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GENERALIZED AUGMENTED-STATE COVARIANCE ANALYSIS FOR SPACEFLIGHT

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The use of linear covariance analysis techniques, also known as LinCov, has been used extensively for more than a half century for spaceflight applications. Originally, its primary purpose was to facilitate navigation analysis. For many past and current applications, the specific implementations only support navigation studies still. When the concept of an augmented-state linear covariance analysis approach was initially introduced that allowed for both navigation and trajectory dispersion analysis, the enhancement was motivated and primarily utilized to support navigation filter tuning and error budget analysis. Relatively few utilize this alternate augmented-state formulation of LinCov due to its additional complexity. The untapped potential of the augmented-state linear covariance analysis technique slowly unfolded in the past two-decades as its capability to rapidly and reliably capture the integrated closed-loop guidance, navigation, and control (GN&C) system performance became more apparent. Even with this dual purpose of generating insights to both navigation errors along with trajectory and delta-v dispersions, the core theoretical development had a heavy emphasis on the impacts of the navigation system and largely neglected the details of the actual guidance, targeting, and control systems. This paper extends the navigation-centric theoretical development by formulating a generalized augmented-state covariance analysis (GAUSCOV) technique that allows for the intricacies of a variety of targeting and control strategies along with ground planning and mission operations to be more formally included in assessing the impacts to spaceflight GN&C system performance.

INTRODUCTION

Over the past several decades, linear covariance analysis, commonly referred to as LinCov, has emerged as a viable technique to generate closed-loop integrated analysis for guidance, navigation, and control (GN&C) systems for spaceflight. Despite its origins as a navigation-only analysis tool, Maybeck[1] introduced an augmented state formulation that enhanced the basic LinCov theory to facilitate filter tuning and design without having to incorporate more burdensome Monte Carlo techniques. Due to its additional complexity, few utilized this alternate augmented-state formulation and largely failed to recognize its dual purpose capability of generating both navigation errors along with trajectory dispersions. When Geller[2] outlined a linear covariance program that extended Maybeck's formulation to support continuous and discrete state feedback for 6 degree-of-freedom vehicle dynamics along with inertial measurements in a closed-loop guidance, navigation, and control (GN&C) system for rendezvous and docking, the untapped potential of the augmented state linear covariance formulation began to be recognized and utilized by the community to support an assortment of critical programs and missions.[3–11]

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Awaking this sleeping technology which had been quietly dormant for decades also began to reveal the lack of features to fully model several subtle yet critical aspects of a vehicles' GN&C system. The current theoretical development of LinCov still has deep navigation-centric roots that does not adequately capture impactful features of guidance, targeting, and control system designs. For example, Christensen[12] recognized detailed features related to the control system for hosted payloads were not available, and extended the current linear covariance theory to model controllers with internal states, filters, and auxiliary control measurements. Although an assortment of different mission phases have utilized the augmented-state LinCov framework by adding different parameter states[9] to add fidelity to sensor, actuator, or environment models, the core theory is largely the same. This paper extends the current LinCov theoretical development by formulating a **g**eneralized **au**gmented-**s**tate **cov**ariance analysis (GAUSCOV) technique that adopts the control system updates proposed by Christensen and extends the framework to accommodate the intricacies of a variety of targeting and mission planning strategies, both ground and onboard, allowing for an assessment of the impacts to spaceflight GN&C system performance.

The development of the GAUSCOV method is initiated by introducing new performance metrics in the first section. Following the formal definition and addition of new state variables, the augmented state vector and nonlinear functional relationships are derived. This foundation facilitates the mathematical formulation to initialize, propagate, update, target, replan, and correct both a generalized augmented-state covariance matrix along with a navigation filter state error covariance matrix.

GENERALIZED AUGMENTED-STATE PERFORMANCE METRICS

To give context to the enhancements to the generalized augmented-state covariance and derive the fundamental equations, there are several performance metrics that must be defined. For the previously developed augmented-state LinCov formulation by Geller,[2] these variables include the true trajectory dispersions $\delta\mathbf{x}$, the navigation dispersions $\delta\hat{\mathbf{x}}$, the true navigation error $\delta\mathbf{e}$, and the onboard navigation error $\delta\hat{\mathbf{e}}$ as depicted in Figure 1.

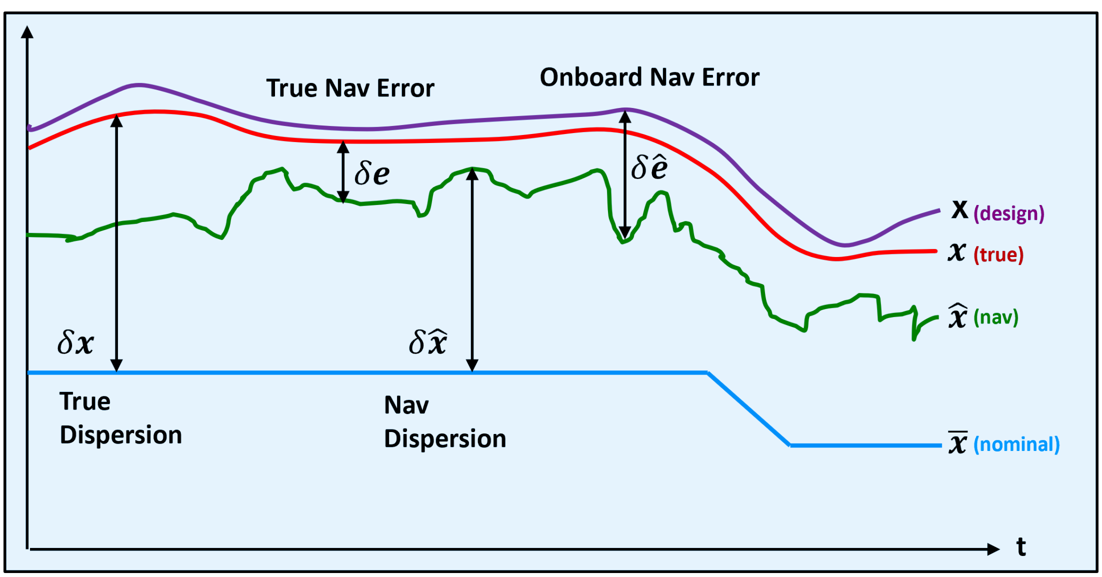


Figure 1. GN&C performance metric variables utilized for augmented-state LinCov analysis

Augmented State and Covariance

The true dispersions $\delta\mathbf{x}$ are defined as the difference between the true state $\mathbf{x}$ and the nominal state $\bar{\mathbf{x}}$. The true state $\mathbf{x}$ is an n-dimensional vector that represents the *real world* environment or actual state.

$$\delta\mathbf{x} \triangleq \mathbf{x} - \bar{\mathbf{x}} \qquad\qquad \mathbf{D} = E\left[\delta\mathbf{x}\delta\mathbf{x}^{\mathrm{T}}\right] \tag{1}$$

The nominal state $\bar{\mathbf{x}}$ is also an n-dimensional vector that represents the implemented desired or reference state with no stochastic elements. The covariance of the environment dispersions, $\mathbf{D}$, indicates how precisely the system can follow a desired trajectory.

The navigation dispersions $\delta\hat{\mathbf{x}}$ are defined as the difference between the navigation state $\hat{\mathbf{x}}$ and the nominal state $\bar{\mathbf{x}}$. The navigation state is an $\hat{n}$-dimensional vector ($\hat{n} < n$) that represents the filter's estimated state.

$$\delta\hat{\mathbf{x}} \triangleq \hat{\mathbf{x}} - \hat{\mathbf{M}}_x\bar{\mathbf{x}} \qquad\qquad \hat{\mathbf{D}} = E\left[\delta\hat{\mathbf{x}}\delta\hat{\mathbf{x}}^{\mathrm{T}}\right] \tag{2}$$

The matrix $\hat{\mathbf{M}}_x$ is an $(\hat{n} \times n)$ matrix that maps the estimated state in terms of the true and nominal state. The covariance of the navigation dispersions, $\hat{\mathbf{D}}$, reflect how precisely the onboard system thinks it can follow a prescribed reference trajectory.

The true navigation error $\delta\mathbf{e}$ is the difference between the environment and navigation states. It is also the difference between the environment and the navigation dispersions.

$$\delta\mathbf{e} \triangleq \hat{\mathbf{M}}_x\mathbf{x} - \hat{\mathbf{x}} = \hat{\mathbf{M}}_x\delta\mathbf{x} - \delta\hat{\mathbf{x}} \qquad\qquad \mathbf{P} = E\left[\delta\mathbf{e}\delta\mathbf{e}^{\mathrm{T}}\right] \tag{3}$$

The covariance of the true navigation error, $\mathbf{P}$, quantifies how precisely the onboard navigation system can estimate the actual state.

The onboard navigation error $\delta\hat{\mathbf{e}}$ itself is never known, but its covariance $\hat{\mathbf{P}}$ is used to develop the onboard navigation filter equations. It is defined as the difference between the design state, x, and the navigation state $\hat{\mathbf{x}}$.

$$\delta\hat{\mathbf{e}} \triangleq \mathsf{x} - \hat{\mathbf{x}} \qquad\qquad \hat{\mathbf{P}} = E\left[\delta\hat{\mathbf{e}}\delta\hat{\mathbf{e}}^{\mathrm{T}}\right] \tag{4}$$

The covariance of the onboard navigation error, $\hat{\mathbf{P}}$, quantifies how precisely the onboard navigation system expects it can determine the actual state. The performance of the onboard navigation system is determined by comparing $\hat{\mathbf{P}}$ to the actual navigation performance $\mathbf{P}$. If the *true* states and the *design* states are assumed to be the same, then the true navigation covariance will equal the onboard navigation covariance.

The covariances of the true dispersions, navigation dispersions, true navigation error, and the onboard navigation error are ultimately used to analyze and assess the performance of a proposed GN&C system. A common approach to obtain these performance metrics is to use a Monte Carlo simulation, where the sample statistics of hundreds or thousands of runs, N, are used to numerically compute the desired covariance matrices.

$$\mathbf{D} = \frac{1}{N-1}\sum\delta\mathbf{x}\delta\mathbf{x}^{\mathrm{T}} \qquad \hat{\mathbf{D}} = \frac{1}{N-1}\sum\delta\hat{\mathbf{x}}\delta\hat{\mathbf{x}}^{\mathrm{T}} \qquad \mathbf{P} = \frac{1}{N-1}\sum\delta\mathbf{e}\delta\mathbf{e}^{\mathrm{T}} \tag{5}$$

The onboard navigation error covariance $\hat{\mathbf{P}}$ is the navigation filter covariance for each run.

