

Analytical Flight Dynamic Model Development for eVTOL Aircraft

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This paper presents an analytical flight dynamic model development method for a multi-rotor eVTOL aircraft in hover and transition flight. The method captures the analytical stability and control sensitivities of rotor unsteady aerodynamic loads about a trim solution generated by a rotorcraft aeromechanics comprehensive analysis tool. The rotor aerodynamic forces include circulatory and non-circulatory components to account for unsteady aerodynamics. The circulatory forces are corrected for unsteady aerodynamics using the time-domain approximation of Theodorsen's function. The integrated flight dynamic model includes the rigid-body aircraft flight dynamics, inflow dynamics, unsteady aerodynamics, and blade pitch and rotor actuator dynamics. Stability analyses shows that the Lift+Cruise aircraft is statically unstable in roll and yaw axes and also dynamically unstable in hover. Closed-loop feedback control is necessary for stable hover flight.

I. Introduction

The flight dynamic motion of a rotorcraft is generally much more complex than that for a fixed-wing aircraft. The complex flight physics of a rotorcraft is difficult to predict precisely. The complex unsteady flow and unsteady wake interactions between rotors and airframe makes the flight dynamic modeling of eVTOL a particularly challenging task. Flight control system design for rotorcraft still largely depend on experimentally-derived flight dynamic modeling using system identification flight experiments. With the emerging Advanced Air Mobility (AAM), there exist a plethora of multi-rotor eVTOL aircraft designs with highly complex aerodynamic configurations. These complex configurations pose significant challenges in stability and control of AAM aircraft. Due to the need for computational efficiency, flight dynamic modeling of eVTOL aircraft still depends on relatively lower fidelity tools such as unsteady vortex lattice method which cannot capture complex unsteady flow fields with viscous dissipation generated by rotor wakes. CFD methods have been coupled to rotorcraft comprehensive analysis tools such as CAMRAD II⁹ for use in analyzing rotorcraft dynamics, but the computational cost is quite high which limits the broad applicability of these high-fidelity tools in conceptual flight dynamic analysis.

In the current practice, general nonlinear flight dynamic models of eVTOL aircraft are generated by rotorcraft comprehensive analysis tools such as CAMRAD II,¹⁰ CHARM,³ and FLIGHTLAB.⁴ These rotorcraft tools are widely used in rotorcraft community. Linearized flight dynamic models are usually extracted from nonlinear models created by rotorcraft comprehensive analysis tools using system identification techniques or numerical differentiation for linearization. Due to nonlinearity, traditional system identification techniques using sinusoidal inputs results in signal distortion which could result in inaccurate identified models.

In this work, we present a general method for estimating the analytical stability and control derivatives for a multi-rotor eVTOL aircraft which can be applied to trim solutions obtained from rotorcraft comprehensive analysis tools. The method affords a computationally efficient process for evaluation of the analytical stability and control derivatives and flight dynamic model development for eVTOL aircraft. A numerical tool has been developed to be used in conjunction with CHARM to automate the process of creating a linear flight dynamic model. The tool is applied to the NASA Lift-Plus-Cruise eVTOL aircraft⁵ in hover and transition flight.

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II. Analytical Flight Dynamic Model Development

In this section, a flight dynamic analysis is performed to develop the analytical stability and control derivatives of an eVTOL aircraft in hover and transition flights. For the analysis purpose, the NASA Lift-Plus-Cruise (LPC) eVTOL aircraft is used.⁵ The development assumes a small perturbation about a nonlinear trim equilibrium solution which is obtained by a trim analysis provided by CHARM which is chosen as the rotorcraft comprehensive analysis tool in this work. Complex unsteady wake interactions between the rotors and fixed-wing airframe are neglected in the present analysis due to the lack of fidelity in the wake models employed by CHARM.

A. Reference Frames

The various reference frames of the LPC vehicle are shown in Figure 1. We assume a rigid blade system with no flapping motion in the present analysis, but this assumption will be relaxed in future work.

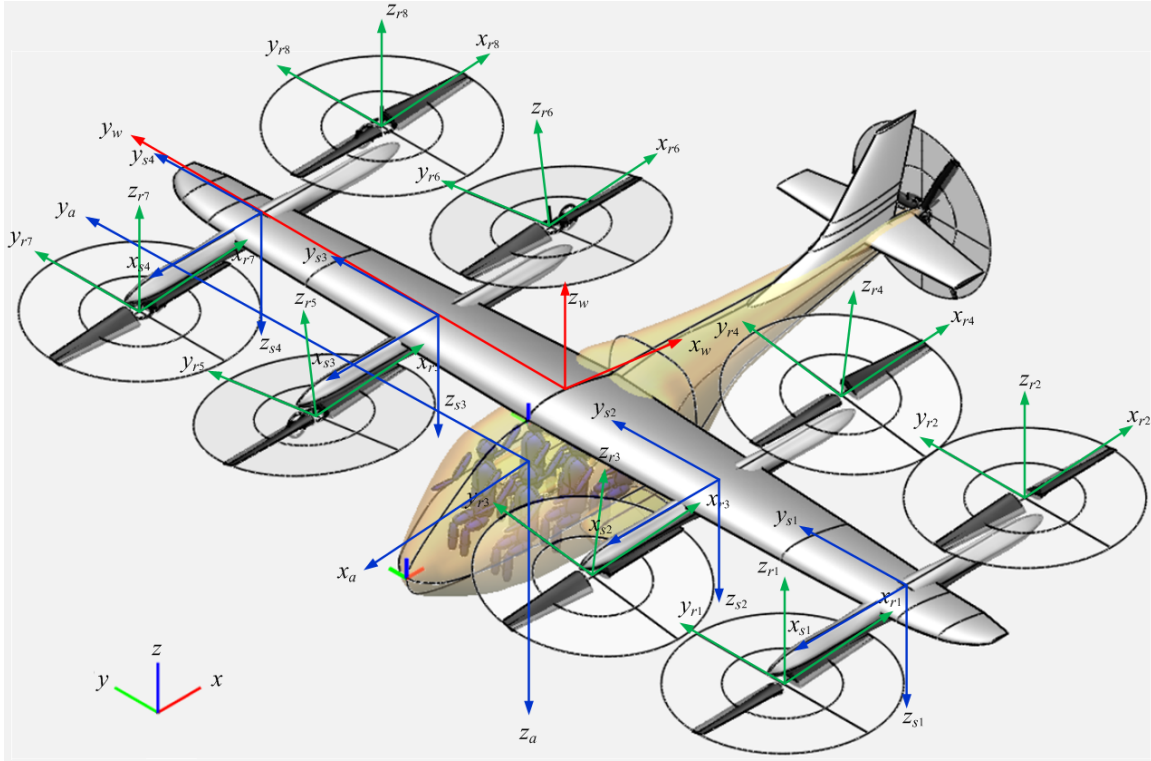


Figure 1. Lift+Cruise Vehicle Reference Frames

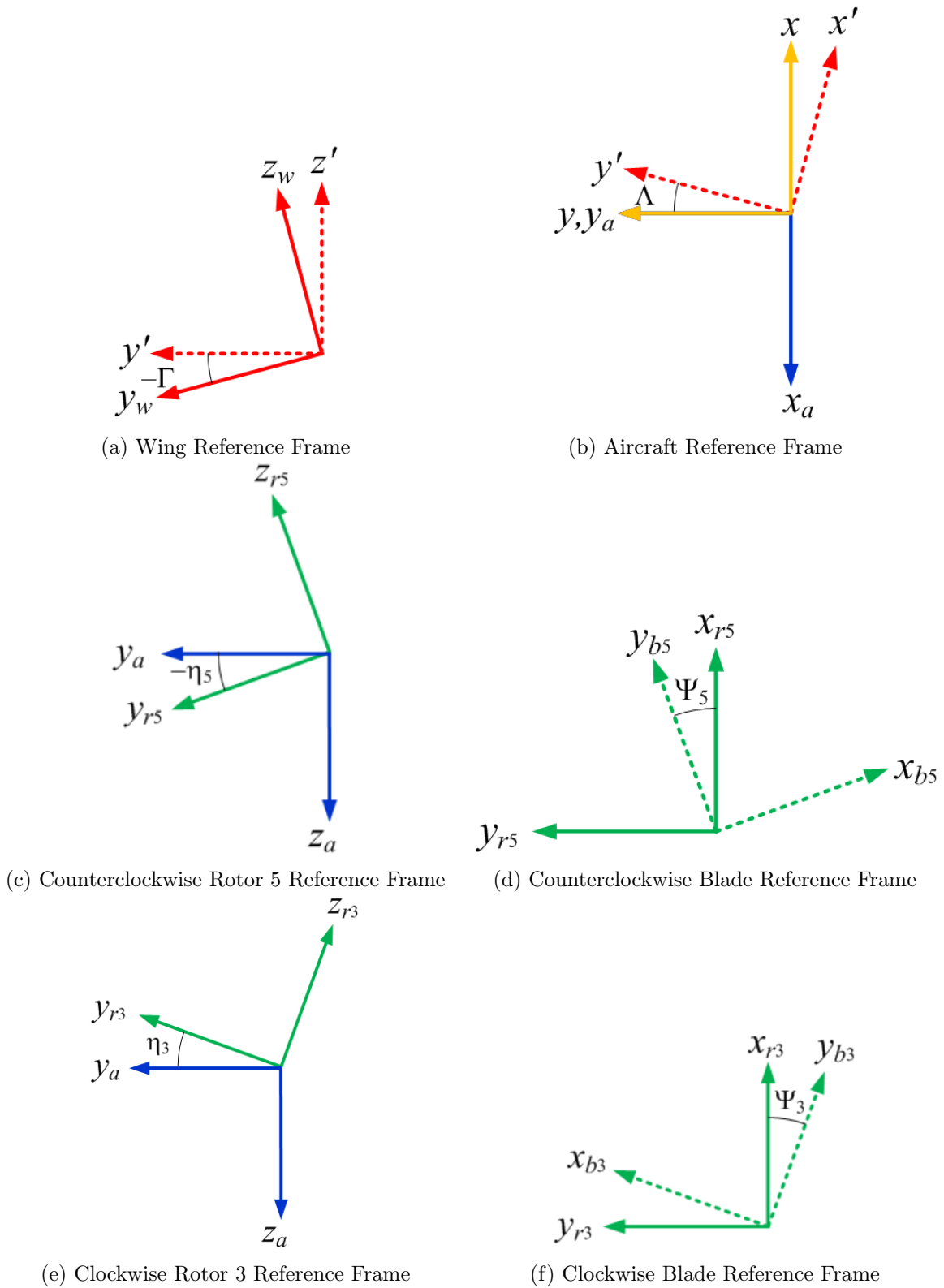


Figure 2. Wing, Aircraft, Rotor, and Blade Reference Frames

The wing has a negative dihedral angle $-\Gamma$ and a positive sweep angle Λ . The wing elastic reference frame is defined by (x_w, y_w, z_w) . The reference frame defined by (x', y', z') is obtained by a rotating (x_w, y_w, z_w)

about $x_w = x'$ by $-\Gamma$ where Γ is a negative value. This is shown in Figure 2(a). The negative value indicates that the dihedral angle is negative or equivalently a positive anhedral angle. Thus,

$$\begin{bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Gamma & -\sin \Gamma \\ 0 & \sin \Gamma & \cos \Gamma \end{bmatrix} \begin{bmatrix} \mathbf{i}_w \\ \mathbf{j}_w \\ \mathbf{k}_w \end{bmatrix} \quad (1)$$

where $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$ are unit vectors along the x' -axis, y' -axis, and z' -axis, respectively.

The z' -axis points in the upward vertical direction in the direction opposite to the aircraft z_a -axis. Referring to Figure 2(b), the (x', y', z') reference frame is related to the aircraft reference frame defined by (x_a, y_a, z_a) as

$$\begin{bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{bmatrix} = \begin{bmatrix} -\cos \Lambda & -\sin \Lambda & 0 \\ -\sin \Lambda & \cos \Lambda & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (2)$$

There are eight lift rotors on the Lift+Cruise vehicles, numbered 1 through 8. Rotors 1, 4, 5, and 8 are rotating in the counterclockwise (CCW) direction as viewed from the top of Figure 1. Rotors 2, 3, 6, and 7 are rotating in the clockwise (CW) direction. The rotors are canted by an angle η_i in the y_a - z_a plane where the positive angle is about the x_r -axis. Referring to Figures 2(c) and (e), the CCW and CW rotor reference frames defined by $(x_{r_i}, y_{r_i}, z_{r_i})$ are related to the aircraft reference frame (x_a, y_a, z_a) by

$$\begin{bmatrix} \mathbf{i}_{r_i} \\ \mathbf{j}_{r_i} \\ \mathbf{k}_{r_i} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos \eta_i & -\sin \eta_i \\ 0 & -\sin \eta_i & -\cos \eta_i \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (3)$$

Referring to Figures 2(d) and (f), the CCW blade reference frames is defined by $(x_{b_i}, y_{b_i}, z_{b_i})$, $i = 1, 4, 5, 8$, where the blade pitch axis is aligned with the y_{b_i} -axis. The CW blade reference frames is defined by $(x_{b_i}, y_{b_i}, z_{b_i})$, $i = 2, 3, 6, 7$, where the blade pitch axis is aligned with the y_{b_i} -axis. The CCW and CW rotations are defined by the angle Ψ_i where the positive angle is about the z_{b_i} -axis. The CCW and CW blade reference frames $(x_{b_i}, y_{b_i}, z_{b_i})$ are related to the rotor reference frames $(x_{r_i}, y_{r_i}, z_{r_i})$ by

$$\begin{bmatrix} \mathbf{i}_{b_i} \\ \mathbf{j}_{b_i} \\ \mathbf{k}_{b_i} \end{bmatrix} = \begin{bmatrix} \sin \Psi_i & \mp \cos \Psi_i & 0 \\ \cos \Psi_i & \pm \sin \Psi_i & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_{r_i} \\ \mathbf{j}_{r_i} \\ \mathbf{k}_{r_i} \end{bmatrix} \quad (4)$$

where the upper sign corresponds to the CCW blade reference frame.

