

Ambiguity Remediation in Launch Vehicles and Morse-Lyapunov-Based Control with Parameter Uncertainties

Matthew Michael Wittal*

NASA Deep Space Logistics Project Office, Kennedy Space Center, FL 32899

Brennan McCann[†]

Johns Hopkins University Applied Physics Laboratory, Laurel, MD 2072

Julio César Benavides[‡]

University of Texas at Arlington, 701 S. Nedderman Drive Arlington, TX 76019

Morad Nazari[§]

Embry-Riddle Aeronautical University, Daytona Beach, FL 32114

Conventional methods for launch vehicle estimation and control are performed primarily using quaternions, which outperform minimal attitude parameterization sets since quaternions have no singularities. Yet quaternions exhibit the disadvantage of non-uniqueness, leaving room for improvement. Launch vehicle attitude dynamics are known to be especially noisy, while position and velocity measurements are usually known with a high degree of precision. In this work, a launch problem is simulated using the dynamic formulation on the special Euclidean group ($SE(3)$) considering varying mass properties. State and parameter estimates are filtered on $SE(3)$ using an unscented Kalman filter (UKF). A state-and-parameter-estimation-error-based Morse-Lyapunov tracking law controller with backstepping is introduced and demonstrated. Finally, the ability of the UKF on $SE(3)$ to overcome ambiguity in attitude and attitude rate estimates is demonstrated in a simulated launch to low Earth orbit, where precise position and velocity estimates are obtained using the coupled formulation.

*Ph.D. Candidate, Embry-Riddle Aeronautical University, Daytona Beach, FL 32114, USA; Robotics and Autonomous Systems Engineer, Granular Mechanics and Regolith Operations Laboratory & Deep Space Logistics, NASA Kennedy Space Center, FL 32899, USA.

[†]Guidance and Control Analyst, Johns Hopkins University Applied Physics Laboratory, Laurel, MD 20723, USA.

[‡]Astrodynamicist, Advanced Astronautics Laboratory, Houston, TX 77058, USA.

[§]Associate Professor, Aerospace Engineering Department, Embry-Riddle Aeronautical University, Daytona Beach, FL 32114, USA.

This work builds upon "Ambiguity Remediation in Space Launch Vehicles with Parameter Uncertainties: A Comparison between Special Euclidean Group and Dual Quaternions", AAS 22-831, presented on Monday, August 8th, 2022, at the AIAA/AAS Astrodynamics Specialist Conference in Charlotte, North Carolina.

I. Nomenclature

$(\hat{\cdot})$	Estimated state or parameter	p	Dimension of the process noise
$(\check{\cdot})$	Unit vector $\in \mathbb{R}^3$	P	Covariance matrix
$(\bar{\cdot})$	Average of a vector	q	dimension of the measurement noise
$\oplus$	Earth-centered Earth-fixed frame	r	Position of the rigid body $\in \mathbb{R}^3$
a	Acceleration $\in \mathbb{R}^3$	R	Rotation matrix $\in \mathbb{R}^{3 \times 3}$
ad	The adjoint mapping between two tangent spaces $\in \mathbb{R}^{6 \times 6}$	t	Time $\in \mathbb{R}$
A	Cross-sectional area of the vehicle $\in \mathbb{R}$	T	Thrust $\in \mathbb{R}^3$
Ad	The adjoint mapping between two frames $\in \mathbb{R}^{6 \times 6}$	u	Control forces $\in \mathbb{R}^6$
$\mathcal{B}$	Body frame	v	Velocity $\in \mathbb{R}^3$
d	A desired or modeled state or parameter	$\mathbb{V}$	Augmented velocity vector $\in \mathbb{R}^6$
D	Force of drag $\in \mathbb{R}^3$	w	width of the rigid body $\in \mathbb{R}$
e	Error in state	W	Weight matrix
f	External forces $\in \mathbb{R}^3$	x	State vector
g	Pose $\in \text{SE}(3)$	z	Measurements
h	Height $\in \mathbb{R}$	α, β	Sigma point tuning factors
I_n	Identity matrix $\in \mathbb{R}^{n \times n}$	γ	Angle of attack
I_{sp}	Specific impulse	ϵ	Error between two states or parameters
$\mathbb{I}$	Augmented inertia tensor $\in \mathbb{R}^{6 \times 6}$	η	Noise standard deviation
J	Inertia $\in \mathbb{R}^{3 \times 3}$	θ	Euler angle
k	Time step	κ	Scalar controller gain
k_{nm}	m th component of the n th controller gain matrix	λ	Longitude
K	Kalman Gain	μ	Gravitational parameter
K_n	The n th controller gain $\in \mathbb{R}^{6 \times 6}$	ρ	Density
l	Length of the rigid body $\in \mathbb{R}$	τ	External forces and torques $\in \mathbb{R}^6$
L	External torques $\in \mathbb{R}^3$	ϕ	Latitude
m	Mass of the rigid body or portion of the rigid body $\in \mathbb{R}$	φ	Retraction function $\mathbb{R}^{2p} \times \text{TSE}(3) \rightarrow \text{TSE}(3)$
$\dot{m}$	Mass flow rate $\in \mathbb{R}$	ψ	Backstepping function $\in \mathbb{R}^6$
$\mathcal{N}$	Inertial frame	ω	Angular velocity $\in \mathbb{R}^3$

II. Introduction

Researchers in rigid body dynamics for aerospace applications traditionally express the problem of orbital motion and attitude separately, treating them as distinct navigation and control problems. However, there are many environments in which these dynamics are inherently coupled, such as those with gravity gradient torques, solar radiation pressure, aerodynamic loads, and in the problem of rendezvous, proximity operations, and docking. Coupled formulations such as that on the special Euclidean group $\text{SE}(3)$ describing rigid body motions overcome this problem by considering the orbit/attitude dynamics simultaneously [1–5]. Furthermore, the use of coupled formulations offers new possibilities for accurate accommodation of dynamic mass properties, in the estimation and control problem.

When considering the problem of spacecraft state estimation, the use of various Kalman filtering methods is

ubiquitous. Previous efforts in this field have focused on demonstrating the efficacy of unscented Kalman filters (UKF) [6–8] and various controllers on $SE(3)$ [3, 9, 10]. Research has been done to demonstrate the behavior of stochastic processes in $SE(3)$ and the robustness of controllers on $SE(3)$ and its tangent bundle $TSE(3)$ [11, 12]. Yet, much work remains to demonstrate these methods in real-world applications where the assumptions of constant or known mass properties may not be realistic. Effort has been invested to demonstrate the breakdown of these assumptions in the launch and landing problem, in which mass properties change rapidly [13–15].

There is a well-known issue in the launch spacecraft problem where noise in attitude and angular velocity far exceeds that in the relatively precise position and velocity components of the pose [15–20]. The formalism on $SE(3)$ and $TSE(3)$ is an ideal candidate to be used in scenarios in which coupled dynamics are considered, capable of accurate position and velocity estimates and accounting for large attitude and angular velocity noise when provided with an adequate dynamic model. Additionally, autonomy for launch vehicles is in its infancy, with nearly all aspects of launch vehicle missions designed using a chronological series of events with a small tolerance to error. As the cost of launch services has continued to drop, the frequency of launches has increased, and the variety of launch environments broadened, the need for highly robust autonomy in launch vehicles has become evident. Thus, it is crucial to develop guidance, navigation, and control techniques that are tolerant to a wide range of configurations and scenarios, are capable of adapting to certain off-nominal scenarios.

In this research, the spacecraft launch problem is modeled on $SE(3)$ and $TSE(3)$. A UKF is employed to accommodate both measurement and process noise in all states, while estimates of mass properties are obtained using a deterministic assumption based on noisy measurements of mass flow rate. The capability of the UKF on $SE(3)$ and $TSE(3)$ to overcome a high degree of attitude and attitude rate ambiguity is assessed. Furthermore, a novel controller that demonstrates asymptotic stability in the presence of varying mass properties is introduced. This paper quantifies the accuracy, computational speed, and control effort of the proposed method and investigates the suitability of the $SE(3)$ formulation for use in launch systems. Specifically, the simulation results consider the operations of a Falcon 9 rocket starting at launch from a known, initial position at Kennedy Space Center and ending at payload deployment. The rocket is modeled as a two-stage launch system fueled by liquid oxygen (O_2) and rocket propellant 1 (RP-1).

The rest of the manuscript is organized as follows. In Section III, preliminaries of rigid body motion formulation on $SE(3)$ and $TSE(3)$ are introduced. The error dynamics are discussed in Section IV. Section V introduces the application of a UKF for state estimation and the methodology for parameter estimation. The estimation-based Morse-Lyapunov-based controller is introduced in Section VI. The case study of a launch problem using a Falcon 9 rocket is described in Section VII. Section VIII provides the numerical simulations and their discussion. The concluding remarks are given in Section IX.

III. Rigid Body Motion Using the Special Euclidean Group

The configuration, or pose (i.e. orbital position and attitude), of a rigid-body space vehicle is

$$g = \begin{bmatrix} R & r \\ 0_{1 \times 3} & 1 \end{bmatrix} \in \text{SE}(3) \quad (1)$$

where $R \in \text{SO}(3)$ is the rotation matrix from a rigid body frame $\mathcal{B}$ to some inertial frame $\mathcal{N}$ and r is the position vector of the center of mass with respect to and expressed in that inertial frame. The rigid body's velocity is

$$\mathbb{V} = [\omega^T, v^T]^T \in \mathbb{R}^6 \quad (2)$$

where $\omega \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ are the angular and translational velocities, respectively, both relative to the inertial frame and expressed in the body frame. From [21], it can be shown that the dynamic equations for a rigid body with variable mass properties can be expressed in TSE(3) as

$$\dot{g} = g \mathbb{V}^\vee \quad (3a)$$

$$\dot{\mathbb{V}} = \mathbb{I}^{-1} \left(\text{ad}_{\mathbb{V}}^* \mathbb{V} + \begin{bmatrix} -\dot{J} (\omega - R_{\mathcal{B}/\Delta} \omega_\Delta) - \dot{m} r_{\mathcal{B}/\Delta}^\times (v - R_{\mathcal{B}/\Delta} v_\Delta) \\ -\dot{m} (v - R_{\mathcal{B}/\Delta} v_\Delta) \end{bmatrix} + \tau + u \right) \quad (3b)$$

where the subscript Δ denotes the change induced by expelled mass. The rotation matrix $R_{\mathcal{B}/\Delta}$ denotes the moving, expelled mass with respect to the body frame. The combination of these terms may be expressed as the exit velocity of the expelled gasses in the form of $v_e = v - R_{\mathcal{B}/\Delta} v_\Delta$. The augmented inertia tensor $\mathbb{I}$ is defined as

$$\mathbb{I} = \begin{bmatrix} J & 0_{3 \times 3} \\ 0_{3 \times 3} & m I_3 \end{bmatrix} \in \mathbb{R}^{6 \times 6}, \quad (4)$$

where $J \in \mathbb{R}^{3 \times 3}$ is the inertia matrix about the center of mass and $m \in \mathbb{R}$ is the mass of the body. It is assumed that $\mathbb{I}$ is time varying but deterministic. That is, the value of $\mathbb{I}$ may be determined at each time step based on the mass flow rate or its estimate. This is elaborated upon in Section VII. The co-adjoint operator is defined as

$$\text{ad}_{\mathbb{V}}^* = \text{ad}_{\mathbb{V}}^T = \begin{bmatrix} -\omega^\times & -v^\times \\ 0_{3 \times 3} & -\omega^\times \end{bmatrix} \in \mathbb{R}^{6 \times 6} \quad (5)$$