This same statistical information can be obtained using the augmented-state linear covariance analysis techniques. LinCov incorporates the non-linear system dynamics models and GN&C algorithms to generate a nominal reference trajectory $\bar{\mathbf{x}}$ which is then used to propagate, update, and correct an onboard navigation covariance matrix $\hat{\mathbf{P}}$ and an augmented state covariance matrix $\mathbf{C}$,

$$\mathbf{C} = E\left[\delta\mathbf{X}\delta\mathbf{X}^{\mathrm{T}}\right] \tag{6}$$

where the augmented state $\delta\mathbf{X}^{\mathrm{T}} = [\delta\mathbf{x}^{\mathrm{T}}\ \delta\hat{\mathbf{x}}^{\mathrm{T}}]$ consists of the true dispersions and the navigation dispersions. Pre- and post-multiplying the augmented state covariance matrix by the following mapping matrices, the covariances for the trajectory dispersions, navigation dispersions, and the navigation error can be obtained.

$$\begin{aligned}
\mathbf{D} &= \left[\, \mathbf{I}_{n\times n},\ \mathbf{0}_{n\times \hat{n}} \,\right] \mathbf{C} \left[\, \mathbf{I}_{n\times n},\ \mathbf{0}_{n\times \hat{n}} \,\right]^{\mathrm{T}} \\
\hat{\mathbf{D}} &= \left[\, \mathbf{0}_{\hat{n}\times n},\ \mathbf{I}_{\hat{n}\times \hat{n}} \,\right] \mathbf{C} \left[\, \mathbf{0}_{\hat{n}\times n},\ \mathbf{I}_{\hat{n}\times \hat{n}} \,\right]^{\mathrm{T}} \\
\mathbf{P} &= \left[\, \mathbf{I}_{\hat{n}\times n},\ -\mathbf{I}_{\hat{n}\times \hat{n}} \,\right] \mathbf{C} \left[\, \mathbf{I}_{\hat{n}\times n},\ -\mathbf{I}_{\hat{n}\times \hat{n}} \,\right]^{\mathrm{T}}
\end{aligned} \tag{7}$$

Generalized Augmented State and Covariance

The GAUSCOV formulation extends previous work via abstraction, grouping qualitatively-similar elements according to common attributes and identifying quantitative differences between states and parameters.[13] In the first movement of abstraction, variables are grouped according to the attribute of knowledge they express. *Nominal* represents the fixed, pre-planned value of a quantity. *Reference* denotes its instantaneous planned value. *Estimated* reflects a system's imperfect knowledge of a quantity, and, finally, *true* expresses its absolute, really-existing value. The second movement separates quantites into modes of dynamic states and static parameters. While the dynamic state variables evolve in time according to a system of differential equations, the static parameters are characterized by their lack of evolution in time, experiencing change exclusively through discrete "events" of the GN&C system, e.g., targeting. The variables modified and added for the GAUSCOV formulation include both dynamic state variables and static parameters as depicted in Figure 2.

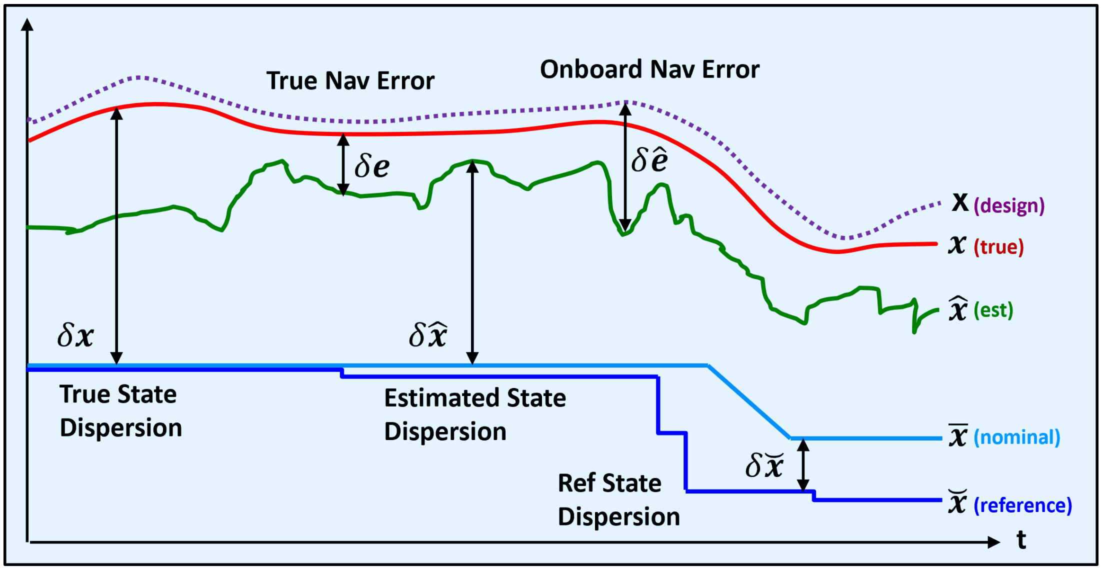
One resulting modification is the generalization of the navigation state $\hat{\mathbf{x}}$ as the *estimated* GN&C state which can be categorized and subdivided into states associated with a navigation filter $\hat{\mathbf{x}}_{nav}$ (i.e., its previous definition), guidance algorithms $\hat{\mathbf{x}}_{gdn}$, targeting $\hat{\mathbf{x}}_{tgt}$, or control logic $\hat{\mathbf{x}}_{con}$ such that $\delta\hat{\mathbf{x}}$ now represents the estimated GN&C state dispersions. The estimated state consists of potential navigation, guidance, targeting, and control states that are distinguished by having dynamical properties, not static values. All members of this set express a GN&C system's knowledge of itself.

$$\hat{\mathbf{x}}^{\mathrm{T}} \triangleq \begin{bmatrix} \hat{\mathbf{x}}_{nav}^{\mathrm{T}} & \hat{\mathbf{x}}_{gdn}^{\mathrm{T}} & \hat{\mathbf{x}}_{tgt}^{\mathrm{T}} & \hat{\mathbf{x}}_{con}^{\mathrm{T}} \end{bmatrix} \tag{8}$$

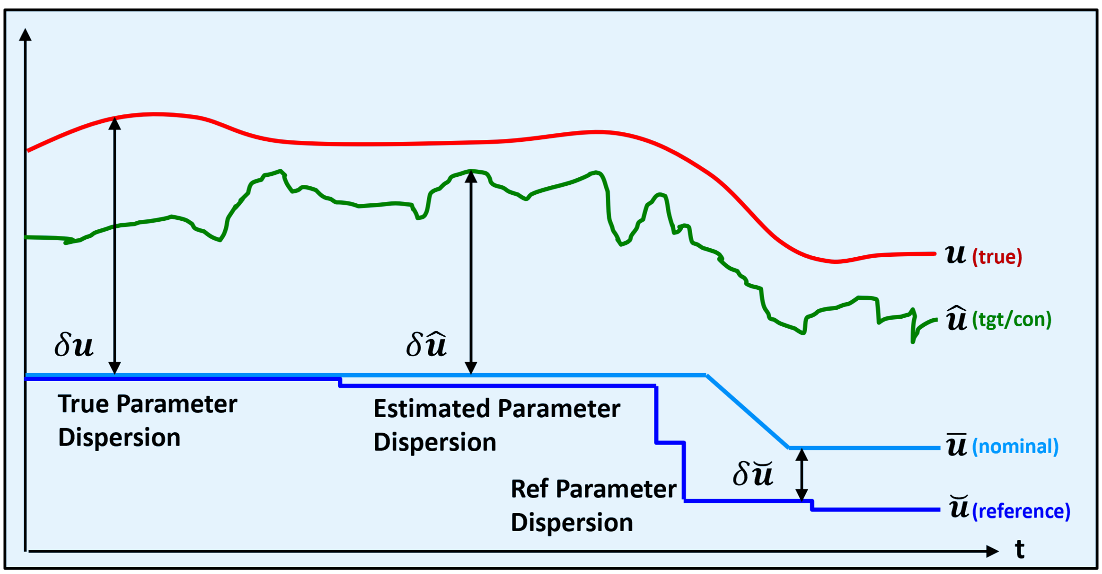
Consequently, the estimated GN&C state is an $\hat{n}$-dimensional vector ($\hat{n} = \hat{n}_{nav} + \hat{n}_{gdn} + \hat{n}_{tgt} + \hat{n}_{con} < n$) that includes all the various *estimated* states utilized by the GN&C system, or collection of GN&C systems. Mathematically, the estimated state dispersions $\delta\hat{\mathbf{x}}$ are defined as the difference between the estimated states $\hat{\mathbf{x}}$ and the nominal state $\bar{\mathbf{x}}$, i.e.,

$$\delta\hat{\mathbf{x}} \triangleq \hat{\mathbf{x}} - \hat{\mathbf{M}}_x\bar{\mathbf{x}} \qquad\qquad \hat{\mathbf{D}} = E\left[\delta\hat{\mathbf{x}}\delta\hat{\mathbf{x}}^{\mathrm{T}}\right] \tag{9}$$

The matrix $\hat{\mathbf{M}}_x$ is an $(\hat{n} \times n)$ matrix that maps the estimated state in terms of the true and nominal state. The covariance of the estimated state dispersions, $\hat{\mathbf{D}}$, reflect how precisely the onboard GN&C system components predict to follow a prescribed reference trajectory.



(a) Dynamics State Variables



(b) Static Parameters

Figure 2. GN&C performance metric variables utilized for generalized augmented-state covariance analysis

The new reference state variable dispersion $\delta\check{\mathrm{x}}$, shown in Figure 2(a), reflects deviations between the instanenous intended quantity and its a priori nominal value. This reference attribute is motivated by events such as in-flight re-optimization or landing site redesignation where typically the reference state and nominal state are the same or nearly identical at the initial simulation epoch. The reference state dispersions $\delta\check{\mathrm{x}}$ are defined as the difference between the reference state $\check{\mathrm{x}}$ and the nominal state $\bar{\mathrm{x}}$.

$$\delta\check{\mathrm{x}} \triangleq \check{\mathrm{x}} - \check{\mathrm{M}}_x\bar{\mathrm{x}} \qquad\qquad \check{\mathbf{D}} = E\left[\delta\check{\mathrm{x}}\delta\check{\mathrm{x}}^{\mathrm{T}}\right] \tag{10}$$

The reference state is an $\check{n}$-dimensional vector ($\check{n} < n$) that represents the current reference profile assumed by ground or the onboard GN&C system. The matrix $\check{\mathbf{M}}_x$ is an ($\check{n} \times n$) matrix that maps the reference state in terms of the true and nominal state. The covariance of the reference state dispersions, $\check{\mathbf{D}}$, reflect how much the reference trajectory has changed or differs from the original nominal profile due to in-flight re-planning decisions.

The associated static parameters of each attribute are all new additions, illustrated in Figure 2(b), which consist of the true parameter dispersion $\delta\mathbf{u}$, the estimated parameter dispersion $\delta\hat{\mathbf{u}}$, and the reference parameter dispersion $\delta\check{\mathbf{u}}$. These parameters may represent physical quantities such as delta-v, thrust, or applied torque for translational burns and attitude maneuvers. Additionally, this set may include more abstract parameters such as coefficients within a steering law, event times, and quantities defining the constraint functions within a targeting problem. The true parameter dispersions $\delta\mathbf{u}$ are defined as the difference between the true parameter $\mathbf{u}$ and the nominal parameter $\bar{\mathbf{u}}$.

$$\delta\mathbf{u} \triangleq \mathbf{u} - \bar{\mathbf{u}} \qquad \mathbf{U} = E\left[\delta\mathbf{u}\delta\mathbf{u}^{\mathrm{T}}\right] \tag{11}$$

The nominal parameters are expressed in an p-dimensional vector. The covariance of the true parameter dispersions, $\mathbf{U}$, reflect how these parameters vary from the original nominal parameters.

