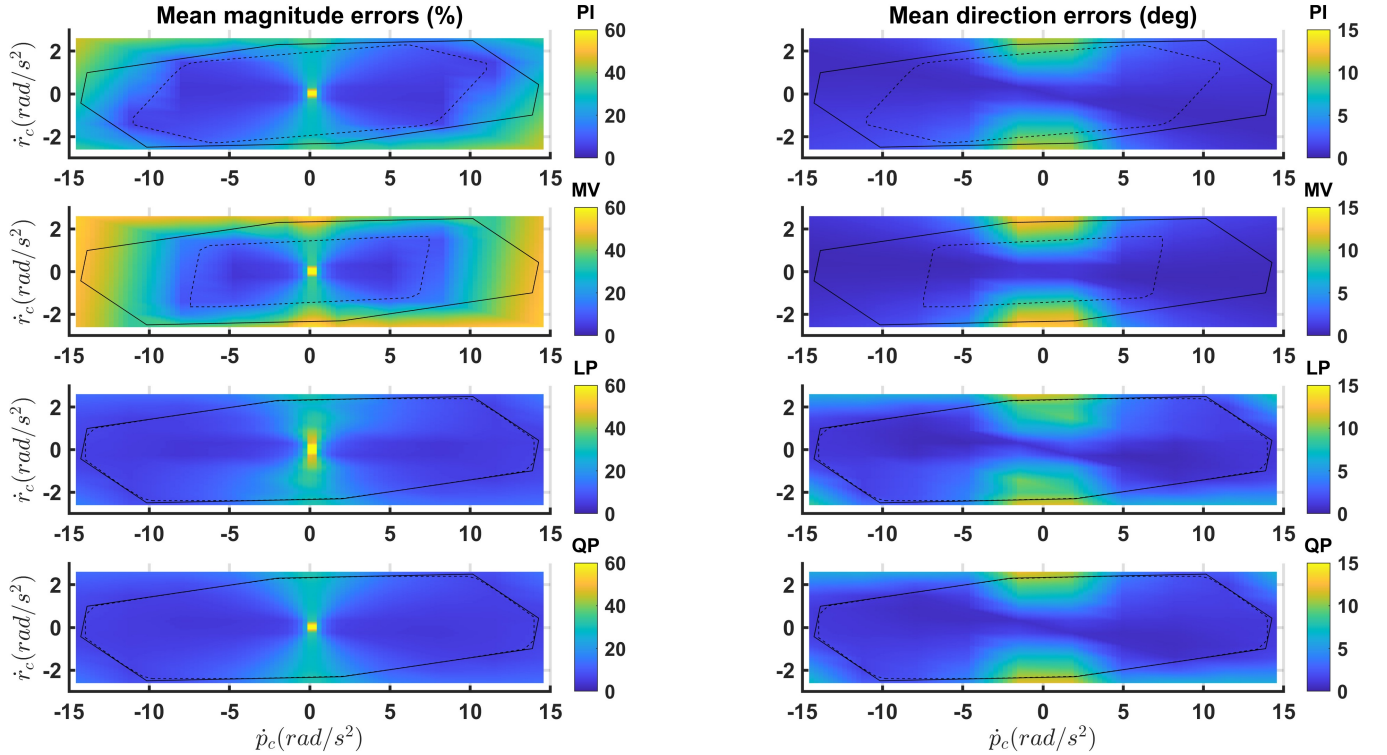


Fig. 4 Mean error parameters.

Figure 5 shows comparisons of the magnitude standard errors expressed as a percentage of moment command. Near the center of the TAMS all four allocators show higher standard error which decreases as the moment command increases in roll magnitude. The LP, and QP allocators show the fastest rate of decrease and the MV allocator shows the slowest rate of decrease. This decrease is due to the moment error being normalized by a larger moment command. As

the moment command increases in the yaw channel the standard error increases, this is attributed to the high uncertainty and the higher control effectiveness and higher uncertainty in the roll channel. The standard direction error shown in Fig. 5 follows a similar trend as the mean direction error shown in Fig. 4 for the same rationale; due to the lesser control effectiveness in the yaw channel a small magnitude error in the yaw channel will result in a large direction error.