In Eq. (3), $\tau = [L^T, f^T]^T \in \mathbb{R}^6$ represents the impressed dynamics acting on the system, with $L \in \mathbb{R}^3$ and $f \in \mathbb{R}^3$ representing all external torques and forces, respectively, both expressed in the $\mathcal{B}$ frame. Furthermore, the augmented

vector of control inputs is given by $u = \begin{bmatrix} u_L^T, u_f^T \end{bmatrix}^T \in \mathbb{R}^6$, where $u_L \in \mathbb{R}^3$ and $u_f \in \mathbb{R}^3$ represent control torques and forces, respectively.

For the purpose of developing the relative motion formulation used in the tracking control design, relative pose between i th and j th bodies is expressed as $g_{i/j} = g_j^{-1} g_i$. The adjoint mapping of this relative configuration is

$$\text{Ad}_{g_{\mathcal{M}/\mathcal{B}}} = \begin{bmatrix} R_{\mathcal{M}/\mathcal{B}} & 0 \\ r_{\mathcal{M}/\mathcal{B}}^\times R_{\mathcal{M}/\mathcal{B}} & R_{\mathcal{M}/\mathcal{B}} \end{bmatrix} \quad (6)$$

The transpose of the adjoint mapping above is defined as

$$\text{Ad}_{g_{\mathcal{M}/\mathcal{B}}}^T = \begin{bmatrix} R_{\mathcal{M}/\mathcal{B}}^T & -R_{\mathcal{M}/\mathcal{B}}^T r_{\mathcal{M}/\mathcal{B}}^\times \\ 0 & R_{\mathcal{M}/\mathcal{B}}^T \end{bmatrix} \quad (7)$$

Furthermore,

$$\text{Ad}_{g_{\mathcal{M}/\mathcal{B}}}^{T^{-1}} = \begin{bmatrix} R_{\mathcal{M}/\mathcal{B}} & R_{\mathcal{M}/\mathcal{B}} \left(R_{\mathcal{M}/\mathcal{B}}^T r_{\mathcal{M}/\mathcal{B}} \right)^\times \\ 0 & R_{\mathcal{M}/\mathcal{B}} \end{bmatrix} \quad (8)$$

IV. Error Dynamics

In the work described herein, two sources of error are assumed: Estimation error defined by $\hat{e} = x - \hat{x}$ and tracking error defined as $e_d = x - x_d$, where x is a generalized true state or parameter, $\hat{x}$ is the estimated state or parameter, and x_d is the desired state or parameter. The difference between these errors can be expressed as $\epsilon_x = \hat{e} - e_d = x - \hat{x} - x + x_d = x_d - \hat{x}$, thus illustrating that the true state is not needed in equations that express the relative state or parameters between estimated and desired elements, and that asymptotic stability about the tracked or desired state is achievable. This may be defined within $\text{SE}(3)$ and expressed as the relative pose between the estimated and desired state, i.e.

$$\begin{aligned} \epsilon_g = g_d^{-1} \hat{g} &= \begin{bmatrix} \hat{R}_d & \epsilon_r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_d^T \hat{R} & R_d^T (\hat{r} - r_d) \\ 0 & 1 \end{bmatrix} \\ \epsilon_{\mathbb{V}} &= \hat{\mathbb{V}} - \text{Ad}_{\epsilon_g} \mathbb{V}_d \end{aligned} \quad (9)$$

where $\text{Ad}_{\epsilon_g} : \text{SE}(3) \rightarrow \mathbb{R}^{6 \times 6}$ is the corresponding adjoint map. The desired states are determined by the reference trajectory, while the desired parameters are determined by an approximate deterministic model, explained in greater detail in Section VII.

According to Eq. (3b), and if it is assumed that the average angular velocity of the expelled mass, i.e. that the angular velocity induced on the body by the expelled gas, is zero, then $\omega_{\mathcal{B}} - R_{\mathcal{B}/\Delta}\omega_{\Delta} = 0$. The error dynamics for the augmented acceleration term can be expressed as

$$\epsilon_{\dot{V}} = \frac{d}{dt} \left(\hat{V} - \text{Ad}_{\epsilon_g} V_d \right) = \dot{\hat{V}} - \text{ad}_{\epsilon_V} \text{Ad}_{\epsilon_g} V_d - \text{Ad}_{\epsilon_g} \dot{V}_d \quad (10)$$

$$= \hat{\mathbb{I}}^{-1} \left(\text{ad}_{\hat{g}}^* \hat{\mathbb{I}} \hat{V} + \hat{\tau} - \text{Ad}_{\epsilon_g} \dot{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^{\times} \hat{v}_e \\ \hat{v}_e \end{bmatrix} \right) - \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \left(\text{ad}_{g_d}^* \mathbb{I}_d V_d + \tau_d - \dot{m}_d \begin{bmatrix} r_{\mathcal{B}/\Delta}^{\times} v_{e,d} \\ v_{e,d} \end{bmatrix} \right) - \text{ad}_{\epsilon_V} \text{Ad}_{\epsilon_g} V_d \quad (11)$$

Similar terms may be grouped to yield

$$\epsilon_{\dot{V}} = \mathbb{I}^{-1} \left(\text{ad}_{\hat{g}}^* \hat{\mathbb{I}} \hat{V} - \text{Ad}_{\epsilon_g} \text{ad}_{g_d}^* \mathbb{I}_d V_d \right) - \epsilon_{\dot{m}} + \epsilon_{\tau} - \text{ad}_{\epsilon_V} \text{Ad}_{\epsilon_g} V_d \quad (12)$$

where

$$\epsilon_{\tau} = \hat{\mathbb{I}}^{-1} \hat{\tau} - \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \tau_d \quad (13)$$

is the error between the estimated and desired inputs and

$$\epsilon_{\dot{m}} = \hat{\mathbb{I}}^{-1} \dot{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^{\times} \hat{v}_e \\ \hat{v}_e \end{bmatrix} - \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \dot{m}_d \begin{bmatrix} r_{\mathcal{B}/\Delta}^{\times} v_{e,d} \\ v_{e,d} \end{bmatrix} \quad (14)$$

is the error between the estimated and desired mass flow rates of the rigid body. It is of interest to define the relationship between ϵ_m and $\epsilon_{\dot{m}}$ by integrating Eq. (14) with respect to time. By performing integration by parts, and assuming that the acceleration of that gas is considered to be instantaneous, the stability of the closed loop can be analyzed.

Lemma IV.1. *The time rate of change of mass tracking error ϵ_m is linear with respect to the error in the mass flow rate $\epsilon_{\dot{m}}$. That is, $\frac{d}{dt} \epsilon_m \equiv \dot{\epsilon}_m \equiv k_3 \epsilon_m + \delta_m \in \mathbb{R}^6$.*

Proof. Eq. (14) can be integrated with respect to time to obtain

$$\epsilon_m = \int \left(\hat{\mathbb{I}}^{-1} \dot{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^{\times} \hat{v}_e \\ \hat{v}_e \end{bmatrix} - \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \dot{m}_d \begin{bmatrix} r_{\mathcal{B}/\Delta}^{\times} v_{e,d} \\ v_{e,d} \end{bmatrix} \right) dt \quad (15)$$

Integration by parts can be applied to the first term of the integrand in Eq. (14). With the knowledge that $\frac{d}{dt} \hat{\mathbb{I}}^{-1} = -\hat{\mathbb{I}}^{-1} \dot{\hat{\mathbb{I}}} \hat{\mathbb{I}}^{-1}$,

it can be shown that the following term inside the integrand can be expressed as

$$\int \hat{\mathbb{I}}^{-1} \dot{\hat{m}} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt = \hat{\mathbb{I}}^{-1} \int \hat{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt - \int \hat{\mathbb{I}}^{-1} \dot{\hat{\mathbb{I}}} \hat{\mathbb{I}}^{-1} \dot{\hat{m}} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt \quad (16)$$

such that

$$\int \dot{\hat{m}} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt = \hat{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} - \int \hat{m} \begin{bmatrix} \hat{v}_{\mathcal{B}/\Delta}^\times \hat{v}_e + \hat{r}_{\mathcal{B}/\Delta}^\times \left(\hat{a} - \dot{\hat{R}}_{\mathcal{B}/\Delta} v_\Delta - \hat{R}_{\mathcal{B}/\Delta} a_\Delta \right) \\ \hat{a} - \dot{\hat{R}}_{\mathcal{B}/\Delta} v_\Delta - \hat{R}_{\mathcal{B}/\Delta} a_\Delta \end{bmatrix} dt \quad (17)$$

using the definition for v_e above and where $\dot{\hat{R}}_{\mathcal{B}/\Delta} = \hat{R}_{\mathcal{B}/\Delta} \hat{\omega}_{\mathcal{B}/\Delta}^\times$, and it is assumed that there is no significant net angular component due to the expelled gasses, i.e. $\hat{\omega}_{\mathcal{B}/\Delta} = 0_{3 \times 1}$ and $\hat{v}_{\mathcal{B}/\Delta}^\times \hat{v}_e = 0_{3 \times 1}$ since $\hat{v}_e \approx \hat{v}_{\mathcal{B}/\Delta}$ and any vector crossed with itself is zero. Furthermore, the assumption can be made that the gasses are expelled at a constant exhaust velocity, thus the acceleration of exhaust gasses is zero. This simplifies the integral in Eq. (15) as

$$\epsilon_m = \hat{\mathbb{I}}^{-1} \hat{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} - \int \hat{\mathbb{I}}^{-1} \dot{\hat{\mathbb{I}}} \hat{\mathbb{I}}^{-1} \dot{\hat{m}} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt \quad (18)$$

