

A PHYSICS-BASED WORK-ENERGY FORMULATION FOR REAL-TIME TRAJECTORY GUIDANCE OF A LUNAR LANDER

Jorge M. Munoz-Burgos,^{*} Peter J. McDonough,[†] and Christopher C. Proctor[‡]

Throughout the years, many researchers have calculated and optimized trajectory solutions for lunar landing systems by employing sophisticated mathematical methods, that include: Hamilton's Principle of Variation, Pontryagin's maximum principle, and well known convex-optimization techniques among others. Many of these approaches typically require expensive computational resources to achieve convergence in the solution. In an effort to reduce complexity and the computational load required to generate real-time guidance commands, a simple Physics-Based Work-Energy approach has been formulated. This approach is based on the dissipation of the mechanical energy of the vehicle to its final desired energy state required to achieve a safe landing. The rocket engine(s) employed during landing (among other maneuvers) dissipates mechanical energy by both doing work against the velocity vector of the vehicle (thus defining the trajectory path), and by jettisoning mass. Therefore, by solving the energy dissipation problem at every step of the maneuver, a much simpler formulation that naturally and quickly attains convergence is obtained. This formulation is not limited to approach, landing, and divert maneuvers, but in principle it can be employed during de-orbiting, braking burn, ascent, as well as orbit insertion.

INTRODUCTION

Trajectory calculation and optimization during lunar de-orbiting, approach, and landing is a relevant and ongoing area of research, particularly in modern times when the U.S. and other countries are conducting efforts to return humans to the moon. Since the 1960's until recently, algorithms that employ fractional-polynomials to well known convex-optimization techniques have been employed in these studies.¹⁻¹¹ Due to the time-critical nature of the maneuvers and the possibility of rapidly changing conditions during an actual lunar-descent (or need of manual control), a simple and computationally fast physics-based approach is presented.

The physics-based approach formulated in this work is based on the energy conservation principle, where a lunar lander approaching the surface with a total amount of mechanical energy (kinetic plus potential) is guaranteed to achieve a soft landing by dissipating its mechanical energy. The lander can only dissipate energy by employing two mechanisms: Jettisoning mass, and by doing work against its velocity vector by directing the thrust-force of the engine. From this simple concept, real-time guidance commands and minimum propellant are calculated.

As an additional feature derived from this formulation, a Physics-Based Lunar Altimeter is suggested in Appendix D as a final approach and descent navigational aid.

^{*}Senior Navigation Engineer, Amentum Space Exploration Group, EV42, NASA MSFC: 4600-4321, Huntsville, AL 35812, USA. Email: jorge.m.munoz-burgos@nasa.gov.

[†]Team Lead, S/C & Veh. Sys., Amentum Space Exploration Group, EV42, NASA MSFC: 4600-5204, Huntsville, AL 35812, USA. Email: peter.j.mcdonough@nasa.gov.

[‡]Senior Aerospace Engineer, Amentum Space Exploration Group, EV42, NASA MSFC: 4600-4323, Huntsville, AL 35812, USA. Email: christopher.c.proctor@nasa.gov.

GENERAL WORK-ENERGY FORMULATION

Fig. 1 illustrates a Lunar Module (LM) with initial kinetic and potential energy on a parking orbit initiates a descent trajectory towards the desired landing location. To land successfully, it is necessary to achieve full energy dissipation at the Landing-Site (LS) in order not to crash.

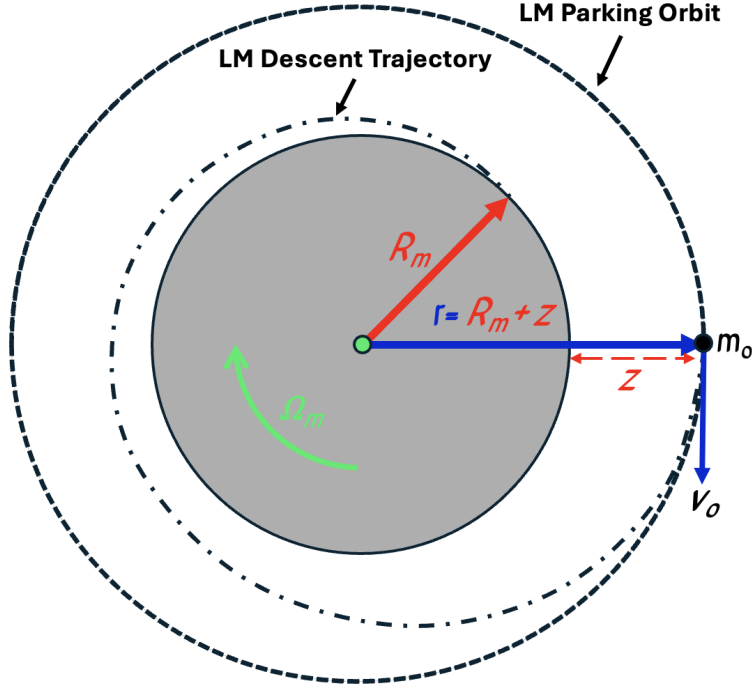


Figure 1 LM Final Approach Trajectory to LS.

In general, the gravitational potential energy expressed in the spherical-harmonics expansion form can be expressed as

$$U(z, \varphi, \lambda) = -m_o \cdot \frac{GM_m}{(R_m + z)} \cdot \sum_{n=0}^{\infty} \sum_{k=0}^n \left[\frac{R_m}{(R_m + z)} \right]^n \cdot P_{n,k}[\sin(\varphi)] \cdot [C_{n,k} \cdot \cos(k \cdot \lambda) + S_{n,k} \cdot \sin(k \cdot \lambda)], \quad (1)$$

where G is Newton's universal gravitational constant, m_o is the initial mass of the lander, M_m and R_m are the mass and the average lunar radius, while Ω_m is the lunar angular rotation, and z is the altitude above the lunar surface. The gravitational potential is also a function of the lunar latitude λ , and longitude φ . From the difference between the potential energies during any point along the trajectory and the LS, a local pseudo-constant value of the gravitational acceleration g can be calculated using

$$g(z, \varphi, \lambda) = \frac{GM_m}{R_m^2 \left(1 + \frac{z}{R_m}\right)} \cdot \left\{ \left[1 + \frac{R_m}{z} \right] \cdot \sum_{n=0}^{\infty} \sum_{k=0}^n P_{n,k}[\sin(\varphi_o)] \cdot [C_{n,k} \cdot \cos(k \cdot \lambda_o) + S_{n,k} \cdot \sin(k \cdot \lambda_o)] \right. \\ \left. - \sum_{n=0}^{\infty} \sum_{k=0}^n \left[\frac{R_m}{(R_m + z)} \right]^n \cdot P_{n,k}[\sin(\varphi)] \cdot [C_{n,k} \cdot \cos(k \cdot \lambda) + S_{n,k} \cdot \sin(k \cdot \lambda)] \right\}. \quad (2)$$

Fig. 2 represents a summary of the physics-based Work-Energy (WE) formulation presented in this work, and illustrates the two energy dissipation mechanisms necessary to achieve a safe landing.

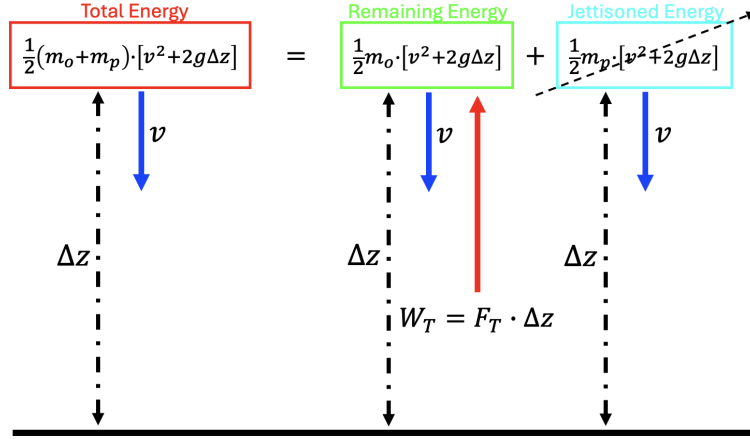


Figure 2 Energy dissipation mechanisms. The jettisoned energy is lost by the mass of propellant m_p exhausted by the engine, while the rest of the energy is dissipated by the work done by the thrust-force F_T of the engine along Δz .

As illustrated on Fig. 2, the WE formulation assumes constant vehicle settings that guarantee full energy dissipation along the total displacement of the trajectory. The assumption of constant settings in the formulation yields analytical solutions that simplifies the calculation of real-time guidance commands.

In this formulation, local right-hand coordinates are conveniently defined as: x -Cross-Range, y -Down-Range, and z -Altitude with respect to the LS as shown in Fig. 3. The start of the approach maneuver is considered to take place when the lander is near the lunar surface, thus validating the assumption of a constant gravitational acceleration g .