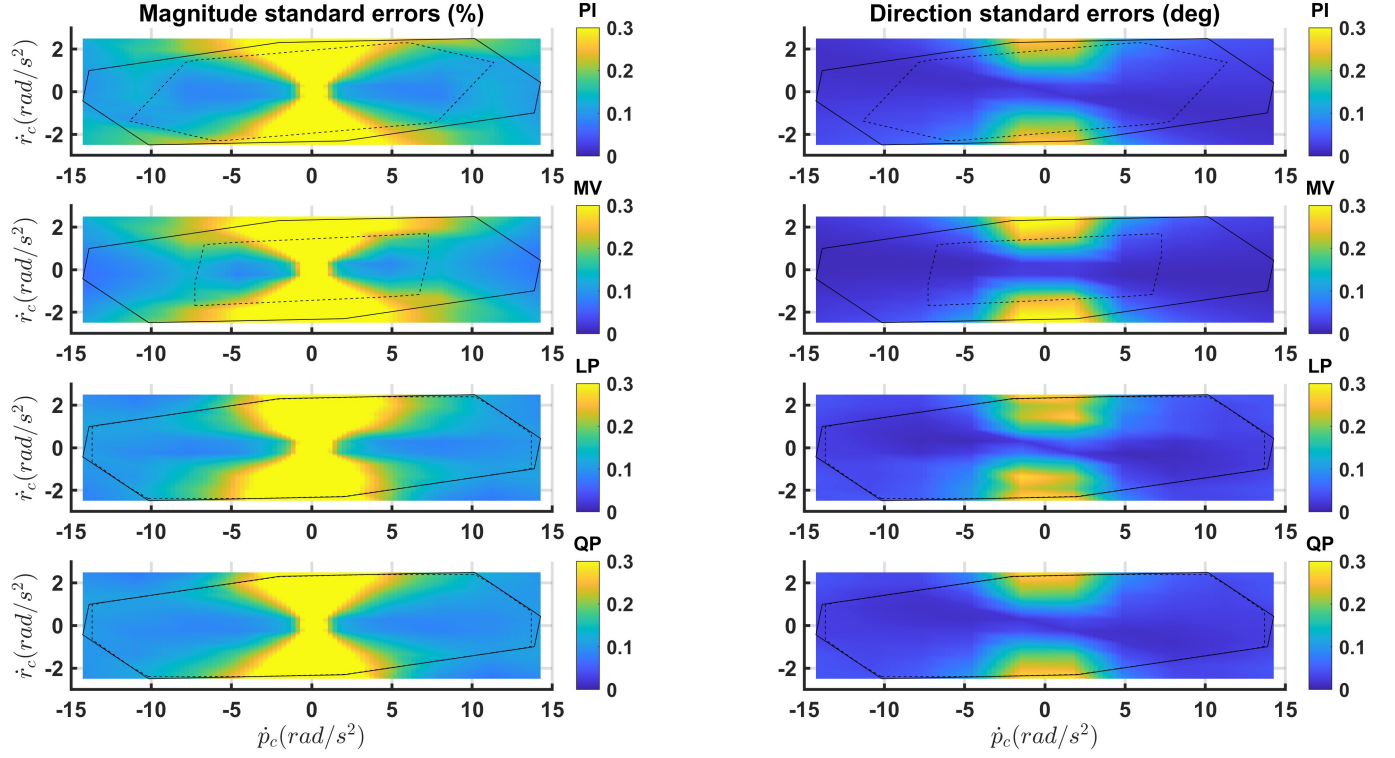


Fig. 5 Standard error parameters.

B. Closed-Loop Allocator Performance

Subsection V.A investigated the static performance of the PI, MV, LP, and QP allocators. The purpose of this section is to compare the closed loop regulation of each allocator. A block diagram of the F-16 AFTI time-domain simulation is shown in Fig. 6; control effector position and rate limits are embedded in the aircraft dynamics model.