Examining the term within the integrand in the above equation, performing matrix algebra yields

$$\int \hat{\mathbb{I}}^{-1} \dot{\hat{\mathbb{I}}} \hat{\mathbb{I}}^{-1} \dot{\hat{m}} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt = \int \begin{bmatrix} \hat{J}^{-1} \dot{\hat{J}} \hat{J}^{-1} \hat{m} & 0_{3 \times 3} \\ 0_{3 \times 3} & \left(\frac{\dot{\hat{m}}}{\hat{m}} \right)^2 \end{bmatrix} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt \quad (19)$$

For the relatively heavy launch vehicle studied in this paper, both the $\hat{J}^{-1} \dot{\hat{J}} \hat{J}^{-1} \hat{m}$ and $\left(\frac{\dot{\hat{m}}}{\hat{m}} \right)^2$ terms are in the order of 1×10^{-15} and 1×10^{-9} in the corresponding SI units, respectively, and thus are neglected. This leaves only the first

term in Eq. (18). Similarly, integration by parts can be applied to the second term in Eq. (15) to yield

$$\begin{aligned}
\int \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \dot{m}_d \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt &= \text{Ad}_{\epsilon_g} \int \mathbb{I}_d^{-1} \dot{m}_d \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt - \dots \\
&\dots \int \text{ad}_{\epsilon_v} \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} m_d \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt \\
&= \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} m_d \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt - \int \text{ad}_{\epsilon_v} \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} m_d \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} dt
\end{aligned} \tag{20}$$

In the expression above, the product of $\text{ad}_{\epsilon_v} \text{Ad}_{\epsilon_g}$ is a higher order term when the integration time step is sufficiently selected small, and hence it is neglected. Therefore, Eq. (20) simplifies as

$$\epsilon_m = \hat{\mathbb{I}}^{-1} \dot{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} - \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} m_d \begin{bmatrix} r_{\mathcal{B}/\Delta, d}^\times v_{e, d} \\ v_{e, d} \end{bmatrix} \tag{21}$$

It could then be stated that

$$\frac{\dot{m}}{\hat{m}} \epsilon_m = \hat{\mathbb{I}}^{-1} \dot{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^\times \hat{v}_e \\ \hat{v}_e \end{bmatrix} - \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \begin{bmatrix} r_{\mathcal{B}/\Delta, d}^\times v_{e, d} \\ v_{e, d} \end{bmatrix} m_d \frac{\dot{m}}{\hat{m}} \tag{22}$$

and subtracting this from Eq. (14) again yields

$$\begin{aligned}
\dot{\epsilon}_m - \frac{\dot{m}}{\hat{m}} \epsilon_m &= \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \begin{bmatrix} r_{\mathcal{B}/\Delta, d}^\times v_{e, d} \\ v_{e, d} \end{bmatrix} \left(\dot{m} \frac{m_d}{\hat{m}} - \dot{m}_d \right) = \delta_m \\
\dot{\epsilon}_m &= \frac{\dot{m}}{\hat{m}} \epsilon_m + \delta_m
\end{aligned} \tag{23}$$

where $\delta_m \in \mathbb{R}^6$ bounds the difference between the mass and mass flow rates based on the noise properties of the system.

The proof is complete. $\square$

Finally, the relative dynamics of a rigid body with a changing center of mass and moments of inertia in $\text{SE}(3)$ are represented in the dynamic model as

$$\dot{\epsilon}_g = \epsilon_g \epsilon_v^\vee \tag{24a}$$

$$\epsilon_{\hat{V}} = \mathbb{I}^{-1} \left(\text{ad}_{\hat{g}}^* \mathbb{I} \hat{V} - \text{Ad}_{\epsilon_g} \text{ad}_{g_d}^* \mathbb{I} V_d \right) - \epsilon_m + \epsilon_\tau - \text{ad}_{\epsilon_V} \text{Ad}_{\epsilon_g} V_d + u \quad (24b)$$

A modified version of the Morse-Lyapunov controller proposed in [9, 12] is introduced and augmented to consider the aforementioned changing mass properties using these dynamics in Section VI.

V. State and Parameter Estimation

The control is first calculated based on the estimated and desired states and parameters and then applied as an external force and torque vector during the integration. State estimation is then performed by a UKF, as outlined by [11]. Parameter estimation is distinct from state estimation and is performed based on the estimated mass flow rate via a throttle sensor.

Algorithm 1 Integration with estimated control on SE(3) with UKF

Given: Dynamical function $f_{\text{con}}(\hat{x}, t)$, a passive dynamical function $f(\hat{x}, u, \chi, t)$, and some measurement function $h(x, \eta)$
Initial true state: $x(g_k, \mathbb{V}_k)$
Initial estimated state: $\hat{x}(\hat{g}_{k|k}, \hat{\mathbb{V}}_{k|k})$
Process and Measurement noise: $\eta_{k,p}, \eta_{k,m}$
Covariance Matrices: $P_{k|k}, Q_k, R_k$
Sigma points, χ
for $k=1, 2, \dots$ **do**
 $u_k = f_{\text{con}}(\hat{x}_{k|k}, \chi)$
 Propagate to obtain *a priori* state estimates $\hat{x}(\hat{g}_{k+1|k+1}, \hat{\mathbb{V}}_{k+1|k+1}), P_{k+1|k+1}$ using $\hat{x}(\hat{g}_{k|k}, \hat{\mathbb{V}}_{k|k}), P_{k|k}, u_k, \eta_{k,p}, t_k$ in the propagation step of the UKF
 Obtain $\hat{x}(\hat{g}_{k+1}, \mathbb{V}_{k+1})$ using the function $f(\hat{x}_{k|k}, u_k, \chi, t)$
 Calculate $y_k = h(x_k, \eta_{k,m})$
 Update from the *a priori* state estimates $\hat{x}(\hat{g}_{k+1|k+1}, \hat{\mathbb{V}}_{k+1|k+1}), P_{k+1|k+1}$ to the *a posteriori* state estimates $\hat{x}(\hat{g}_{k+1}, \mathbb{V}_{k+1}), P_{k+1}$ using the update step in the UKF
 Measure mass flow rate at time step t_k to estimate $\hat{u}_k$. Then use $\hat{u}_k$ to update the center of mass, moments of inertia, and total mass based on estimated fuel burn
end for

The overall process is summarized in Algorithm 1, where the difference between the dynamical function $f_{\text{con}}(\hat{x}, t)$ and the passive dynamical function $f(\hat{x}, u, \chi, t)$ is that the former is used to calculate the control vector $u \in \mathbb{R}^6$ using a predictive model of mass flow rate as a function of throttle elaborated upon in Section V.B, while the latter takes in that control input as a disturbance force and torque. In other words, the control obtained by the dynamical function $f_{\text{con}}(\hat{x}, t)$ is used in the passive dynamical function $f(\hat{x}, u, \chi, t)$ to propagate the estimated states in the propagation step of the UKF. This is important since the use of the same dynamics $f_{\text{con}}(\hat{x}, t)$ within the propagation step of the UKF could result in volatile behavior in the state estimates, which are sensitive to the initial guess at each time step. The integration of the true dynamics using the function $f(\hat{x}, u, \chi, t)$ is performed using the variational integrator described in [22]. For the details on the UKF on TSE(3), including information regarding the propagation and update steps, the reader is referred to [4, 11, 13].

A. State Estimation using a UKF

The UKF performs state estimation in three steps: The sigma point selection step, the *a priori* state estimation in the prediction step, and the *a posteriori* estimate generation via the measurement update step. In the sigma point selection step, the sigma points are selected such that they are $2q + 1$ sigma points where q is the dimension of the measurement model such that $P_{k|k} \in \mathbb{R}^{2q \times 2q}$ is the total system covariance and p is the dimension of process noise such that $Q_k \in \mathbb{R}^{2p \times 2p}$ is the noise covariance. These dimensions are reflected in the sigma points through the matrix $\chi_p \in \mathbb{R}^{2p \times (4p+1)}$ of $4p + 1$ sigma-point column vectors $\chi_{p,i}$ and the matrix $\chi_q \in \mathbb{R}^{2q \times (4q+1)}$ of $4q + 1$ sigma column vectors $\chi_{q,i}$. Additionally, the weight matrices are defined as

$$\begin{aligned} W_p^{(m)} &= [W_{p,0}^{(m)}, W_{p,1}^{(m)}, \dots, W_{p,4p}^{(m)}] \\ W_p^{(c)} &= [W_{p,0}^{(c)}, W_{p,1}^{(c)}, \dots, W_{p,4p}^{(c)}] \end{aligned} \quad (25)$$

where

$$\begin{aligned} W_{p,0}^{(m)} &= \frac{\Lambda_p}{\Lambda_p + p}, \quad W_{p,0}^{(c)} = \frac{\Lambda_p}{\Lambda_p + p} + (1 - \alpha^2 + \beta) \\ W_{p,i}^{(m)} &= W_{p,i}^{(c)} = \frac{1}{2(\Lambda_p + p)} \quad (i = 1, 2, \dots, 4p) \end{aligned} \quad (26)$$

This same procedure is repeated for $W_q^{(m)}$ and $W_q^{(c)}$ except for using the corresponding $\Lambda_q = (\alpha^2 - 1)q$ instead of $\Lambda_p = (\alpha^2 - 1)p$. Next, an *a priori* prediction of the state at time step $\hat{x}_{k+1|k}$ is generated based on the dynamic model and the state $\hat{x}_{k|k}$ and covariance $P_{k|k}$ and the sigma point vectors $\chi_{q,i}, \chi_{p,i}$. Since the states of the system are in TSE(3), a retraction function $\varphi : \mathbb{R}^{2p} \times \text{TSE}(3) \rightarrow \text{TSE}(3)$ is introduced [23, 24], which is a smooth, arbitrarily chosen function that encodes the mean and covariance noise in the Lie group and its tangent bundle. The retraction function is used to encode the sigma points onto the manifold, while its inverse is used to decode the sigma points from the manifold. Given the current optimal estimated state $\hat{x}_{k|k}$, the first set of sigma points are retracted into the manifold and then used to propagate the system dynamics starting from $\hat{x}_{k|k}$. Finally, once $\hat{x}_{k+1|k}$ and $P_{k+1|k}$ are calculated, an *a posteriori* measurement is generated with the help of new measurements $z_k \in \mathbb{R}^n$. The same retraction function maps the measurements in Euclidean space onto the TSE(3) manifold to correct the estimated states and obtain new states and covariances $\hat{x}_{k+1|k+1}$ and $P_{k+1|k+1}$. The *a posteriori* state estimate may be expressed as

$$\hat{x}_{k+1|k+1} = \varphi(\hat{x}_{k+1|k}, K_{k+1} z_{k+1}) \quad (27)$$

where K_{k+1} is the Kalman gain and $\tilde{Z}$ is the residual, i.e. the discrepancy, between the estimated measurement $\bar{z}_{k+1}$ from the *a priori* predictions and the actual measurement z_{k+1} . It is clear that K_{k+1} acts as a weighting factor for the residual.