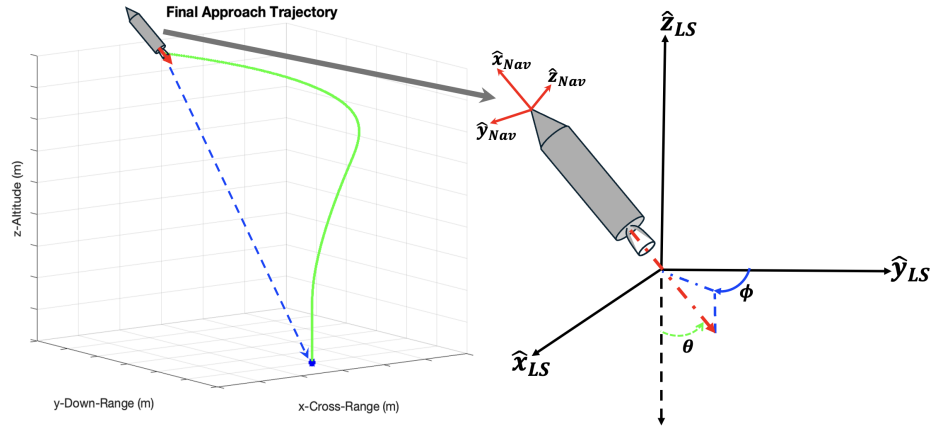


Figure 3 Definition of the Local LS Coordinates for the Approach Trajectory. The y -Down-Range direction may also be defined along the horizontal component of the vehicle's velocity vector. The estimated future position of the LS when landing may also be employed as reference coordinates when considered as a non-inertial frame.

From the definition of LS-coordinates shown in Fig. 3, the attitude θ is defined as the angle between the axial axis of the rocket and the z -axis, while the yaw ϕ is defined as the right-hand rotation angle on the xy -plane along the negative z -axis.

The mass-flow rate expelled from the rocket engine is defined as

$$\frac{dm}{dt} = \dot{m} = T_{hrtl} \cdot \dot{m}_{max} , \quad (3)$$

where $\dot{m} \leq 0$, and the engine throttle is defined as $0 \leq T_{hrtl} \leq 1$, and the magnitude of $\dot{m}_{max}$ represents the maximum mass-flow rate possible at full throttle. For a constant mass-flow rate during a certain time interval t , the space-craft mass as a function of time is given by

$$m(t) = m_o + \dot{m} \cdot t , \quad (4)$$

with initial mass m_o at $t = 0$. The magnitude of the rocket engine thrust-force as derived by Konstantin E. Tsiolkovsky is

$$F_T = -v_e \cdot \dot{m} , \quad (5)$$

with the effective exhaust velocity $v_e = g_o \cdot I_{sp}$, the standard gravity $g_o = 9.80665 \text{ m/s}^2$, and the engine's specific impulse I_{sp} . The maximum thrust-force possible is given by

$$F_{T_{max}} = -v_e \cdot \dot{m}_{max} . \quad (6)$$

With respect to the LS coordinates frame, the braking force that the engine exerts on the rocket is given by

$$\begin{aligned} \mathbf{F}_T &= F_{T_x} \cdot \hat{\mathbf{i}} + F_{T_y} \cdot \hat{\mathbf{j}} + F_{T_z} \cdot \hat{\mathbf{k}} \\ &= \dot{m} \cdot v_e \sin(\phi) \sin(\theta) \cdot \hat{\mathbf{i}} + \dot{m} \cdot v_e \cos(\phi) \sin(\theta) \cdot \hat{\mathbf{j}} - [m \cdot g + \dot{m} \cdot v_e \cos(\theta)] \cdot \hat{\mathbf{k}} . \end{aligned} \quad (7)$$

The components of the acceleration vector are obtained by dividing Eq. (7) by the vehicle's mass as a function of time defined by Eq. (4)

$$a_x(t) = \frac{\dot{m}}{m} \cdot v_e \sin(\phi) \sin(\theta) , \quad (8a)$$

$$a_y(t) = \frac{\dot{m}}{m} \cdot v_e \cos(\phi) \sin(\theta) , \quad (8b)$$

$$a_z(t) = -g - \frac{\dot{m}}{m} \cdot v_e \cos(\theta) . \quad (8c)$$

Eqs. (8a) through (8c) are integrated with respect to time to obtain the velocity components

$$v_x(t) = v_{x_o} + v_e \sin(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (9a)$$

$$v_y(t) = v_{y_o} + v_e \cos(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (9b)$$

$$v_z(t) = v_{z_o} - g \cdot t - v_e \cos(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (9c)$$

that are also integrated to obtain the vector components of the position in the form

$$x(t) = x_o + [v_{x_o} - v_e \sin(\phi) \sin(\theta)] \cdot t + v_e \sin(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot [1 + \frac{\dot{m}}{m_o} \cdot t] \cdot \ln [1 + \frac{\dot{m}}{m_o} \cdot t] , \quad (10a)$$

$$y(t) = y_o + [v_{y_o} - v_e \cos(\phi) \sin(\theta)] \cdot t + v_e \cos(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot [1 + \frac{\dot{m}}{m_o} \cdot t] \cdot \ln [1 + \frac{\dot{m}}{m_o} \cdot t] , \quad (10b)$$

$$z(t) = z_o + [v_{z_o} + v_e \cos(\theta)] \cdot t - \frac{1}{2} \cdot g \cdot t^2 - v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot [1 + \frac{\dot{m}}{m_o} \cdot t] \cdot \ln [1 + \frac{\dot{m}}{m_o} \cdot t] . \quad (10c)$$

It can be shown that by combining Eqs. (9a) through (9c), with Eqs.(10a) through (10c), the time-intervals for each of the vector components to achieve their final states are given by

$$\tau_x = \frac{\left\{ \frac{m_o}{\dot{m}} \cdot [v_{x_o} - v_x(\tau_x)] - [x_o - x(\tau_x)] \right\}}{[v_x(\tau_x) - v_e \sin(\phi) \sin(\theta)]} , \quad (11a)$$

$$\tau_y = \frac{\left\{ \frac{m_o}{\dot{m}} \cdot [v_{y_o} - v_y(\tau_y)] - [y_o - y(\tau_y)] \right\}}{[v_y(\tau_y) - v_e \cos(\phi) \sin(\theta)]} , \quad (11b)$$

$$\tau_z = - \left\{ \frac{1}{g} \cdot [v_z(\tau_z) + v_e \cos(\theta)] + \frac{m_o}{\dot{m}} \right\} \pm \sqrt{\left\{ \frac{1}{g} \cdot [v_z(\tau_z) + v_e \cos(\theta)] + \frac{m_o}{\dot{m}} \right\}^2 - \frac{2}{g} \cdot \left\{ [z_o - z(\tau_z)] - \frac{m_o}{\dot{m}} \cdot [v_{z_o} - v_z(\tau_z)] \right\}} , \quad (11c)$$

where: $x(\tau_x)$, $y(\tau_y)$, $z(\tau_z)$, $v_x(\tau_x)$, $v_y(\tau_y)$, and $v_z(\tau_z)$ represent the final components of the position and velocity vectors of the vehicle at the end of the maneuver (e.g. landing).

WORK-ENERGY TRAJECTORY APPROACH

The initial and final energy states of the lunar lander during the approach maneuver can be expressed as

$$E_o = \frac{1}{2} \cdot m_o \cdot [\mathbf{v}_o^2 + 2 \cdot g \cdot z_o] , \quad (12a)$$

$$E_f = \frac{1}{2} \cdot (m_o - m_p) \cdot [\mathbf{v}_f^2 + 2 \cdot g \cdot z_f] , \quad (12b)$$

where the horizontal components of the velocity are set to zero on the landing in order to avoid a crash, and the vertical velocity is set to a small descent velocity v_{z_f} (negative) at a final target altitude z_f , therefore

$$\mathbf{v}_o^2 = v_{x_o}^2 + v_{y_o}^2 + v_{z_o}^2 , \quad (13a)$$

$$\mathbf{v}_f^2 = v_{z_f}^2 . \quad (13b)$$

By assuming constant conditions along the trajectory, the work done by the engine thrust along the displacement vector $\Delta \mathbf{r}$ is given by

$$W_T = \mathbf{F}_T \cdot \Delta \mathbf{r} . \quad (14)$$

The second energy dissipation mechanism is the jettisoning of mass (in the form of propellant) with the assumption of a constant mass-flow $\dot{m}$ in the form

$$E_{Jet} = \frac{1}{2} \cdot m_P \cdot [\bar{\mathbf{v}}^2 + 2 \cdot g \cdot \bar{z}] , \quad (15)$$