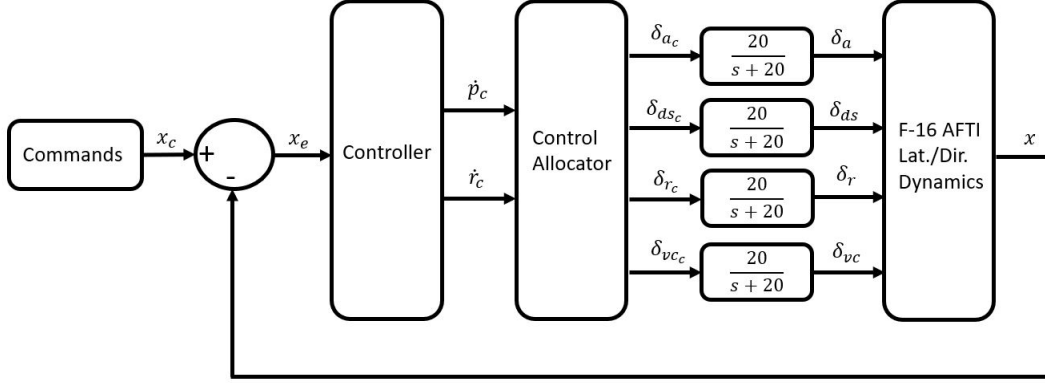


Fig. 6 Block diagram of time-domain simulation.

The initial conditions are as follows: $\beta = 30$ deg, $p = 0$ deg/s, $r = 0$ deg/s, $\phi = 0$ deg. With the nominal set of control gains, the closed-loop system returned to the equilibrium within 6 seconds. The states and control effector time histories are shown in Fig. 7 and Fig. 9 with all four allocators and the no allocator (NA) case. Figure 10 represents a “zoomed in” view of Fig. 9. The NA case was generated by feeding the virtual control commands directly into the F-16 AFTI aircraft model; this case represents the perfect allocator where all commanded moments are realized instantly. Fig. 7 shows that all five cases yield similar performance, the differences between the four allocators and the NA case is shown in Fig. 8. The subtle differences are caused by each allocator commanding a unique control deflection; due to the modeled actuator lag, the control deflection commanded by each allocator will be realized in a different time, thus resulting in subtle differences tracking performance. During the nominal simulation, no allocator realizes effector saturation which can be seen in Fig. 9.

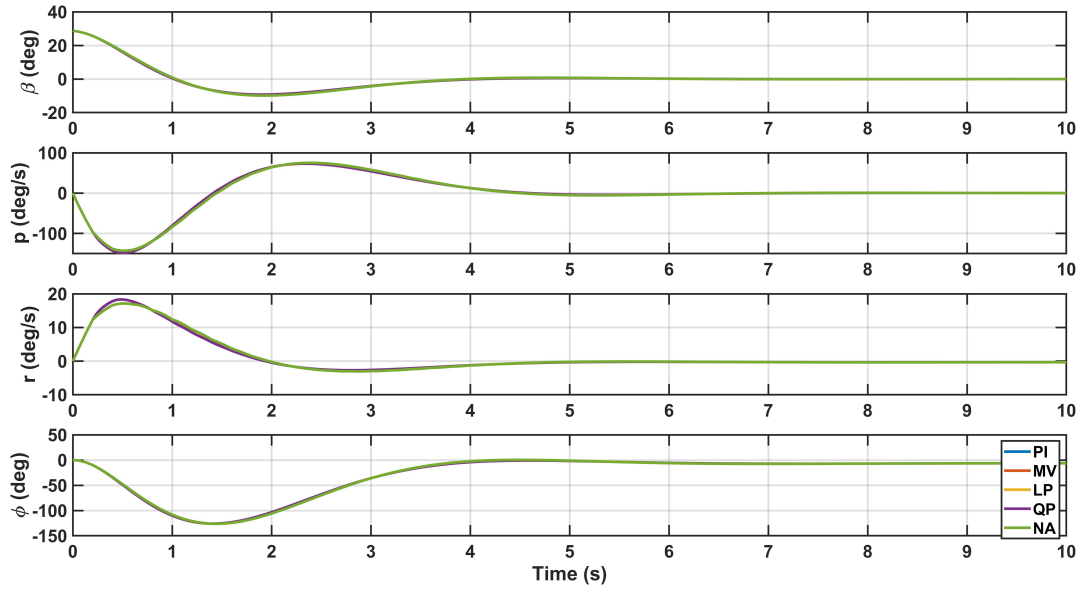


Fig. 7 Nominal tracking performance for all allocators.