In particular, when the measurement is corrupted and T assumes large values, the Kalman gain and the residual weight are low. Therefore, this gain allows one to have an optimal estimate weight for the measurement received based on its reliability. Finally, the *a posteriori* error covariance matrix is computed as

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1} P_{zz,k+1} K_{k+1}^T \quad (28)$$

B. Parameter Estimation Using Thrust Measurement

Parameter estimation is performed by measuring the mass flow rate of the rocket with knowledge of thruster performance and vehicle geometry. Suppose the total fuel mass, the fuel-to-oxidizer ratio, and the geometry of the fuel tanks are all known. In that case, thrust measurements can be converted to estimated mass flow rate and, in turn, extended to estimated total mass, center of mass, and moments of inertia. Actuator noise exists and it is converted into an estimate of mass flow rate $\dot{m}$, and thus a difference between reference (or desired) mass m_d and estimated mass $\hat{m}$ accumulates throughout the burn time but remains bounded as the noise is Gaussian in nature. The method of mass property estimation in this manner depends on vehicle geometry, where the fill height of each tank is reduced based on the mass flow rate at each time step. The location and geometry of the tanks are assumed to be known. Reference mass is determined through the same method as estimated mass, but using the desired thrust as commanded by the Morse-Lyapunov controller instead of measured thrust. Specifically, the noisy measurement of thrust at each time step is converted into mass flow rate and subtracted from the mass at the preceding time step, thus generating an estimated mass at each time step $k + 1$, e.g. $\hat{m}_{k+1} = \hat{m}_k + \dot{m}_k dt$ where

$$\dot{m}_k = \frac{\|T\| - (p_e - p_o)A_e}{\|v_e\|} \quad (k = 1, 2, \dots, \text{number of time steps}) \quad (29)$$

where T is the thrust vector, p_e is the exit pressure, p_o is the ambient pressure, $\|v_e\|$ is the magnitude of the velocity of expelled gasses discussed earlier, and A_e is the nozzle exit area. This estimated mass is then used to determine mass properties in the same way as described above. Mass error is reset to zero after separation of the first stage as the mass in the second stage has not changed since its initial true state and is thus known.

VI. State-and-Parameter-Error-Based Morse-Lyapunov Adaptive Controller on TSE(3) Utilizing the Backstepping Technique

Using the notation described in Section IV, the dynamics in Eq. (3b) may be expressed as a tracking controller as

$$\dot{\epsilon}_v = \mathbb{I}^{-1} \left(\text{ad}_g^* \hat{\mathbb{V}} - \text{Ad}_{\epsilon_g} \text{ad}_{g_d}^* \mathbb{V}_d \right) + \epsilon_\tau - \dot{\epsilon}_m + u - \text{ad}_{\epsilon_v} \text{Ad}_{\epsilon_g} \mathbb{V}_d \quad (30)$$

Lemma VI.1. *The state- and parameter-estimation-based Morse-Lyapunov controller is defined as*

$$u = -\mathbb{I}^{-1} \left(\text{ad}_{\hat{g}}^* \mathbb{I} \hat{V} - \text{Ad}_{\epsilon_g} \text{ad}_{g_d}^* \mathbb{I} V_d \right) - \epsilon_\tau + \dot{\epsilon}_m + \text{ad}_{\epsilon_V} \text{Ad}_{\epsilon_g} V_d - \mathbb{I} K_1 \dot{l} - \mathbb{I} K_2 \psi - \mathbb{I} \kappa [0_{1 \times 3}^T (\epsilon_R \epsilon_r)^T]^T \in \mathbb{R}^6 \quad (31)$$

results in asymptotic stability of the closed loop in the presence of changing mass properties, where $K_1 = \text{blkdiag}(k_{11}I_3, k_{12}I_3) \in \mathbb{R}^6 \geq 0$ and $K_2 = \text{blkdiag}(k_{21}I_3, k_{22}I_3) \in \mathbb{R}^{6 \times 6} \geq 0$ are positive definite diagonal gain matrices and $\kappa \in \mathbb{R}$ is a scalar gain coefficient. The backstepping function is defined as

$$\psi = \epsilon_V + K_1 l(\epsilon_g) \in \mathbb{R}^6 \quad (32)$$

where

$$l(\epsilon_g) = [s^T(\epsilon_R), \epsilon_r^T]^T \in \mathbb{R}^6 \quad (33)$$

such that

$$s(\epsilon_R) = \sum_{i=1}^3 a_i \left(\epsilon_R^T \check{e}_i \right)^\times \check{e}_i \in \mathbb{R}^3 \quad (34)$$

and $a \in \mathbb{R}^3$ is a matrix of tunable gain values, where $1 \leq a_1 < a_2 < a_3$ and $\check{e}_i$ ($i = 1, 2, 3$) are unit vectors aligned with the vehicle's principal axes.

Proof. Taking the derivative of Eq. (32) yields

$$\dot{\psi} = \dot{\epsilon}_V + K_1 \dot{l}(\epsilon_g) \quad (35)$$

From Eqs. (21), (30), and (31) it can be shown that

$$\dot{\psi} = K_2 \psi - \mathbb{I} \kappa [0_{3 \times 1}^T, (\epsilon_R \epsilon_r)^T]^T \quad (36)$$

where ϵ_R is the upper-left 3×3 partition of ϵ_g and ϵ_r is the upper-right 3×1 partition of ϵ_g . Next, define a Lyapunov equation $V = V_1 + V_2$ such that

$$\begin{aligned} V_1 &= \psi^T P \psi + \kappa (1 - k_{12}) r^T P_{22} r \\ V_2 &= \epsilon_m^T P \epsilon_m \end{aligned} \quad (37)$$

where $P \geq 0$ and diagonal. It has been shown in [9] that $\dot{V}_1 < 0$, so it remains to demonstrate that $\dot{V}_2 < 0$. As shown in Lemma 1, $\dot{\epsilon}_m = \epsilon_{\dot{m}}$. Therefore,

$$\dot{V}_2 = 2 \epsilon_m^T P \epsilon_{\dot{m}} \quad (38)$$

Since $\epsilon_{\dot{m}} = \frac{\dot{m}}{\hat{m}}\epsilon_m + \delta_m$ and the values ϵ_m and $\epsilon_{\dot{m}}$ are assumed deterministic at each time step, Eq. (38) becomes

$$\dot{V}_2 = 2\epsilon_m^T P \left(\frac{\dot{m}}{\hat{m}}\epsilon_m + \delta_m \right) \quad (39)$$

where δ_m is defined in Eq. (23). It can further be stated that

$$\dot{V}_2 = 2\epsilon_m^T P \left(\frac{\dot{m}}{\hat{m}}I_6 + \delta_m(\epsilon_m^T \epsilon_m)^{-1} \epsilon_m^T \right) \epsilon_m < 0 \quad (40)$$

Note that $\min(\|\epsilon_m\|) = 0$ at the start of the simulation and approaches zero at convergence, and $\|\delta_{m,\max}\|$ is dominated by the $(\frac{\dot{m}}{\hat{m}}\frac{m_d}{\hat{m}} - \dot{m}_d)$ term which, assuming a Gaussian distribution in the mass flow rate uncertainty, becomes $(\frac{\dot{m}}{\hat{m}} - \dot{m}_d)$ as $t \rightarrow \infty$. Each system has a different upper bound, but in each case, $(\dot{m}_d(1 + \eta_m) - \dot{m}_d) = \dot{m}_d\eta_m$ where η_m is the standard deviation of the mass flow rate noise. The result is that the terms in the parentheses can be expressed as

$$\frac{\dot{m}}{\hat{m}}I_6 + \dot{m}_d\eta_m \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \begin{bmatrix} r_{\mathcal{B}/\Delta,d}^{\times} v_{e,d} \\ v_{e,d} \end{bmatrix} (\epsilon_m^T \epsilon_m)^{-1} \epsilon_m^T < 0 \quad (41)$$

All that remains is to examine the vector product between the bracketed term and ϵ_m . The bracketed expression multiplied by the second term in Eq. (21) is a vector multiplied by the transpose of itself, resulting in a positive definite matrix. This only leaves the bracketed term multiplied by the first term in Eq. (21) which may be rewritten as

$$\begin{aligned} & \dot{m}_d\eta_m \frac{1}{\|\epsilon_m\|^2} \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \begin{bmatrix} r_{\mathcal{B}/\Delta,d}^{\times} v_{e,d} \\ v_{e,d} \end{bmatrix} \left(\hat{\mathbb{I}}^{-1} \hat{m} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^{\times} \hat{v}_e \\ \hat{v}_e \end{bmatrix} \right)^T \\ &= \dot{m}_d\eta_m \frac{\hat{m}}{\|\epsilon_m\|^2} \text{Ad}_{\epsilon_g} \mathbb{I}_d^{-1} \begin{bmatrix} r_{\mathcal{B}/\Delta,d}^{\times} v_{e,d} \\ v_{e,d} \end{bmatrix} \begin{bmatrix} \hat{r}_{\mathcal{B}/\Delta}^{\times} \hat{v}_e \\ \hat{v}_e \end{bmatrix}^T \hat{\mathbb{I}}^{-T} \in \mathbb{R}^{6 \times 6} \\ &= \dot{m}_d\eta_m \frac{\hat{m}}{\|\epsilon_m\|^2} \text{Ad}_{\epsilon_g} \begin{bmatrix} J_d^{-1} \left(r_{\mathcal{B}/\Delta,d}^{\times} v_{e,d} \right) \left(\hat{r}_{\mathcal{B}/\Delta}^{\times} \hat{v}_e \right)^T \hat{J}^{-T} & J_d^{-1} \frac{(r_{\mathcal{B}/\Delta,d}^{\times} v_{e,d}) \hat{v}_e^T}{\hat{m}} \\ \frac{v_{e,d} (\hat{r}_{\mathcal{B}/\Delta}^{\times} \hat{v}_e)^T}{m_d} \hat{J}^{-T} & \frac{v_{e,d} \hat{v}_e^T}{m_d \hat{m}} \end{bmatrix} \in \mathbb{R}^{6 \times 6} \end{aligned} \quad (42)$$