where the jettisoned propellant mass during the interval τ with constant $\dot{m}$ is given by

$$m_P = -\tau \cdot \dot{m} . \quad (16)$$

The average square of the velocity $\bar{\mathbf{v}}^2$ and average altitude displacement $\bar{z}$ in Eq. (15), are calculated using the average integral functions

$$\bar{v}^2 = \frac{1}{\tau} \int_0^\tau dt \cdot v^2(t) , \quad (17a)$$

$$\bar{z} = \frac{1}{\tau} \int_0^\tau dt \cdot z(t) . \quad (17b)$$

By plugging in the velocities from Eqs. (9a) through (9c) into Eq. (17a), and by integrating along the interval τ , the average square of the velocity components are obtained

$$\begin{aligned} \bar{v}_x^2 = & v_{x_o}^2 + 2 \cdot v_e^2 \sin^2(\phi) \sin^2(\theta) - 2 \cdot v_{x_o} v_e \sin(\phi) \sin(\theta) \\ & + \frac{1}{\tau} \cdot v_e \sin(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\ & \times \left\{ 2 \cdot [v_{x_o} - v_e \sin(\phi) \sin(\theta)] + v_e \sin(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\ & \times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] , \end{aligned} \quad (18a)$$

$$\begin{aligned} \bar{v}_y^2 = & v_{y_o}^2 + 2 \cdot v_e^2 \cos^2(\phi) \sin^2(\theta) - 2 \cdot v_{y_o} v_e \cos(\phi) \sin(\theta) \\ & + \frac{1}{\tau} \cdot v_e \cos(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\ & \times \left\{ 2 \cdot [v_{y_o} - v_e \cos(\phi) \sin(\theta)] + v_e \cos(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\ & \times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] , \end{aligned} \quad (18b)$$

$$\begin{aligned} \bar{v}_z^2 = & v_{z_o}^2 + 2 \cdot v_e^2 \cos^2(\theta) + 2 \cdot v_{z_o} v_e \cos(\theta) \\ & + g\tau \cdot \left\{ \frac{1}{3} \cdot g\tau - [v_{z_o} + \frac{1}{2} \cdot v_e \cos(\theta)] + \frac{1}{\tau} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \right\} \\ & + \frac{1}{\tau} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \\ & \times \left\{ g \cdot \left[\tau - \frac{m_o}{\dot{m}} \right] - 2 \cdot [v_{z_o} + v_e \cos(\theta)] + v_e \cos(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\} \\ & \times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] . \end{aligned} \quad (18c)$$

And by substituting in the altitude from Eq. (10c) into Eq. (17b), and by integrating along the time interval τ , the average altitude displacement is obtained in the form

$$\begin{aligned} \bar{z} = z_o + \frac{1}{2} \cdot \tau \cdot \left[v_{z_o} + \frac{3}{2} \cdot v_e \cos(\theta) \right] - \frac{1}{6} \cdot g \cdot \tau^2 \\ + \frac{1}{2} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left\{ 1 - \frac{1}{\tau} \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right]^2 \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau \right] \right\}. \end{aligned} \quad (19)$$

Therefore, the total energy dissipated by both the work done by the engine and mass jettisoning is expressed as

$$E_o - E_f = W_T + E_{Jet}. \quad (20)$$

To employ the WE approach to calculate optimal guidance commands, the problem is simplified to 2D. This is due to the initial definition of the local LS coordinates in Fig. 3, where the net cross-range displacement $\Delta x = 0$. Fig. 4 illustrates the separation of the problem between horizontal and vertical components.

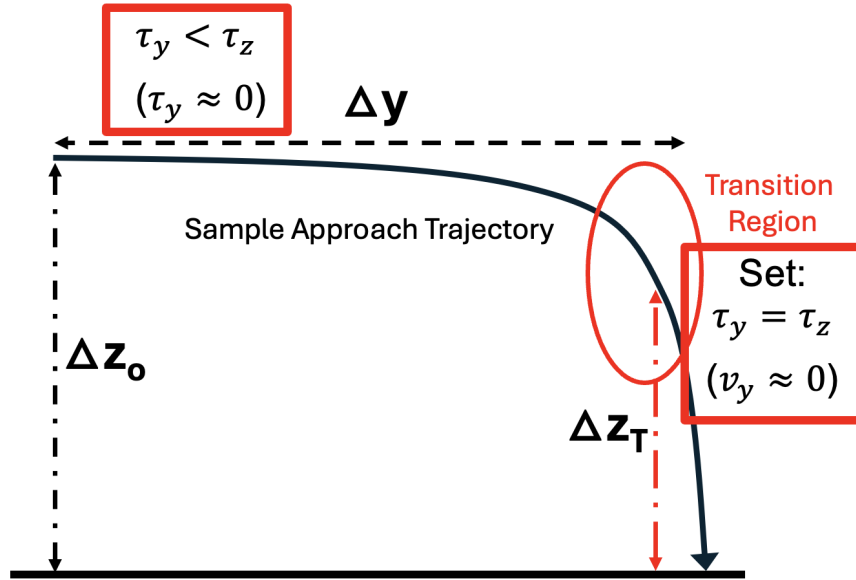


Figure 4 Separation of the approach trajectory into predominantly horizontal and vertical stages. In general $\tau_y < \tau_z$ [or till $\tau_y \approx 0$], which means that the horizontal energy (horizontal velocity) is dissipated before the vertical (landing) in order to avoid a crash.

Therefore, the problem is simplified by dividing Eq. (20) into both horizontal and vertical components in the form

$$E_{O_y} - E_{f_y}^0 = F_y \cdot \Delta y + E_{Jet_y}, \quad (21a)$$

$$E_{O_z} - E_{f_z} = F_z \cdot \Delta z + E_{Jet_z}, \quad (21b)$$

where the jettisoned energy is represented by

$$E_{Jet_y} = \frac{1}{2} \cdot m_{P_y} \cdot \bar{v}_y^2, \quad (22a)$$

$$E_{Jet_z} = \frac{1}{2} \cdot m_{P_z} \cdot [\bar{v}_z^2 + 2 \cdot g \cdot \bar{z}], \quad (22b)$$

and the mass-loss with respect to horizontal and vertical stages during the approach given by

$$m_{P_y} = -\tau_y \cdot \dot{m}, \quad (23a)$$

$$m_{P_z} = -\tau_z \cdot \dot{m}. \quad (23b)$$

The horizontal and vertical force components during the approach are calculated using

$$F_y = \frac{(E_{Oy} - E_{Jet_y})}{(y_f - y_o)}, \quad (24a)$$

$$F_z = \frac{(E_{Oz} - E_{fz} - E_{Jet_z})}{|z_f - z_o|}, \quad (24b)$$

that yield an optimal attitude approach angle given by

$$\theta_{Opt.} = \tan^{-1} \left[\frac{F_y}{F_z \cdot \cos(\phi)} \right]. \quad (25)$$

It can be shown that by combining Eqs. (9a) through (11c) that two different solutions for the yaw angle ϕ are found, the first one is time-dependent and more applicable during the predominantly horizontal stage, and the second one more applicable during the quasi-vertical stage of the approach

$$\begin{aligned} \phi &= \tan^{-1} \left\{ \frac{\left[\frac{[x(\tau) - x_o] - v_{x_o} \cdot \tau_x}{[y(\tau) - y_o] - v_{y_o} \cdot \tau_y} \right] \cdot \left[\frac{\tau_y - \frac{m_o}{\dot{m}} \cdot \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right) \cdot \ln \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right)}{\left[\tau_x - \frac{m_o}{\dot{m}} \cdot \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right) \cdot \ln \left(1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right) \right]} \right]}{\left[\frac{\tau_y^2 \cdot [x(\tau) - x_o] - v_{x_o} \cdot \tau_x}{\tau_x^2 \cdot [y(\tau) - y_o] - v_{y_o} \cdot \tau_y} \right]} \right\} \\ &\approx \tan^{-1} \left\{ \frac{\tau_y^2 \cdot [x(\tau) - x_o] - v_{x_o} \cdot \tau_x}{\tau_x^2 \cdot [y(\tau) - y_o] - v_{y_o} \cdot \tau_y} \right\}, \end{aligned} \quad (26a)$$

$$\phi = \tan^{-1} \left\{ \frac{\left[\frac{(x_f - x_o) - \frac{m_o}{\dot{m}} \cdot (v_{x_f} - v_{x_o})}{(y_f - y_o) - \frac{m_o}{\dot{m}} \cdot (v_{y_f} - v_{y_o})} \right]}{\left[\frac{(x_f - x_o) - \frac{m_o}{\dot{m}} \cdot (v_{x_f} - v_{x_o})}{(y_f - y_o) - \frac{m_o}{\dot{m}} \cdot (v_{y_f} - v_{y_o})} \right]} \right\}. \quad (26b)$$

In general, the engine throttle is calculated using

$$T_{hrtl} = \frac{F_z}{F_{T_{max.}} \cdot \cos \left(\theta_o + \frac{1}{2} \cdot \omega_\theta \cdot \Delta t \right)}, \quad (27)$$

where the throttle is always calculated with respect to the vertical force F_z , to prioritize full vertical energy dissipation at every point of the maneuver and avoid a crash.