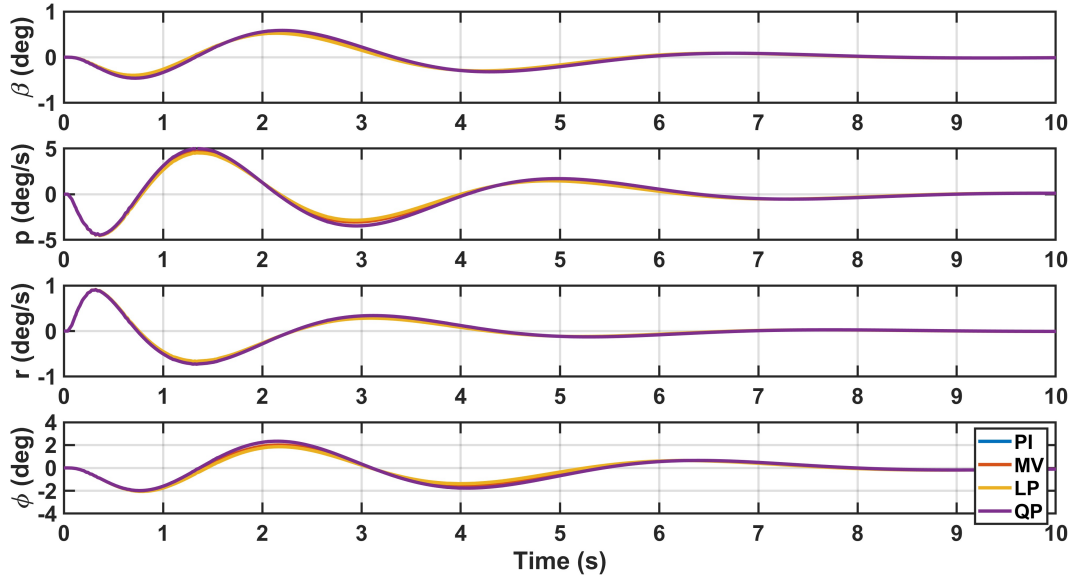


Fig. 8 Difference between NA.

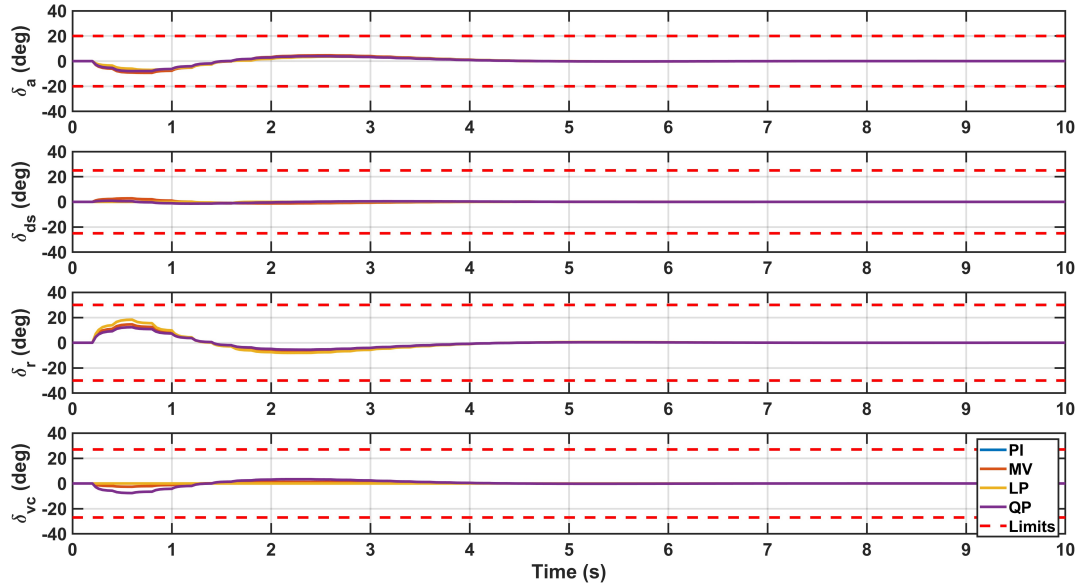


Fig. 9 Nominal control surface deflections.