The rotor support reference frames are defined by $(x_{s_i}, y_{s_i}, z_{s_i})$, $i = 1, 2, 3, 4$ in the same orientation as the aircraft reference frame (x_a, y_a, z_a) . Therefore,

$$\begin{bmatrix} \mathbf{i}_{s_i} \\ \mathbf{j}_{s_i} \\ \mathbf{k}_{s_i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (5)$$

The aircraft body station, buttline station, and waterline station are defined by the global reference frame (x, y, z) as shown in Figure 1. The global reference frame (x, y, z) is related to the aircraft reference frame (x_a, y_a, z_a) as

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (6)$$

The aircraft center of gravity (CG) is located at (x_{cg}, y_{cg}, z_{cg}) in the global reference frame (x, y, z) . The origin of the wing elastic reference frame is located at $(x_{w_0}, y_{w_0}, z_{w_0})$ relative to the CG location with respect to the global reference frame (x, y, z) . The origins of the rotor support reference frames are located at $(x_{s_{i0}}, y_{s_{i0}}, z_{s_{i0}})$ relative to the CG location with respect to the global reference frame (x, y, z) . The origins of the rotor and blade reference frames are located at $(x_{r_{i0}}, y_{r_{i0}}, z_{r_{i0}})$ relative to the CG location with respect to the global reference frame (x, y, z) .

B. Air Velocity and Acceleration

The lift forces are created by the air velocity and air acceleration due to the motion of the aircraft and the rotors. The circulatory lift force on an airfoil due to circulation is created by the downwash at the three-quarter chord collocation point. The non-circulatory lift force due to the inertial acceleration of the air is also accounted in the model.

1. Air Velocity Analysis

The air velocity on a rigid blade section is computed by a coordinate transformation from the global reference frame (x, y, z) to the blade reference frames $(x_{b_i}, y_{b_i}, z_{b_i})$ as

$$\mathbf{v}_{b_i} = \bar{\mathbf{v}}_{b_i} + \mathbf{v}_{r_i} \quad (7)$$

where $\bar{\mathbf{v}}_{b_i}$ is the air velocity due to the aircraft motion and $\mathbf{v}_{r_i}$ is the relative air velocity due to the blade motion in the rotor reference frame. These air velocities are given by

$$\bar{\mathbf{v}}_{b_i} = \mathbf{v} + \boldsymbol{\omega} \times (\mathbf{r}_{r_i} + \Delta\mathbf{r}_{b_i}) \quad (8)$$

$$\mathbf{v}_{r_i} = \mathbf{v}_i + \boldsymbol{\omega}_{b_i} \times \Delta\mathbf{r}_{b_i} \quad (9)$$

where

- $\mathbf{v} = -u\mathbf{i}_a - v\mathbf{j}_a - w\mathbf{k}_a = u\mathbf{i} - v\mathbf{j} + w\mathbf{k}$ is the air velocity opposite to the aircraft velocity,
- $\boldsymbol{\omega} = -p\mathbf{i}_a - q\mathbf{j}_a - r\mathbf{k}_a = p\mathbf{i} - q\mathbf{j} + r\mathbf{k}$ is the air angular velocity opposite to the aircraft angular velocity,
- $\mathbf{v}_i = -v_{p_i}\mathbf{k}_{r_i}$ is the inflow velocity normal to the rotor plane,
- $\boldsymbol{\omega}_{b_i} = \mp\dot{\theta}_{b_i}\mathbf{j}_{b_i} - \Omega_i\mathbf{k}_{b_i}$ is the air angular velocity on the rotor blades opposite to the rotor blade angular velocity,
- $\dot{\theta}_{b_i}\mathbf{j}_{b_i}$ is the blade pitch velocity about the blade pitch axis,
- $\Omega_i\mathbf{k}_{b_i}$ is the blade angular velocity,
- $\mathbf{r}_{r_i}$ is a position vector from the aircraft CG to the origin of a rotor reference frame, and
- $\Delta\mathbf{r}_{b_i}$ is a position vector from the origin of a rotor reference frame to a point on a blade section.

The airfoil coordinate is defined along the blade chord direction, pointing from the leading edge ξ at $\xi = 0$ to the trailing edge at $\xi = c_b$. The velocity at the collocation point at the three-quarter chord establishes the equivalent lift circulation. Thus, the position vectors are given by

$$\mathbf{r}_{r_i} = \Delta x_{r_{i0}}\mathbf{i} + \Delta y_{r_{i0}}\mathbf{j} + \Delta z_{r_{i0}}\mathbf{k} \quad (10)$$

$$\Delta\mathbf{r}_{b_i} = (\xi - \xi_e)\mathbf{i}_{b_i} + r'\mathbf{j}_{b_i} \quad (11)$$

where

- $(\Delta x_{r_{i0}}, \Delta y_{r_{i0}}, \Delta z_{r_{i0}}) = (x_{r_{i0}} - x_{cg}, y_{r_{i0}} - y_{cg}, z_{r_{i0}} - z_{cg})$ is the coordinate of the origin of the rotor i -th reference frame relative to the aircraft CG,
- ξ is the blade chord coordinate from 0 at the leading edge to c_b at the trailing edge,
- ξ_e is the location of the pitch center, and
- r' is the blade radial station.

The air velocity due to the aircraft motion is obtained in the blade reference frame $(x_{b_i}, y_{b_i}, z_{b_i})$ as

$$\bar{\mathbf{v}}_{b_i} = \bar{v}_{x_{b_i}}\mathbf{i}_{b_i} + \bar{v}_{y_{b_i}}\mathbf{j}_{b_i} + \bar{v}_{z_{b_i}}\mathbf{k}_{b_i} \quad (12)$$

where

$$\begin{aligned} \bar{v}_{x_{b_i}} &= (u - q\Delta z_{r_{i0}} - r\Delta y_{r_{i0}}) \sin \Psi_i \pm (v + p\Delta z_{r_{i0}} - r\Delta x_{r_{i0}}) \cos \eta_i \cos \Psi_i \\ &\mp (w + p\Delta y_{r_{i0}} + q\Delta x_{r_{i0}}) \sin \eta_i \cos \Psi_i \mp qr' \sin \eta_i^* \mp rr' \cos \eta_i \end{aligned} \quad (13)$$

$$\begin{aligned} \bar{v}_{y_{b_i}} &= (u - q\Delta z_{r_{i0}} - r\Delta y_{r_{i0}}) \cos \Psi_i \mp (v + p\Delta z_{r_{i0}} - r\Delta x_{r_{i0}}) \cos \eta_i \sin \Psi_i \\ &\pm (w + p\Delta y_{r_{i0}} + q\Delta x_{r_{i0}}) \sin \eta_i \sin \Psi_i \pm q(\xi - \xi_e) \sin \eta_i \pm r(\xi - \xi_e) \cos \eta_i \end{aligned} \quad (14)$$

$$\begin{aligned} \bar{v}_{z_{b_i}} &= \pm (v + p\Delta z_{r_{i0}} - r\Delta x_{r_{i0}}) \sin \eta_i \pm (w + p\Delta y_{r_{i0}} + q\Delta x_{r_{i0}}) \cos \eta_i \\ &- p[(\xi - \xi_e) \cos \Psi_i - r' \sin \Psi_i] \pm (q \cos \eta_i - r \sin \eta_i) [(\xi - \xi_e) \sin \Psi_i + r' \cos \Psi_i] \end{aligned} \quad (15)$$

The relative air velocity due to the blade motion is

$$\mathbf{v}_{r_i} = \Omega_i r' \mathbf{i}_{b_i} - \Omega_i (\xi - \xi_e) \mathbf{j}_{b_i} \mp [v_{p_i} - \dot{\theta}_{b_i} (\xi - \xi_e)] \mathbf{k}_{b_i} \quad (16)$$

The tangential, radial, and normal airspeeds on a rotor blade section are obtained

$$\begin{bmatrix} V_t(r', \xi, t) \\ V_r(r', \xi, t) \\ V_z(r', \xi, t) \end{bmatrix} = \begin{bmatrix} \bar{v}_{x_{b_i}} + \Omega_i r' \\ \bar{v}_{y_{b_i}} - \Omega_i (\xi - \xi_e) \\ \pm \bar{v}_{z_{b_i}} - v_{p_i} + \dot{\theta}_{b_i} (\xi - \xi_e) \end{bmatrix} \quad (17)$$

The inflow velocity is prescribed by⁶

$$v_{p_i} = v_{0,p_i} + v_{s,p_i} \frac{r'}{R} \sin \Psi_i + v_{c,p_i} \frac{r'}{R} \cos \Psi_i \quad (18)$$

The Pitt-Peters three-state dynamic inflow model is used to describe the inflow velocity⁷

$$\mathbf{M}_{p_i} \dot{\mathbf{v}}_{p_i} = -\mathbf{L}_{p_i}^{-1} \frac{\mathbf{v}_{p_i}}{R\Omega} + \mathbf{C}_{p_i} \quad (19)$$

where $\mathbf{v}_{p_i} = [v_{0,p_i} \quad v_{s,p_i} \quad v_{c,p_i}]^\top$, $\mathbf{C}_{p_i} = [C_{T_{r_i}} \quad C_{l_{r_i}} \quad C_{m_{r_i}}]^\top$, $\mathbf{M}_{p_i} = \frac{1}{R\Omega_i^2} \text{diag} \left(\left[\frac{8}{3\pi} \quad -\frac{16}{45\pi} \right] \right)$ is the apparent mass matrix, and $\mathbf{L}_{p_i} = \frac{R^2 \Omega_i^2}{\bar{v}_{p_i}} \text{diag} \left(\left[\frac{1}{2} \quad -1 \quad -1 \right] \right)$ for hover and $\mathbf{L}_{p_i} = \frac{R^2 \Omega_i^2}{V} \begin{bmatrix} \frac{64}{75\pi} & 0 & -\frac{1}{2} \\ 0 & \frac{64}{45\pi} & 0 \\ \frac{2}{5} & 0 & 0 \end{bmatrix}$ is the reciprocal of the sensitivity matrix. The thrust and moment coefficients are defined as

$$C_{T_{r_i}} = \frac{T_{r_i}}{\rho_\infty \pi R^4 \Omega_i^2} \quad (20)$$

$$C_{l_{r_i}} = \frac{l_{r_i}}{\rho_\infty \pi R^5 \Omega_i^2} \quad (21)$$

$$C_{m_{r_i}} = \frac{m_{r_i}}{\rho_\infty \pi R^5 \Omega_i^2} \quad (22)$$

2. Air Acceleration Analysis

The air acceleration creates inertial forces, also called non-circulatory forces. The air acceleration is computed as

$$\mathbf{a}_{b_i} = \left(\frac{d\mathbf{v}_{b_i}}{dt} \right)_{b_i} + (\boldsymbol{\omega} + \boldsymbol{\omega}_{b_i}) \times \mathbf{v}_{b_i} \quad (23)$$

where $\left(\frac{d}{dt} \right)_{b_i}$ denotes differentiation with respect to the blade reference frame.

We are interested in the vertical acceleration component in the rotor thrust $\mathbf{k}_{r_i}$ -direction which contributes to the non-circulatory force. It is given by

$$\begin{aligned} a_{z_{b_i}} &= \pm \dot{\bar{v}}_{z_{b_i}} - \left[\dot{v}_{p_i} - \ddot{\theta}_{b_i} (\xi - \xi_e) \right] \pm (p \sin \Psi_i \pm q \cos \eta_i \cos \Psi_i \mp r \sin \eta_i \cos \Psi_i) \left[\bar{v}_{y_{b_i}} - \Omega_i (\xi - \xi_e) \right] \\ &\mp \left(p \cos \Psi_i \mp q \cos \eta_i \sin \Psi_i \pm r \sin \eta_i \sin \Psi_i \mp \dot{\theta}_{b_i} \right) \left(\bar{v}_{x_{b_i}} + \Omega_i r' \right) \end{aligned} \quad (24)$$

C. Unsteady Aerodynamic Forces

The unsteady aerodynamic forces comprise forces due to lift circulation, also called circulatory forces, and non-circulatory forces which are inertial reaction forces exerted by the air.

1. Circulatory Lift Analysis

The air velocity $\mathbf{v}_b(r', \frac{3c_b}{4}, t)$ at the three quarter-chord location is obtained by letting $\xi - \xi_e = \frac{3c_b}{4} - \xi_e = e_{cb}$ where e_{cb} is the distance between the three-quarter-chord and the pitch axis, positive when the three quarter-chord location is aft of the pitch axis.

The blade angle of attack is defined as

$$\alpha_{b_i} \left(r', \frac{3c_b}{4}, t \right) = \tan^{-1} \frac{V_z \left(r', \frac{3c_b}{4}, t \right)}{V_t \left(r', \frac{3c_b}{4}, t \right)} \quad (25)$$

The circulatory lift is given by

$$\frac{dl_{c,b_i}}{d\Psi_i} = c_l \frac{1}{2} \rho_\infty V^2 (r', \xi_e, t) c_b \quad (26)$$

where $c_b(r')$ is the blade chord.