Each term in the matrix is dominated by inverse mass and inertia coefficients, on the order of 1×10^{-9} , varying by one or two orders of magnitude in either direction depending on the system. This is further compounded by the $\frac{1}{\|\epsilon_m\|^2}$ which is on the order of 1×10^{-6} . Thus, the entire second term may be neglected, as the ratio of mass flow rate to total mass is far more negative (on the order of 1×10^{-3} , give or take an order of magnitude depending on the system). This

simplifies $\dot{V}_2$ to

$$\dot{V}_2 = 2 \frac{\dot{m}}{\hat{m}} \epsilon_m^T P \epsilon_m < 0 \quad (43)$$

□

VII. Case Study: Falcon 9 Launch Scenario

In this scenario, a Falcon 9 is initialized in the topocentric-horizon coordinate system at the ground station (GS) of reference located at Kennedy Space Center [25]. A transformation from one frame to another may be defined as the product of one or more rotation matrices. Specifically, $R_{\oplus/\text{GS}} = R_1(\lambda)R_3(\phi)$, where λ is the latitude and ϕ is the longitude of the ground station, and subscripts ‘1’ and ‘3’ denote rotations about the first and third axes, respectively. This transformation maps vectors from the Earth-centered Earth-fixed (ECEF) frame, denoted by ‘ $\oplus$ ’, into the ground station frame, which is assumed to be the same as the Falcon 9’s body frame at the start of the simulation. Also, another transformation is needed to map the $\oplus$ frame into the Earth-centered inertial (ECI) frame, denoted by ‘ $\mathcal{N}$ ’. The initial values for the true and estimated pose and velocities (i.e., initial true and estimated states) are all assumed to be zero with respect to the ground station. of the reference trajectory and initial pose of the estimated and true state are all initialized at this initial position and zero velocity with respect to the ground station. Launch complex 40 (LC-40) is located at $\phi = 28^\circ 33.72'$ latitude and $\lambda = 80^\circ 34.630'$ longitude. Furthermore, the true and estimated attitudes of the rocket are assumed to be aligned with the coordinates of the ground station. The initial position and velocity of the Falcon 9 with respect to the ECI coordinates is calculated by

$$\begin{aligned} {}^{\mathcal{N}}r_0 &= (r_e + h_{slc40})R_{\mathcal{N}/\oplus} [\cos \phi \cos \lambda, \cos \phi \sin \lambda, \sin \phi]^T \in \mathbb{R}^3 \\ {}^{\mathcal{N}}v_0 &= {}^{\oplus}\omega_e \times {}^{\oplus}r_0 = \|\omega_e\|(r_e + h_{slc40}) [-\cos \phi \sin \lambda, \cos \phi \cos \lambda, 0]^T \in \mathbb{R}^3 \end{aligned} \quad (44)$$

where r_e is the mean radius of the Earth and h_{slc40} is the altitude of LC-40 above mean equator, which is assumed to be equal to zero, ${}^{\oplus}\omega_e = (2\pi/T_e) \hat{e}_3$ is the rotational velocity of the Earth relative to the $\oplus$ frame, where T_e is the period of one rotation of the Earth about its rotational axis $\check{e}_3$, and $R_{\mathcal{N}/\oplus} = R_3(\theta_e)$ is the matrix that transforms vectors from the $\oplus$ coordinates to ECI coordinates and a function of the Earth rotation angle, $\theta_e(T_u)$. The Earth rotation angle (θ_e) is a linear function of T_u , which is the number of solar days that have transpired between January 1, 2000, at noon and the time of launch.

A. Point Mass Reference Model

The translational accelerations in the ECI coordinate system are expressed as

$$\ddot{r} = \frac{T(t) + D}{m(t)} + a_g(r) \in \mathbb{R}^3 \quad (45)$$

where $T(t) \in \mathbb{R}^3$ is the thrust vector as a function of time, $D \in \mathbb{R}^3$ is the atmospheric drag vector, $m(t)$ is the Falcon 9's mass as a function of time, and $a_g(\cdot) \in \mathbb{R}^3$ is the acceleration due to gravity. The force components, i.e. f from τ in Eq. (3b), can be related to the acceleration $\ddot{r}$ above by $f = m\ddot{r} - T(t)$. The thrust vector is calculated by

$$T = \|T(t)\| (\cos \alpha {}^\oplus\check{e}_1 + \sin \alpha {}^\oplus\check{e}_2) \in \mathbb{R}^3 \quad (46)$$

where $\|T(t)\|$ is the thrust magnitude as a function of time ${}^\oplus\check{e}_1$ and ${}^\oplus\check{e}_2$ are unit vectors pointing in the radial direction and tangential direction in the ECEF frame, respectively. These are calculated using the rocket's position with respect to the Earth (${}^\oplus r \in \mathbb{R}^3$), i.e.

$$\begin{aligned} {}^\oplus\check{e}_1 &= \frac{{}^\oplus r}{\|{}^\oplus r\|} \\ {}^\oplus\check{e}_2 &= \frac{({}^\oplus r \times {}^\oplus v) \times {}^\oplus r}{\|({}^\oplus r \times {}^\oplus v) \times {}^\oplus r\|} \end{aligned} \quad (47)$$

Note that the angle α is a piece-wise function that points in the gravity direction until the pitchover maneuver, from then until main engine cutoff (MECO) and again between second engine start (SES), the thrust vector is calculated by $T = -\|T(t)\| {}^\mathcal{N}\check{v}$ such that ${}^\mathcal{N}\check{v}$ is the velocity of the spacecraft in the inertial frame, and then at a fixed angle until second engine cutoff (SECO).

The atmospheric drag D is modeled as

$$D = -\frac{1}{2} c_d \rho a_r \|v_{ta}\| v_{ta} \in \mathbb{R}^3 \quad (48)$$

such that $v_{ta} = {}^\oplus\omega_e \times {}^\oplus r$ and $v_{ta} = \|v_{ta}\|$ and where c_d is the drag coefficient, ρ is the atmospheric density, A_r is the Falcon 9's reference area, $v_{ta} \in \mathbb{R}^3$ is the true air velocity. The drag coefficient and the reference area are treated as constants while the atmospheric density is calculated as a function of altitude above the mean equator using the ARDC 1959 Standard Atmosphere Model.

B. Rigid Body Model

The mass values listed above describe the deterministic, modeled mass. The rigid body is modeled as a collection of hollow and rigid cylinders as illustrated in Figure 1, where the orange and blue sections represent RP-1 and liquid oxygen fuel, respectively, and the dark grey cylinders represent solid sections with unchanging mass. As the fuel is depleted, the cylinders corresponding to the fuel decrease in height with respect to the bottom of the rocket linearly with time. When the burn time of each respective stage is completed, these stages are no longer considered in the integrated rocket model, simulating a stage separation. The stages have an empty, 'dry' mass that is considered separate from the

fuel and are modeled as hollow cylinders with a wall thickness of 1 cm. The center of mass is likewise calculated at each time step. At each time step, the rocket equation is implemented with known, constant values for specific impulse I_{sp} and sea level gravitational acceleration $g_0 = 9.807 \text{ m/s}^2$. From the values mentioned above, the mass flow rate is obtained and the mass change in each fuel cylinder is then determined using the oxidizer-to-fuel ratio. Then, the moments of inertia, center of mass, and total mass are updated accordingly.

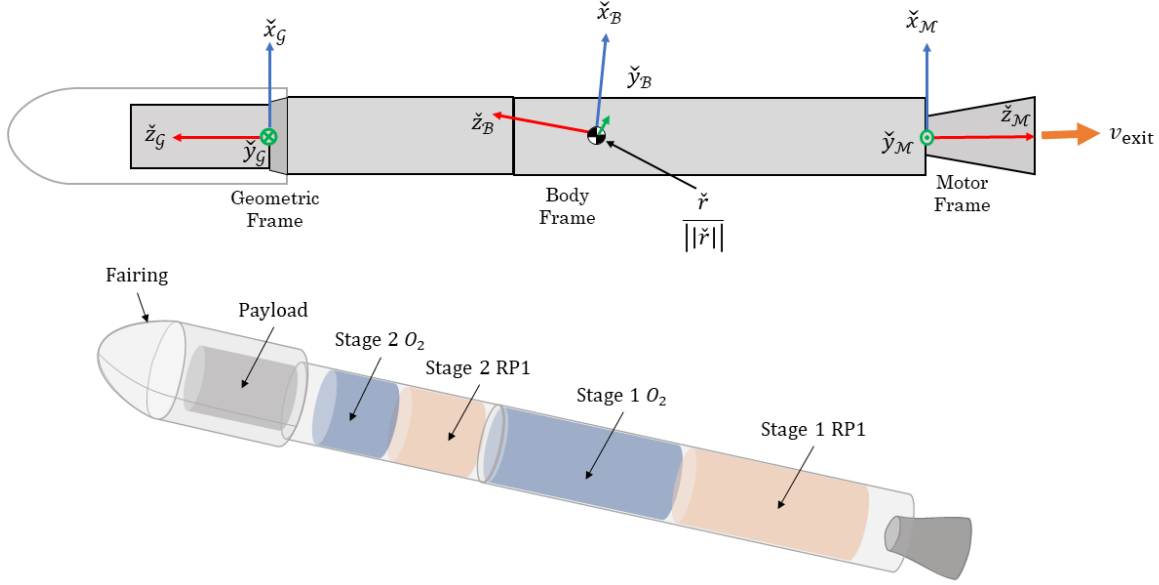


Fig. 1 Falcon 9 launch vehicle dimensions and mass breakdown.

For the rigid body simulation, gravity gradient torque is considered and may be expressed as

$$L = 3 \frac{\mu}{\|r\|^3} \left(1 + \frac{3}{m(t)\|r\|^2} \left(J(t) + \frac{1}{2} \text{tr} \left(J(t) - 5^B \check{r}^T J(t)^B \check{r} \right) I_3 \right) \right) r \quad (49)$$

where μ is the gravitational parameter of the central body and ${}^B \check{r} = R^T \frac{r}{\|r\|}$.