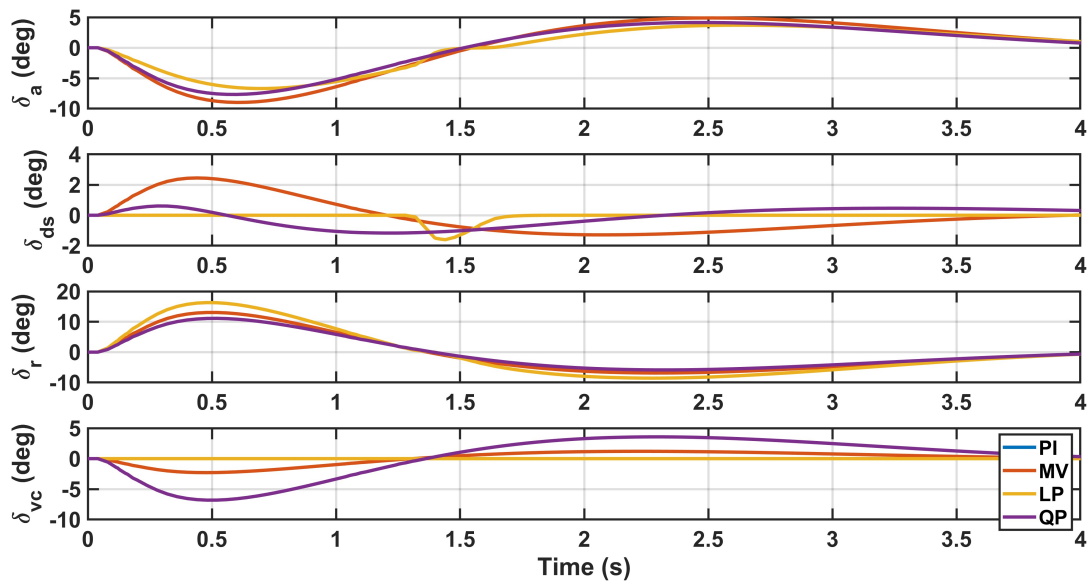


Fig. 10 Nominal control surface deflections.

To highlight the disparity in performance in the presence of effector saturation, the control system gains were increased to four times their nominal value. The tracking performance and control surface deflections are shown in Fig. 11 and Fig. 13. The differences between each allocator and the NA case are shown in Fig. 12. Figure 14 represents a “zoomed in” view of Fig. 13.

Initially there are large transient errors in the roll channel, which can be seen by p and ϕ in Fig. 12. This is expected because the aircraft ailerons, which have a large contribution to rolling control effectiveness saturate which limits the aircraft’s ability to regulate the roll channel. The yawing channel is most effected by the rudder, which also saturates leading to tracking errors in r . The tracking errors in r are less significant than the tracking errors in p and ϕ because the rudder control effectiveness is not as large as the aileron control effectiveness.

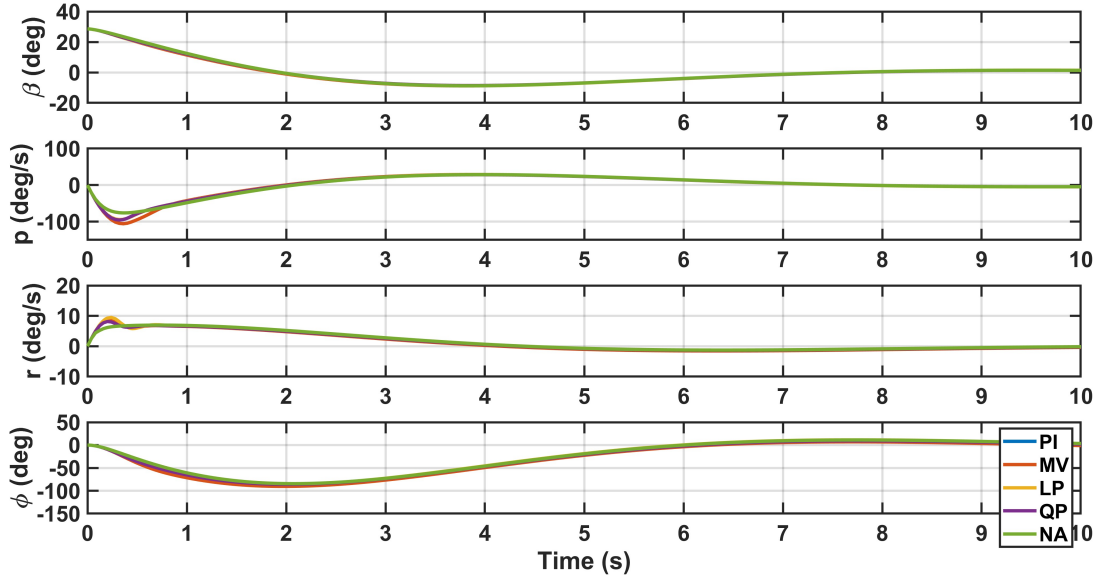


Fig. 11 Tracking performance for increased gain case.