The estimated parameter dispersions $\delta\hat{\mathbf{u}}$ are defined as the difference between the estimated parameter $\hat{\mathbf{u}}$ and the nominal parameter $\bar{\mathbf{u}}$. The estimated parameter is an $\hat{p}$-dimensional vector ($\hat{p} < p$) that represents the estimated or onboard parameters.

$$\delta\hat{\mathbf{u}} \triangleq \hat{\mathbf{u}} - \hat{\mathbf{M}}_u\bar{\mathbf{u}} \qquad \hat{\mathbf{U}} = E\left[\delta\hat{\mathbf{u}}\delta\hat{\mathbf{u}}^{\mathrm{T}}\right] \tag{12}$$

The matrix $\hat{\mathbf{M}}_u$ is an ($\hat{p} \times p$) matrix that maps the estimated parameters in terms of the true or nominal parameters. The covariance of the estimated parameter dispersions, $\hat{\mathbf{U}}$, reflect how much the estimated parameters vary from their nominal values.

The reference parameter dispersions $\delta\check{\mathbf{u}}$ are defined as the difference between the reference parameters $\check{\mathbf{u}}$ and the nominal parameters $\bar{\mathbf{u}}$. The reference parameters is an $\check{p}$-dimensional vector ($\check{p} < p$) that represents the current reference input required by the onboard GN&C system to follow the latest reference profile.

$$\delta\check{\mathbf{u}} \triangleq \check{\mathbf{x}} - \check{\mathbf{M}}_u\bar{\mathbf{x}} \qquad \check{\mathbf{U}} = E\left[\delta\check{\mathbf{u}}\delta\check{\mathbf{u}}^{\mathrm{T}}\right] \tag{13}$$

The matrix $\check{\mathbf{M}}_u$ is an ($\check{p} \times p$) matrix that maps the reference parameters in terms of the true or nominal parameters. The covariance of the reference parameter dispersions, $\check{\mathbf{U}}$, reflect how much the updated reference parameters vary from the original nominal values.

The generalized augmented state vector is defined as the concatenation of the dynamical states and static parameters of all attributes, i.e.,

$$\boldsymbol{\mathcal{X}}^T = \begin{bmatrix}\mathbf{x}^{\mathrm{T}} & \hat{\mathbf{x}}^{\mathrm{T}} & \check{\mathbf{x}}^{\mathrm{T}} & \mathbf{u}^{\mathrm{T}} & \hat{\mathbf{u}}^{\mathrm{T}} & \check{\mathbf{u}}^{\mathrm{T}}\end{bmatrix} \tag{14}$$

The generalized augmented state covariance matrix $\boldsymbol{\mathcal{C}}$ includes the true state dispersions $\delta\mathbf{x}$, estimated state dispersions $\delta\hat{\mathbf{x}}$, reference state dispersions $\delta\check{\mathbf{x}}$, true parameter dispersions $\delta\mathbf{u}$, estimated parameter dispersions $\delta\hat{\mathbf{u}}$, and the reference parameter dispersions $\delta\check{\mathbf{u}}$,

$$\boldsymbol{\mathcal{C}} = E\left[\delta\boldsymbol{\mathcal{X}}\delta\boldsymbol{\mathcal{X}}^{\mathrm{T}}\right] \tag{15}$$

with the following augmented state dispersion definition, $\delta\boldsymbol{\mathcal{X}}^{\mathrm{T}} = [\delta\mathbf{x}^{\mathrm{T}}\ \delta\hat{\mathbf{x}}^{\mathrm{T}}\ \delta\check{\mathbf{x}}^{\mathrm{T}}\ \delta\mathbf{u}^{\mathrm{T}}\ \delta\hat{\mathbf{u}}^{\mathrm{T}}\ \delta\check{\mathbf{u}}^{\mathrm{T}}]$. Pre- and post-multiplying the generalized augmented state covariance matrix by the simple mapping matrices produce collection of relevant covariance matrices are obtained, e.g.,

$$\begin{aligned}
\mathbf{D} &= [\,\mathbf{I}_{n\times n},\ \mathbf{0}_{n\times\hat{n}},\ \mathbf{0}_{n\times\check{n}},\ \mathbf{0}_{n\times p},\ \mathbf{0}_{n\times\hat{p}},\ \mathbf{0}_{n\times\check{p}}\,]\,\boldsymbol{\mathcal{C}}\,[\,\mathbf{I}_{n\times n},\ \mathbf{0}_{n\times\hat{n}},\ \mathbf{0}_{n\times\check{n}},\ \mathbf{0}_{n\times p},\ \mathbf{0}_{n\times\hat{p}},\ \mathbf{0}_{n\times\check{p}}\,]^{\mathrm{T}} \\
\mathbf{P} &= \left[\,\hat{\mathbf{M}}_x,\ -\mathbf{I}_{\hat{n}\times\hat{n}},\ \mathbf{0}_{\hat{n}\times\check{n}},\ \mathbf{0}_{\hat{n}\times p},\ \mathbf{0}_{\hat{n}\times\hat{p}},\ \mathbf{0}_{\hat{n}\times\check{p}}\,\right]\boldsymbol{\mathcal{C}}\left[\,\hat{\mathbf{M}}_x,\ -\mathbf{I}_{\hat{n}\times\hat{n}},\ \mathbf{0}_{\hat{n}\times\check{n}},\ \mathbf{0}_{\hat{n}\times p},\ \mathbf{0}_{\hat{n}\times\hat{p}},\ \mathbf{0}_{\hat{n}\times\check{p}}\,\right]^{\mathrm{T}} \\
\check{\mathbf{U}} &= [\,\mathbf{0}_{\check{p}\times n},\ \mathbf{0}_{\check{p}\times\hat{n}},\ \mathbf{0}_{\check{p}\times\check{n}},\ \mathbf{0}_{\check{p}\times p},\ \mathbf{0}_{\check{p}\times\hat{p}},\ \mathbf{I}_{\check{p}\times\check{p}}\,]\,\boldsymbol{\mathcal{C}}\,[\,\mathbf{0}_{\check{p}\times n},\ \mathbf{0}_{\check{p}\times\hat{n}},\ \mathbf{0}_{\check{p}\times\check{n}},\ \mathbf{0}_{\check{p}\times p},\ \mathbf{0}_{\check{p}\times\hat{p}},\ \mathbf{I}_{\check{p}\times\check{p}}\,]^{\mathrm{T}}
\end{aligned}$$

It is important to recognize that when a Kalman filter is utilized in a LinCov formulation, operations on the estmated state require a Kalman gain, which in turn requires the navigation filter state error covariance $\hat{\mathbf{P}}$. Hence the event operations equations for $\hat{\mathbf{P}}$ are required for a GAUSCOV analysis tool when a Kalman filter is utilized for navigation. Other estimation algorithms such as a Luenbuerger filter will not require the equations for $\hat{\mathbf{P}}$.

The development of the propagation, update, and correction equations for $\hat{\mathbf{x}}$ and $\hat{\mathbf{P}}$ require a navigation filter *design model*. To distinguish the filter design model from the truth model and the navigation algorithm, the filter design model state and the design model measurements are denoted by x and z_i , in sans serif font, to distinguish them from the true state $\mathbf{x}$ and true measurement $\mathbf{z}$, or from the estimated state $\hat{\mathbf{x}}$.

EVENT-BASED OPERATIONS ON AUGMENTED STATE AND COVARIANCE

The definitions of the nominal, reference, estimated, and true attributes of the dynamic states and static parameters lead into the development of operations that express their connectedness through GN&C system events. These events correspond to key functions of the GN&C system and flight control personnel. The order of events as outlined in this document represents a recommended chronology for a discrete-time LinCov analysis tool. For each event, the (generally) nonlinear equations governing operations on the generalized augmented-state vector $\boldsymbol{\mathcal{X}}$ are developed. The linear variational equations governing $\delta\boldsymbol{\mathcal{X}}$ rely on linearization along the nominal trajectory $\boldsymbol{\mathcal{X}}^*(t)$,

$$\boldsymbol{\mathcal{X}}^{*\mathrm{T}} = \begin{bmatrix}\mathbf{x}^{*\mathrm{T}} & \hat{\mathbf{x}}^{*\mathrm{T}} & \check{\mathbf{x}}^{*\mathrm{T}} & \mathbf{u}^{*\mathrm{T}} & \hat{\mathbf{u}}^{*\mathrm{T}} & \check{\mathbf{u}}^{*\mathrm{T}}\end{bmatrix} = \begin{bmatrix}\bar{\mathbf{x}}^{\mathrm{T}} & \bar{\mathbf{x}}^{\mathrm{T}} & \bar{\mathbf{x}}^{\mathrm{T}} & \bar{\mathbf{u}}^{\mathrm{T}} & \bar{\mathbf{u}}^{\mathrm{T}} & \bar{\mathbf{u}}^{\mathrm{T}}\end{bmatrix} \tag{16}$$

where $*$ denotes the operation of evaluating the nominal value of the quantity. Since $\delta\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}} - \boldsymbol{\mathcal{X}}^*$ is to first-order a zero-mean vector (assuming the nominal trajectory is the mean trajectory), these linear maps form the necessary event operations on the generalized augmented-state dispersion covariance $\boldsymbol{\mathcal{C}}$ and the estimated filter state error covariance $\hat{\mathsf{P}}$.

Discrete Propagation

The Discrete Propagation event models the flow of the dynamic states from an initial time t_i to a final time t_{i+1}. The true parameters $\mathbf{u}^{i+1}$, e.g., steering law coefficients or burn event times, propagated through this duration are a function of the current true parameters $\mathbf{u}^i$ (only as far as those that are not implicated in the propagation), the estimated parameters $\hat{\mathbf{u}}^i$, the true state $\mathbf{x}^i$ (e.g., true vehicle orientation), and actuation error noise $\boldsymbol{\eta}_d$ with covariance $\mathbf{S}_{\boldsymbol{\eta}_d}$ through the function $\mathbf{w}_d$, i.e.,

$$\mathbf{u}^{i+1} = \mathbf{u}^{i'} = \mathbf{w}_d(\mathbf{x}^i, \mathbf{u}^i, \hat{\mathbf{u}}^i, \boldsymbol{\eta}_d) \tag{17}$$

These parameters do not evolve according to differential equations but are assumed to be instantaneously realized at the start of the propagation duration. The function $\mathbf{w}_d$ only depends on $\mathbf{u}^i$ as a means to pass forward parameters that are not involved in the propagation.

The propagated true state $\mathbf{x}^{i+1}$ is related to the current true state $\mathbf{x}^i$, the true parameters through $\mathbf{w}_d$, and process noise $\boldsymbol{\omega}_d$ through the nonlinear function $\boldsymbol{g}_d$, i.e.,

$$\mathbf{x}^{i+1} = \boldsymbol{g}_d(\mathbf{x}^i, \mathbf{w}_d(\mathbf{x}^i, \mathbf{u}^i, \hat{\mathbf{u}}^i, \boldsymbol{\eta}_d), \boldsymbol{\omega}_d) \tag{18}$$

The noise $\boldsymbol{\omega}_d$ with a covariance $\mathbf{S}_{\boldsymbol{\omega}_d}$ represents any noise on the discrete propagation of the dynamic states, e.g., system uncertainties and disturbance accelerations. In addition to the position, velocity, and attitude of a spacecraft, the true state $\mathbf{x}_i$ may also include states for GN&C system modeling, including biases, scale factors, and misalignments for sensors and actuators, which are frequently represented as exponentially-correlated random variables (ECRVs) in time.