The lift coefficient is given by

$$c_l = c_{l_0} + c_{l_\alpha} \alpha + c_{l_{\alpha^2}} \alpha^2 \quad (27)$$

where α is the angle of attack on a blade section which is defined as

$$\alpha = \theta_{b_i}(r', t) + \alpha_{b_i} \left(r', \frac{3c_b}{4}, t \right) \quad (28)$$

The blade pitch angle is given by

$$\theta_{b_i}(r', t) = \bar{\theta}_{b_i}(r') + \tilde{\theta}_{b_i}(r', \Psi, t) \quad (29)$$

where $\tilde{\theta}_{b_i}(r', t)$ is the blade pitch control input which can be expressed as

$$\tilde{\theta}_{b_i} = \sum_{n=0}^N \theta_{s_{i,n}} \sin n\Omega t + \theta_{c_{i,n}} \cos n\Omega t \quad (30)$$

The circulatory drag is modeled as

$$\frac{dd_{c,b_i}}{d\Psi_i} = c_d \frac{1}{2} \rho_\infty V^2 (r', \xi_e, t) c_b \quad (31)$$

where the drag coefficient c_d is given by

$$c_d = c_{d_0} + c_{d_\alpha} \alpha + c_{d_{\alpha^2}} \alpha^2 \quad (32)$$

The circulatory force on a blade section in the blade reference frame is expressed as

$$\mathbf{f}_{c,b_i} = \frac{df_{x_{c,b_i}}}{d\Psi_i} \mathbf{i}_{b_i} + \frac{df_{z_{c,r_i}}}{d\Psi_i} \mathbf{k}_{r_i} \quad (33)$$

where

$$\frac{df_{x_{c,b_i}}}{d\Psi_i} = -\frac{dl_{c,b_i}}{d\Psi_i} \sin \alpha_{b_i} + \frac{dd_{c,b_i}}{d\Psi_i} \cos \alpha_{b_i} \quad (34)$$

$$\frac{df_{z_{c,r_i}}}{d\Psi_i} = \frac{dl_{c,b_i}}{d\Psi_i} \cos \alpha_{b_i} + \frac{dd_{c,b_i}}{d\Psi_i} \sin \alpha_{b_i} \quad (35)$$

The circulatory force on a blade section in the aircraft reference frame is

$$\mathbf{f}_{c,b_i} = \frac{df_{x_{c,a_i}}}{d\Psi_i} \mathbf{i}_a + \frac{df_{y_{c,a_i}}}{d\Psi_i} \mathbf{j}_a + \frac{df_{z_{c,a_i}}}{d\Psi_i} \mathbf{k}_a \quad (36)$$

where

$$\frac{df_{x_{c,a_i}}}{d\Psi_i} = -\frac{df_{x_{c,b_i}}}{d\Psi_i} \sin \Psi_i \quad (37)$$

$$\frac{df_{y_{c,a_i}}}{d\Psi_i} = \mp \frac{df_{x_{c,b_i}}}{d\Psi_i} \cos \eta_i \cos \Psi_i - \frac{df_{z_{c,b_i}}}{d\Psi_i} \sin \eta_i \quad (38)$$

$$\frac{df_{z_{c,a_i}}}{d\Psi_i} = \pm \frac{df_{x_{c,b_i}}}{d\Psi_i} \sin \eta_i \cos \Psi_i - \frac{df_{z_{c,b_i}}}{d\Psi_i} \cos \eta_i \quad (39)$$

The circulatory force on a blade section in the rotor reference frame is

$$\mathbf{f}_{c,b_i} = \frac{df_{x_{c,r_i}}}{d\Psi_i} \mathbf{i}_{r_i} + \frac{df_{y_{c,r_i}}}{d\Psi_i} \mathbf{j}_{r_i} + \frac{df_{z_{c,r_i}}}{d\Psi_i} \mathbf{k}_{r_i} \quad (40)$$

where

$$\frac{df_{x_{c,r_i}}}{d\Psi_i} = \frac{df_{x_{c,b_i}}}{d\Psi_i} \sin \Psi_i \quad (41)$$

$$\frac{df_{y_{c,r_i}}}{d\Psi_i} = \mp \frac{df_{x_{c,b_i}}}{d\Psi_i} \cos \Psi_i \quad (42)$$

$$\frac{df_{z_{c,r_i}}}{d\Psi_i} = \frac{df_{z_{c,b_i}}}{d\Psi_i} \quad (43)$$

The circulatory rotor thrust force in the rotor reference frame is

$$\frac{d^2 T_{c,r_i}}{dr' d\Psi_i} = \frac{df_{z_{c,r_i}}}{d\Psi_i} = \frac{dl_{c,b_i}}{d\Psi_i} \cos \alpha_{b_i} + \frac{dd_{c,b_i}}{d\Psi_i} \sin \alpha_{b_i} \quad (44)$$

The position vector of the quarter-chord point on a blade from the origin of the rotor reference frame is expressed as

$$\Delta \mathbf{r}_{b_i} \left(\frac{c_b}{4} \right) = r' \mathbf{j}_{b_i} - e_b \mathbf{i}_{b_i} \quad (45)$$

where $e_b = \xi_e - \frac{c_b}{4}$ is the offset between the blade elastic center and the aerodynamic center at the quarter-chord location.

The circulatory rotor moment about the origin of the i -th rotor reference frame is

$$\mathbf{M}_{c,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \Delta \mathbf{r}_{b_i} \left(\frac{c_b}{4} \right) \times \mathbf{f}_{c,b_i} d\Psi_i dr' = -l_{c,r_i} \mathbf{i}_{r_i} + m_{c,r_i} \mathbf{j}_{r_i} - Q_{c,r_i} \mathbf{k}_{r_i} \quad (46)$$

where

$$l_{c,r_i} = \mp \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} (r' \sin \Psi_i + e_b \cos \Psi_i) \frac{df_{z_{c,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (47)$$

$$m_{c,r_i} = -\frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} (r' \cos \Psi_i - e_b \sin \Psi_i) \frac{df_{z_{c,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (48)$$

$$Q_{c,r_i} = \pm \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} r' \frac{df_{x_{c,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (49)$$

with $N_b = 2$ is the number of blades.

Note that the negative sign in front of the rolling moment l_{c,r_i} and yawing moment or rotor torque Q_{c,r_i} reflects the sign convention in flight dynamics where the positive rolling moment points forward and positive yawing moment points down.

The circulatory rotor force acting on the aircraft due to the i -th rotor is obtained as

$$\mathbf{F}_{c,a_i} = X_{c,a_i} \mathbf{i}_a + Y_{c,a_i} \mathbf{j}_a + Z_{c,a_i} \mathbf{k}_a \quad (50)$$

where

$$X_{c,a_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{x_{c,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (51)$$

$$Y_{c,a_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{y_{c,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (52)$$

$$Z_{c,a_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{z_{c,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (53)$$

The circulatory rotor moment about the aircraft CG due to the i -th rotor is obtained as

$$\mathbf{M}_{c,a_i} = \mathbf{M}_{c,r_i} + \mathbf{r}_{r_i} \times \mathbf{F}_{c,r_i} = l_{c,a_i} \mathbf{i}_a + m_{c,a_i} \mathbf{j}_a + n_{c,a_i} \mathbf{k}_a \quad (54)$$

where

$$l_{c,a_i} = l_{c,r_i} + Y_{c,a_i} \Delta z_{r_{i0}} + Z_{c,a_i} \Delta y_{r_{i0}} \quad (55)$$

$$m_{c,a_i} = m_{c,r_i} \cos \eta_i + Q_{c,r_i} \sin \eta_i - X_{c,a_i} \Delta z_{r_{i0}} + Z_{c,a_i} \Delta x_{r_{i0}} \quad (56)$$

$$n_{c,a_i} = -m_{c,r_i} \sin \eta_i + Q_{c,r_i} \cos \eta_i - X_{c,a_i} \Delta y_{r_{i0}} - Y_{c,a_i} \Delta x_{r_{i0}} \quad (57)$$

2. Non-Circulatory Lift Analysis

Theodorsen's unsteady aerodynamic theory for an oscillating airfoil in pitch and plunge provides the non-circulatory forces and moments as functions of the air acceleration.⁸ In this work, we extend the Theodorsen's theory to account for a general blade motion. The air vertical acceleration component can be expressed as

$$a_{z_{b_i}}(r', \xi, t) = a_{z_{0,b_i}}(r', t) + a_\xi(r', t)(\xi - \xi_e) \quad (58)$$

The time derivative of the velocity potential due to the air acceleration according to Theodorsen's theory is given by⁹

$$\frac{d\phi}{dt} = \sqrt{\xi(c_b - \xi)} \left[a_{z_{0,b_i}} + a_\xi \left(\frac{\xi}{2} + \frac{c_b}{4} - \xi_e \right) \right] \quad (59)$$

The non-circulatory lift contribution is computed as

$$\frac{dl_{nc,b_i}}{d\Psi_i} = 2\rho_\infty \int_0^{c_b} \frac{\partial \phi}{\partial t} d\xi = \frac{\pi}{4} \rho_\infty c_b^2 \left[a_{z_{0,b_i}} + a_\xi \left(\frac{c_b}{2} - \xi_e \right) \right] = \frac{\pi}{4} \rho_\infty c_b^2 a_{z_{b_i}} \left(r', \frac{c_b}{2}, t \right) \quad (60)$$

The non-circulatory pitching moment contribution about the blade pitch axis is computed as

$$\begin{aligned} \frac{dm_{nc,b_i}}{d\Psi_i} &= \frac{dm_{ac,b_i}}{d\Psi_i} - 2\rho_\infty \int_0^c \frac{\partial \phi}{\partial t} (\xi - \xi_e) d\xi \\ &= \frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{4} \rho_\infty c_b^2 \left\{ \left(\frac{c_b}{2} - \xi_e \right) a_{z_{0,b_i}} + a_\xi \left[\left(\frac{c_b}{2} - \xi_e \right)^2 + \frac{1}{32} c_b^2 \right] \right\} \\ &= \frac{dm_{ac,b_i}}{d\Psi_i} - e_{mb} \frac{dl_{nc,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \end{aligned} \quad (61)$$

where $e_{mb} = \frac{c_b}{2} - \xi_e$ is the offset between the blade pitch axis and the mid-chord location and $\frac{dm_{ac,b_i}}{d\Psi_i}$ is the pitching moment about the aerodynamic center of a blade section given by

$$\frac{dm_{ac,b_i}}{d\Psi_i} = c_{mac} \frac{1}{2} \rho_\infty V^2 (r', \xi_e, t) c_b^2 \quad (62)$$

The pitching moment about the aerodynamic center does not depend on lift, hence it is a non-circulatory pitching moment. Moreover, it is a function of the effective blade mean camber angle which is expressed as

$$\frac{dz}{d\xi} = -\tan^{-1} \frac{V_z}{V_t} \approx -\alpha_{b_i}^*(r', t) - \alpha_\xi(r', t)(\xi - \xi_e) \quad (63)$$

where

$$\alpha_\xi(r', t) = \frac{\bar{V}_t}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_z}{\partial(\xi - \xi_e)} - \frac{\bar{V}_z}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_t}{\partial(\xi - \xi_e)} \quad (64)$$

Using the transformation $\xi = \frac{c_b}{2}(1 - \cos \phi)$, the pitching moment coefficient about the quarter-chord aerodynamic center is evaluated as¹⁰

$$c_{m_{ac}} = \frac{1}{2} \int_0^\pi \frac{dz}{dx} (\cos 2\theta - \cos \theta) d\theta = -\frac{\pi}{8} c_b \alpha_\xi \quad (65)$$

We assume the non-circulatory drag is zero. Then, the non-circulatory force on a blade section in the blade reference frame is expressed as

$$\mathbf{f}_{nc, b_i} = \frac{df_{x_{nc, b_i}}}{d\Psi_i} \mathbf{i}_{b_i} \pm \frac{df_{z_{nc, b_i}}}{d\Psi_i} \mathbf{k}_{b_i} \quad (66)$$

where

$$\frac{df_{x_{nc, b_i}}}{d\Psi_i} = -\frac{dl_{nc, b_i}}{d\Psi_i} \sin \alpha_{b_i} \quad (67)$$

$$\frac{df_{z_{nc, b_i}}}{d\Psi_i} = \frac{dl_{nc, b_i}}{d\Psi_i} \cos \alpha_{b_i} \quad (68)$$

The non-circulatory force on a blade section in the aircraft reference frame is

$$\mathbf{f}_{nc, b_i} = \frac{df_{x_{nc, a_i}}}{d\Psi_i} \mathbf{i}_a + \frac{df_{y_{nc, a_i}}}{d\Psi_i} \mathbf{j}_a + \frac{df_{z_{nc, a_i}}}{d\Psi_i} \mathbf{k}_a \quad (69)$$

where

$$\frac{df_{x_{nc, a_i}}}{d\Psi_i} = -\frac{df_{x_{nc, b_i}}}{d\Psi_i} \sin \Psi_i \quad (70)$$

$$\frac{df_{y_{nc, a_i}}}{d\Psi_i} = \mp \frac{df_{x_{nc, b_i}}}{d\Psi_i} \cos \eta_i \cos \Psi_i - \frac{df_{z_{nc, b_i}}}{d\Psi_i} \sin \eta_i \quad (71)$$

$$\frac{df_{z_{nc, a_i}}}{d\Psi_i} = \pm \frac{df_{x_{nc, b_i}}}{d\Psi_i} \sin \eta_i \cos \Psi_i - \frac{df_{z_{nc, b_i}}}{d\Psi_i} \cos \eta_i \quad (72)$$

The non-circulatory force on a blade section in the rotor reference frame is

$$\mathbf{f}_{nc, b_i} = \frac{df_{x_{c, r_i}}}{d\Psi_i} \mathbf{i}_{r_i} + \frac{df_{y_{c, r_i}}}{d\Psi_i} \mathbf{j}_{r_i} + \frac{df_{z_{c, r_i}}}{d\Psi_i} \mathbf{k}_{r_i} \quad (73)$$

where

$$\frac{df_{x_{nc, r_i}}}{d\Psi_i} = \frac{df_{x_{nc, b_i}}}{d\Psi_i} \sin \Psi_i \quad (74)$$

$$\frac{df_{y_{nc, r_i}}}{d\Psi_i} = \mp \frac{df_{x_{nc, b_i}}}{d\Psi_i} \cos \Psi_i \quad (75)$$

$$\frac{df_{z_{nc, r_i}}}{d\Psi_i} = \frac{df_{z_{nc, b_i}}}{d\Psi_i} \quad (76)$$

The non-circulatory rotor thrust force in the rotor reference frame is

$$\frac{d^2 T_{nc, r_i}}{dr' d\Psi_i} = \frac{df_{z_{nc, r_i}}}{d\Psi_i} = \frac{df_{z_{nc, b_i}}}{d\Psi_i} = \frac{dl_{nc, b_i}}{d\Psi_i} \cos \alpha_{b_i} \quad (77)$$

The position vector of the mid-chord point on a blade from the origin of the rotor reference frame is expressed as

$$\Delta \mathbf{r}_{b_i} \left(\frac{c}{2} \right) = r' \mathbf{j}_{b_i} + e_{mb} \mathbf{i}_{b_i} \quad (78)$$