VIII. Numerical Simulations and Discussion

To generate the reference trajectory, a simulated Falcon 9 is launched due east from SLC-40 at Cape Canaveral Space Force Station, Florida. The reference trajectory is divided into eight events listed in Table 1.

A pitchover maneuver is set based on the specific conditions of each launch and it occurs at the time at which the rocket stops thrusting directly against the gravity vector and begins applying tangential velocity. At this point, the launch vehicle is at the highest angle of attack. The Falcon 9's thrust magnitude depends on staging. For the first stage, the thrust magnitude is modeled as a linear curve with a magnitude ranging from $7686 < \|T\| < 8227 \text{ kN}$ during a

burn time of $t_b = 162$ s. For the second stage, the thrust magnitude is modeled as a constant value of 981 kN during a burn time of 397 s. An estimated I_{sp} of 300 N, a standard gravity of $g_0 = 9.81$ m/s², and an oxidizer-to-fuel ratio of $\gamma_{O_2,RP-1} = 2.56$ is used.

The initial mass of the Falcon 9 at launch is 553,600 kg, which is the sum of the fully fueled mass of the first stage (433,100 kg), the fully fueled mass of the second stage (111,500 kg), the mass of the payload fairing housing (4,400 kg), and the mass of the payload (4,600 kg). During the time from Launch to MECO, the first-stage propellant is assumed to burn at a constant mass flow rate of 2536.42 kg/s. At MECO, the first stage is shut down and the vehicle's mass is calculated to be 185,700 kg. Three seconds later, during main engine separation (MES), the first-stage structure is jettisoned making the rocket's mass 120,500 kg. This is also the rocket's mass as the second stage ignites eight seconds later during SES. During the time from SES to SECO, the second-stage propellant is assumed to burn at a constant mass flow rate of 270.781 kg/s and engines are gimbaled at a fixed angle of 3.8° in the reference trajectory. At the same time, the payload fairing doors are jettisoned and the mass of the vehicle becomes 108,048 kg. At SECO, the second stage is shut down and the vehicle's mass is 15,419 kg. The second stage is jettisoned and, at the final time step, the only mass that remains is that of the payload. The sequence of these events can be seen in Table 1, and the vehicle properties are listed in Table 2.

The thrust dynamics of the Falcon 9 depend on staging. The assumptions used in this model is that full thrust is applied during the first stage, and the transition between the first and second stage occurs instantaneously. This model treats the rocket as having a single thruster on each stage with a total thrust of 7607 kN and 934 kN for the first and second stages, respectively.

The initial velocity of the Falcon 9 with respect to the Earth in ECI coordinates is calculated by $v = R_{\oplus}^{\oplus} v$. A comparison between the reference, true, and estimated trajectories is shown in Figure 3 and 4 where the point mass reference trajectory, true trajectory, and estimated trajectory are shown in black, green, and blue, respectively. Although similar, small variations in the final state are visible among each trajectory. The controller used for the numerical simulation uses the aforementioned reference model for both state and mass properties relative to the estimated state and mass properties to correct for the perceived error. Prior to the pitchover maneuver, the controller is not active,

Table 1 Sequence of events for the launch vehicle simulation.

Sequence	Name	Acronym	Time (min)
1	launch	n/a	0
2	pitchover maneuver	t_{pitch}	1.3
3	main engine cutoff	MECO	2.42
4	main engine separation	MES	2.46
5	second engine start	SES	2.6
6	second engine cutoff	SECO	9

Table 2 Rocket properties

Stage 1 Values	
first-stage empty mass (kg)	$m_{s1} = 25600$
propellant mass (kg)	$m_{p1} = 395700$
O-F ratio	$\gamma_{O_2,RP-1} = 2.56$
$\dot{m}$ (kg/s)	$\dot{m} = 2321.6$
dimensions (m)	$r = 3.65, l = 36.83, r_{th} = 0.01$
inertia (kg m ²)	$J_0 = \text{diag} ([7604291, 7604291, 2975979])$
Stage 2 Values	
empty mass (kg)	$m_{s2} = 2900$
propellant mass (kg)	$m_{p2} = 92670$
O-F ratio	$\gamma_{O_2,RP-1} = 2.56$
$\dot{m}$ (kg/s)	$\dot{m} = 233.4$
dimensions (m)	$r = 3.65, l = 9.1, r_{th} = 0.01$
inertia (kg m ²)	$J = \text{diag} ([884642, 576846, 655827])$
payload values	
mass (kg)	$m_{pl} = 4600$
dimensions (m)	$l = 4, w = 4, h = 6$
inertia (kg m ²)	$J = \frac{m}{12} \text{diag} ([40679, 40679, 62192])$
noise properties	
η_r (km)	± 1
η_v (km/s)	± 0.1
η_a (km/s ²)	± 0.001
η_θ (rad)	± 0.5
η_ω (rad/s)	± 0.05
$\eta_{\dot{\omega}}$ (rad/s ²)	± 0.005

using instead simple radial thrust in the ECEF frame. This allows uncertainty to accumulate in both the states and parameters before activating the controller. This is visible in Figure 4, where the error in flight path angle is corrected upon controller activation, with the peak error occurring at the pitchover maneuver. The selected gain coefficients in Eqs. (31) are

$$k_{11} = 0.095, k_{12} = 0.001, k_{21} = 1 \times 10^{-6}, k_{22} = 2, \text{ and } \kappa = 0.001 \quad (50)$$

with

$$a_1 = 1.1, a_2 = 1.2 \text{ and } a_3 = 1.3. \quad (51)$$

In Figure 2, the ability of the UKF on SE(3) to overcome high levels of ambiguity in attitude and attitude rate is demonstrated, with the estimated states remaining well within the 3σ bounds. Furthermore, as seen in Figure 5, the controller is able to track the reference trajectory despite large errors in thrust and mass estimates based on the error described by Eqs. (14) and (21), and mass discontinuities due to staging. Actuator noise is a percentage of total thrust

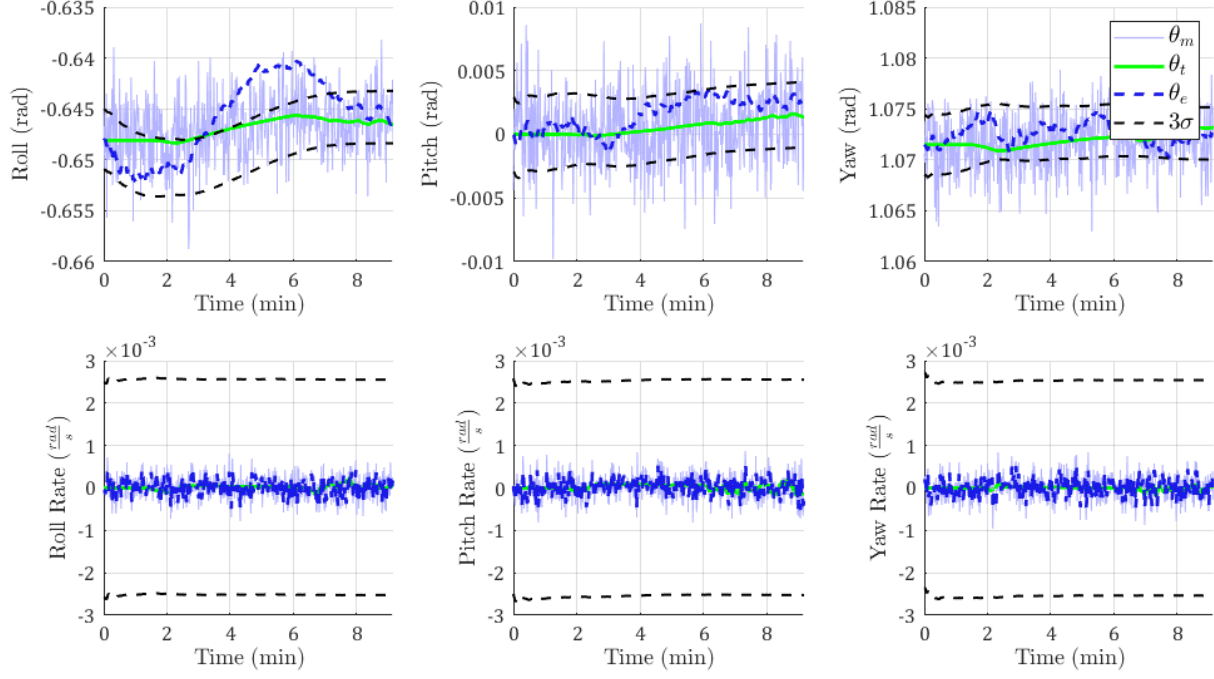


Fig. 2 The measured, true, and estimated attitude of the rocket with 3σ bounds.