In its simplest form, the propagation of the estimated state $\hat{\mathbf{x}}^{i+1}$ depends only on the initial estimated state and estimated parameters via a nonlinear function, expressed as

$$\hat{\mathbf{x}}^{i+1} = \hat{\boldsymbol{g}}_d(\hat{\mathbf{x}}^i, \hat{\mathbf{u}}^i)$$

However, in model replacement scenarios or those where inertial measurements drive closed-loop continous control, e.g., using inertial measurement units (IMUs), the propagation of the estimated state additionally depends on the true state and noise term $\hat{\boldsymbol{\omega}}_d$, i.e.,

$$\hat{\mathbf{x}}^{i+1} = \hat{\boldsymbol{g}}_d(\mathbf{x}^i, \hat{\mathbf{x}}^i, \hat{\mathbf{u}}^i, \hat{\boldsymbol{\omega}}_d) \tag{19}$$

where $\hat{\boldsymbol{\omega}}_d$ represents discrete-time zero-mean white noise with covariance $\mathbf{S}_{\hat{\boldsymbol{\omega}}_d}$. This complication is necessary to accommodate measurement feedback over the propagation duration. Note that $\boldsymbol{\omega}_d$ and $\hat{\boldsymbol{\omega}}_d$ are statistically independent.

The remaining relations of Discrete Propagation are straightforward. The propagation of the reference state to $\check{\mathbf{x}}^{i+1}$ depends simply on the prior reference state $\check{\mathbf{x}}^i$ and parameters $\check{\mathbf{u}}^i$, and a similar relation holds for the nominal future nominal state and its prior states and parameters, i.e.,

$$\check{\mathbf{x}}^{i+1} = \check{\boldsymbol{g}}_d(\check{\mathbf{x}}^i, \check{\mathbf{u}}^i) \tag{20}$$
$$\bar{\mathbf{x}}^{i+1} = \bar{\boldsymbol{g}}_d(\bar{\mathbf{x}}^i, \bar{\mathbf{u}}^{i+}) \tag{21}$$

The parameters are by definition static and retain their values over the propagation duration, expressed mathematically as

$$\hat{\mathbf{u}}^{i+1} = \hat{\mathbf{u}}^i \tag{22}$$
$$\check{\mathbf{u}}^{i+1} = \check{\mathbf{u}}^i \tag{23}$$
$$\bar{\mathbf{u}}^{i+1} = \bar{\mathbf{u}}^i \tag{24}$$

Changes in these parameters solely result from instantaneous events.

Discrete Propagation affects the generalized augmented state covariance $\boldsymbol{\mathcal{C}}$ through a linear variational mapping. Equations (17), (18), (19), (20), (22), and (23) are linearized on the nominal states and parameters to produce the following true, estimated, and reference state and parameter dispersion relations

$$\delta\mathbf{x}^{i+1} = \mathbf{\Phi}_i^{i+1}\delta\mathbf{x}^i + \frac{\partial \boldsymbol{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \hat{\mathbf{u}}^i}\delta\hat{\mathbf{u}}^i + \frac{\partial \boldsymbol{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \boldsymbol{\eta}_d}\boldsymbol{\eta}_d + \frac{\partial \boldsymbol{g}_d}{\partial \boldsymbol{\omega}_d}\boldsymbol{\omega}_d \tag{25}$$

$$\delta\hat{\mathbf{x}}^{i+1} = \frac{\partial \hat{\boldsymbol{g}}_d}{\partial \mathbf{x}^i}\delta\mathbf{x}^i + \hat{\mathbf{\Phi}}_i^{i+1}\delta\hat{\mathbf{x}}^i + \frac{\partial \hat{\boldsymbol{g}}_d}{\partial \hat{\mathbf{u}}^i}\delta\hat{\mathbf{u}}^i + \frac{\partial \hat{\boldsymbol{g}}_d}{\partial \hat{\boldsymbol{\omega}}_d}\hat{\boldsymbol{\omega}}_d \tag{26}$$

$$\delta\check{\mathbf{x}}^{i+1} = \check{\mathbf{\Phi}}_i^{i+1}\delta\check{\mathbf{x}}^i + \frac{\partial \check{\boldsymbol{g}}_d}{\partial \check{\mathbf{u}}^i}\delta\check{\mathbf{u}}^i \tag{27}$$

$$\delta\mathbf{u}^{i+1} = \frac{\partial \mathbf{w}_d}{\partial \mathbf{x}^i}\delta\mathbf{x}^i + \frac{\partial \mathbf{w}_d}{\partial \mathbf{u}^i}\delta\mathbf{u}^i + \frac{\partial \mathbf{w}_d}{\partial \hat{\mathbf{u}}^i}\delta\hat{\mathbf{u}}^i + \frac{\partial \mathbf{w}_d}{\partial \boldsymbol{\eta}_d}\boldsymbol{\eta}_d \tag{28}$$

$$\delta\hat{\mathbf{u}}^{i+1} = \delta\hat{\mathbf{u}}^i \tag{29}$$

$$\delta\check{\mathbf{u}}^{i+1} = \delta\check{\mathbf{u}}^i \tag{30}$$

The dependency of $\mathbf{w}_d$ on $\mathbf{u}^i$ is assumed to be purely for the purpose of preserving existing parameters that do not relate to the current state propagation. Therefore, the term $\frac{\partial \boldsymbol{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \mathbf{u}^i}\delta\mathbf{u}^i$ that would otherwise appear in a straightforward linearization is equal to the zero matrix $\mathbf{0}$ and disappears.

These equations are subsequently arranged in matrix form to produce the generalized augmented state vector $\delta\boldsymbol{\mathcal{X}}^{i+1}$ as

$$\delta\boldsymbol{\mathcal{X}}^{i+1} = \mathbf{A}_d\delta\boldsymbol{\mathcal{X}}^i + \mathbf{B}_d\boldsymbol{\omega}_d + \hat{\mathbf{B}}_d\hat{\boldsymbol{\omega}}_d + \tilde{\mathbf{B}}_d\boldsymbol{\eta}_d$$

where, noting the subscripts of the identity $\mathbf{I}$ and zero $\mathbf{0}$ matrices are discarded for brevity,

$$\mathbf{A}_d = \begin{bmatrix} \mathbf{\Phi}_i^{i+1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \boldsymbol{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \hat{\mathbf{u}}^i} & \mathbf{0} \\ \frac{\partial \hat{\boldsymbol{g}}_d}{\partial \mathbf{x}^i} & \hat{\mathbf{\Phi}}_i^{i+1} & \mathbf{0} & \mathbf{0} & \frac{\partial \hat{\boldsymbol{g}}_d}{\partial \hat{\mathbf{u}}^i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \check{\mathbf{\Phi}}_i^{i+1} & \mathbf{0} & \mathbf{0} & \frac{\partial \check{\boldsymbol{g}}_d}{\partial \check{\mathbf{u}}^i} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{31}$$

$$\mathbf{B}_d = \begin{bmatrix} \frac{\partial \boldsymbol{g}_d}{\partial \boldsymbol{\omega}_d} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \hat{\mathbf{B}}_d = \begin{bmatrix} \mathbf{0} \\ \frac{\partial \hat{\boldsymbol{g}}_d}{\partial \hat{\boldsymbol{\omega}}_d} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tilde{\mathbf{B}}_d = \begin{bmatrix} \frac{\partial \boldsymbol{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \boldsymbol{\eta}_d} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{32}$$

The $\mathbf{\Phi}_i^{i+1}$ matrices, decorated with the corresponding attribute markers, denote the state transition matrices (STMs) from t_i to t_{i+1}. The generalized augmented-state dispersion covariance after a Discrete Propagation event is consequently given as

$$\boxed{\boldsymbol{\mathcal{C}}^{i+1} = \mathbf{A}_d\boldsymbol{\mathcal{C}}^i\mathbf{A}_d^{\mathrm{T}} + \mathbf{B}_d\mathbf{S}_{\boldsymbol{\omega}_d}\mathbf{B}_d^{\mathrm{T}} + \hat{\mathbf{B}}_d\mathbf{S}_{\hat{\boldsymbol{\omega}}_d}\hat{\mathbf{B}}_d^{\mathrm{T}} + \tilde{\mathbf{B}}_d\mathbf{S}_{\hat{\boldsymbol{\eta}}_d}\tilde{\mathbf{B}}_d^{\mathrm{T}}} \tag{33}$$

If noise terms or dependencies do not exist for a particular scenario, simplifying assumptions may be trivially applied to reduce the complexity of the event operations.

The propagation model for the navigation filter design state dynamics is

$$\mathsf{x}_{i+1} = \mathsf{g}_d(\mathsf{x}^i, \mathsf{w}_d(\mathsf{x}^i, \hat{\mathbf{u}}^i, \tilde{\boldsymbol{\eta}}_d), \tilde{\boldsymbol{\omega}}_d) \tag{34}$$

where $\mathbf{x}^i$ is the design model filter state, $\tilde{\boldsymbol{\omega}}_d$ is random white process noise with covariance $\mathbf{S}_{\tilde{\omega}_d}$, and $\tilde{\boldsymbol{\eta}}_d$ is the random white actuation error noise with covariance $\mathbf{S}_{\tilde{\eta}_d}$. It is important to recognize that *design model* is dependent on estimated controls that are expressed through the estimated parameters $\hat{\mathbf{u}}^i$. For a linear Kalman filter, Eq. (34) must be linearized about the nominal trajectory

$$\delta\mathbf{x}^{i+1} = \tilde{\boldsymbol{\Phi}}_i^{i+1}\delta\mathbf{x}^i + \frac{\partial \mathbf{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \hat{\mathbf{u}}^i}\delta\hat{\mathbf{u}}^i + \frac{\partial \mathbf{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \tilde{\boldsymbol{\eta}}_d}\tilde{\boldsymbol{\eta}}_d + \frac{\partial \mathbf{g}_d}{\partial \tilde{\boldsymbol{\omega}}_d}\tilde{\boldsymbol{\omega}}_d \tag{35}$$

where $\delta\mathbf{x}^i$ and $\hat{\mathbf{u}}^i$ are dispersions from their nominal trajectory associated with $\boldsymbol{\mathcal{X}}^*$, and the partial derivatives are evaluated along the nominal reference trajectory.