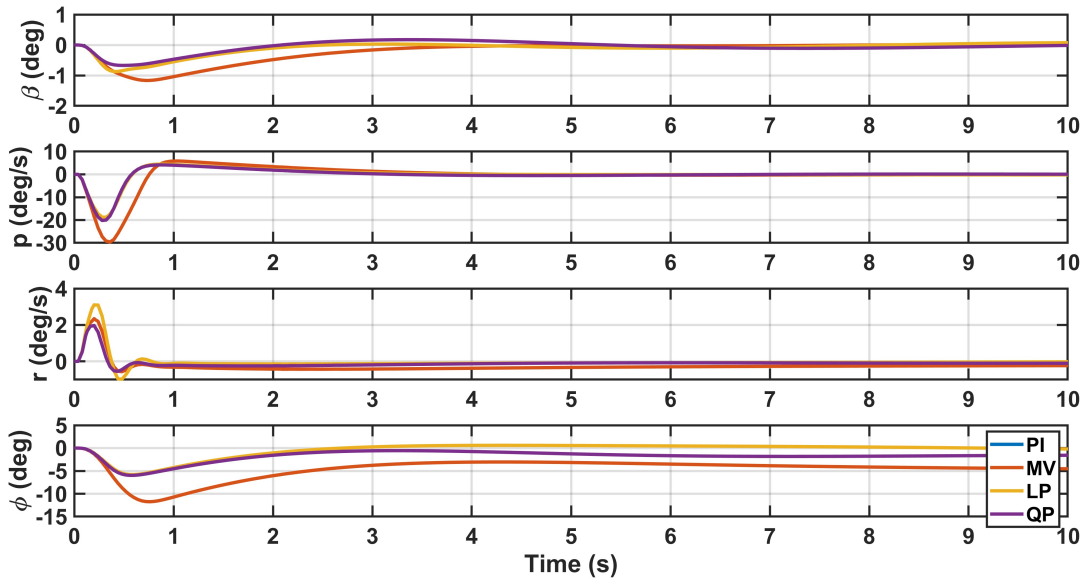


Fig. 12 Difference between NA for increased gain case.

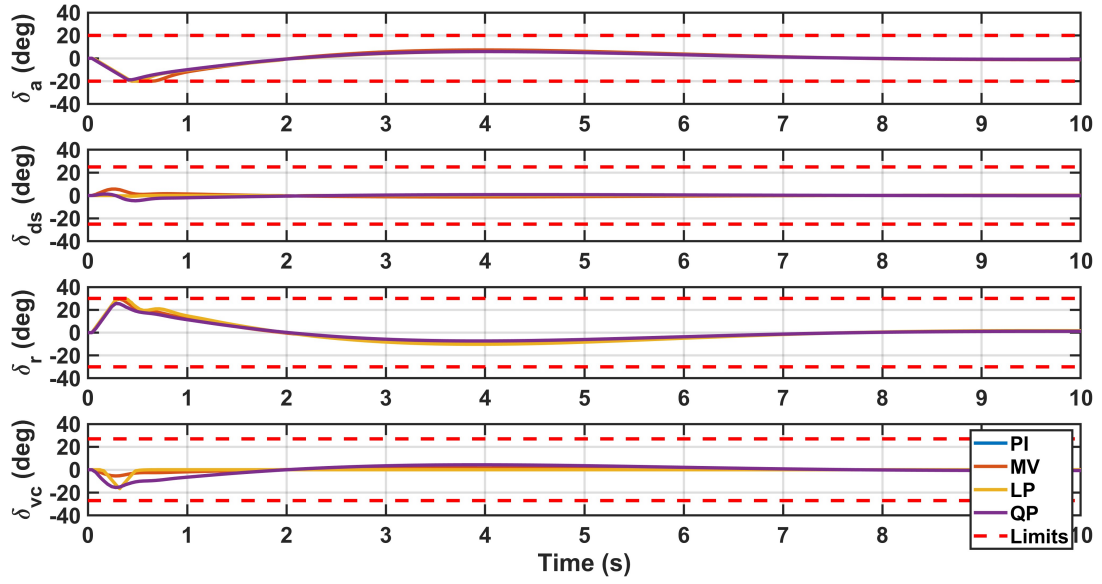


Fig. 13 Control surface deflections for increased gain case.