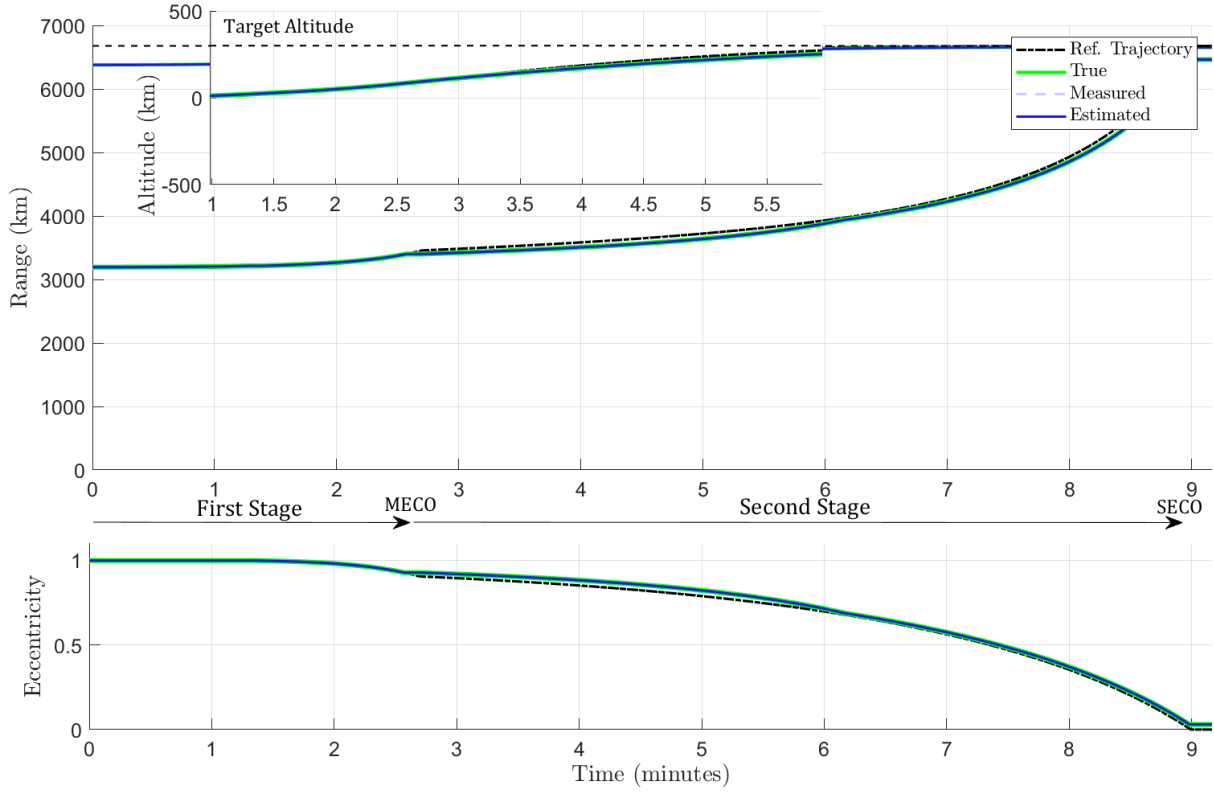


Fig. 3 The range, altitude, and eccentricity of the point mass reference model, the true rigid body, and the estimated rigid body.

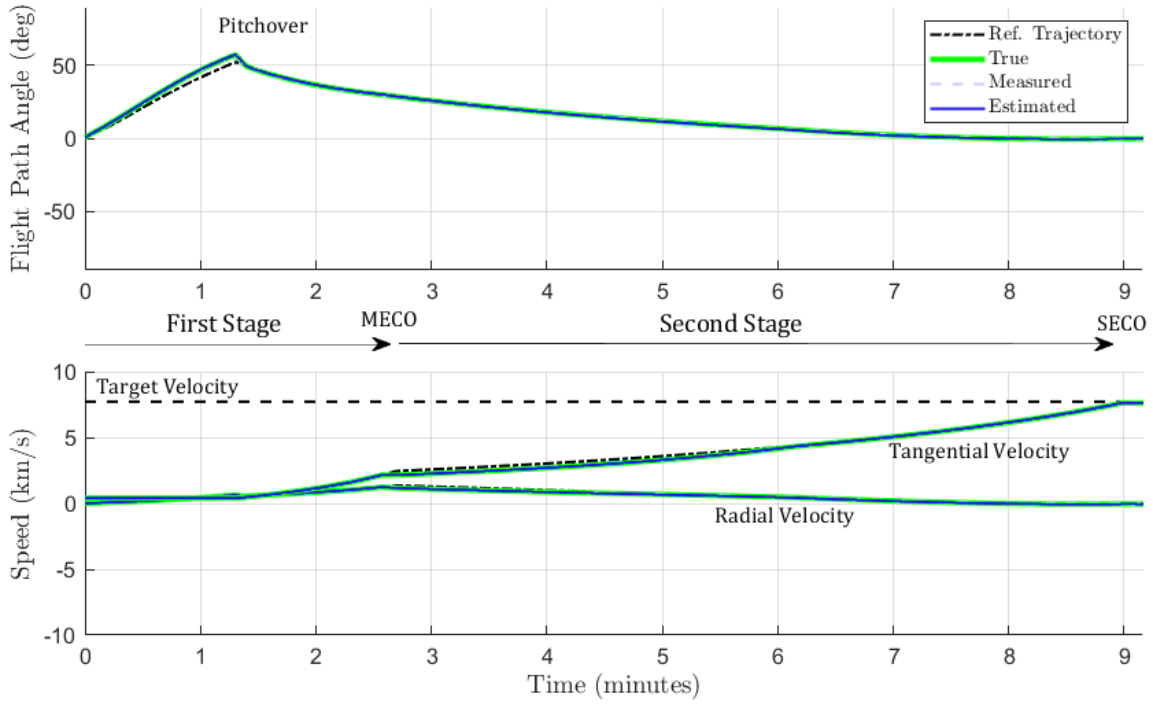


Fig. 4 The angle of attack and velocity components of the point mass reference model, the true rigid body, and the estimated rigid body.

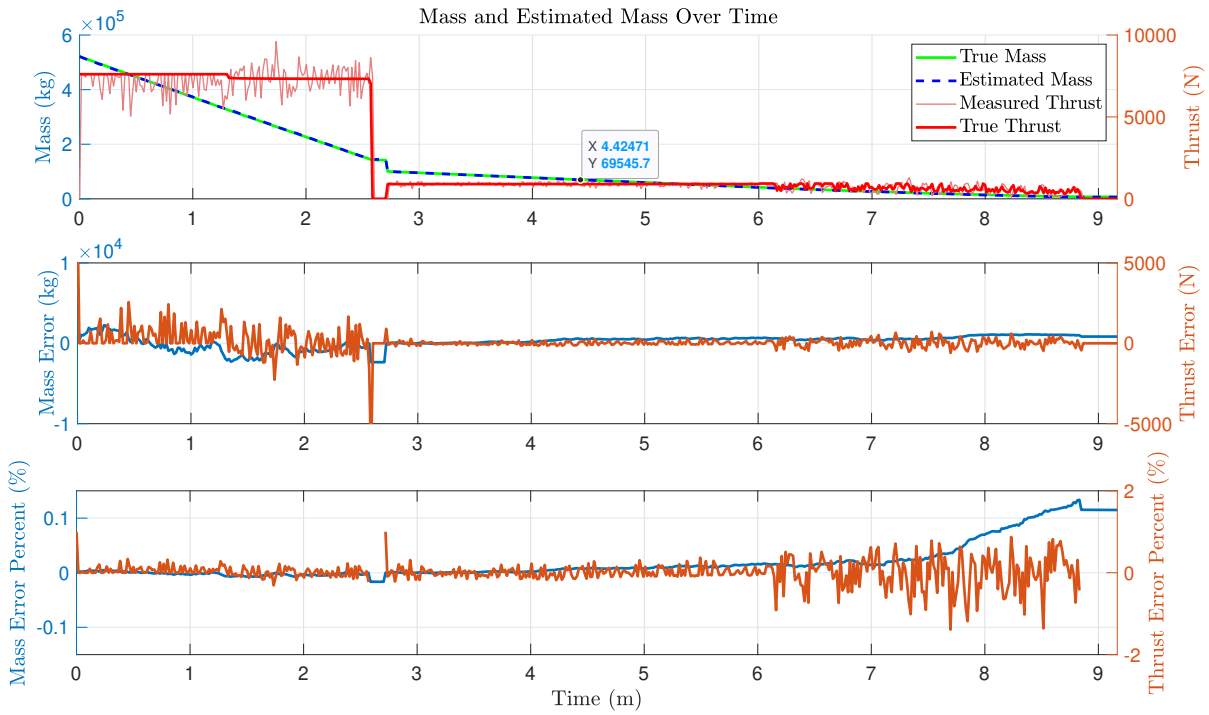


Fig. 5 Thrust and total mass over the course of the simulation.

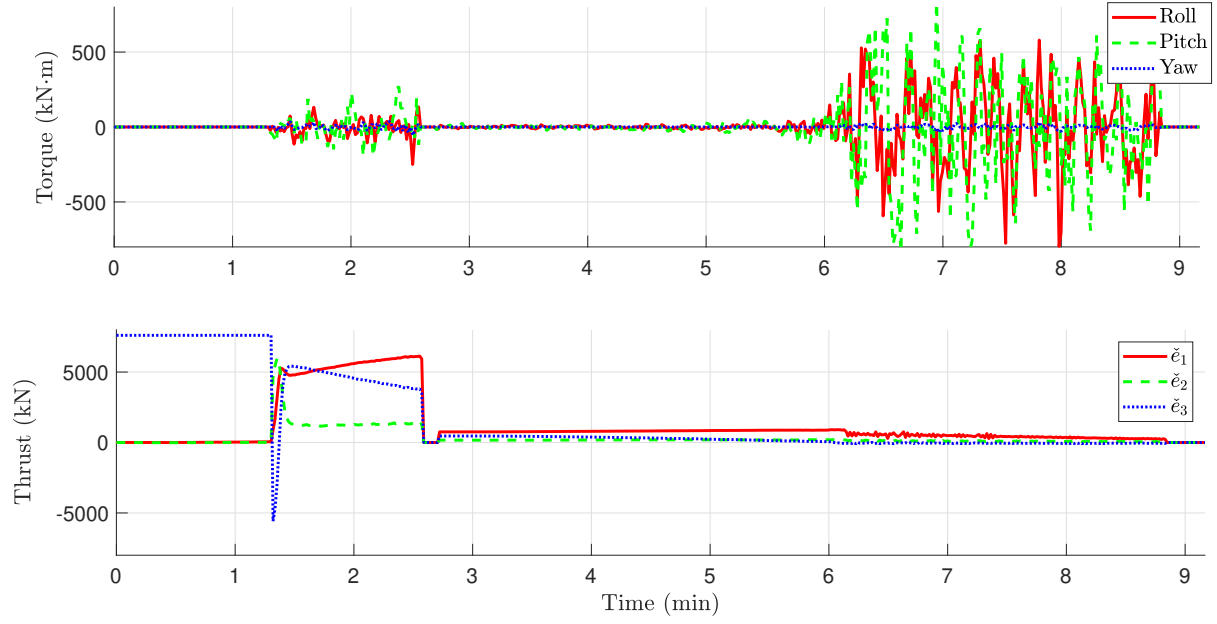


Fig. 6 Control input about all six degrees of freedom of motion over time.

and thus is significantly less in the second stage when compared with the first stage. The activation of the controller at 1.3 minutes into the simulation is visible as a change in thrust in the first plot of the figure, and then MECO followed by stage separation, and second engine start. The change in thrust at around 6.4 minutes is due to the estimated state converging with the reference trajectory, resulting in a throttling of the engines. The top plot in Figure 3 demonstrates the range and altitude of the rocket over time, with the desired circular orbit at 300 km in altitude indicated by a black dotted line. Since the destination orbit is circular, the eccentricity is tracked in the lower plot in Figure 3, which nearly approaches zero at the end of the simulation, with a small error in position and velocity. The true (green) and estimated (blue) states are nearly overlapped with one another as well as the reference trajectory (black dotted line). Finally, the control input about each degree of freedom is plotted in Figure 6, demonstrating that the control input is effectively constrained within the performance limits of the Falcon 9 vehicle.