The horizontal and vertical force components given by Eqs. (24a) and (24b) are calculated recursively by first setting the mass-flow rate $\dot{m}$ at an arbitrarily initial value within the throttle limits, and by setting both propellant masses in Eqs. (23a) and (23b) to zero. The initial optimal attitude $\theta_{Opt.}$ is calculated using Eq. (25) by first setting the yaw angle $\phi = 0$. A new value for the engine throttle is obtained from Eq. (27) and a mass-flow rate $\dot{m}$ is calculated using Eq. (3). These values are employed to calculate both the horizontal and vertical time intervals in Eqs. (11b) and (11c), and that are used to calculate: the propellant masses with Eqs. (23a) and (23b), the average square of the horizontal and vertical velocities in Eqs. (18b) and (18c), and the average altitude displacement in Eq. (19). These values are employed in Eqs. (22a) and (22b) to calculate the energy lost by mass jettisoning. The whole calculation is repeated until convergence is achieved, and the optimal horizontal and vertical forces together with optimal attitude and yaw [Eq. (26a)] angles are obtained.

Fig. 5 summarizes the recursive calculation cycle employed in obtaining optimal guidance commands. In general, it has been observed that between 5 and 8 iterations are enough to achieve convergence. In the work presented here, a default value of 10 iterations was chosen.

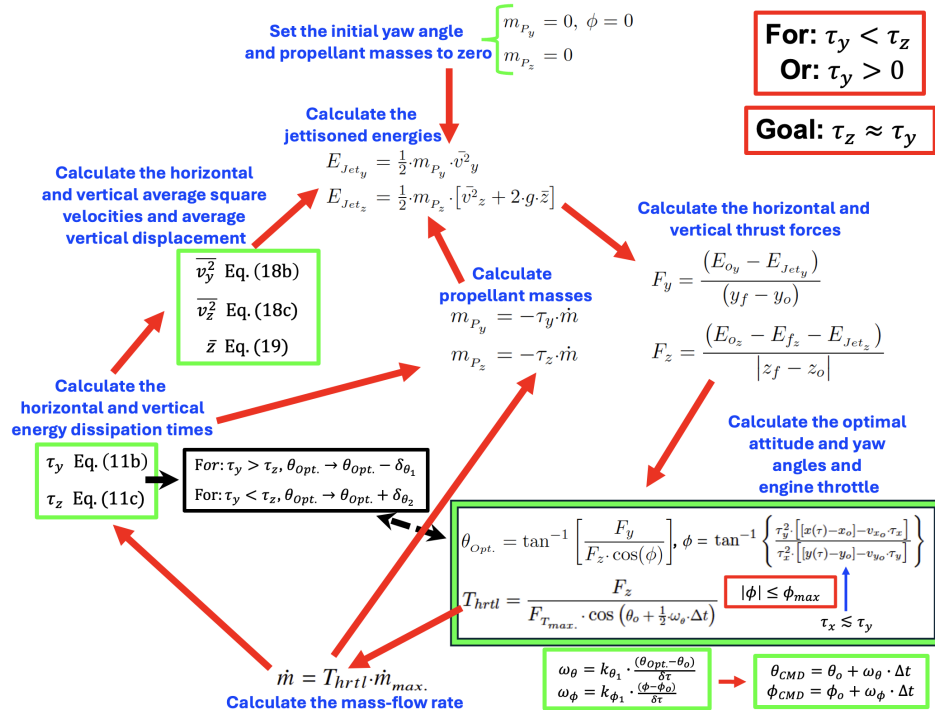


Figure 5 Summary of the WE calculation cycle during $\tau_y < \tau_z$ [$\tau_y > 0$]. Both times are adjusted to converge [$\tau_z \approx \tau_y$] as much as possible by making small adjustments [δ_{θ_1} and δ_{θ_2}] to the optimal attitude angle $\theta_{Opt.}$, where $\delta_{\theta_1} = c_1 \cdot |\tau_z - \tau_y| / \tau_z$ and $\delta_{\theta_2} = c_2 \cdot |\tau_z - \tau_y| / \tau_y$, with c_1 and c_2 determined heuristically. When employed, these small attitude corrections may cause the down-range target to be undershoot, therefore a small forward bias added to the final down-range target can help to compensate. In this work, the yaw angle is limited to $|\phi| \leq 15^\circ$ in order not to overcorrect both cross-range position and velocity. The differential time Δt is defined by the frequency of the guidance commands calculation [$\Delta t = 0.1 \text{ sec}$]. The angular rate constants k_{θ_1} and k_{ϕ_1} , and $\tau_x = 0.78 \cdot \tau_y$ are obtained heuristically. The time interval $\delta \tau$ may be set equal to τ_y , or to the constant Δt .¹²

The WE cycle illustrated in Fig. 5 is performed as long as $\tau_y < \tau_z$ [or till $\tau_y \approx 0$], or $\tau_z > \tau_{Rot.}$, where $\tau_{Rot.}$ is given by Eq. (49), and derived in Appendix A. For the cases when either $\tau_y \approx \tau_z$ or $\tau_z \approx \tau_{Rot.}$ conditions are achieved, both $\tau_o = \tau_y = \tau_z$ are set, and a countdown of $\tau = \tau_o - \Delta t$ for every future guidance calculation is employed.

Fig. 6 summarizes the recursive calculation cycle for the case when $\tau_y = \tau_z = \tau$.

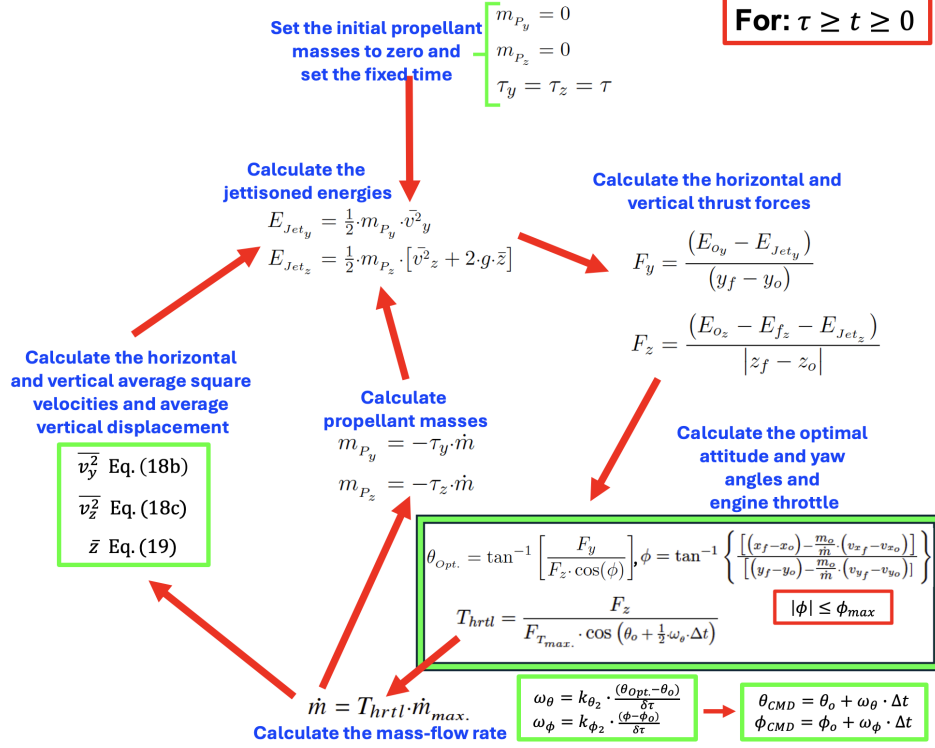


Figure 6 Summary of the WE calculation cycle when $\tau_y = \tau_z = \tau$. During every guidance calculation, the remaining time for landing is reduced by $\tau = \tau_o - \Delta t$. The optimal yaw angle is calculated using Eq. (26b), and is limited to $|\phi| \leq 15^\circ$. The angular rate constants k_{θ_2} and k_{ϕ_2} are also obtained heuristically. As an alternative for the heuristic approach to obtain the angular-rate for the attitude, ω_θ can also be calculated using Eq. (47) derived in Appendix A. For the case when approaching the LS $[(y_f - y_o) \rightarrow 0]$, the optimal attitude and yaw angles are calculated using Eqs. (55) and (56) derived in Appendix A.