The non-circulatory rotor moment about the origin of the i -th rotor reference frame is

$$\begin{aligned}\mathbf{M}_{nc,r_i} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[\Delta \mathbf{r}_{b_i} \left(\frac{c_b}{2} \right) \times \mathbf{f}_{nc,b_i} + \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \mathbf{j}_{b_i} \right] d\Psi_i dr' \\ &= -l_{nc,r_i} \mathbf{i}_{r_i} + m_{nc,r_i} \mathbf{j}_{r_i} - Q_{nc,r_i} \mathbf{k}_{r_i}\end{aligned}\quad (79)$$

where

$$l_{nc,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[\mp (r' \sin \Psi_i - e_{mb} \cos \Psi_i) \frac{df_{z_{nc,b_i}}}{d\Psi_i} - \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \cos \Psi_i \right] d\Psi_i dr' \quad (80)$$

$$m_{nc,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[- (r' \cos \Psi_i + e_{mb} \sin \Psi_i) \frac{df_{z_{nc,b_i}}}{d\Psi_i} \pm \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \sin \Psi_i \right] d\Psi_i dr' \quad (81)$$

$$Q_{nc,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \pm r' \frac{df_{x_{nc,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (82)$$

The non-circulatory force acting on the aircraft due to the i -th rotor is obtained as

$$\mathbf{F}_{nc,a_i} = X_{nc,a_i} \mathbf{i}_a + Y_{nc,a_i} \mathbf{j}_a + Z_{nc,a_i} \mathbf{k}_a \quad (83)$$

where

$$X_{nc,a_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{x_{nc,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (84)$$

$$Y_{nc,a_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{y_{nc,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (85)$$

$$Z_{nc,a_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{z_{nc,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (86)$$

The non-circulatory moment about the aircraft CG due to the i -th rotor is obtained as

$$\mathbf{M}_{nc,a_i} = \mathbf{M}_{nc,r_i} + \mathbf{r}_{r_i} \times \mathbf{F}_{nc,r_i} = l_{nc,a_i} \mathbf{i}_a + m_{nc,a_i} \mathbf{j}_a + n_{nc,a_i} \mathbf{k}_a \quad (87)$$

where

$$l_{nc,a_i} = l_{nc,r_i} + Y_{nc,a_i} \Delta z_{r_{i0}} + Z_{nc,a_i} \Delta y_{r_{i0}} \quad (88)$$

$$m_{nc,a_i} = m_{nc,r_i} \cos \eta_i + Q_{nc,r_i} \sin \eta_i - X_{nc,a_i} \Delta z_{r_{i0}} + Z_{nc,a_i} \Delta x_{r_{i0}} \quad (89)$$

$$n_{nc,a_i} = -m_{nc,r_i} \sin \eta_i + Q_{nc,r_i} \cos \eta_i - X_{nc,a_i} \Delta y_{r_{i0}} - Y_{nc,a_i} \Delta x_{r_{i0}} \quad (90)$$

D. Stability and Control Derivative Analysis

To construct a linearized flight dynamic model of eVTOL aircraft, the partial derivatives of the forces and moments are evaluated.

1. Partial Derivatives of Circulatory Forces and Moment

We write the lift force as

$$\frac{dl_{c,b_i}}{d\Psi_i} = \frac{dl_{c,b_i}}{d\Psi_i} \left(V \left(\mathbf{x}_a, \mathbf{v}_{p_i}, \dot{\theta}_{b_i}, \Omega_i \right), c_l \left(M(V), \alpha \left(\tilde{\theta}_{b_i}, \alpha_{b_i}(V) \right) \right) \right) \quad (91)$$

where $\mathbf{x}_a(t) = \begin{bmatrix} u(t) & v(t) & w(t) & p(t) & q(t) & r(t) \end{bmatrix}^\top$ is the aircraft state vector and $\tilde{\theta}_{b_i}(t)$, $\dot{\theta}_{b_i}(t)$, and $\Omega_i(t)$ are the control variables.

Let $x(t)$ be a place holder variable for a state or control variable. The partial derivative of $\frac{dl_{c,b_i}}{d\Psi_i}$ with respect to $x(t)$ is evaluated as

$$\frac{\partial}{\partial x} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) = \frac{\partial}{\partial V} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) \frac{\partial V}{\partial x} + \frac{\partial}{\partial c_l} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) \left(\frac{\partial c_l}{\partial M} \frac{\partial M}{\partial x} + \frac{\partial c_l}{\partial \alpha} \frac{\partial \alpha}{\partial x} \right) \quad (92)$$

The partial derivatives of $\frac{dl_{c,b_i}}{d\Psi_i}$ are

$$\frac{\partial}{\partial V} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) = \bar{c}_l \frac{1}{2} \rho_\infty 2\bar{V} c_b \quad (93)$$

$$\frac{\partial}{\partial c_l} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) = \frac{1}{2} \rho_\infty \bar{V}^2 c_b \quad (94)$$

where

$$\bar{V} = \sqrt{\bar{V}_t^2 + \bar{V}_r^2 + \bar{V}_z^2} \quad (95)$$

The partial derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial M}{\partial x}$ are

$$\frac{\partial V}{\partial x} = \frac{\bar{V}_t}{\bar{V}} \frac{\partial V_t}{\partial x} + \frac{\bar{V}_r}{\bar{V}} \frac{\partial V_r}{\partial x} + \frac{\bar{V}_z}{\bar{V}} \frac{\partial V_z}{\partial x} \quad (96)$$

$$\frac{\partial M}{\partial x} = \frac{1}{a(h)} \frac{\partial V}{\partial x} = \frac{1}{a(h)} \frac{\partial V}{\partial x} \quad (97)$$

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha_w}{\partial x} + \frac{\partial \alpha_{s_i}}{\partial x} + \frac{\partial \alpha_{b_i}}{\partial x} \quad (98)$$

$$\frac{\partial \alpha_{b_i}}{\partial x} = \frac{\bar{V}_t}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_z}{\partial x} - \frac{\bar{V}_z}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_t}{\partial x} \quad (99)$$

The Prandtl-Glauert compressibility correction formula is given by¹⁰

$$c_l = \frac{c_l^*}{\sqrt{1 - M^2} + \frac{c_l^*}{\pi \mathcal{R} \epsilon}} \quad (100)$$

where c_l^* is the ideal incompressible lift coefficient, $\mathcal{R}$ is the aspect ratio, and ϵ is the span efficiency factor.

The partial derivative $\frac{\partial c_l}{\partial M}$ is evaluated as

$$\frac{\partial c_l}{\partial M} = \frac{c_l M}{\sqrt{1 - M^2} \left(\sqrt{1 - M^2} + \frac{c_l^*}{\pi \mathcal{R} \epsilon} \right)} \quad (101)$$

c_l^* can be solved in terms of c_l as

$$c_l^* = \frac{c_l \sqrt{1 - M^2}}{1 - \frac{c_l}{\pi \mathcal{R} \epsilon}} \quad (102)$$

Upon substitution, we obtain

$$\frac{\partial c_l}{\partial M} = \frac{c_l (1 - k_{b_i} c_l) M}{1 - M^2} \quad (103)$$

where

$$k_{b_i} = \frac{1}{\pi \mathcal{R} \epsilon} = \frac{1}{c_l} - \frac{\sqrt{1 - M^2}}{c_l^*} \quad (104)$$

The partial derivative $\frac{\partial c_d}{\partial M}$ is evaluated as

$$\frac{\partial c_d}{\partial M} = \frac{\partial c_d}{\partial \alpha} \frac{\partial \alpha}{\partial c_l} \frac{\partial c_l}{\partial M} = \frac{\bar{c}_{d_\alpha} \bar{c}_l (1 - k_{b_i} \bar{c}_l) M}{\bar{c}_{l_\alpha} (1 - M^2)} \quad (105)$$

The partial derivative of $\frac{dl_{c,b_i}}{d\Psi_i}$ is

$$\frac{\partial}{\partial x} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) = \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{\partial f_{b_i}}{\partial x} \quad (106)$$

where

$$\frac{d\bar{l}_{c,b_i}}{d\Psi_i} = \bar{c}_l \frac{1}{2} \rho_\infty \bar{V}^2 c_b \quad (107)$$

$$\frac{\partial f_{b_i}}{\partial x} = \frac{k_{c_l}}{\bar{V}} \frac{\partial V}{\partial x} + \frac{\bar{c}_{l_\alpha}}{\bar{c}_l} \left(\frac{\partial \alpha_w}{\partial x} + \frac{\partial \alpha_{s_i}}{\partial x} + \frac{\partial \alpha_{b_i}}{\partial x} \right) \quad (108)$$

$$k_{c_l} = 2 + \frac{(1 - k_{b_i} \bar{c}_l) M^2}{1 - M^2} \quad (109)$$

$$\bar{c}_{l_\alpha} = c_{l_\alpha} + 2c_{l_\alpha^2} \bar{\alpha} \quad (110)$$

The partial derivative of $\frac{dd_{c,b_i}}{d\Psi_i}$ is

$$\frac{\partial}{\partial x} \left(\frac{dd_{c,b_i}}{d\Psi_i} \right) = \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \frac{\partial g_{b_i}}{\partial x} \quad (111)$$

where

$$\frac{d\bar{d}_{c,b_i}}{d\Psi_i} = \bar{c}_d \frac{1}{2} \rho_\infty \bar{V}^2 c_b \quad (112)$$

$$\frac{\partial g_{b_i}}{\partial x} = \frac{k_{c_d}}{\bar{V}} \frac{\partial V}{\partial x} + \frac{\bar{c}_{d_\alpha}}{\bar{c}_d} \left(\frac{\partial \alpha_w}{\partial x} + \frac{\partial \alpha_{s_i}}{\partial x} + \frac{\partial \alpha_{b_i}}{\partial x} \right) \quad (113)$$

$$k_{c_d} = 2 + \frac{\bar{c}_{d_\alpha} \bar{c}_l (1 - k_{b_i} \bar{c}_l) M^2}{\bar{c}_{l_\alpha} \bar{c}_d (1 - M^2)} \quad (114)$$

$$\bar{c}_{d_\alpha} = c_{d_\alpha} + 2c_{d_{\alpha^2}} \bar{\alpha} \quad (115)$$

We express $\frac{\partial f_{b_i}}{\partial x}$, $\frac{\partial g_{b_i}}{\partial x}$, and $\frac{\partial \alpha_{b_i}}{\partial x}$ as

$$\frac{\partial f_{b_i}}{\partial x} = \frac{\partial f_{0,b_i}}{\partial x} + \frac{\partial f_{s,b_i}}{\partial x} \sin \Psi_i + \frac{\partial f_{c,b_i}}{\partial x} \cos \Psi_i \quad (116)$$

$$\frac{\partial g_{b_i}}{\partial x} = \frac{\partial g_{0,b_i}}{\partial x} + \frac{\partial g_{s,b_i}}{\partial x} \sin \Psi_i + \frac{\partial g_{c,b_i}}{\partial x} \cos \Psi_i \quad (117)$$

$$\frac{\partial \alpha_{b_i}}{\partial x} = \frac{\partial \alpha_{0,b_i}}{\partial x} + \frac{\partial \alpha_{s,b_i}}{\partial x} \sin \Psi_i + \frac{\partial \alpha_{c,b_i}}{\partial x} \cos \Psi_i \quad (118)$$

2. Partial Derivatives of Rotor and Aircraft Circulatory Forces and Moments

The partial derivatives of the circulatory axial and normal forces in the blade reference frame are

$$\frac{\partial}{\partial x} \left(\frac{df_{x_c,b_i}}{d\Psi_i} \right) = -\frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{\partial f_{b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \frac{\partial g_{b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \frac{d\bar{f}_{z_c,b_i}}{d\Psi_i} \frac{\partial \alpha_{b_i}}{\partial x} \quad (119)$$

$$\frac{\partial}{\partial x} \left(\frac{df_{z_c,b_i}}{d\Psi_i} \right) = \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{\partial f_{b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \frac{\partial g_{b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \frac{d\bar{f}_{x_c,b_i}}{d\Psi_i} \frac{\partial \alpha_{b_i}}{\partial x} \quad (120)$$

where

$$\frac{d\bar{f}_{x_c,b_i}}{d\Psi_i} = -\frac{d\bar{l}_{c,b_i}}{d\Psi_i} \sin \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \cos \bar{\alpha}_{b_i} \quad (121)$$

$$\frac{d\bar{f}_{z_c,b_i}}{d\Psi_i} = \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \cos \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \sin \bar{\alpha}_{b_i} \quad (122)$$

The partial derivatives of the integrated circulatory thrust force and rotor moments which affect the

inflow velocity are computed as

$$\begin{aligned}\frac{\partial T_{c,r_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{df_{z_c,b_i}}{d\Psi_i} \right) d\Psi_i dr' \\ &= \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_c,b_i} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr'\end{aligned}\quad (123)$$

$$\begin{aligned}\frac{\partial l_{c,r_i}}{\partial x} &= \mp \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \int_0^{2\pi} \frac{\partial}{\partial x} \left[(r' \sin \Psi_i + e_b \cos \Psi_i) \frac{df_{z_c,b_i}}{d\Psi_i} \right] d\Psi_i dr \\ &= \mp \frac{1}{2} \int_{R_i}^R r' \left(\bar{l}_{c,b_i} \frac{\partial f_{s,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{s,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_c,b_i} \frac{\partial \alpha_{s,b_i}}{\partial x} \right) dr' \\ &\quad \mp \frac{1}{2} \int_{R_i}^R e_b \left(\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_c,b_i} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr'\end{aligned}\quad (124)$$

$$\begin{aligned}\frac{\partial m_{c,r_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left[-(r' \cos \Psi_i - e_b \sin \Psi_i) \frac{df_{z_c,b_i}}{d\Psi_i} \pm \frac{dm_{c,b_i}}{d\Psi_i} \sin \Psi_i \right] d\Psi_i dr' \\ &= -\frac{1}{2} \int_{R_i}^R r' \left(\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_c,b_i} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr' \\ &\quad + \frac{1}{2} \int_{R_i}^R e_b \left(\bar{l}_{c,b_i} \frac{\partial f_{s,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{s,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_c,b_i} \frac{\partial \alpha_{s,b_i}}{\partial x} \right) dr'\end{aligned}\quad (125)$$

$$\begin{aligned}\frac{\partial Q_{c,r_i}}{\partial x} &= \pm \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(r' \frac{df_{x_c,b_i}}{d\Psi_i} \right) d\Psi_i dr' \\ &= \pm \int_{R_i}^R r' \left(-\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z_c,b_i} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr'\end{aligned}\quad (126)$$

where

$$\bar{l}_{c,b_i} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{l}_{c,b_i}}{\partial \Psi_i} d\Psi_i \quad (127)$$

$$\bar{d}_{c,b_i} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{d}_{c,b_i}}{\partial \Psi_i} d\Psi_i \quad (128)$$

$$\bar{f}_{x_c,b_i} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{f}_{x_c,b_i}}{\partial \Psi_i} d\Psi_i \quad (129)$$

$$\bar{f}_{z_c,b_i} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{f}_{z_c,b_i}}{\partial \Psi_i} d\Psi_i \quad (130)$$