Taking the expected value of the above linearized design model produces the linear filter state propagation algorithm,

$$\delta\hat{\mathbf{x}}^{i+1} = \tilde{\boldsymbol{\Phi}}_i^{i+1}\delta\hat{\mathbf{x}}^i + \frac{\partial \mathbf{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \hat{\mathbf{u}}^i}\delta\hat{\mathbf{u}}^i \tag{36}$$

where $\delta\hat{\mathbf{x}}^i = E[\delta\mathbf{x}^i] = E[\mathbf{x}^i - \mathbf{x}^{*i}]$ and $\delta\hat{\mathbf{u}}^i$ are used to propagate the filter state. Using Eqs. 35 and 36 , the filter estimation error $\hat{\mathbf{e}}^{i+1} = \delta\mathbf{x}^{i+1} - \delta\hat{\mathbf{x}}^{i+1}$ is given by

$$\hat{\mathbf{e}}^{i+1} = \tilde{\boldsymbol{\Phi}}_i^{i+1}\hat{\mathbf{e}}^i + \frac{\partial \mathbf{g}_d}{\partial \mathbf{w}_d}\frac{\partial \mathbf{w}_d}{\partial \tilde{\boldsymbol{\eta}}_d}\tilde{\boldsymbol{\eta}}_d + \frac{\partial \mathbf{g}_d}{\partial \tilde{\boldsymbol{\omega}}_d}\tilde{\boldsymbol{\omega}}_d \tag{37}$$

and the filter state error covariance propagation algorithm is obtained by substituting Eq. 155 into $\hat{\mathbf{P}}^{i+1} = E[(\hat{\mathbf{e}}^{i+1})(\hat{\mathbf{e}}^{i+1})^{\mathrm{T}}]$,

$$\boxed{\hat{\mathbf{P}}^{i+1} = \tilde{\boldsymbol{\Phi}}\hat{\mathbf{P}}^i\tilde{\boldsymbol{\Phi}}^{\mathrm{T}} + \boldsymbol{G}_{\tilde{\eta}_d}\mathbf{S}_{\tilde{\eta}_d}\boldsymbol{G}_{\tilde{\eta}_d}^{\mathrm{T}} + \boldsymbol{G}_{\tilde{\omega}_d}\mathbf{S}_{\tilde{\omega}_d}\boldsymbol{G}_{\tilde{\omega}_d}^{\mathrm{T}}} \tag{38}$$

where notation of the matrices has been simplified.

Measurement Update

A instantaneous Measurement Update event provides a set of operations to model the incorporation of measurement information into the estimated state and covariance. When a measurement is available at time t_z, the true measurement $\mathbf{z}$ is a function of the true state $\mathbf{x}^{z-}$ and the true sensor measurement noise $\boldsymbol{\eta}_z$, i.e.,

$$\mathbf{z} = \mathbf{h}(\mathbf{x}^{z-}, \boldsymbol{\eta}_z) \tag{39}$$

and the estimated measurement $\hat{\mathbf{z}}$ is a function of the estimated state, defined as

$$\hat{\mathbf{z}} = \hat{\mathbf{h}}(\hat{\mathbf{x}}^{z-}) \tag{40}$$

The zero-mean white noise $\boldsymbol{\eta}_z$ has covariance $\mathbf{S}_{\boldsymbol{\eta}_z}$.

The estimated state $\hat{\mathbf{x}}^{z+}$ at time t_{z+} after a measurement is processed is a function of the true and estimated measurements and is, therefore, expressible as

$$\hat{\mathbf{x}}^{z+} = \hat{\boldsymbol{g}}_z(\mathbf{x}^{z-}, \hat{\mathbf{x}}^{z-}, \boldsymbol{\eta}_z) \tag{41}$$

All parameters, as well as the true, reference, and nominal states, remain unchanged and are, therefore, expressed across the Measurement Update event as

$$\mathbf{x}^{z+} = \mathbf{x}^{z-} \tag{42}$$
$$\check{\mathbf{x}}^{z+} = \check{\mathbf{x}}^{z-} \tag{43}$$
$$\bar{\mathbf{x}}^{z+} = \bar{\mathbf{x}}^{z-} \tag{44}$$
$$\mathbf{u}^{z+} = \mathbf{u}^{z-} \tag{45}$$
$$\hat{\mathbf{u}}^{z+} = \hat{\mathbf{u}}^{z-} \tag{46}$$
$$\check{\mathbf{u}}^{z+} = \check{\mathbf{u}}^{z-} \tag{47}$$
$$\bar{\mathbf{u}}^{z+} = \bar{\mathbf{u}}^{z-} \tag{48}$$

Equations (41), (42), (43), (45), (46), and (47) are linearized about the nominal trajectory to produce the following true, estimated, and reference state and parameter dispersion variational equations,

$$\delta\mathbf{x}^{z+} = \delta\mathbf{x}^{z-} \tag{49}$$
$$\delta\hat{\mathbf{x}}^{z+} = \frac{\partial\hat{\boldsymbol{g}}_z}{\partial\mathbf{x}}\delta\mathbf{x}^{z-} + \frac{\partial\hat{\boldsymbol{g}}_z}{\partial\hat{\mathbf{x}}}\delta\hat{\mathbf{x}}^{z-} + \frac{\partial\hat{\boldsymbol{g}}_z}{\partial\boldsymbol{\eta}_z}\boldsymbol{\eta}_z \tag{50}$$
$$\delta\check{\mathbf{x}}^{z+} = \delta\check{\mathbf{x}}^{z-} \tag{51}$$
$$\delta\mathbf{u}^{z+} = \delta\mathbf{u}^{z-} \tag{52}$$
$$\delta\hat{\mathbf{u}}^{z+} = \delta\hat{\mathbf{u}}^{z-} \tag{53}$$
$$\delta\check{\mathbf{u}}^{z+} = \delta\check{\mathbf{u}}^{z-} \tag{54}$$

These equations are subsequently arranged in matrix form to produce the generalized augmented state vector $\delta\boldsymbol{\mathcal{X}}^{z+}$ as

$$\delta\boldsymbol{\mathcal{X}}^{z+} = \mathbf{A}_z\delta\boldsymbol{\mathcal{X}}^{z-} + \mathbf{B}_z\boldsymbol{\eta}_z$$

where, noting the subscripts of the identity $\mathbf{I}$ and zero $\mathbf{0}$ matrices are discarded for brevity,

$$\mathbf{A}_z = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial\hat{\boldsymbol{g}}_z}{\partial\mathbf{x}} & \frac{\partial\hat{\boldsymbol{g}}_z}{\partial\hat{\mathbf{x}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{B}_z = \begin{bmatrix} \mathbf{0} \\ \frac{\partial\hat{\boldsymbol{g}}_z}{\partial\boldsymbol{\eta}_z} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{55}$$

For implementations that utilize the Kalman filter for measurement updates, a derivation of operations across the event begins with an examination of the navigation filter and its design state. The nonlinear design model for the measurement is given by

$$\mathsf{z} = \mathsf{h}(\mathsf{x}^{z-}, \tilde{\boldsymbol{\eta}}_z) \tag{56}$$

where $\tilde{\boldsymbol{\eta}}_z$ denotes the zero-mean white measurement noise with covariance $\mathbf{S}_{\tilde{\boldsymbol{\eta}}_z}$. Linearization about the nominal trajectory yields the linearized measurement design model, i.e.,

$$\delta\mathsf{z} = \mathsf{H}\delta\mathsf{x}^{z-} + \mathsf{D}\tilde{\boldsymbol{\eta}}_z \tag{57}$$

where $\mathbf{H} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}$ and $\mathbf{D} = \frac{\partial \mathbf{h}}{\partial \tilde{\boldsymbol{\eta}}_z}$. The expected value of design model measurement variation gives

$$\delta \hat{\mathbf{z}} = \mathbf{H} \delta \hat{\mathbf{x}}^{z-} \tag{58}$$

where $\delta \hat{\mathbf{x}} = E[\delta \mathbf{x}] = E[\mathbf{x} - \mathbf{x}^*]$. The update to the expected value after the measurement is provided by the Kalman filter via the measurement innovation. i.e.,

$$\delta \hat{\mathbf{x}}^{z+} = \delta \hat{\mathbf{x}}^{z-} + \hat{\mathbf{K}} \left[\delta \mathbf{z} - \delta \hat{\mathbf{z}} \right] \tag{59}$$

$$\delta \hat{\mathbf{x}}^{z+} = \delta \hat{\mathbf{x}}^{z-} + \hat{\mathbf{K}} \left[\mathbf{H} \delta \mathbf{x}^{z-} + \mathbf{D} \tilde{\boldsymbol{\eta}}_z - \mathbf{H} \delta \hat{\mathbf{x}}^{z-} \right] \tag{60}$$

where the Kalman gain of the design filter $\hat{\mathbf{K}}$ is defined

$$\hat{\mathbf{K}} = \hat{\mathbf{P}}^{z-} \mathbf{H} \left[\mathbf{H} \hat{\mathbf{P}}^{z-} \mathbf{H}^{\mathrm{T}} + \mathbf{D} \mathbf{S}_{\boldsymbol{\eta}_z} \mathbf{D}^{\mathrm{T}} \right]^{-1} \tag{61}$$

as a function of the prior design model estimation error covariance $\hat{\mathbf{P}}^{z-} = E[(\hat{\mathrm{e}}^{z-})(\hat{\mathrm{e}}^{z-})^{\mathrm{T}}]$. The design model estimation error covariance incorporates the measurement update via the Kalman fliter, i.e.,

$$\boxed{\hat{\mathbf{P}}^{z+} = \left[\mathbf{I} - \hat{\mathbf{K}} \mathbf{H} \right] \hat{\mathbf{P}}^{z+} \left[\mathbf{I} - \hat{\mathbf{K}} \mathbf{H} \right]^{\mathrm{T}} + \hat{\mathbf{K}} \mathbf{D} \mathbf{S}_{\tilde{\boldsymbol{\eta}}_z} \mathbf{D}^{\mathrm{T}} \hat{\mathbf{K}}^{\mathrm{T}}} \tag{62}$$

Returning to the generalized augmented state vector, the estimated state after the measurement update is a linear function of the prior estimate and the measurement innovation described mathematically as

$$\begin{aligned} \delta \hat{\mathbf{x}}^{z+} &= \delta \hat{\mathbf{x}}^{z-} + \hat{\mathbf{K}} \left[\delta \mathbf{z} - \delta \hat{\mathbf{z}} \right] && (63) \\ \delta \hat{\mathbf{x}}^{z+} &= \delta \hat{\mathbf{x}}^{z-} + \hat{\mathbf{K}} \left[\mathbf{H} \delta \mathbf{x} + \hat{\mathbf{D}} \boldsymbol{\eta}_z - \mathbf{H} \hat{\delta} \hat{\mathbf{x}} \right] && (64) \end{aligned}$$

where $\mathbf{H} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}$, $\hat{\mathbf{H}} = \frac{\partial \hat{\mathbf{h}}}{\partial \hat{\mathbf{x}}}$, and $\hat{\mathbf{D}} = \frac{\partial \mathbf{h}}{\partial \boldsymbol{\eta}_z}$. The Kalman gain $\hat{\mathbf{K}}$ is assumed equal to the design model Kalman gain $\hat{\mathbf{K}}$ padded with rows of zeros to augment the matrix for estimated states that are not associated with the navigation filter.