IX. Conclusions

This paper has demonstrated the performance of a state-and-parameter-estimation-based Morse-Lyapunov controller with backstepping considering noisy measurements and parameter uncertainties. Specifically, the performance of the controller is studied in the presence of parameter uncertainties in total mass, the center of mass, and moments of inertia. An unscented Kalman filter (UKF) has been introduced in special Euclidean space and its tangent bundle to filter noisy state estimates in attitude, position, velocity, and angular velocity. Furthermore, noisy measurements of thrust have been considered to generate an estimated mass model that differs from the reference model. The controller

was able to autonomously navigate the rocket to the desired circular low Earth orbit with very low error despite these noisy quantities in both states and mass flow rate. This shows that asymptotic stability is possible even in the presence of changing mass properties so long as there is some understanding of how those properties are changing and some method of measuring those mass properties. This work has also demonstrated that very noisy attitude and attitude rate components can be overcome when considering coupled dynamics provided that there exist precise position and velocity measurements. This ambiguity can be overcome via the consideration of coupled rotational/attitude dynamics. This results in the accurate measurement of attitude and attitude rate even when the noise is much greater than the change in state. This is significant due to the inherently noisy nature of attitude and attitude rate measurements in launch vehicles, which has historically been a barrier to autonomy. Coupled dynamics, such as those described on the special Euclidean group, are thus able to overcome this ambiguity and accurately estimate the attitude and attitude even when the noise is many times greater than the change in those states.

Future work will seek to extend this controller to other scenarios in which mass properties are known to vary, such as the case with slosh for on-orbit refuelers [26–28], asteroid sample return, and logistics services with tug-cargo spacecraft configurations. Furthermore, it is of interest to the authors to examine other methods of varying inertia properties and apply a version of the UKF and discriminative Kalman filter for slosh and mass property estimation to include a comprehensive state and parameter estimation scheme in the presence of noise.

X. Acknowledgments

The authors would like to acknowledge Bill Benson of NASA’s Launch Services Program for proposing remediation of ambiguity in geometric mechanic framework as a subject of study.

References

- [1] Muller, A., and Terze, Z., “The significance of the configuration space Lie group for the constraint satisfaction in numerical time integration of multi body systems,” *Mechanism and Machine Theory*, Vol. 82, 2014, pp. 173–202. <https://doi.org/10.1016/j.mechmachtheory.2014.06.014>.
- [2] Lee, T., McClamroch, H., and Leok, M., “Optimal attitude control for a rigid body with symmetry,” *Proceedings of the American Control Conference*, 2007, pp. 1073–1078. <https://doi.org/10.1109/ACC.2007.4282362>.
- [3] Lee, T., Leok, M., and McClamroch, N., “Geometric Tracking Control of a Quadrotor on $SE(3)$,” *49th IEEE Conference on Decision and Control*, 2010. <https://doi.org/10.1109/CDC.2010.5717652>.
- [4] McCann, B. S., Nazari, M., Wittal, M. M., and Smith, J., “Rigid Body Pose Estimation on $TSE(3)$ for Spacecraft with Unknown Moments of Inertia,” *AIAA SciTech 2023 Forum*, 2023. <https://doi.org/10.2514/6.2023-2643>.

- [5] Seo, D., Nazari, M., and Arabi, E., "Rigid-Body Noncertainty Equivalence Adaptive Control on the Tangent Bundle of $SE(3)$," *Journal of Guidance, Control, and Dynamics*, Vol. 46, No. 10, 2023, pp. 1892–1900. <https://doi.org/10.2514/1.G006702>.
- [6] Cao, L., and Li, H., "Unscented predictive variable structure filter for satellite attitude estimation with model errors when using low precision sensors," *Acta Astronautica*, Vol. 127, 2016, pp. 505–513. <https://doi.org/10.1016/j.actaastro.2016.06.045>.
- [7] Vandyke, M., Schwartz, J., and Hall, C., "Unscented Kalman Filtering for spacecraft attitude state and parameter estimation," *Advances in the Astronautical Sciences*, Vol. 119, 2004.
- [8] Crassidis, J., and Markley, L., "Unscented Filtering for Spacecraft Attitude Estimation," *Journal of Guidance Control and Dynamics*, Vol. 26, 2003. <https://doi.org/10.2514/6.2003-5484>.
- [9] Nazari, M., Maadani, M., Butcher, E. A., and Yucelen, T., "Morse-Lyapunov-Based Control of Rigid Body Motion on $TSE(3)$ via backstepping," *AIAA Scitech Forum*, 2018. <https://doi.org/10.2514/6.2018-0602>.
- [10] Wang, J., Butcher, E. A., and Yucelen, T., "Consensus Estimation of Rigid Body Motion Using Lie Algebra $se(3)$ and the Consensus Extended Kalman Filter," *AIAA SciTech 2019 Forum*, 2019. <https://doi.org/10.2514/6.2019-1163>.
- [11] Mangiacapra, G., Wittal, M., Capello, E., and Nazari, M., "Unscented Kalman filter and control on $TSE(3)$ with application to spacecraft dynamics," *Nonlinear Dynamics*, 2022. <https://doi.org/10.1007/s11071-022-07293-x>.
- [12] McCann, B., and Nazari, M., "Control and maintenance of fully-constrained and underconstrained rigid body motion on Lie groups and their tangent bundles," *Journal of Geometric Mechanics*, Vol. 14, No. 1, 2022, pp. 29–55. <https://doi.org/10.3934/jgm.2022002>.
- [13] Wittal, M. M., McCann, B., Benavides, J., and Nazari, M., "Ambiguity Remediation in Launch Vehicles with Parameter Uncertainties: A Comparison between Special Euclidean Group and Dual Quaternion," *AIAA/AAS Astrodynamics Specialist Conference*, Vol. 22-623, 2022.
- [14] Dutta, S., Way, D. W., Zumwalt, C. H., and Blette, D. J., "Postflight Assessment of Mars 2020 Entry, Descent, and Landing Simulation," *Journal of Spacecraft and Rockets*, Vol. 0, No. 0, 0, pp. 1–11. <https://doi.org/10.2514/1.A35771>, URL <https://doi.org/10.2514/1.A35771>.
- [15] Lee, M. W., Shea, P. R., Pinier, J. T., and Chan, D. T., "Uncertainty Quantification of Artemis I Space Launch System Integrated Aerodynamics Databases," *Journal of Spacecraft and Rockets*, Vol. 0, No. 0, 0, pp. 1–10. <https://doi.org/10.2514/1.A35844>, URL <https://doi.org/10.2514/1.A35844>.
- [16] Braun, B., and Barf, J., "Image processing based horizon sensor for estimating the orientation of sounding rockets, launch vehicles and spacecraft," *CAES Space Journal*, Vol. 15, 2023, pp. 509–533. <https://doi.org/10.1007/s12567-022-00461-0>.
- [17] Winkler, S., Wiedermann, G., and Gockel, W., "Gyro-stellar Attitude Estimation Considering Measurement Noise Correlation and Time-Variant Relative Sensor Misalignment," *International Astronautical Congress, Glasgow, Scotland*, 2008.

- [18] Winkler, S., “Observability of Star Tracker / Gyro Based Attitude Estimation Considering Time-Variant Sensor Misalignment,” *Advances in Aerospace Guidance, Navigation and Control*, edited by F. Holzapfel and S. Theil, Springer Berlin Heidelberg, Berlin, Heidelberg, 2011, pp. 373–384. https://doi.org/10.1007/978-3-642-19817-5_29.
- [19] Jang, J.-W. J., Alaniz, A., Hall, R., et al., “Design of Launch Vehicle Flight Control Systems Using Ascent Vehicle Stability Analysis Tool,” *AIAA Guidance, Navigation, and Control Conference 2011*, 2011. <https://doi.org/10.2514/6.2011-6652>.
- [20] Briese, L. E., Acquatella B, P., and Schnepfer, K., “Multidisciplinary modeling and simulation framework for launch vehicle system dynamics and control,” *Acta Astronautica*, Vol. 170, 2020, pp. 652–664. <https://doi.org/10.1016/j.actaastro.2019.08.022>.
- [21] McCann, B., V. Gunter, H., Nazari, M., and Henderson, T., “Generalized Formalism for Variable-Mass-Property Spacecraft Dynamics Utilizing Rigid Body Motion on Lie Groups,” *AIAA/AAS Astrodynamics Specialist Conference*, Vol. 22-790, 2022.
- [22] Lee, T., Leok, M., and McClamroch, N. H., “Lie group variational integrators for the full body problem in orbital mechanics,” *Celestial Mechanics and Dynamical Astronomy*, Vol. 98, 2007, p. 121–144. <https://doi.org/10.1007/s10569-007-9073-x>.
- [23] Brossard, M., Barrau, A., and Bonnabel, S., “A code for unscented Kalman filtering on manifolds (UKF-M),” *International Conference on Robotics and Automation (ICRA)*, 2020, arXiv:2002.00878.
- [24] Absil, P.-A., Mahony, R., and Sepulchre, R., “Optimization Algorithms on Matrix Manifolds,” *Princeton University Press*, 2008, ISBN: 978-0-691-13298-3.
- [25] McCarthy, D. D., “Using UTC to determine Earth’s rotation angle,” *AIAA/AAS Astrodynamics Specialist Conference*, Vol. 11-666, 2011.
- [26] Fagetti, M., McCann, B., Nazari, M., Wittal, M., and Smith, J., “Generalized Sloshing Equations of Motion for Any Equivalent Mechanical Model on TSE(3),” *AAS Astrodynamics Specialist Conference*, , No. 23-339, 2023.
- [27] Fagetti, M., McCann, B., Nazari, M., Wittal, M., and Smith, J., “Mass and Inertia Property Estimation on TSE(3) in the Presence of Sloshing Liquid,” *AAS Astrodynamics Specialist Conference*, , No. 23-321, 2023.
- [28] Wittal, M., Dziedzic, W., Pitt, J., McCann, B., Fagetti, M., and Nazari, M., “Modeling Low-G Slosh Using Negative Mass,” *AAS Rocky Mountain GN&C Conference, paper 24-096*, 2024.