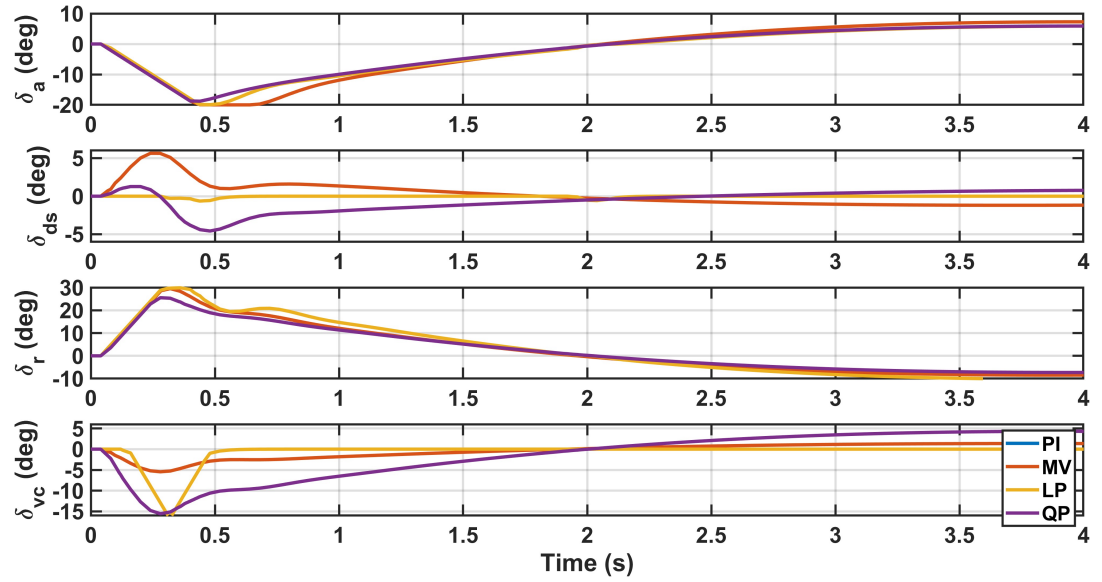


Fig. 14 Control surface deflections for increased gain case.

The performance of the increased gain case is quantified by Table 3 and Table 4, where the allocation error is defined as the difference in states between runs with allocator and NA. Both the maximum error and the integrated (cumulative) error over the 10s simulation window are reported. The larger the maximum allocation error for a given allocator the larger the unexpected transient response. In the case of piloting, could lead to degraded handling qualities. The larger cumulative allocation error for a given allocator the more control effort is needed. A feedback controller would need to apply more effort to eliminate control allocation errors. Or in a piloted case, this could result in degraded handling qualities. In general, the PI, LP, and QP allocators show better performance in both maximum allocation error and cumulative allocation error as compared to the MV allocator. This is expected because the MV design is more conservative, therefore the allocator realizes a smaller PAMS. The LP and QP allocator accumulate slightly less error in β and ϕ , due to the brief period when the controller commands a moment which is realizable by the LP and QP allocators but not the PI allocator.

Table 3 Maximum allocation errors with nominal B matrix.

State	PI	MV	LP	QP
β (deg)	0.79	1.29	0.95	0.79
p (deg/s)	22.70	32.26	21.55	22.68
r (deg/s)	2.27	2.71	3.45	2.27
ϕ (deg)	6.90	13.00	6.77	6.84

Table 4 Cumulative allocation errors with nominal B matrix.

State	PI	MV	LP	QP
β (deg·s)	2.84	3.71	2.78	2.83
p (deg)	19.17	28.89	19.63	19.08
r (deg)	2.14	3.97	2.12	2.13
ϕ (deg·s)	26.13	51.74	21.20	25.93

To illustrate the effects of varying control effectiveness on allocator performance, 100 Monte Carlo simulations were performed with variations in the control effectiveness matrix while the control gains are set to four times the nominal value. To quantify the performance of the allocators across the Monte Carlo simulations, the mean maximum allocation error and the mean cumulative allocation error was taken for each allocator. The results are shown in Table 5 and Table 6. The PI, LP, and QP allocators in this case show similar performance against variations in the control effectiveness matrix. The MV allocator, while designed to be robust to variations in the control effectiveness matrix, can not realize large moment commands therefore performs worse when the gain is increased.