Consequently, the matrices $\mathbf{A}_z$ and $\mathbf{B}_z$ take the form

$$\mathbf{A}_z = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{K}} \mathbf{H} & \mathbf{I} - \hat{\mathbf{K}} \hat{\mathbf{H}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{B}_z = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{K}} \hat{\mathbf{D}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{65}$$

Thus, the generalized augmented-state dispersion covariance after a Measurement Update event is given as

$$\boxed{\mathcal{C}^{z+} = \mathbf{A}_z \mathcal{C}^{z-} \mathbf{A}_z^{\mathrm{T}} + \mathbf{B}_z \mathbf{S}_{\boldsymbol{\eta}_z} \mathbf{B}_z^{\mathrm{T}}} \tag{66}$$

Reference Replanning

In space flight operations, the reference trajectory frequently begins with its definition along the pre-launch nominal trajectory and is updated throughout the mission to mitigate the inevitable effects of dispersions. These reference replanning activities may include re-optimization to reduce corrective propellant expenditure or the redesigation of a landing site to avoid natural hazards. For simplicity, reference replanning is decomposed into two instantaneous events, Estimate Incorporation and Reference Planning. By introducing the reference states and parameters in this work, analysis on dispersions between the truth and pre-flight nominal ($\delta\mathbf{x}$) and dispersions between the truth and in-flight reference ($\delta\mathbf{x} - \delta\check{\mathbf{x}}$), including their correlations, is possible.

Estimate Incorporation The first event, Estimate Incorporation, expresses the incorporation of current estimated states and parameters into the reference states and parameters. Frequently, replanning events occur with information regarding the GN&C system's knowledge of itself (*estimated* knowledge), so particular reference quantities after the Estimate Incorporation at t_{r+} are overwritten by their associated members in the estimated states and parameters with the selection matrices, $\mathbf{N}_x$ and $\mathbf{N}_u$, respectively, according to

$$\check{\mathbf{x}}^{r+} = \mathbf{N}_x\hat{\mathbf{x}}^{r-} + (\mathbf{I} - \mathbf{N}_x\mathbf{N}_x^{\dagger})\check{\mathbf{x}}^{r-} \quad (67)$$

$$\check{\mathbf{u}}^{r+} = \mathbf{N}_u\hat{\mathbf{u}}^{r-} + (\mathbf{I} - \mathbf{N}_u\mathbf{N}_u^{\dagger})\check{\mathbf{u}}^{r-} \quad (68)$$

where the dagger † denotes the pseudo-inverse. The remaining quantities remain unchanged, i.e.,

$$\mathbf{x}^{r+} = \mathbf{x}^{r-} \quad (69)$$

$$\hat{\mathbf{x}}^{r+} = \hat{\mathbf{x}}^{r-} \quad (70)$$

$$\bar{\mathbf{x}}^{r+} = \bar{\mathbf{x}}^{r-} \quad (71)$$

$$\mathbf{u}^{r+} = \mathbf{u}^{r-} \quad (72)$$

$$\hat{\mathbf{u}}^{r+} = \hat{\mathbf{u}}^{r-} \quad (73)$$

$$\bar{\mathbf{u}}^{r+} = \bar{\mathbf{u}}^{r-} \quad (74)$$

and the dispersion relations are expressed as

$$\delta\mathbf{x}^{r+} = \delta\mathbf{x}^{r-} \quad (75)$$

$$\delta\hat{\mathbf{x}}^{r+} = \delta\hat{\mathbf{x}}^{r-} \quad (76)$$

$$\delta\check{\mathbf{x}}^{r+} = \mathbf{N}_x\delta\hat{\mathbf{x}}^{r-} + (\mathbf{I} - \mathbf{N}_x\mathbf{N}_x^{\dagger})\delta\check{\mathbf{x}}^{r-} \quad (77)$$

$$\delta\mathbf{u}^{r+} = \delta\mathbf{u}^{r-} \quad (78)$$

$$\delta\hat{\mathbf{u}}^{r+} = \delta\hat{\mathbf{u}}^{r-} \quad (79)$$

$$\delta\check{\mathbf{u}}^{r+} = \mathbf{N}_u\delta\hat{\mathbf{u}}^{r-} + (\mathbf{I} - \mathbf{N}_u\mathbf{N}_u^{\dagger})\delta\check{\mathbf{u}}^{r-} \quad (80)$$

The linear dispersion relations are expressed in terms of the generalized augmented-state dispersion vector as

$$\delta\boldsymbol{\mathcal{X}}^{r+} = \mathbf{A}_r\delta\boldsymbol{\mathcal{X}}^{r-} \quad (81)$$

where

$$\mathbf{A}_r = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_x & \mathbf{I} - \mathbf{N}_x\mathbf{N}_x^\dagger & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{N}_u & \mathbf{I} - \mathbf{N}_u\mathbf{N}_u^\dagger \end{bmatrix} \tag{82}$$

resulting in the generalized augmented-state dispersion covariance after the Estimate Incorporation event,

$$\boxed{\boldsymbol{\mathcal{C}}^{r+} = \mathbf{A}_r \boldsymbol{\mathcal{C}}^{r-} \mathbf{A}_r^{\mathrm{T}}} \tag{83}$$

Because only the reference quantites are affected by the Estimate Incorporation step, the estimation error covariance is unchanged, i.e.,

$$\boxed{\hat{\mathsf{P}}^{r+} = \hat{\mathsf{P}}^{r-}} \tag{84}$$

Reference Planning The second phase, Reference Replanning, involves the update of reference parameters to satisfy mission constraints. This process may involve the adjustment of reference event times, targeting problem constants, finite burn parameters,or delta-v impulses away from their nominal values. The reference parameters $\check{\mathbf{u}}^{p+}$ after the replanning at time t_p are dependent on the estimated state, reference state, reference parameters, and a discrete noise $\boldsymbol{\eta}_p$ with covariance $\mathbf{S}_{\boldsymbol{\eta}_p}$ through the function $\mathbf{w}_p$, i.e.,

$$\check{\mathbf{u}}^{p+} = \mathbf{w}_p(\hat{\mathbf{x}}^{p-}, \check{\mathbf{x}}^{p-}, \check{\mathbf{u}}^{p-}, \boldsymbol{\eta}_p) \tag{85}$$

The Reference Planning event is purely expressed through changes in reference parameters, so the remaining states and parameters are unchanged at time t_p through replanning, i.e.,

$$\mathbf{x}^{p+} = \mathbf{x}^{p-} \tag{86}$$
$$\hat{\mathbf{x}}^{p+} = \hat{\mathbf{x}}^{p-} \tag{87}$$
$$\check{\mathbf{x}}^{p+} = \check{\mathbf{x}}^{p-} \tag{88}$$
$$\bar{\mathbf{x}}^{p+} = \bar{\mathbf{x}}^{p-} \tag{89}$$
$$\mathbf{u}^{p+} = \mathbf{u}^{p-} \tag{90}$$
$$\hat{\mathbf{u}}^{p+} = \hat{\mathbf{u}}^{p-} \tag{91}$$
$$\bar{\mathbf{u}}^{p+} = \bar{\mathbf{u}}^{p-} \tag{92}$$

The replanning process is assumed to ideally involve the satisfaction of a constraint equation, $\boldsymbol{f}_p = \mathbf{0}$, which is a function of estimated state, reference state, and reference parameters. However, the numerical process involved in the determination of appropriate parameters $\check{\mathbf{u}}^p$ imply that the actual constraint equation will settle on a zero-mean noise value, i.e.,

$$\boldsymbol{f}_p(\hat{\mathbf{x}}^p, \check{\mathbf{x}}^p, \check{\mathbf{u}}^p) = \boldsymbol{\eta}_p \tag{93}$$
$$E[\boldsymbol{\eta}_p] = \mathbf{0} \tag{94}$$

The nominal trajectory $\boldsymbol{\mathcal{X}}^*$ is assumed to satisfy the replanning constraint equation, so the variation in the planning constraint function off of the nominal ($\boldsymbol{f}_p^* = \mathbf{0}$) before replanning is given by

$$\delta \boldsymbol{f}_p^{p-} = \frac{\partial \boldsymbol{f}_p}{\partial \hat{\mathbf{x}}} \delta\hat{\mathbf{x}}^{p-} + \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{x}}} \delta\check{\mathbf{x}}^{p-} + \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}} \delta\check{\mathbf{u}}^{p-} \tag{95}$$

This variation represents the error in the constraint vector that must be corrected.

To compute the reference parameters $\check{\mathbf{u}}^{p+}$ coming out of the Reference Planning event, a first-order differential corrections scheme is employed. The change in the replanning constraint function across the event is related to the update in the reference parameters through

$$\boldsymbol{f}_p^{p+} - \boldsymbol{f}_p^{p-} = \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}(\check{\mathbf{u}}^{p+} - \check{\mathbf{u}}^{p-}) \tag{96}$$

While the resulting constraint vector $\boldsymbol{f}_p^{p+}$ is ideally equal to the zero vector $\mathbf{0}$, the numerical update process renders it equal to $\boldsymbol{\eta}_p$, i.e.,

$$\boldsymbol{\eta}_p - \boldsymbol{f}_p^{p-} = \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}(\check{\mathbf{u}}^{p+} - \check{\mathbf{u}}^{p-}) \tag{97}$$

Rearranging the above equation and converting to differential form with respect to the nominal trajectory yields

$$\delta \boldsymbol{f}_p^{p-} = -\frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}(\delta\check{\mathbf{u}}^{p+} - \delta\check{\mathbf{u}}^{p-}) + \boldsymbol{\eta}_p \tag{98}$$

Substituting Equation (95) into Equation (98) produces a mathematical relationship between the induced constraint error and the applied reference parameter correction, i.e.,

$$-\frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}(\delta\check{\mathbf{u}}^{p+} - \delta\check{\mathbf{u}}^{p-}) + \boldsymbol{\eta}_p = \frac{\partial \boldsymbol{f}_p}{\partial \hat{\mathbf{x}}}\delta\hat{\mathbf{x}}^{p-} + \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{x}}}\delta\check{\mathbf{x}}^{p-} + \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}\delta\check{\mathbf{u}}^{p-} \tag{99}$$

By solving for the applied correction, a expression for $\delta\check{\mathbf{u}}^{p+}$ emerges as

$$\delta\check{\mathbf{u}}^{p+} = \delta\check{\mathbf{u}}^{p-} - \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}^{\dagger}\left(\frac{\partial \boldsymbol{f}_p}{\partial \hat{\mathbf{x}}}\delta\hat{\mathbf{x}}^{p-} + \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{x}}}\delta\check{\mathbf{x}}^{p-} + \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}\delta\check{\mathbf{u}}^{p-}\right) + \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}^{\dagger}\boldsymbol{\eta}_p \tag{100}$$

and the remaining dispersion relations follow via

$$\begin{aligned}
\delta\mathbf{x}^{p+} &= \delta\mathbf{x}^{p-} && (101)\\
\delta\hat{\mathbf{x}}^{p+} &= \delta\hat{\mathbf{x}}^{p-} && (102)\\
\delta\check{\mathbf{x}}^{p+} &= \delta\check{\mathbf{x}}^{p-} && (103)\\
\delta\mathbf{u}^{p+} &= \delta\mathbf{u}^{p-} && (104)\\
\delta\hat{\mathbf{u}}^{p+} &= \delta\hat{\mathbf{u}}^{p-} && (105)
\end{aligned}$$