The partial derivatives of the aircraft integrated circulatory forces are evaluated as

$$\begin{aligned}\frac{\partial X_{c,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(-\frac{df_{x,c,b_i}}{d\Psi_i} \right) \sin \Psi_i d\Psi_i dr' \\ &= -\frac{1}{2} \int_{R_i}^R \left(-\bar{l}_{c,b_i} \frac{\partial f_{s,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{s,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z,c,b_i} \frac{\partial \alpha_{s,b_i}}{\partial x} \right) dr'\end{aligned}\quad (131)$$

$$\begin{aligned}\frac{\partial Y_{c,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\mp \frac{df_{x,c,b_i}}{d\Psi_i} \cos \eta_i \cos \Psi_i - \frac{df_{z,c,b_i}}{d\Psi_i} \sin \eta_i \right) d\Psi_i dr' \\ &= \mp \frac{1}{2} \cos \bar{\eta}_i \int_{R_i}^R \left(-\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z,c,b_i} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr' \\ &\quad - \sin \bar{\eta}_i \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x,c,b_i} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr'\end{aligned}\quad (132)$$

$$\begin{aligned}\frac{\partial Z_{c,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\pm \frac{df_{x,c,b_i}}{d\Psi_i} \sin \eta_i \cos \Psi_i - \frac{df_{z,c,b_i}}{d\Psi_i} \cos \eta_i \right) d\Psi_i dr' \\ &= \pm \frac{1}{2} \sin \bar{\eta}_i \int_{R_i}^R \left(-\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z,c,b_i} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr' \\ &\quad - \cos \bar{\eta}_i \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x,c,b_i} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr'\end{aligned}\quad (133)$$

The partial derivatives of the aircraft integrated circulatory moments are evaluated as

$$\frac{\partial l_{c,a_i}}{\partial x} = \frac{\partial l_{c,r_i}}{\partial x} + \frac{\partial Y_{c,a_i}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{c,a_i}}{\partial x} \Delta y_{r_{i0}} \quad (134)$$

$$\frac{\partial m_{c,a_i}}{\partial x} = \frac{\partial m_{c,r_i}}{\partial x} \cos \bar{\eta}_i + \frac{\partial Q_{c,r_i}}{\partial x} \sin \bar{\eta}_i - \frac{\partial X_{c,r_i}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{c,r_i}}{\partial x} \Delta x_{r_{i0}} \quad (135)$$

$$\frac{\partial n_{c,a_i}}{\partial x} = -\frac{\partial m_{c,r_i}}{\partial x} \sin \bar{\eta}_i + \frac{\partial Q_{c,r_i}}{\partial x} \cos \bar{\eta}_i - \frac{\partial X_{c,r_i}}{\partial x} \Delta y_{r_{i0}} - \frac{\partial Y_{c,r_i}}{\partial x} \Delta x_{r_{i0}} \quad (136)$$

Let $\mathbf{v}_p(t) = \begin{bmatrix} \mathbf{v}_{p_1}^\top(t) & \mathbf{v}_{p_2}^\top(t) & \cdots & \mathbf{v}_{p_{n_{rotor}}}^\top(t) \end{bmatrix}^\top$ where $\mathbf{v}_{p_i}(t) = \begin{bmatrix} v_{0,p_i}(t) & v_{s,p_i}(t) & v_{c,p_i}(t) \end{bmatrix}^\top$, $\mathbf{u}_\theta(t) = \begin{bmatrix} \tilde{\boldsymbol{\theta}}_b(t) & \dot{\boldsymbol{\theta}}_b(t) \end{bmatrix}^\top$ where $\tilde{\boldsymbol{\theta}}_b(t) = \begin{bmatrix} \tilde{\theta}_{b_1}(t) & \tilde{\theta}_{b_2}(t) & \cdots & \tilde{\theta}_{b_{n_{rotor}}}(t) \end{bmatrix}^\top$, and $\boldsymbol{\Omega}(t) = [\Omega_1(t) \ \Omega_2(t) \ \cdots \ \Omega_{n_{rotor}}(t)]^\top$. We sum all the aircraft integrated circulatory force and moment contributions by the rotors and express them in terms of the partial derivatives as

$$\mathbf{F}_{c,a} = \sum_{i=1}^{n_{rotor}} \mathbf{F}_{c,a_i} = \frac{\partial \mathbf{F}_{c,a}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{c,a}}{\partial \mathbf{v}_p} \mathbf{v}_p + \frac{\partial \mathbf{F}_{c,a}}{\partial \mathbf{u}_\theta} \mathbf{u}_\theta + \frac{\partial \mathbf{F}_{c,a}}{\partial \boldsymbol{\Omega}} \boldsymbol{\Omega} \quad (137)$$

$$\mathbf{M}_{c,a} = \sum_{i=1}^{n_{rotor}} \mathbf{M}_{c,a_i} = \frac{\partial \mathbf{M}_{c,a}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{c,a}}{\partial \mathbf{v}_p} \mathbf{v}_p + \frac{\partial \mathbf{M}_{c,a}}{\partial \mathbf{u}_\theta} \mathbf{u}_\theta + \frac{\partial \mathbf{M}_{c,a}}{\partial \boldsymbol{\Omega}} \boldsymbol{\Omega} \quad (138)$$

In addition, the rotor circulatory thrust and moments acting at the origin of the i -th rotor reference frame are expressed in terms of the partial derivatives as

$$T_{c,r_i} = \frac{\partial T_{c,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial T_{c,r_i}}{\partial \mathbf{v}_{p_i}} \mathbf{v}_{p_i} + \frac{\partial T_{c,r_i}}{\partial \mathbf{u}_{\theta_i}} \mathbf{u}_{\theta_i} + \frac{\partial T_{c,r_i}}{\partial \Omega_i} \Omega_i \quad (139)$$

$$\mathbf{M}_{c,r_i} = \frac{\partial \mathbf{M}_{c,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{c,r_i}}{\partial \mathbf{v}_{p_i}} \mathbf{v}_{p_i} + \frac{\partial \mathbf{M}_{c,r_i}}{\partial \mathbf{u}_{\theta_i}} \mathbf{u}_{\theta_i} + \frac{\partial \mathbf{M}_{c,r_i}}{\partial \Omega_i} \Omega_i \quad (140)$$

Note that the aircraft circulatory force and moment contributions by the airframe including wings, horizontal tail, and vertical tails as well as the control surfaces should also be included for both hover and especially for transition flight. These contributions are well-developed and are not included here.

3. Partial Derivatives of Non-Circulatory Forces and Moment

We write the non-circulatory lift force and the quarter-chord pitching moment as

$$\frac{dl_{nc,b_i}}{d\Psi_i} = \frac{\pi}{4} \rho_\infty c_b^2 a_{z_{b_i}} \left(\mathbf{x}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{v}}_{p_i}, \dot{\theta}_i, \ddot{\theta}_{b_i} \right) \quad (141)$$

$$\frac{dm_{ac,b_i}}{d\Psi_i} = c_{mac} \frac{1}{2} \rho_\infty V^2 \left(\mathbf{x}_a, \mathbf{v}_{p_i}, \dot{\theta}_{b_i}, \Omega_i \right) c_b^2 \quad (142)$$

The partial derivatives of $\frac{dl_{nc,b_i}}{d\Psi_i}$ and $\frac{dm_{ac,b_i}}{d\Psi_i}$ are evaluated as

$$\frac{\partial}{\partial x} \left(\frac{dl_{nc,b_i}}{d\Psi_i} \right) = \frac{\pi}{4} \rho_\infty c_b^2 \frac{\partial a_{z_{b_i}}}{\partial x} \quad (143)$$

$$\frac{\partial}{\partial x} \left(\frac{dm_{ac,b_i}}{d\Psi_i} \right) = \frac{\partial c_{mac}}{\partial x} \frac{1}{2} \rho_\infty \bar{V}^2 c_b^2 = \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{c_b}{\bar{c}_l} \frac{\partial c_{mac}}{\partial x} \quad (144)$$

where $x(t)$ is a place holder variable.

The partial derivatives of the non-circulatory axial and normal forces in the blade reference frame are

$$\frac{\partial}{\partial x} \left(\frac{df_{x_{nc,b_i}}}{d\Psi_i} \right) = - \frac{\partial}{\partial x} \left(\frac{dl_{nc,b_i}}{d\Psi_i} \right) \sin \bar{\alpha}_{b_i} = - \frac{\pi}{4} \rho_\infty c_b^2 \sin \bar{\alpha}_{b_i} \frac{\partial a_{z_{b_i}}}{\partial x} \quad (145)$$

$$\frac{\partial}{\partial x} \left(\frac{df_{z_{nc,b_i}}}{d\Psi_i} \right) = \frac{\partial}{\partial x} \left(\frac{dl_{nc,b_i}}{d\Psi_i} \right) \cos \bar{\alpha}_{b_i} = \frac{\pi}{4} \rho_\infty c_b^2 \cos \bar{\alpha}_{b_i} \frac{\partial a_{z_{b_i}}}{\partial x} \quad (146)$$

We express $\frac{\partial a_{z_{b_i}}}{\partial x}$, $\frac{\partial a_\xi}{\partial x}$, and $\frac{\partial c_{mac}}{\partial x}$ as

$$\frac{\partial a_{z_{b_i}}}{\partial x} = \frac{\partial a_{0,z_{b_i}}}{\partial x} + \frac{\partial a_{s,z_{b_i}}}{\partial x} \sin \Psi_i + \frac{\partial a_{c,z_{b_i}}}{\partial x} \cos \Psi_i \quad (147)$$

$$\frac{\partial a_\xi}{\partial x} = \frac{\partial a_{0,\xi}}{\partial x} + \frac{\partial a_{s,\xi}}{\partial x} \sin \Psi_i + \frac{\partial a_{c,\xi}}{\partial x} \cos \Psi_i \quad (148)$$

$$\frac{\partial c_{mac}}{\partial x} = \frac{\partial c_{0,mac}}{\partial x} + \frac{\partial c_{s,mac}}{\partial x} \sin \Psi_i + \frac{\partial c_{c,mac}}{\partial x} \cos \Psi_i \quad (149)$$

The partial derivative of the integrated non-circulatory thrust force and rotor moments are computed as

$$\frac{\partial T_{nc,r_i}}{\partial x} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{df_{z_{nc,b_i}}}{d\Psi_i} \right) d\Psi_i dr' = \frac{\pi}{4} \rho_\infty \int_{R_i}^R c_b^2 \cos \bar{\alpha}_{b_i} \frac{\partial a_{z_{0,b_i}}}{\partial x} dr' \quad (150)$$

$$\begin{aligned} \frac{\partial l_{nc,r_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[\mp (r' \sin \Psi_i - e_{mb} \cos \Psi_i) \frac{df_{z_{nc,b_i}}}{d\Psi_i} \right. \\ &\quad \left. - \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \cos \Psi_i \right] d\Psi_i dr' \\ &= \mp \frac{\pi}{8} \rho_\infty \int_{R_i}^R \left(r' \frac{\partial a_{s,z_{b_i}}}{\partial x} - e_{mb} \frac{\partial a_{c,z_{b_i}}}{\partial x} \right) c_b^2 \cos \bar{\alpha}_{b_i} dr' \\ &\quad - \frac{1}{2} \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{c_b}{\bar{c}_l} \frac{\partial c_{c,mac}}{\partial x} - \frac{\pi}{128} \rho_\infty c_b^4 \frac{\partial a_{c,\xi}}{\partial x} \right) dr' \end{aligned} \quad (151)$$

$$\begin{aligned} m_{nc,r_i} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[- (r' \cos \Psi_i + e_{mb} \sin \Psi_i) \frac{df_{x_{nc,b_i}}}{d\Psi_i} \right. \\ &\quad \left. \pm \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \sin \Psi_i \right] d\Psi_i dr' \\ &= - \frac{\pi}{8} \rho_\infty \int_{R_i}^R \left(r' \frac{\partial a_{c,z_{b_i}}}{\partial x} + e_{mb} \frac{\partial a_{s,z_{b_i}}}{\partial x} \right) c_b^2 \cos \bar{\alpha}_{b_i} dr' \\ &\quad \pm \frac{1}{2} \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{c_b}{\bar{c}_l} \frac{\partial c_{s,mac}}{\partial x} - \frac{\pi}{128} \rho_\infty c_b^4 \frac{\partial a_{s,\xi}}{\partial x} \right) dr' \end{aligned} \quad (152)$$

$$Q_{nc,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \pm r' \frac{df_{x_{nc,b_i}}}{d\Psi_i} d\Psi_i dr' = \mp \frac{\pi}{4} \rho_\infty \int_{R_i}^R r' \frac{\partial a_{0,z_{b_i}}}{\partial x} c_b^2 \sin \bar{\alpha}_{b_i} dr' \quad (153)$$