Notice that during both of the WE calculation cycles summarized in Figs. 5 and 6, all commanded angular rates ω_θ and ω_ϕ are calculated with respect to the local LS coordinates defined in Fig. 3. The Navigation-Body angular-rates (Roll, Pitch, and Yaw) of the lander (as illustrated on the right of Fig. 3) are calculated using

$$\omega_{LS}^{Nav.} = -\omega_\phi \cdot \cos(\theta) \cdot \hat{x}_{Nav.} + \omega_\theta \cdot \hat{y}_{Nav.} - \omega_\phi \cdot \sin(\theta) \cdot \hat{z}_{Nav.}, \quad (28)$$

which is based on the solution expressed in Eq. (71), and derived in Appendix C.

BRAKING-BURN WORK-ENERGY MANEUVER

Since the energy conservation principle law is universal, the WE approach may also be employed into calculating guidance commands for orbital and sub-orbital type maneuvers, such as de-orbiting, and/or braking-burn. Fig. 7 illustrates the formulation of the WE approach for the braking-burn sub-orbital maneuver.

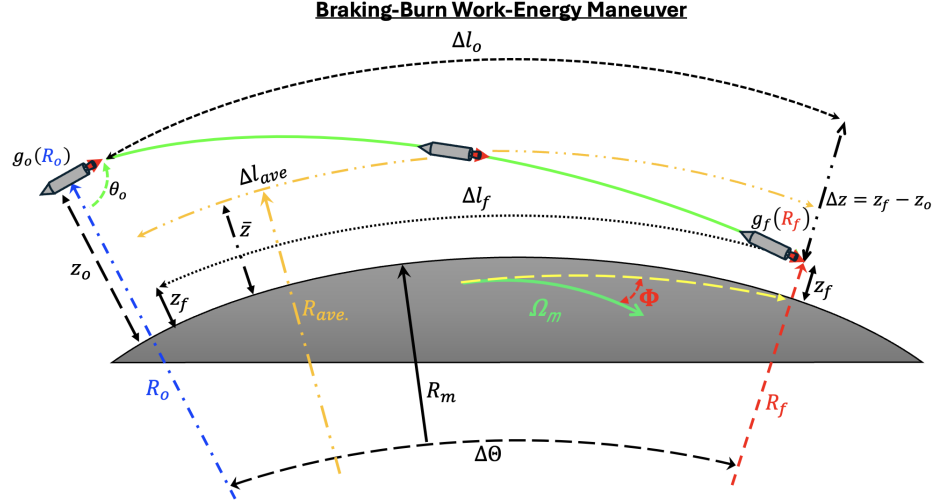


Figure 7 The initial and final gravitational accelerations $g_o(R_o)$ and $g_f(R_f)$ are a function of the radial distances R_o and R_f with respect to the lunar center of mass, and R_m is the local radius of the moon. Gravity may also be function of initial and final latitudes and longitudes [see Eq. (2)]. The upper and lower down-range displacements are a function of the total angular displacement $\Delta\Theta$: $\Delta l_o = R_o \cdot \Delta\Theta$ and $\Delta l_f = R_f \cdot \Delta\Theta$. The total vertical displacement is defined by Δz , while the angle Φ represents the angle between the ground-track of the trajectory and the lunar angular rotation Ω_m . $R_{ave.}$ is the average radius along the trajectory that is employed in the calculation of the total horizontal displacement $\Delta l_{ave.} = R_{ave.} \cdot \Delta\Theta$.

By taking into account the curvature of the trajectory and the variation of the gravity field (as illustrated in Fig. 7), the initial calculation for the average gravitational acceleration and down-range displacement may be approximated using

$$g_{ave.} \approx \frac{1}{2} \cdot [g_o(R_o) + g_f(R_f)] , \quad (29a)$$

$$R_{ave.} \approx \frac{1}{2} \cdot [R_o + R_f] , \quad (29b)$$

$$\Delta l_{ave.} \approx \frac{1}{2} \cdot [R_o + R_f] \cdot \Delta\Theta . \quad (29c)$$

For further calculations, the average altitude $\bar{z}$ along the trajectory is calculated using Eq. (19), and the average radius with respect to the lunar center of mass is given by $R_{ave.} = R_m + \bar{z}$. The average gravitational acceleration together with both the cross/down-range displacements during the down-range trajectory time interval τ_y , and that include the lunar rotation contribution with respect to the local latitude angle λ are calculated using

$$g_{ave.} = g(R_{ave.}) , \quad (30a)$$

$$\Delta x = \tau_y \cdot \Omega_m \cdot R_{ave.} \cdot \cos(\lambda) \sin(\Phi) , \quad (30b)$$

$$\Delta y = R_{ave.} \cdot [\Delta\Theta + \tau_y \cdot \Omega_m \cdot \cos(\lambda) \cos(\Phi)] . \quad (30c)$$

Figure 8 Summary of the WE calculation employed during the braking-burn sub-orbital maneuver. In this case, the vertical time is adjusted to be approximately equal to the down-range time [$\tau_z \approx \tau_y$] during the whole trajectory by making small adjustments [δ_{θ_1} and δ_{θ_2}] to the optimal attitude angle $\theta_{\text{Opt.}}$, where $\delta_{\theta_1} = \mathbf{c}_1 \cdot |\tau_z - \tau_y| / \tau_z$ and $\delta_{\theta_2} = \mathbf{c}_2 \cdot |\tau_z - \tau_y| / \tau_y$, with $\mathbf{c}_1$ and $\mathbf{c}_2$ determined heuristically. The throttle is obtained from the horizontal force F_y , due to the high amount of kinetic energy along the horizontal direction needed to be dissipated during orbital and sub-orbital maneuvers. The x -Cross-Range is corrected with the yaw angle ϕ calculated using Eq. (26a). In this work, ϕ is limited to $|\phi| \leq 5^\circ$, with cross-range time set to $\tau_x = 0.1 \cdot \tau_y$.

PONTRYAGIN'S MAXIMIZATION PRINCIPLE AND ENGINE EFFICIENCY

Since the WE formulation directly calculates both the necessary engine throttle (or engine thrust-force), and the engine firing time-length in order to accomplish the required energy dissipation, Pontryagin's maximization²⁻⁴ is easily implemented into the calculation. This is accomplished by defining a minimum throttle threshold (or range of thresholds determined according to the mission) to ignite the engine, below this threshold the engine remains idle. The lower the throttle threshold for ignition, the longer the burn-time needed, and more the propellant employed in the maneuver. The higher the throttle threshold for ignition, the shorter the burn-time, and less propellant needed. Therefore, by increasing the throttle threshold (equivalent to delaying the ignition of the engine), the more propellant efficient the burn becomes at the cost of time, thus achieving the highest propellant efficiency at 100% minimum throttle threshold. Fig. 9 illustrates Pontryagin's maximization principle as implemented into the WE calculation.

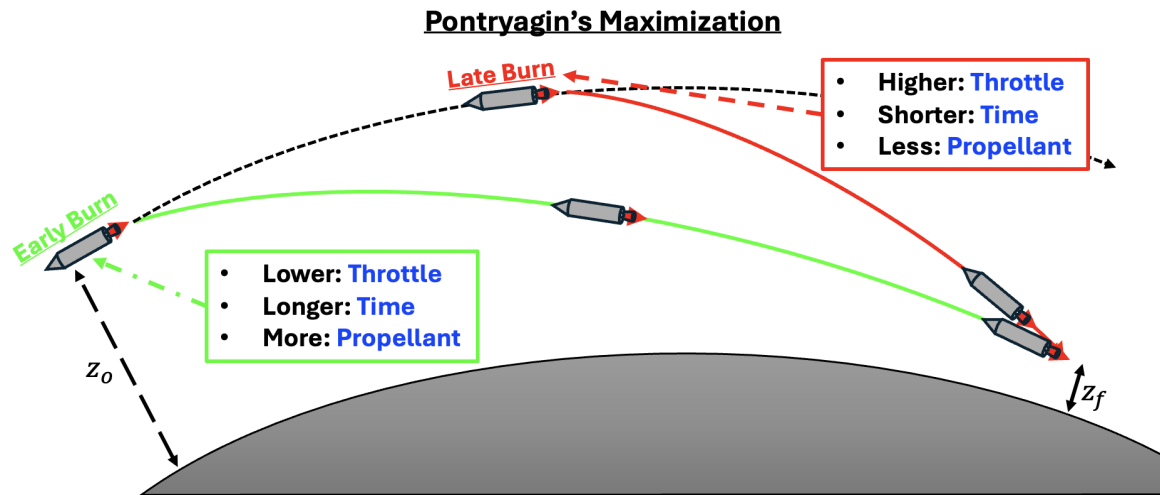


Figure 9 Illustration of Pontryagin's maximization²⁻⁴ as implemented into the WE formulation. In this case, the maximization is no longer determined by a time-consuming convex-optimization calculation (where time is minimized in order to minimize propellant consumption). Instead, the optimization is determined from a crew-resource/risk management perspective. Just as propellant consumption is an important and finite resource to take into account to achieve the mission objectives, time also is. In case of an engine failure or misfire, having extra time for the crew to troubleshoot or ignite an alternate engine at a later time and still accomplish the mission is as important as saving propellant. By solely considering Pontryagin's maximization principle, the most propellant efficient throttle threshold would yield 100%, this would minimize both propellant consumption and time, but would reduce control-authority, and would not allow further time for further corrective maneuvering and troubleshooting, thus increasing the risk of mission failure in case of a mishap. The WE formulation has the advantage of real-time calculation of the necessary throttle and engine firing time-length (therefore amount of total propellant required) in order to successfully accomplish the maneuver with the use of a few simple analytical expressions.