These equations are subsequently arranged in matrix form to produce the generalized augmented state vector $\delta\boldsymbol{\mathcal{X}}^{p+}$ as

$$\delta\boldsymbol{\mathcal{X}}^{p+} = \mathbf{A}_p\delta\boldsymbol{\mathcal{X}}^{p-} + \mathbf{B}_p\boldsymbol{\eta}_p \tag{106}$$

$$\mathbf{A}_p = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}^{\dagger}\frac{\partial \boldsymbol{f}_p}{\partial \hat{\mathbf{x}}} & -\frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}^{\dagger}\frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{x}}} & \mathbf{0} & \mathbf{0} & \mathbf{I} - \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}^{\dagger}\frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}} \end{bmatrix}, \quad \mathbf{B}_p = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \frac{\partial \boldsymbol{f}_p}{\partial \check{\mathbf{u}}}^{\dagger} \end{bmatrix} \tag{107}$$

as well as its associated covariance matrix

$$\boxed{\boldsymbol{\mathcal{C}}^{p+} = \mathbf{A}_p \boldsymbol{\mathcal{C}}^{p-} \mathbf{A}_p^{\mathrm{T}} + \mathbf{B}_p \mathbf{R}_{\boldsymbol{\eta}_p} \mathbf{B}_p^{\mathrm{T}}} \tag{108}$$

Because only the reference quantites are affected by the Reference Planning step, the estimation error covariance is unchanged, i.e.,

$$\boxed{\hat{\mathbf{P}}^{p+} = \hat{\mathbf{P}}^{p-}} \tag{109}$$

Targeting

Preparing for a correction in its dispersions, a GN&C system employs a Targeting event to compute estimated parameters, e.g., a commanded (or *estimated*) delta-v. This formulation allows for a displacement in time between events that update the reference, estimated, and true spaces, permitting the real-life possiblity of accumulating noise and measurements between these events. The interval between targeting and execution may be as small as a single targeting cycle, or as long as a ground-based maneuver planning cycle, as is the case for maneuvers planned and up-linked from the ground. These estimated parameters after targeting $\hat{\mathbf{u}}^{\tau+}$ are updated as a function of the estimated state, reference states, reference parameters, and a numerical noise term $\boldsymbol{\eta}_\tau$ with covariance $\mathbf{S}_{\boldsymbol{\eta}_\tau}$ prior to targeting at t_τ, i.e.,

$$\hat{\mathbf{u}}^{\tau+} = \hat{\mathbf{w}}_\tau(\hat{\mathbf{x}}^{\tau-}, \check{\mathbf{x}}^{\tau-}, \check{\mathbf{u}}^{\tau-}, \boldsymbol{\eta}_\tau) \tag{110}$$

At the time of Targeting, the update to parameters exists only in our estimated or commanded knowledge and is not yet realized in the *true*, physical realm. Neither is the targeting process affecting states and parameters in our pre-planned nominal or in-flight reference. Therefore, the remaining dispersions are unchanged through targeting, i.e.,

$$\begin{aligned}
\mathbf{x}^{\tau+} &= \mathbf{x}^{\tau-} && (111)\\
\hat{\mathbf{x}}^{\tau+} &= \hat{\mathbf{x}}^{\tau-} && (112)\\
\check{\mathbf{x}}^{\tau+} &= \check{\mathbf{x}}^{\tau-} && (113)\\
\bar{\mathbf{x}}^{\tau+} &= \bar{\mathbf{x}}^{\tau-} && (114)\\
\mathbf{u}^{\tau+} &= \mathbf{u}^{\tau-} && (115)\\
\check{\mathbf{u}}^{\tau+} &= \check{\mathbf{u}}^{\tau-} && (116)\\
\bar{\mathbf{u}}^{\tau+} &= \bar{\mathbf{u}}^{\tau-} && (117)
\end{aligned}$$

Since closed-form solutions are, in general, not available, the zero-mean white noise $\boldsymbol{\eta}_\tau$ is introduced to represent the residual constraint error, bounded by a convergence tolerance, after the numerical targeting process converges to a particular solution, satisfying

$$\begin{aligned}
\boldsymbol{f}_\tau(\hat{\mathbf{x}}, \check{\mathbf{x}}, \hat{\mathbf{u}}, \check{\mathbf{u}}) &= \boldsymbol{\eta}_\tau && (118)\\
E[\boldsymbol{\eta}_\tau] &= \mathbf{0} && (119)
\end{aligned}$$

At the instant before targeting, the variation in the constraint vector, representing a violation of the constraints, is given by

$$\delta \boldsymbol{f}_\tau^{\tau-} = \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{x}}} \delta \hat{\mathbf{x}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{x}}} \delta \check{\mathbf{x}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}} \delta \hat{\mathbf{u}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{u}}} \delta \check{\mathbf{u}}^{\tau-} \tag{120}$$

The update to the estimated parameters, remembering these may be delta-v vectors at a later time or some other anticipated control parameter,

$$\boldsymbol{f}_\tau^{\tau+} - \boldsymbol{f}_\tau^{\tau-} = \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}(\hat{\mathbf{u}}^{\tau+} - \hat{\mathbf{u}}^{\tau-}) \tag{121}$$

$$\boldsymbol{\eta}_\tau - \boldsymbol{f}_\tau^{\tau-} = \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}(\hat{\mathbf{u}}^{\tau+} - \hat{\mathbf{u}}^{\tau-}) \tag{122}$$

noting the expected value of post-targeting constraint vector is given by $E[\boldsymbol{f}_\tau^{\tau+}] = E[\boldsymbol{\eta}_\tau] = \mathbf{0}$. Rearranging this equation to prepare for substitution into Equation (120) yields

$$\delta \boldsymbol{f}_\tau^{\tau-} = -\frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}(\delta\hat{\mathbf{u}}^{\tau+} - \delta\hat{\mathbf{u}}^{\tau-}) + \boldsymbol{\eta}_\tau \tag{123}$$

Completing this substitution produces a relation between the induced targeting constraint error before Targeting and the applied correction in $\delta\hat{\mathbf{u}}^{\tau+}$ via

$$-\frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}(\delta\hat{\mathbf{u}}^{\tau+} - \delta\hat{\mathbf{u}}^{\tau-}) + \boldsymbol{\eta}_\tau = \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{x}}}\delta\hat{\mathbf{x}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{x}}}\delta\check{\mathbf{x}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}\delta\hat{\mathbf{u}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{u}}}\delta\check{\mathbf{u}}^{\tau-} \tag{124}$$

Solving for the new estimated parameters $\delta\hat{\mathbf{u}}^{\tau+}$ generates the expression

$$\delta\hat{\mathbf{u}}^{\tau+} = \delta\hat{\mathbf{u}}^{\tau-} - \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}^\dagger \left(\frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{x}}}\delta\hat{\mathbf{x}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{x}}}\delta\check{\mathbf{x}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}\delta\hat{\mathbf{u}}^{\tau-} + \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{u}}}\delta\check{\mathbf{u}}^{\tau-}\right) + \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}^\dagger \boldsymbol{\eta}_\tau \tag{125}$$

along with the remaining dispersions relations

$$\delta\mathbf{x}^{\tau+} = \delta\mathbf{x}^{\tau-} \tag{126}$$

$$\delta\hat{\mathbf{x}}^{\tau+} = \delta\hat{\mathbf{x}}^{\tau-} \tag{127}$$

$$\delta\check{\mathbf{x}}^{\tau+} = \delta\check{\mathbf{x}}^{\tau-} \tag{128}$$

$$\delta\mathbf{u}^{\tau+} = \delta\mathbf{u}^{\tau-} \tag{129}$$

$$\delta\check{\mathbf{u}}^{\tau+} = \delta\check{\mathbf{u}}^{\tau-} \tag{130}$$

These equations are subsequently arranged in matrix form to produce the generalized augmented state vector $\delta\boldsymbol{\mathcal{X}}^{\tau+}$ as

$$\delta\boldsymbol{\mathcal{X}}^{\tau+} = \mathbf{A}_\tau \delta\boldsymbol{\mathcal{X}}^{\tau-} + \mathbf{B}_\tau \boldsymbol{\eta}_\tau \tag{131}$$

$$\mathbf{A}_\tau = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}^\dagger \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{x}}} & -\frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}^\dagger \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{x}}} & \mathbf{0} & \mathbf{I} - \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}^\dagger \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}} & -\frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}^\dagger \frac{\partial \boldsymbol{f}_\tau}{\partial \check{\mathbf{u}}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{B}_\tau = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \frac{\partial \boldsymbol{f}_\tau}{\partial \hat{\mathbf{u}}}^\dagger \\ \mathbf{0} \end{bmatrix} \tag{132}$$

as well as its associated covariance matrix

$$\boldsymbol{\mathcal{C}}^{\tau+} = \mathbf{A}_\tau \boldsymbol{\mathcal{C}}^{\tau-} \mathbf{A}_\tau^{\mathrm{T}} + \mathbf{B}_\tau \mathbf{R}_{\eta_\tau} \mathbf{B}_\tau^{\mathrm{T}} \tag{133}$$

Because the true and estimated state quantities are unaffected by the Targeting event, the estimation error covariance is unchanged, i.e.,

$$\hat{\mathsf{P}}^{\tau+} = \hat{\mathsf{P}}^{\tau-} \tag{134}$$

A realization of these equations for a GAUSCOV analysis tool is presented in the Generalized Reference Targeting document.[14]

Instantaneous Actuation

Dynamical states are instantaneously actuated according to the parameters through an Instantaneous Actuation event. This event marks the *true* or physical realization of the *estimated* or commanded parameters. For example, a truly-applied delta-v is a function of its commanded (*estimated*) value, the true orientation of the vehicle, and noise associated with maneuver exectution errors. Another example is "resetting the clock" to account for variable event timing, e.g., parameters of time-of-ignition for a burn or the time associated with an imposed constraint. Mathematically, the true parameter $\mathbf{u}^{c+}$ after actuation at time t_c is given by

$$\mathbf{u}^{c+} = \mathbf{w}_c(\mathbf{x}^{c-}, \mathbf{u}^{c-}, \hat{\mathbf{u}}^{c-}, \boldsymbol{\eta}_c) \tag{135}$$

with zero-mean white noise $\boldsymbol{\eta}_c$ with covariance $\mathbf{S}_{\boldsymbol{\eta}_c}$.