The partial derivatives of the aircraft integrated non-circulatory forces are evaluated as

$$\frac{\partial X_{nc,a_i}}{\partial x} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(-\frac{df_{x_{nc,b_i}}}{d\Psi_i} \right) \sin \Psi_i d\Psi_i dr' = \frac{\pi}{8} \rho_\infty \int_{R_i}^R c_b^2 \frac{\partial a_{s,z_{b_i}}}{\partial x} \sin \bar{\alpha}_{b_i} dr' \quad (154)$$

$$\begin{aligned} \frac{\partial Y_{nc,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\mp \frac{df_{x_{nc,b_i}}}{d\Psi_i} \cos \eta_i \cos \Psi_i - \frac{df_{z_{nc,b_i}}}{d\Psi_i} \sin \eta_i \right) d\Psi_i dr' \\ &= \frac{\pi}{8} \rho_\infty \int_{R_i}^R c_b^2 \left[\pm \frac{\partial a_{c,z_{b_i}}}{\partial x} \sin \bar{\alpha}_{b_i} \cos \bar{\eta}_i - 2 \frac{\partial a_{0,z_{b_i}}}{\partial x} \cos \bar{\alpha}_{b_i} \sin \bar{\eta}_i \right] dr' \end{aligned} \quad (155)$$

$$\begin{aligned} \frac{\partial Z_{nc,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\pm \frac{df_{x_{nc,b_i}}}{d\Psi_i} \sin \eta_i \cos \Psi_i - \frac{df_{z_{nc,b_i}}}{d\Psi_i} \cos \eta_i \right) d\Psi_i dr' \\ &= \frac{\pi}{8} \rho_\infty \int_{R_i}^R c_b^2 \left[\mp \frac{\partial a_{c,z_{b_i}}}{\partial x} \sin \bar{\alpha}_{b_i} \sin \bar{\eta}_i - 2 \frac{\partial a_{0,z_{b_i}}}{\partial x} \cos \bar{\alpha}_{b_i} \cos \bar{\eta}_i \right] dr' \end{aligned} \quad (156)$$

The partial derivatives of the aircraft integrated non-circulatory moments are evaluated as

$$\frac{\partial l_{nc,a_i}}{\partial x} = \frac{\partial l_{nc,r_i}}{\partial x} + \frac{\partial Y_{nc,a_i}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{nc,a_i}}{\partial x} \Delta y_{r_{i0}} \quad (157)$$

$$\frac{\partial m_{nc,a_i}}{\partial x} = \frac{\partial m_{nc,r_i}}{\partial x} \cos \bar{\eta}_i + \frac{\partial Q_{nc,r_i}}{\partial x} \sin \bar{\eta}_i - \frac{\partial X_{nc,r_i}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{nc,r_i}}{\partial x} \Delta x_{r_{i0}} \quad (158)$$

$$\frac{\partial n_{nc,a_i}}{\partial x} = -\frac{\partial m_{nc,r_i}}{\partial x} \sin \bar{\eta}_i + \frac{\partial Q_{nc,r_i}}{\partial x} \cos \bar{\eta}_i - \frac{\partial X_{nc,r_i}}{\partial x} \Delta y_{r_{i0}} - \frac{\partial Y_{nc,r_i}}{\partial x} \Delta x_{r_{i0}} \quad (159)$$

We sum all the aircraft integrated non-circulatory force and moment contributions by the rotors and express them in terms of the partial derivatives as

$$\mathbf{F}_{nc,a} = \sum_{i=1}^{n_{rotor}} \mathbf{F}_{nc,a_i} = \frac{\partial \mathbf{F}_{nc,a}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{nc,a}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{F}_{nc,a}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \frac{\partial \mathbf{F}_{nc,a}}{\partial \dot{\boldsymbol{\theta}}} \dot{\boldsymbol{\theta}} \quad (160)$$

$$\mathbf{M}_{nc,a} = \sum_{i=1}^{n_{rotor}} \mathbf{M}_{nc,a_i} = \frac{\partial \mathbf{M}_{nc,a}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{nc,a}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{M}_{nc,a}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \frac{\partial \mathbf{M}_{nc,a}}{\partial \dot{\boldsymbol{\theta}}} \dot{\boldsymbol{\theta}} \quad (161)$$

In addition, the rotor non-circulatory thrust and moments acting at the origin of the i -th rotor reference frame are expressed in terms of the partial derivatives as

$$T_{nc,r_i} = \frac{\partial T_{nc,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial T_{nc,r_i}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial T_{nc,r_i}}{\partial \dot{\mathbf{v}}_{p_i}} \dot{\mathbf{v}}_{p_i} + \frac{\partial T_{nc,r_i}}{\partial \dot{\boldsymbol{\theta}}_i} \dot{\boldsymbol{\theta}}_i \quad (162)$$

$$\mathbf{M}_{nc,r_i} = \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\mathbf{v}}_{p_i}} \dot{\mathbf{v}}_{p_i} + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\boldsymbol{\theta}}_i} \dot{\boldsymbol{\theta}}_i \quad (163)$$

Note that the aircraft non-circulatory force and moment contributions should be included for both hover and transition flight. These contributions show up as the apparent mass derivatives such as $C_{m_{\dot{\alpha}}}$ for fixed-wing flight dynamics.

III. Integrated Flight Dynamic Model of eVTOL Aircraft

An integrated flight dynamic model of an eVTOL aircraft includes

- An unsteady aerodynamic model which models the phase delay and amplitude attenuation by a state-space realization of the frequency-domain Theodorsen's function,
- A flight dynamic model of the rigid-body aircraft,
- A rotor dynamic inflow model, and
- A closed-loop servo-actuator dynamic model of the control effectors which includes a closed-loop PID controller that tracks an actuator command.

Each of these elements are described as follows.

A. Unsteady Aerodynamic Correction

The unsteady aerodynamic effect of a rotor in motion is much more complex than that of a fixed wing. The dynamic inflow models are generally used in flight dynamic modeling of rotors to capture the unsteady aerodynamics of the induced flow. Since the flight dynamic motion of a rigid-body rotorcraft or aircraft involves very low frequencies, the dynamic inflow models are sufficient. For motions that involve high frequency dynamics such as structural modes when those modes are coupled to the rigid-body modes, the unsteady aerodynamic behaviors created by the high frequency motion which involved pitching and plunging of the entire rotor are less well-understood. Theodorsen's unsteady aerodynamic theory for an oscillating airfoil in free air is well-known,⁸ but there exist gaps in the theoretical knowledge of unsteady aerodynamics of a rotating airfoil in a pitching and plunging motion. The unsteady aerodynamic response of a rotor can be estimated by conducting forced oscillation numerical experiments using CFD^{6, 11, 12} but such studies are computationally very expensive. In this study, we propose to use the Theodorsen's theory to capture some semblance of unsteady flow on an airfoil in a rotating period motion subject to high-frequency dynamic motion in the vertical plane. While the present study pertains to flight dynamic model development, the use of the Theodorsen's theory creates a general framework for future consideration of high frequency dynamic motion such as coupled flight dynamic motion with aeroelasticity for eVTOL aircraft. The simplicity of the Theodorsen's theory facilitates the analytical flight dynamic model development readily. Other methods such as the Leishman's method¹³ are also considered.

The circulatory lift according to Theodorsen⁸ is given by

$$\frac{dl_{c,b_i}}{d\Psi} = [\bar{c}_l + C(k) c_{l_\alpha} \alpha] \frac{1}{2} \rho V^2 c_b \quad (164)$$

where $C(k)$ is the Theodorsen's function which adjusts the lift expression to correct for unsteady aerodynamics. The Theodorsen's function $C(k)$ is a complex-valued transcendental function given by

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} = F(k) + iG(k) \quad (165)$$

where $H_0^{(2)}(k)$ and $H_1^{(2)}(k)$ are the Hankel functions.

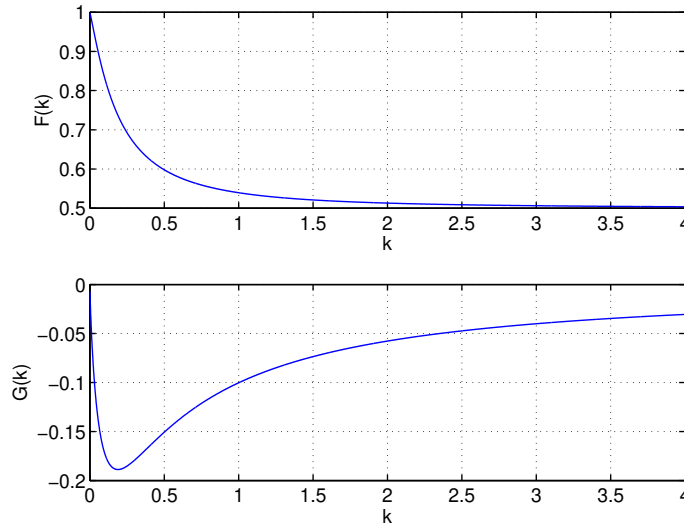


Figure 3. Theodorsen's Function

The functions $F(k)$ and $G(k)$ are plotted in Fig. 3. It is noted that $F(0) = 1$ and $G(k) = 0$ for steady-state aerodynamics when $\omega = 0$. As $k \rightarrow \infty$, $F(k) \rightarrow \frac{1}{2}$ and $G(k) \rightarrow 0$. So the magnitude of the unsteady circulatory lift coefficient is always less than the magnitude of the steady-state lift coefficient.

In the time domain, the unsteady lift may be approximated by a rational fraction approximation in the form

$$\tilde{c}_l = c_{l_\alpha} C(k) \alpha = c_{l_\alpha} \left(r_0 + \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} \right) \alpha \quad (166)$$

where $r_0 = 0.5$, $r_1 = 0.08851$, $r_2 = 0.003620$, $p_1 = -0.2278$, and $p_2 = -0.03249$.

Let

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{2\bar{V}_t}{\bar{c}_b} \left(\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \alpha \right) \quad (167)$$

Then, the unsteady lift coefficient is obtained as

$$\tilde{c}_l = c_{l_\alpha} (r_0 \alpha + r_1 x_1 + r_2 x_2) \quad (168)$$

We now generalize the unsteady aerodynamic analysis for the aerodynamic force acting on the aircraft. We write the aircraft circulatory force and moment expressions in the time domain with the unsteady aerodynamic correction as

$$\begin{aligned} C(k) \mathbf{F}_{c,a_i} &= \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_p \mathbf{y}_p) \\ &\quad + \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_\theta \mathbf{y}_\theta) + \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{\Omega}} (r_0 \mathbf{\Omega} + \mathbf{r}_\Omega \mathbf{y}_\Omega) \end{aligned} \quad (169)$$

$$\begin{aligned} C(k) \mathbf{M}_{c,a_i} &= \frac{\partial \mathbf{M}_{c,a_i}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \frac{\partial \mathbf{M}_{c,a_i}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_p \mathbf{y}_p) \\ &\quad + \frac{\partial \mathbf{M}_{c,a_i}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_\theta \mathbf{y}_\theta) + \frac{\partial \mathbf{M}_{c,a_i}}{\partial \mathbf{\Omega}} (r_0 \mathbf{\Omega} + \mathbf{r}_\Omega \mathbf{y}_\Omega) \end{aligned} \quad (170)$$

where $\mathbf{r}_{(a,p,q,\theta,\Omega)} = \begin{bmatrix} r_1 \mathbf{I} & r_2 \mathbf{I} \end{bmatrix}$ with $\mathbf{I}$ as an identity matrix of appropriate dimension, and $\mathbf{y}_{(a,p,q,\theta,\Omega)}$ are the aerodynamic lag state vectors described by

$$\dot{\mathbf{y}}_a = \mathbf{A}_a \mathbf{y}_a + \mathbf{B}_a \mathbf{x}_a \quad (171)$$

$$\dot{\mathbf{y}}_p = \mathbf{A}_p \mathbf{y}_p + \mathbf{B}_p \mathbf{v}_p \quad (172)$$

$$\dot{\mathbf{y}}_\theta = \mathbf{A}_\theta \mathbf{y}_\theta + \mathbf{B}_\theta \mathbf{u}_\theta \quad (173)$$

$$\dot{\mathbf{y}}_\Omega = \mathbf{A}_\Omega \mathbf{y}_\Omega + \mathbf{B}_\Omega \mathbf{\Omega} \quad (174)$$

with $\mathbf{A}_{(a,p,q,\theta,\Omega)} = \frac{2\bar{V}_t}{\bar{c}_b} \text{diag} \left(\begin{bmatrix} p_1 \mathbf{I} & p_2 \mathbf{I} \end{bmatrix} \right)$ and $\mathbf{B}_{(a,p,q,\theta,\Omega)} = \frac{2\bar{V}_t}{\bar{c}_b} \begin{bmatrix} \mathbf{I} & \mathbf{I} \end{bmatrix}^\top$.

Note that for rigid-body flight dynamic motion at very low frequencies, the quasi-steady aerodynamic assumption is generally valid. Therefore, $C(k) \approx 1$. This results in a simplification with $r_0 = 1$ and $r_1 = r_2 = 0$ which effectively removes the aerodynamic lag states $\mathbf{y}_{(a,p,q,\theta,\Omega)}$ from the model.

B. Flight Dynamic Model

The nonlinear flight dynamic equation of a rigid-body aircraft is described by

$$m_a \mathbf{I}_3 \dot{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{v} = C(k) \mathbf{F}_{c,a} + \mathbf{F}_{nc,a} + \mathbf{F}_b \quad (175)$$

$$\mathbf{I}_a \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_a \boldsymbol{\omega} = C(k) \mathbf{M}_{c,a} + \mathbf{M}_{nc,a} + \mathbf{M}_b \quad (176)$$

The body and propulsive force and moment vectors $\mathbf{F}_b$ and $\mathbf{M}_b$ are given by

$$\mathbf{F}_b = \begin{bmatrix} T - m_a g \sin \theta \\ m_a g \cos \theta \sin \phi \\ m_a g \cos \theta \cos \phi \end{bmatrix} \quad (177)$$

$$\mathbf{M}_b = \begin{bmatrix} l_T \\ m_T \\ n_T \end{bmatrix} \quad (178)$$

where T denotes forward flight thrust force, l_T , m_T , and n_T are thrust-induced moments, $\theta(t)$ is the aircraft pitch angle, and $\phi(t)$ is the aircraft bank angle.