Another important aspect to be considered in order to minimize propellant consumption, is the actual engine efficiency and performance. In the ideal case, the general assumption is that the thrust-force of the engine is a linear function with respect to the mass flow with slope given by a constant effective exhaust velocity v_e [see Eq. (5)] as formulated by Konstantin E. Tsiolkovsky. However, when engine performance is measured, it is found that it is better described as a combination of affine and/or polynomial functions as illustrated in Fig. 10

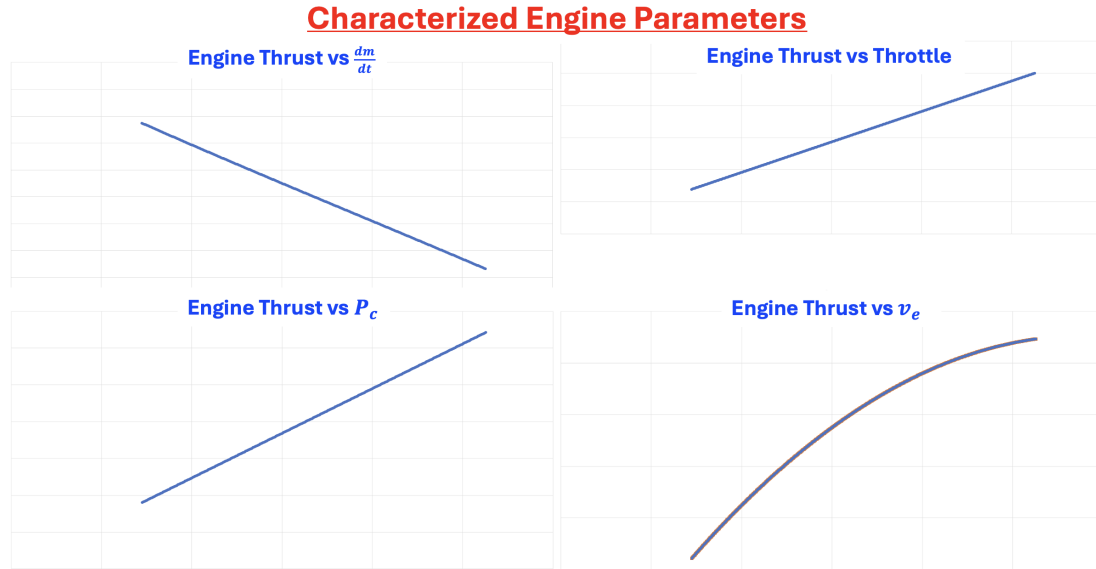


Figure 10 Arbitrary sample of a rocket engine characterization. The figure shows the relationships between engine thrust to negative mass-flow, engine throttle, chamber pressure, and effective exhaust velocity v_e . Most quantities are parametrized as affine functions, while the effective exhaust velocity is parametrized as a polynomial. Notice that for this specific example, the effective exhaust velocity attains its highest value at maximum throttle, this represents a maximum thrust-force in relation to mass-flow. Therefore, for this specific example, the maximum engine performance and propellant efficiency is attained at maximum throttle.

Real engine characterization is easily implemented into the WE formulation, since this approach calculates the total engine thrust required to accomplish the desired energy dissipation, the thrust-force is recursively employed into the engine parametrization to obtain engine throttle, effective exhaust velocity, and mass-flow, that in turn are plugged back into the calculation cycle.

MONTE-CARLO BRAKING-BURN AND APPROACH TRAJECTORIES

The WE formulation has been implemented into the NASA Marshall Space Flight Center Generalized Aerospace Simulation in SIMULINK Framework (GLASS) lunar-lander 6DoF simulator.¹² The GLASS simulator encompasses all maneuvers from lunar de-orbiting to landing, and includes navigation and sensor errors in its computations. In order to protect sensitive proprietary data, the comparisons presented here are purely qualitative rather than quantitative. In this work, trajectories that include both braking-burn and final approach maneuvers are calculated and compared to results from other methods also implemented in the GLASS simulation, such as: Fractional-Polynomial (FP) Apollo braking-burn powered descent guidance algorithm,^{10,11} and a 6-DoF Quadratic-Programming (QP) Convex-Optimization guidance algorithm.^{7,8}

The QP algorithm is employed during flight to generate minimum time trajectories (as a proxy for minimum propellant).⁷ This algorithm is called at fixed time intervals during the descent to capture off nominal conditions and changing vehicle mass properties. The underlying optimal control problem can be stated as

$$\begin{aligned}
& \min_{\mathbf{x}(t), \mathbf{u}(t)} t \\
& \text{subject to: } \dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t)) , \\
& \mathbf{x}_0 = \mathbf{x}(0) , \\
& \mathbf{x} \in \mathcal{X} , \\
& \mathbf{u} \in \mathcal{U} , \\
& \mathbf{x}_{t_f} = \mathbf{x}(t_f) ,
\end{aligned} \tag{33}$$

where the state, $\mathbf{x}$ and input, $\mathbf{u}$, belong to the closed sets $\mathcal{X}$ and $\mathcal{U}$ respectively. To solve the QP underlying optimal control problem during every call [50Hz], a direct transcription is employed by making the following assumptions (for both QP and FP guidance):

1. The LS is considered an inertial frame.
2. The vehicle can be represented by rigid body equations of motion.
3. The dynamics are linearized and represented as a discrete time system.
4. The moment of inertia of the vehicle is unchanging over a single solution.
5. The system constraints can be represented as linear functions of state and control.

In contrast with both: FP and QP algorithm, the WE approach does not assume that the LS is in an inertial frame as illustrated in Figs. 1 and 7. Therefore, the WE guidance calculation is based on the state of the LS at the moment of landing. Both the FP and WE are called every tenth of a second [10Hz] to provide real-time guidance commands. Both WE and QP employ the same minimum throttle threshold value and similar final target conditions during the braking-burn maneuver, while the FP employs a very different parametrization.

In this section, a Monte-Carlo (MC) calculation that consists of 101 (1 nominal case plus 100 seeds) is employed. All methods are compared and evaluated with respect to reaching the determined boundary limits around the LS, as well as achieving both horizontal and vertical velocity limits at touchdown. Landing within the determined velocity limits are a very important parameter since those limits represent a margin of safety for the lander not to crash-land into the lunar surface.

The first comparison includes a perfect navigation and vehicle mass case, where navigation and mass-determination errors are not included. This comparison helps to evaluate the performance of the guidance algorithms without being “contaminated” by navigation errors. The second comparison is performed with the baseline navigation and mass that includes errors in both navigation and vehicle mass-determination, This helps to evaluate both robustness and resilience of the methods when varying navigational performance is present. Figs. 11, 12, and 13 show the MC results for the final landing performance of the modeled Lunar-Lander (LM) employed in this work, while Figs. 14 and 15 show the baseline Nav./Mass Braking-Burn and Approach trajectories.