Table 5 Mean maximum allocation errors.

State	PI	MV	LP	QP
β (deg)	0.77	1.21	0.93	0.77
p (deg/s)	21.56	30.27	20.11	21.57
r (deg/s)	2.27	2.78	3.52	2.28
ϕ (deg)	6.21	11.37	6.21	6.18

Table 6 Mean cumulative allocation errors.

State	PI	MV	LP	QP
β (deg·s)	2.68	3.48	2.65	2.68
p (deg)	18.80	27.20	19.10	18.73
r (deg)	1.97	3.64	2.07	1.97
ϕ (deg·s)	23.66	44.29	19.73	23.57

The cumulative control effort during the Monte Carlo simulations was also examined. To show the control effort of each allocator relative to the NA allocator case, the difference between realized moment and commanded moment is integrated over the 10s simulation and the results are shown as a non-Gaussian probability density function (PDF). These PDFs are shown in Fig. 15 and Fig. 16. The PI and QP allocators show nearly identical control effort, and the least overall. However for cases with sustained large moment commands the QP allocator is more efficient and will be able to realize large sustained moments better. The PI and QP allocators are nearly identical because inside the PAMS the PI and the QP allocator command nearly the same control deflection. The MV and the LP utilize a higher control effort.

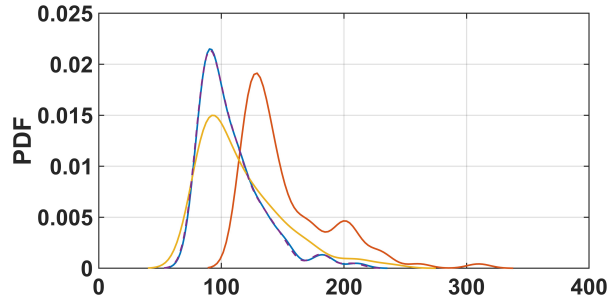


Fig. 15 Cumulative $\dot{p}$ errors (deg/s).

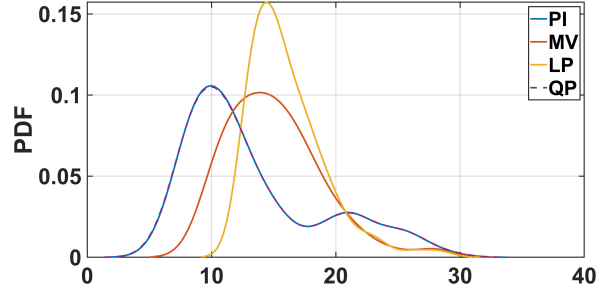


Fig. 16 Cumulative $\dot{r}$ errors (deg/s).

VI. Conclusions

In this study, several control allocation approaches were compared for systems with parametric model uncertainty in the control effectiveness matrix. Two generalized inverse approaches, the psuedo-inverse and the minimum-variance were considered. Additionally, two on-line optimization-based approaches were considered, linear and quadratic programming.

First, the static performance of the allocators was examined with a time domain tracking simulation. In the absence of dispersions the LP and QP allocators yield the largest PAMS. The MV allocator which was originally designed for robustness to model uncertainties yielded the smallest PAMS, as expected. This implies that the MV allocator is well-suited to handle model dispersions, but might limit vehicle performance during aggressive maneuvers. When dispersions are considered the LP and QP allocators show the smallest mean magnitude errors near the edges of the TAMS, this is because the LP and QP allocators have the highest efficiency. The four allocation approaches exhibited similar direction error properties.

Second, the closed-loop response of each allocator was examined. Under nominal conditions it was shown that all allocators have the same performance. The control system gain was then increased such that the controller would command moments outside the PAMS of each allocator. The LP and QP allocators showed lesser cumulative allocation error in this case. This implies that the LP and QP allocators are better suited for vehicles where large maneuvers are desired. Additionally the robustness to model dispersions was examined in a closed loop configuration. It was found that the PI, and QP allocators both had the least allocation error and the least cumulative control effort.

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