We are interested in the linearized equations of motion about the equilibrium condition given by $\bar{\sigma} = \mathbf{0}$ and $\omega = \mathbf{0}$. Upon linearization, we obtain

$$m_a \mathbf{I}_3 \dot{\mathbf{v}} = \frac{\partial \mathbf{F}}{\partial \omega} \omega + C(k) \mathbf{F}_{c,a} + \mathbf{F}_{nc,a} + \frac{\partial \mathbf{F}_b}{\partial \sigma} \sigma \quad (179)$$

$$\bar{\mathbf{I}}_a \dot{\omega} = C(k) \mathbf{M}_{c,a} + \mathbf{M}_{nc,a} \quad (180)$$

$$\dot{\sigma} = \omega = \bar{\mathbf{H}} \mathbf{x}_a \quad (181)$$

where $\sigma(t) = \begin{bmatrix} \phi(t) & \theta(t) & \psi(t) \end{bmatrix}^\top$, $\psi(t)$ is the heading angle, $\bar{\mathbf{I}}_a = \text{diag}(\mathbf{I}_a)$, $\frac{\partial \mathbf{F}}{\partial \omega} = \begin{bmatrix} 0 & -\bar{w} & \bar{v} \\ \bar{w} & 0 & -\bar{u} \\ -\bar{v} & \bar{u} & 0 \end{bmatrix}$,

$$\frac{\partial \mathbf{F}_b}{\partial \sigma} = \begin{bmatrix} 0 & -m_a g & 0 \\ m_a g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \bar{\mathbf{H}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Let $\mathbf{S}_a = \begin{bmatrix} \mathbf{F}_a^\top & \mathbf{M}_a^\top \end{bmatrix}^\top$ and $\mathbf{M}_v = \text{diag} \left(\begin{bmatrix} m_a \mathbf{I}_3 & \bar{\mathbf{I}}_a \end{bmatrix} \right)$. The linearized equation of motion is expressed with the unsteady aerodynamic correction as

$$\begin{aligned} \left(\mathbf{M}_v - \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_a} \right) \dot{\mathbf{x}}_a - \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p - \frac{\partial \mathbf{S}_{na}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta &= \frac{\partial \mathbf{S}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) \\ &+ \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \frac{\partial \mathbf{S}_{c,a}}{\partial \Omega} (r_0 \Omega + \mathbf{r}_a \mathbf{y}_\Omega) + \frac{\partial \mathbf{S}_{nc,a}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{S}_b}{\partial \sigma} \sigma \end{aligned} \quad (182)$$

where $\frac{\partial \mathbf{S}}{\partial \mathbf{x}_a} = \begin{bmatrix} \mathbf{0} & \frac{\partial \mathbf{F}}{\partial \omega} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\frac{\partial \mathbf{S}_b}{\partial \sigma} = \begin{bmatrix} \frac{\partial \mathbf{F}_b}{\partial \sigma} \\ \mathbf{0} \end{bmatrix}$.

C. Dynamic Inflow Model

The linearized Pitt-Peters three-state dynamic inflow model is given by

$$\mathbf{M}_p \dot{\mathbf{v}}_p = -\mathbf{L}_p^{-1} \mathbf{v}_p + C(k) \mathbf{C}_{c,p} + \mathbf{C}_{nc,p} \quad (183)$$

Applying the unsteady aerodynamic correction by the Theodorsen's theory yields

$$\begin{aligned} -\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \left(\mathbf{M}_p - \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} \right) \dot{\mathbf{v}}_p - \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta &= -\mathbf{L}_p^{-1} \mathbf{v}_p + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) \\ &+ \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \frac{\partial \mathbf{C}_{c,p}}{\partial \Omega} (r_0 \Omega + \mathbf{r}_a \mathbf{y}_\Omega) + \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} \mathbf{x}_a \end{aligned} \quad (184)$$

D. Actuator Dynamic Models

The control effectors include the aerodynamic control surfaces $\delta_e(t)$, $\delta_a(t)$, and $\delta_r(t)$, the blade pitch angle θ_{b_i} , and the rotor speed Ω_i . For hover, the blade pitch and rotor speed are used as control effectors.

1. Blade Pitch Actuator Dynamic Model

The blade pitch angle is driven by a DC drive motor which is modeled as

$$\mathbf{J}_b \ddot{\theta}_b + \mathbf{G}_b \dot{\theta}_b = \mathbf{m}_b + \mathbf{T}_b \quad (185)$$

where $\mathbf{J}_b$ is the motor inertia matrix, $\mathbf{G}_b$ is a matrix representing the motor friction windage loss, $\mathbf{m}_b$ is the blade pitching moment vector about the pitch axis, and $\mathbf{T}_b$ is the motor torque vector.

The blade pitch actuator dynamic model is also expressed as

$$\mathbf{M}_b \dot{\mathbf{u}}_\theta = \mathbf{S}_b \mathbf{u}_\theta + \mathbf{B}_b C(k) \mathbf{m}_{c,b} + \mathbf{B}_b \mathbf{m}_{nc,b} + \mathbf{B}_b \mathbf{T}_b \quad (186)$$

where $\mathbf{M}_b = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_b \end{bmatrix}$, $\mathbf{S}_b = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & -\mathbf{G}_b \end{bmatrix}$, and $\mathbf{B}_b = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$.

The circulatory and non-circulatory blade pitching moments are computed as

$$m_{c,b_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} e_b \frac{df_{z_{c,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (187)$$

$$m_{nc,b_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left(\frac{dm_{ac,b_i}}{d\Psi_i} - e_{mb} \frac{dl_{nc,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) d\Psi_i dr' \quad (188)$$

The partial derivatives of the circulatory and non-circulatory pitching moments are obtained as

$$\frac{\partial m_{c,b_i}}{\partial x} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{dm_{c,b_i}}{d\Psi_i} \right) d\Psi_i dr' = N_b \int_{R_i}^R \bar{l}_{c,b_i} e_b \frac{\partial f_{0,b_i}}{\partial x} dr' \quad (189)$$

$$\frac{\partial m_{nc,b_i}}{\partial x} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{dm_{ac,b_i}}{d\Psi_i} - e_{mb} \frac{dl_{nc,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) d\Psi_i dr' \quad (190)$$

$$= N_b \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{c_b}{\bar{c}_l} \frac{\partial c_{0,m_{ac}}}{\partial x} dr' - \frac{\pi}{4} \rho_\infty c_b^2 e_{mb} \frac{\partial a_{0,z_{b_i}}}{\partial x} - \frac{\pi}{128} \rho_\infty c_b^4 \frac{\partial a_{0,\xi}}{\partial x} \right) dr' \quad (191)$$

The blade pitch angle is to be controlled by a PI controller. The control motor torque is given by

$$\mathbf{T}_b = \mathbf{u}_b - \mathbf{K}_{b,p} \mathbf{u}_\theta - \mathbf{K}_{b,i} \mathbf{e}_b \quad (192)$$

where $\mathbf{e}_b(t)$ is the blade pitch angle integral error and the controller gain matrices $\mathbf{K}_{b,p}$ and $\mathbf{K}_{b,i}$ are computed by the linear quadratic regulator (LQR) optimal control.

The error dynamics are given by

$$\dot{\mathbf{e}}_b = \mathbf{C}_b \mathbf{u}_\theta - \boldsymbol{\theta}_{b,c} \quad (193)$$

where $\mathbf{C}_b = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$.

The equation of blade pitch dynamics is expressed with the unsteady aerodynamic correction as

$$\begin{aligned} -\mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a - \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \left(\mathbf{M}_b - \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{u}}_\theta} \right) \dot{\mathbf{u}}_\theta &= \mathbf{S}_b \mathbf{u}_\theta + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) \\ &+ \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \boldsymbol{\Omega}} (r_0 \boldsymbol{\Omega} + \mathbf{r}_a \mathbf{y}_\Omega) \\ &+ \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \mathbf{x}_a} \mathbf{x}_a + \mathbf{B}_b \mathbf{u}_b - \mathbf{B}_b \mathbf{K}_{b,p} \mathbf{u}_\theta - \mathbf{B}_b \mathbf{K}_{b,i} \mathbf{e}_b \end{aligned} \quad (194)$$

2. Rotor Actuator Dynamic Model

The rotor speed is driven by a DC drive motor which is modeled as

$$\mathbf{J}_r \dot{\boldsymbol{\Omega}} + \mathbf{B}_r \boldsymbol{\Omega} = \mathbf{m}_r + \mathbf{T}_r \quad (195)$$

where $\mathbf{J}_r$ is the motor inertia matrix, $\mathbf{B}_r$ is the motor friction matrix, $\mathbf{m}_r = \begin{bmatrix} \pm Q_1 & \pm Q_2 & \cdots & \pm Q_{n_{rotor}} \end{bmatrix}$ is the rotor torque vector, and $\mathbf{T}_r$ is the motor torque vector.

The rotor speed is to be controlled by a PI controller. The control motor torque is given by

$$\mathbf{T}_r = \mathbf{u}_r - \mathbf{K}_{r,p} \boldsymbol{\Omega} - \mathbf{K}_{r,i} \mathbf{e}_r \quad (196)$$

where $\mathbf{u}_r(t)$ is the motor torque control, $\mathbf{e}_r(t)$ is the rotor speed integral error and the controller gain matrices $\mathbf{K}_{r,p}$ and $\mathbf{K}_{r,i}$ are computed by the LQR optimal control.

The error dynamics are given by

$$\dot{\mathbf{e}}_r = \boldsymbol{\Omega} - \boldsymbol{\Omega}_c \quad (197)$$

The equation of the rotor dynamics is expressed with the unsteady aerodynamic correction as

$$\begin{aligned} -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a - \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p - \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta + \mathbf{J}_r \dot{\boldsymbol{\Omega}} &= -\mathbf{B}_r \boldsymbol{\Omega} + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) \\ &+ \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \frac{\partial \mathbf{m}_{c,r}}{\partial \boldsymbol{\Omega}} (r_0 \boldsymbol{\Omega} + \mathbf{r}_a \mathbf{y}_\Omega) + \frac{\partial \mathbf{m}_{nc,r}}{\partial \mathbf{x}_a} \mathbf{x}_a + \mathbf{u}_r - \mathbf{K}_{r,p} \boldsymbol{\Omega} - \mathbf{K}_{r,i} \mathbf{e}_r \end{aligned} \quad (198)$$

E. State Space Model

We form the integrated flight dynamic model by coupling together the flight dynamic model, dynamic inflow model, actuator dynamic model, and unsteady aerodynamic model. The integrated flight dynamic model is expressed compactly in a physical state-space form as

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{S}\mathbf{x} + \mathbf{T}\mathbf{u} \quad (199)$$

In the physical state-space form, $\mathbf{M}$ is the mass matrix, $\mathbf{S}$ is the force-moment stability derivative matrix, and $\mathbf{T}$ is the force-moment control derivative matrix.

By inverting the mass matrix $\mathbf{M}$, we obtain the standard state-space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \mathbf{v} \quad (200)$$

The state-space equation can also be expressed in a partitioned form as

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_r \\ \dot{\mathbf{x}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{rr} & \mathbf{S}_{rc} \\ \mathbf{S}_{cr} & \mathbf{S}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_c \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{rb} & \mathbf{T}_{rr} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_r \end{bmatrix} + \begin{bmatrix} \mathbf{v}_c \\ \mathbf{0} \end{bmatrix} \quad (201)$$

where $\mathbf{x}_r(t) = \left[\boldsymbol{\sigma}^\top(t) \quad \mathbf{x}_a^\top(t) \quad \mathbf{v}_p^\top(t) \quad \mathbf{u}_\theta^\top(t) \quad \boldsymbol{\Omega}^\top(t) \quad \mathbf{e}_b^\top(t) \quad \mathbf{e}_r^\top(t) \right]^\top$, $\mathbf{x}_c(t) = \left[\mathbf{y}_a^\top(t) \quad \mathbf{y}_p^\top(t) \quad \mathbf{y}_q^\top(t) \quad \mathbf{y}_\theta^\top(t) \quad \mathbf{y}_\Omega^\top(t) \right]^\top$, and $\mathbf{v}_c(t) = \left[\mathbf{0}^\top \quad \mathbf{0}^\top \quad \mathbf{0}^\top \quad \mathbf{0}^\top \quad \mathbf{0}^\top \quad \tilde{\boldsymbol{\theta}}_{b,c}^\top(t) \quad \boldsymbol{\Omega}_c^\top(t) \right]^\top$ is the blade pitch and rotor speed command. For aircraft state tracking control, we set $\mathbf{v}_c = \mathbf{0}$ to maintain trim blade pitch and rotor speed. We then obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (202)$$

The mass matrices are given by

$$\mathbf{M}_{rr} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v - \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_a} & -\frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{v}}_p} & -\frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_a} & \mathbf{M}_p - \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} & -\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_a} & -\mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{v}}_p} & \mathbf{M}_b - \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_a} & -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{v}}_p} & -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{J}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (203)$$

The stability derivative matrices are given by

$$\mathbf{S}_{rr} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{H}} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{F}_b}{\partial \boldsymbol{\sigma}} & \frac{\partial \mathbf{S}}{\partial \mathbf{x}_a} + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{S}_{nc,a}}{\partial \mathbf{x}_a} & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{v}_p} r_0 & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{u}_\theta} r_0 \\ \mathbf{0} & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} r_0 - \mathbf{L}_p^{-1} & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} r_0 \\ \mathbf{0} & \mathbf{B}_b \left(\frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{m}_{nc,b}}{\partial \mathbf{x}_a} \right) & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{v}_p} r_0 & \mathbf{B}_b \left(\frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{u}_\theta} r_0 + \mathbf{S}_b - \mathbf{K}_{b,p} \right) \\ \mathbf{0} & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{m}_{nc,r}}{\partial \mathbf{x}_a} & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{v}_p} r_0 & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{u}_\theta} r_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_b \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{S}_{c,a}}{\partial \boldsymbol{\Omega}} r_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{C}_{c,p}}{\partial \boldsymbol{\Omega}} r_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \boldsymbol{\Omega}} r_0 & -\mathbf{B}_b \mathbf{K}_{b,i} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{m}_{c,r}}{\partial \boldsymbol{\Omega}} r_0 - \mathbf{B}_r - \mathbf{K}_{r,p} & \mathbf{0} & -\mathbf{K}_{r,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (204)$$

$$\mathbf{S}_{rc} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_a} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{v}_p} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_q} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \Omega} \mathbf{r}_a \\ \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_q} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \Omega} \mathbf{r}_a \\ \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_a} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{v}_p} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_q} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \Omega} \mathbf{r}_a \\ \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_a} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{v}_p} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_q} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \Omega} \mathbf{r}_a \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (205)$$

$$\mathbf{S}_{cr} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_\theta & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_\Omega & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (206)$$

$$\mathbf{S}_{cc} = \begin{bmatrix} \mathbf{A}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_\theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_\Omega \end{bmatrix} \quad (207)$$

The control derivative matrices are given by

$$\mathbf{T}_{rb} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{B}_b \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (208)$$

$$\mathbf{T}_{rr} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (209)$$

IV. Linearized Flight Dynamic Model of Lift-Plus-Cruise eVTOL Aircraft in Hover

The Lift-Plus-Cruise (LPC) eVTOL aircraft is a six-passenger aircraft⁴ as shown in Fig. 1. It employs 8 vertical lift rotors for eVTOL and an aft push fan. In forward flight, the LPC vehicle operates like a conventional fixed-wing aircraft with the 8 vertical lift rotors stopped and locked in place. Two versions of the Lift+Cruise vehicle are conceptualized: a turboprop version and an all-electric version.⁴ The all-electric Lift+Cruise vehicle has a design gross weight of 8,395 lb and cruises at an average 100 knots. This model is used in the current study.