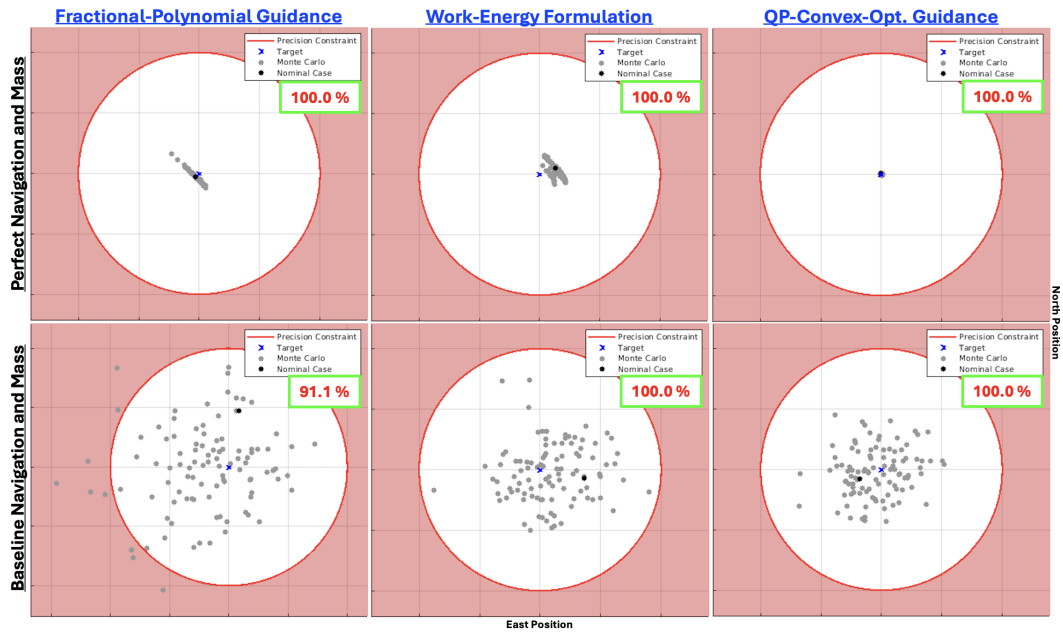


Figure 11 Position performance with respect to LS boundaries. All three methods achieve 100% when perfect Nav./Mass is present, while the FP method degrades to 91.1% when baseline Nav./Mass is present in comparison to both WE and QP, that still achieve 100%.

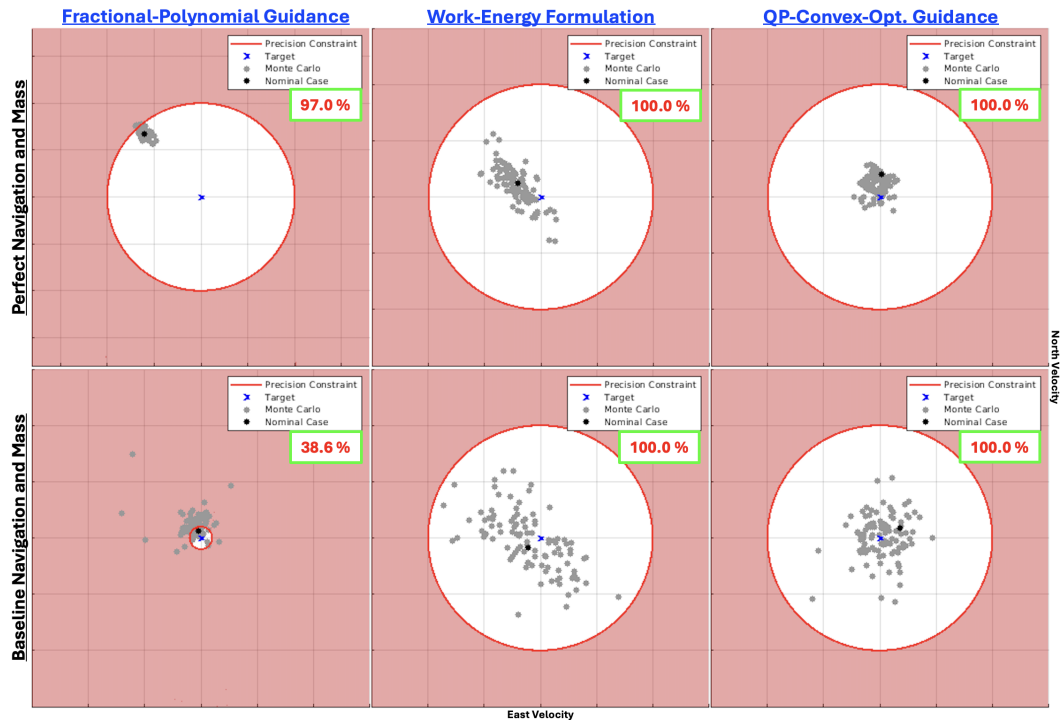


Figure 12 Horizontal velocity performance with respect to LS limits. Both WE and QP methods achieve 100% when perfect Nav./Mass is present, while FP achieves 97%. When baseline Nav./Mass is present, the FP method further degrades to 38.6%, while both WE and QP still achieve 100%.

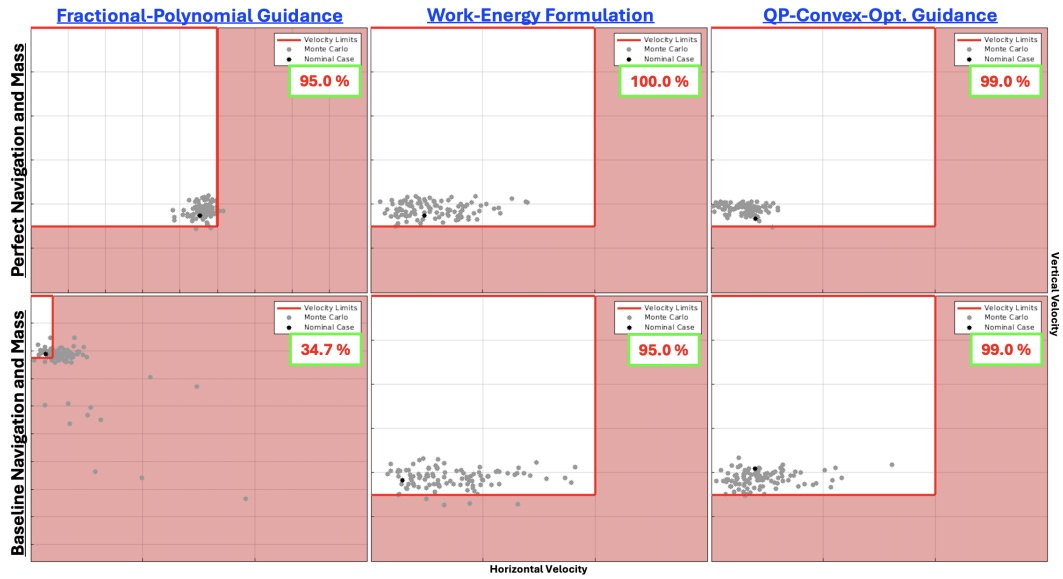


Figure 13 Horiz. vs vert. velocity performance with respect to LS limits. FP and QP methods achieve below 100% (95% and 99%) when perfect Nav./Mass is present, while WE achieves 100%. When baseline Nav./Mass is present, the FP method further degrades to 34.7%, WE degrades to 95%, while QP does not change.

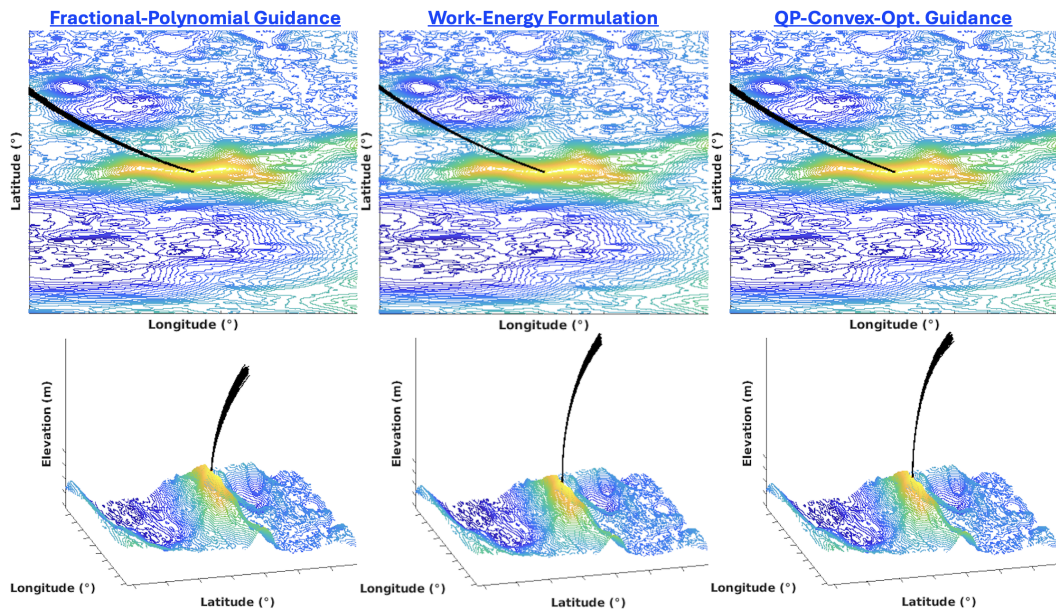


Figure 14 Braking-Burn trajectories calculated using FP, WE, and QP methods. The FP trajectory has a lower target altitude in comparison to WE and QP with very similar final targets, thus both WE and QP yield very similar profiles.