The actuation is mapped into the dynamical state according to the correspondingly-decorated nonlinear function $\boldsymbol{g}_c$, i.e.,

$$\begin{aligned}
\mathbf{x}^{c+} &= \boldsymbol{g}_c(\mathbf{x}^{c-}, \mathbf{w}_c(\mathbf{x}^{c-}, \mathbf{u}^{c-}, \hat{\mathbf{u}}^{c-}, \boldsymbol{\eta}_c)) && (136)\\
\hat{\mathbf{x}}^{c+} &= \hat{\boldsymbol{g}}_c(\hat{\mathbf{x}}^{c-}, \hat{\mathbf{u}}^{c-}) && (137)\\
\check{\mathbf{x}}^{c+} &= \check{\boldsymbol{g}}_c(\check{\mathbf{x}}^{c-}, \check{\mathbf{u}}^{c-}) && (138)\\
\bar{\mathbf{x}}^{c+} &= \bar{\boldsymbol{g}}_c(\bar{\mathbf{x}}^{c-}, \bar{\mathbf{u}}^{c-}) && (139)
\end{aligned}$$

while the other parameters remain unaffected, i.e.,

$$\begin{aligned}
\hat{\mathbf{u}}^{c+} &= \hat{\mathbf{u}}^{c-} && (140)\\
\check{\mathbf{u}}^{c+} &= \check{\mathbf{u}}^{c-} && (141)\\
\bar{\mathbf{u}}^{c+} &= \bar{\mathbf{u}}^{c-} && (142)
\end{aligned}$$

Linearization of the nonlinear state and parameter functions yields a set of linear variational relations, i.e.,

$$\begin{aligned}
\delta\mathbf{x}^{c+} &= \left(\frac{\partial \boldsymbol{g}_c}{\partial \mathbf{x}} + \frac{\partial \boldsymbol{g}_c}{\partial \mathbf{w}_c}\frac{\partial \mathbf{w}_c}{\partial \mathbf{x}}\right)\delta\mathbf{x}^{c-} + \frac{\partial \boldsymbol{g}_c}{\partial \mathbf{w}_c}\frac{\partial \mathbf{w}_c}{\partial \hat{\mathbf{u}}}\delta\hat{\mathbf{u}}^{c-} + \frac{\partial \boldsymbol{g}_c}{\partial \mathbf{w}_c}\frac{\partial \mathbf{w}_c}{\partial \boldsymbol{\eta}_c}\boldsymbol{\eta}_c && (143)\\
\delta\hat{\mathbf{x}}^{c+} &= \frac{\partial \hat{\boldsymbol{g}}_c}{\partial \hat{\mathbf{x}}}\delta\hat{\mathbf{x}}^{c-} + \frac{\partial \hat{\boldsymbol{g}}_c}{\partial \hat{\mathbf{u}}}\delta\hat{\mathbf{u}}^{c-} && (144)\\
\delta\check{\mathbf{x}}^{c+} &= \frac{\partial \check{\boldsymbol{g}}_c}{\partial \check{\mathbf{x}}}\delta\check{\mathbf{x}}^{c-} + \frac{\partial \check{\boldsymbol{g}}_c}{\partial \check{\mathbf{u}}}\delta\check{\mathbf{u}}^{c-} && (145)\\
\delta\mathbf{u}^{c+} &= \frac{\partial \mathbf{w}_c}{\partial \mathbf{x}}\delta\mathbf{x}^{c-} + \frac{\partial \mathbf{w}_c}{\partial \mathbf{u}}\delta\mathbf{u}^{c-} + \frac{\partial \mathbf{w}_c}{\partial \hat{\mathbf{u}}}\delta\hat{\mathbf{u}}^{c-} + \frac{\partial \mathbf{w}_c}{\partial \boldsymbol{\eta}_c}\boldsymbol{\eta}_c && (146)\\
\delta\hat{\mathbf{u}}^{c+} &= \delta\hat{\mathbf{u}}^{c-} && (147)\\
\delta\check{\mathbf{u}}^{c+} &= \delta\check{\mathbf{u}}^{c-} && (148)
\end{aligned}$$

The dependency of $\mathbf{w}_c$ on $\mathbf{u}^{c-}$ is assumed to be purely for the purpose of preserving existing parameters that do not relate to the current actuation. Therefore, the term $\frac{\partial \boldsymbol{g}_c}{\partial \mathbf{w}_c}\frac{\partial \mathbf{w}_c}{\partial \mathbf{u}}\delta\mathbf{u}^{c-}$ that would otherwise appear in a straightforward linearization of Equation (143) is equal to the zero matrix $\mathbf{0}$ and disappears.

These equations are subsequently arranged in matrix form to produce the generalized augmented state vector $\delta\boldsymbol{\mathcal{X}}^{c+}$ as

$$\delta\boldsymbol{\mathcal{X}}^{c+} = \mathbf{A}_c\delta\boldsymbol{\mathcal{X}}^{c-} + \mathbf{B}_c\boldsymbol{\eta}_c \tag{149}$$

$$\mathbf{A}_c = \begin{bmatrix} \frac{\partial \boldsymbol{g}_c}{\partial \mathbf{x}} + \frac{\partial \boldsymbol{g}_c}{\partial \mathbf{w}_c}\frac{\partial \mathbf{w}_c}{\partial \mathbf{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \boldsymbol{g}_c}{\partial \mathbf{w}_c}\frac{\partial \mathbf{w}_c}{\partial \hat{\mathbf{u}}} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \hat{\boldsymbol{g}}_c}{\partial \hat{\mathbf{x}}} & \mathbf{0} & \mathbf{0} & \frac{\partial \hat{\boldsymbol{g}}_c}{\partial \hat{\mathbf{u}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{\partial \check{\boldsymbol{g}}_c}{\partial \check{\mathbf{x}}} & \mathbf{0} & \mathbf{0} & \frac{\partial \check{\boldsymbol{g}}_c}{\partial \check{\mathbf{u}}} \\ \frac{\partial \mathbf{w}_c}{\partial \mathbf{x}} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{w}_c}{\partial \mathbf{u}} & \frac{\partial \mathbf{w}_c}{\partial \hat{\mathbf{u}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{B}_c = \begin{bmatrix} \frac{\partial \boldsymbol{g}_c}{\partial \mathbf{w}_c}\frac{\partial \mathbf{w}_c}{\partial \boldsymbol{\eta}_c} \\ \mathbf{0} \\ \mathbf{0} \\ \frac{\partial \mathbf{w}_c}{\partial \boldsymbol{\eta}_c} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{150}$$

as well as its associated covariance matrix

$$\boxed{\boldsymbol{\mathcal{C}}^{c+} = \mathbf{A}_c \boldsymbol{\mathcal{C}}^{c-} \mathbf{A}_c^{\mathrm{T}} + \mathbf{B}_c \mathbf{S}_{\boldsymbol{\eta}_c} \mathbf{B}_c^{\mathrm{T}}} \tag{151}$$

The navigation design state model for Instantaneous Actuation is given by

$$\mathsf{x}^{c+} = \mathsf{g}_c(\mathsf{x}^{c-}, \mathsf{w}_c(\mathsf{x}^{c-}, \hat{\mathbf{u}}^{c-}, \tilde{\boldsymbol{\eta}}_c)) \tag{152}$$

where w_c represents to design model expression of the actuation parameters, and the noise $\tilde{\boldsymbol{\eta}}_c$ is zero-mean white noise, independent of other variables. Linearization of the design state produces

$$\delta \mathsf{x}^{c+} = \left(\frac{\partial \mathsf{g}_c}{\partial \mathsf{x}} + \frac{\partial \mathsf{g}_c}{\partial \mathsf{w}_c} \frac{\partial \mathsf{w}_c}{\partial \mathsf{x}} \right) \delta \mathsf{x}^{c-} + \frac{\partial \mathsf{g}_c}{\partial \mathsf{w}_c} \frac{\partial \mathsf{w}_c}{\partial \hat{\mathbf{u}}} \delta \hat{\mathbf{u}}^{c-} + \frac{\partial \mathsf{g}_c}{\partial \mathsf{w}_c} \frac{\partial \mathsf{w}_c}{\partial \tilde{\boldsymbol{\eta}}_c} \tilde{\boldsymbol{\eta}}_c \tag{153}$$

Taking the expected value of the above linearized design model produces the linear filter state propagation algorithm,

$$\delta \hat{\mathsf{x}}^{c+} = \left(\frac{\partial \mathsf{g}_c}{\partial \mathsf{x}} + \frac{\partial \mathsf{g}_c}{\partial \mathsf{w}_c} \frac{\partial \mathsf{w}_c}{\partial \mathsf{x}} \right) \delta \hat{\mathsf{x}}^{c-} + \frac{\partial \mathsf{g}_c}{\partial \mathsf{w}_c} \frac{\partial \mathsf{w}_c}{\partial \hat{\mathbf{u}}} \delta \hat{\mathbf{u}}^{c-} \tag{154}$$

where $\delta \hat{\mathsf{x}} = E[\delta \mathsf{x}] = E[\mathsf{x} - \mathsf{x}^*]$ and $\delta \hat{\mathbf{u}}^i$ are used to correct the filter state. To reduce the burden of notation the partial derivatives evaluated on the reference $\tilde{\boldsymbol{\Phi}}_c = \left(\frac{\partial \mathsf{g}_c}{\partial \mathsf{x}} + \frac{\partial \mathsf{g}_c}{\partial \mathsf{w}_c} \frac{\partial \mathsf{w}_c}{\partial \mathsf{x}} \right)$.

Using Eqs. 153 and 154, the filter estimation error $\mathbf{e}^{c+} = \delta \mathsf{x}^{c+} - \delta \hat{\mathsf{x}}^{c+}$ is given by

$$\mathbf{e}^{c+} = \tilde{\boldsymbol{\Phi}}_c \mathbf{e}^{c-} + \frac{\partial \mathsf{g}_c}{\partial \mathsf{w}_c} \frac{\partial \mathsf{w}_c}{\partial \tilde{\boldsymbol{\eta}}_c} \tilde{\boldsymbol{\eta}}_c \tag{155}$$

and the filter state error covariance propagation algorithm is obtained by substituting Eq. 155 into $\hat{\mathsf{P}}^{i+1} = E[(\mathbf{e}^{i+1})(\mathbf{e}^{i+1})^{\mathrm{T}}]$,

$$\boxed{\hat{\mathsf{P}}^{c+} = \tilde{\boldsymbol{\Phi}}_c \hat{\mathsf{P}}^{c-} \tilde{\boldsymbol{\Phi}}_c^{\mathrm{T}} + \boldsymbol{G}_{\tilde{\boldsymbol{\eta}}_c} \mathbf{S}_{\tilde{\boldsymbol{\eta}}_c} \boldsymbol{G}_{\tilde{\boldsymbol{\eta}}_c}^{\mathrm{T}}} \tag{156}$$

with compacted matrix notation where $\mathbf{S}_{\tilde{\boldsymbol{\eta}}_c}$ is the covariance of $\tilde{\boldsymbol{\eta}}_c$.

CONCLUSIONS

In this work, the augmented state formulation is generalized to accommodate an expanded set of GN&C system quantities and events via a two-step abstraction. First, the qualitative delineation of different categories of knowledge (true, estimated, reference, nominal, and design) are developed. Second, the quantitative difference of the evolution in time for dynamical and static quantites of various GN&C system is used to delineate between states and parameters, respectively. This document introduces a generalized augmented-state vector consisting of states and parameters with associated knowledge attributes through the generalized augmented state vector $\boldsymbol{\mathcal{X}}$. The nonlinear functional relationships defining the operations on this generalized augmented state are presented and then linearized to obtain the core equations required for operations on dispersions and their covariance within a Generalized Augmented-State Covariance (GAUSCOV) analysis tool. Specifically, this work serves as a preliminary reference document for the architecture of the Swift Integrated GN&C and Mission Analysis (SIGMA) tool.

The order of events in this document comprises a recommended chronology for implementing a Generalized Augmented-State Covariance (GAUSCOV) analysis tool. The concept of the Reference Replanning event is introduced to model the effects of re-establishing the in-flight reference trajectory while maintaining its covariance relative to the original nominal and manifesting correlations to the estimated state. Compared to previous work, the Targeting event is separated from the Instantaneous Actuation, allowing the effect of a time-delay between targeting and execution to be quantified. By breaking functions of the GN&C system into "atomic" events, a separation of events in time is possible.

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