An aerodynamic trim analysis for hover flight at 4000 ft altitude is performed in CHARM.³ The trim solution is used to compute the analytical derivatives of the forces and moments created by the rotors. In addition, the non-circulatory forces and moments generated by the aircraft lifting surfaces are also computed.

Figure 4 shows the rotor thrust radial distributions of the eight rotors. The front rotors 1, 3, 5, 7 are trimmed at higher thrust values than the aft rotors 2, 4, 6, 8.

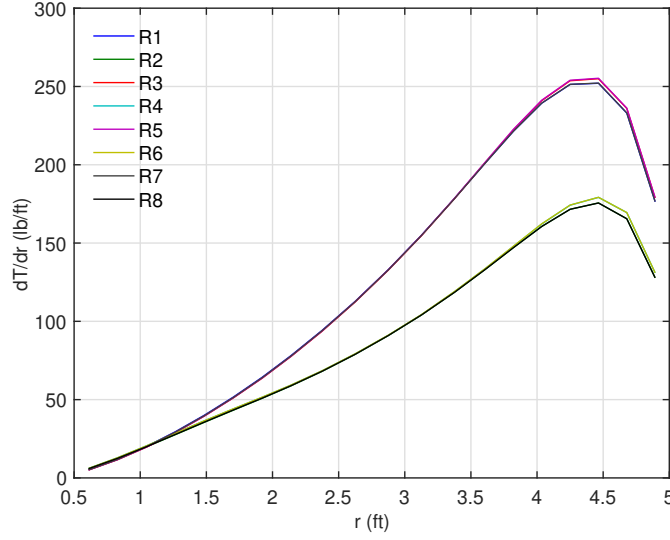


Figure 4. Rotor Thrust Distributions

A MATLAB script is created to read in the CHARM trim solution and automatically compute the analytical stability and control derivatives of the LPC aircraft developed in this study. The linearized flight dynamic model of the LPC aircraft in hover contains 236 states including 8 aircraft states excluding the heading angle, 24 dynamic inflow states, 16 blade pitch actuator states, 8 rotor speed actuator dynamic states, 8 blade pitch integral error states, 8 rotor speed integral error states, and 164 unsteady aerodynamic lag states. There are 16 control inputs: 8 rotor speed commands and 8 blade pitch angle commands.

Static stability of the Lift+Cruise aircraft is examined. Key static and dynamic stability derivatives are computed and are shown in Table 1. The force and moment stability derivatives are normalized with respect to $\rho_{\infty}\pi R^4\Omega^2$ and $\rho_{\infty}\pi R^5\Omega^2$, respectively. Longitudinal stability requires stability derivative $C_{m_w} < 0$, lateral stability requires $C_{l_v} < 0$, and directional stability requires $C_{n_v} > 0$. The Lift+Cruise aircraft is statically stable in pitch but unstable in roll and yaw. The dynamic derivatives C_{m_q} , C_{l_p} , and C_{n_r} are basic rate damping derivatives which are required to be negative. The Lift+Cruise aircraft satisfies this basic rate damping requirement.

C_{m_w}	$-1.9859 \cdot 10^{-4}$	C_{m_q}	$-1.9938 \cdot 10^{-4}$
C_{l_v}	$4.6984 \cdot 10^{-6}$	C_{l_p}	-0.01818
C_{n_v}	$-2.9123 \cdot 10^{-7}$	C_{n_r}	-0.004848

Table 1. Stability Derivatives of All-Electric Lift+Cruise Aircraft

The eigenvalues of the aircraft 6-degree-of-freedom (6 DoF) rigid-body modes in hover are shown in Table 2. The vehicle has two unstable modes: phugoid mode and spiral mode. The Lift+Cruise aircraft is dynamically unstable. The differences in the eigenvalues between the integrated model and the 6-DoF model alone are noticeable for some of the modes such as the Dutch-roll mode and the spiral mode. The short-period mode is overdamped.

Rigid-Body Mode	Integrated Model	6-DoF + Inflow Model	6 DoF Model
Phugoid	$0.09327 \pm i0.2142$	$0.09368 \pm i0.2139$	$0.08926 \pm i0.2126$
Short-Period	$-0.06723, -0.2810$	$-0.06755, -0.2860$	$-0.07994, -0.2932$
Roll	-0.04541	-0.04523	-0.04567
Spiral	0.08947	0.08861	0.1155
Dutch-Roll	$-0.1776 \pm i0.05671$	$-0.1749 \pm i0.06245$	$-0.2106 \pm i0.09380$

Table 2. Eigenvalues of Rigid-Body Modes of All-Electric Lift+Cruise Aircraft

Figure 5 is the root locus plot of the eigenvalues of the Lift+Cruise aircraft. The unsteady aerodynamic lag states have the largest negative values which imply that the unsteady aerodynamic effect is mostly decoupled from the aircraft response. That is, the aerodynamic response is quasi-steady. The dynamic inflow states have eigenvalues in the mid-range from -100 to -60 . There are two groups of eigenvalues in the range between -30 and -20 associated with unsteady aerodynamics and inflow dynamics. The blade pitch dynamics are in a frequency range of up to 4 rad/sec. The rotor dynamics have frequencies up to 1.2 rad/sec.

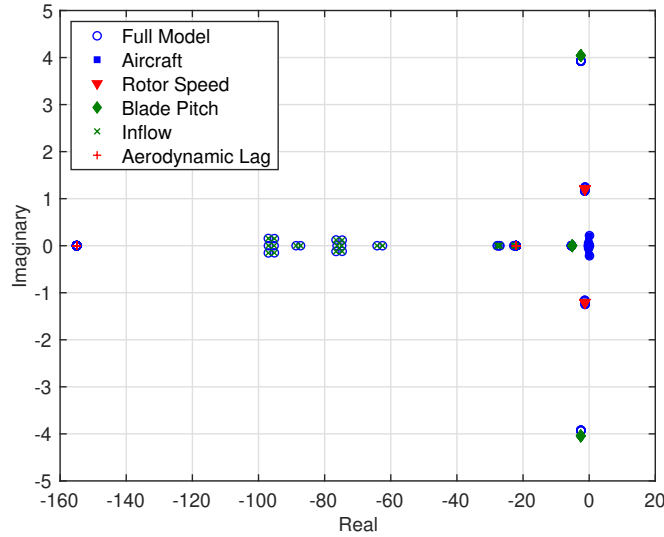


Figure 5. Eigenvalues of Lift+Cruise Aircraft

The Lift+Cruise aircraft requires feedback stabilization for hover operation. Since the unsteady aerodynamic lag states and dynamic inflow states have large negative values, their dynamics are essentially decoupled from the aircraft and actuator dynamics. A reduced-order control can be designed using the reduced-order state $\mathbf{x}_o(t) = \left[\boldsymbol{\sigma}^\top(t) \quad \mathbf{x}_a^\top(t) \quad \mathbf{u}_\theta^\top(t) \quad \boldsymbol{\Omega}^\top(t) \quad \mathbf{e}_b^\top(t) \quad \mathbf{e}_r^\top(t) \right]^\top$. It is assumed that these state variables are directly measured or observable for state reconstruction through a Kalman filter state estimation. The feedback gain matrix is computed by the linear quadratic regulator (LQR) optimal control. The reduced-order gain matrix is then used in the full-order model to evaluate the vehicle stability.

Figure 6 and 7 show the stable closed-loop response of the Lift+Cruise aircraft due to an initial pitch angle disturbance $\theta(0) = 10^\circ$. The aircraft response mostly settles to zero within 100 sec except for the yaw rate which tends to zero very slowly. This is due to the lack of a heading angle control implementation.

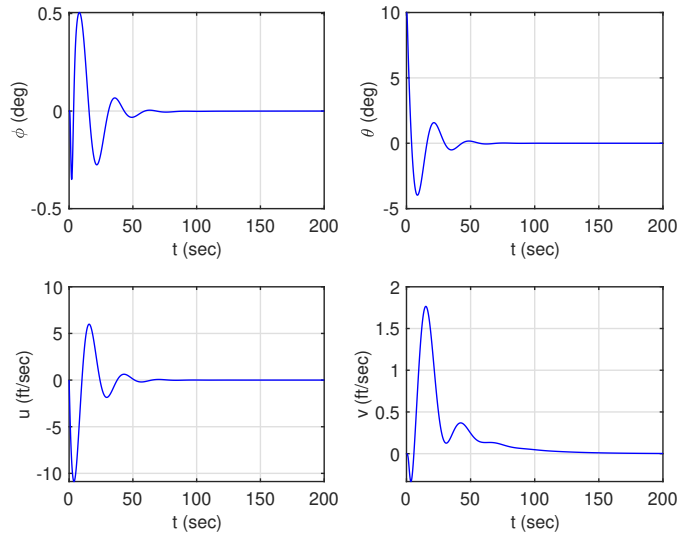


Figure 6. Closed-Loop Response of Lift+Cruise Aircraft

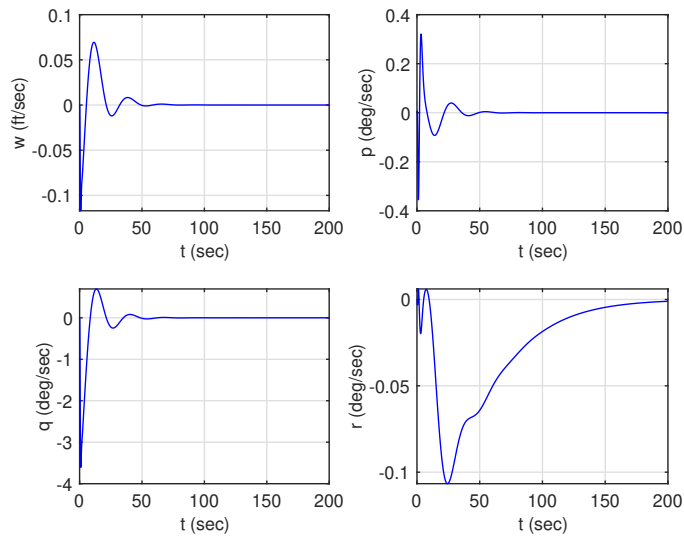


Figure 7. Closed-Loop Response of Lift+Cruise Aircraft

Figure 8 shows the blade pitch and rotor speed responses while being regulated by the PI controllers. The blade pitch changes up to 2.2° and the rotor speed changes up to 13 rpm.

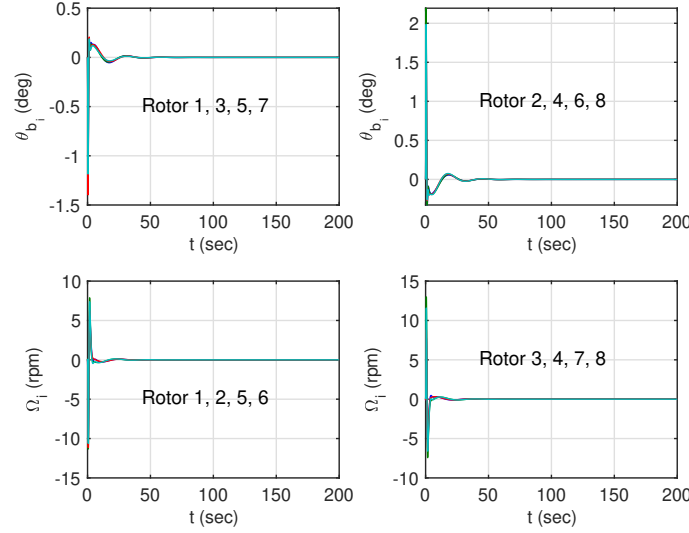


Figure 8. Blade Pitch and Rotor Speed Responses of Lift+Cruise Aircraft

V. Conclusion

This paper presents an analytical model development of flight dynamics of a Lift+Cruise eVTOL aircraft. The analytical model accounts the full vehicle motion in the rotor forces. The non-circulatory forces are accounted for in the model to capture unsteady aerodynamics. The circulatory forces are corrected for unsteady aerodynamics using the time-domain approximation of Theodorsen's function. The integrated flight dynamic model includes the rigid-body aircraft flight dynamics, inflow dynamics, unsteady aerodynamics, and blade pitch and rotor actuator dynamics. The flight dynamic model shows that the Lift+Cruise aircraft is unstable in hover. The unstable modes are the phugoid mode and spiral mode. Stabilization is accomplished by feedback control. A reduced-order LQR controller is designed to enable the vehicle to achieve closed-loop stability.

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