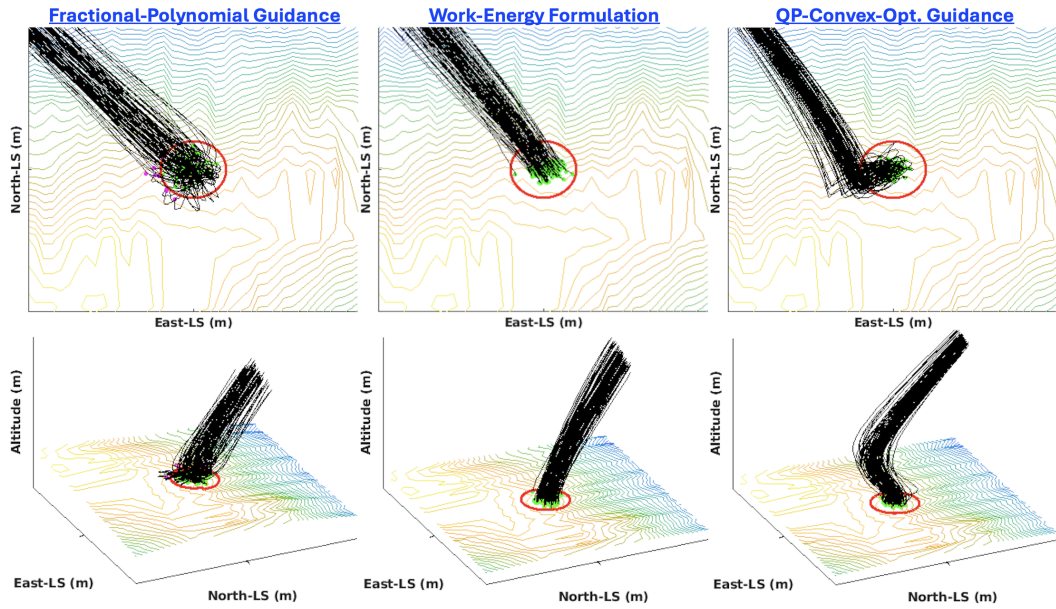


Figure 15 Approach trajectories calculated using FP, WE, and QP methods. The FP trajectory begins at a lower altitude in comparison to WE and QP with very similar initial conditions and performance, with the exception of the final corrective turn performed by the QP.

The three methods are compared with respect to propellant consumption, the total spent propellant mass after the approach maneuver is compared to the initial propellant mass before braking-burn. In order to accurately compare the three methods, only the cases that achieved both LS position boundaries and velocity limits are included.

In this work, the FP calculation commands a low throttle during the braking-burn, and a full throttle during most of the duration of the approach maneuver. This high maximum throttle setting yields the lowest net propellant consumption compared WE and QP, while greatly reducing control-authority thus inhibiting the possibility of further corrections. Therefore in this work, FP yields the lowest performance with respect to the LS boundary limits. Further tuning of the FP method inside the GLASS simulation may be required in order to improve performance. Fig. 16 shows the net propellant consumption for all three methods.

Both WE and QP employ the same minimum throttle threshold in order to ignite the engine during the braking-burn maneuver. However, it is observed that WE yield a monotonically increase in the commanded throttle during braking burn, and a short initially constant, followed by monotonically reduction of the commanded throttle during the approach maneuver, while the QP yields an irregularly oscillatory throttle commands. The monotonically increasing/decreasing commanded throttle from the WE formulation, allows astronauts to easily follow these commands during manual control.

Propellant consumption is easily reduced by increasing the throttle threshold during the braking-burn maneuver, Fig. 17 show the reduction of both consumed propellant and maneuvering time as a function of minimum throttle threshold.

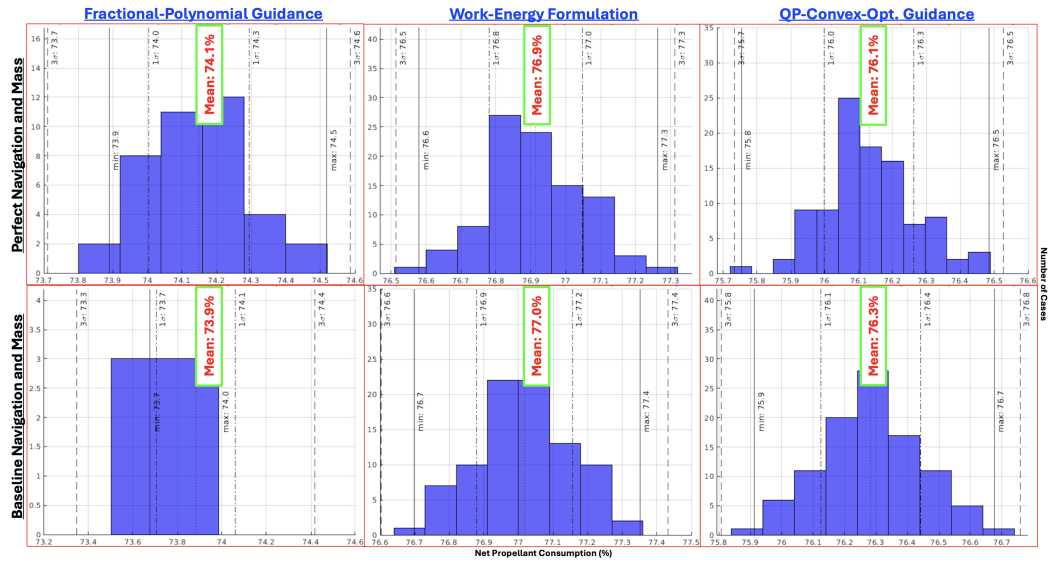


Figure 16 Net propellant consumption during braking-burn and approach maneuvers for both perfect and baseline Nav./Mass cases. Due to the maximum throttle commands by the FP calculation during the approach maneuver, the FP yields the lowest propellant consumption, while at the same time yielding the lowest performance with respect to the LS position and velocity constrains. Both WE and QP yield very similar propellant consumption with differences lower than 1%.

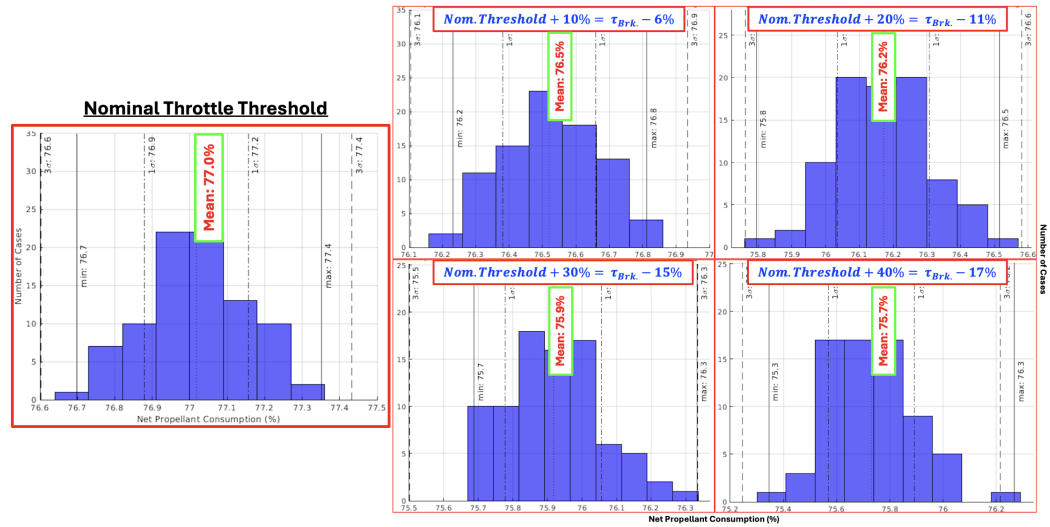


Figure 17 Net propellant consumption as a function of increasing the minimum throttle threshold. The threshold is increased by a tenth of a percentage every time, while the highest threshold is kept below 100%. The calculation includes both braking-burn and approach maneuvers, while the throttle threshold is only increased during braking-burn. The time reduction percentage is calculated in reference to the nominal braking-burn maneuvering time. Notice that by increasing the threshold by 10%, the propellant consumption is reduced by 0.5%, and the maneuver time is reduced by 6%. For this example, the maximum propellant reduction is obtained when increasing the threshold by 40%, propellant consumption is reduced by 1.3%, while the maneuvering time is reduced by 17%.

WORK-ENERGY-BASED REAL-TIME DIVERT-SPACE DETERMINATION

The WE formulation is applied in the real-time calculation of the divert space around the LS that satisfies the energy conservation principle. A divert landing is a maneuver that is performed when astronauts may consider the current LS to be unfeasible or unsafe to reach, and an alternate LS must be determined. Fig. 18 illustrates in general the divert maneuver mechanism.

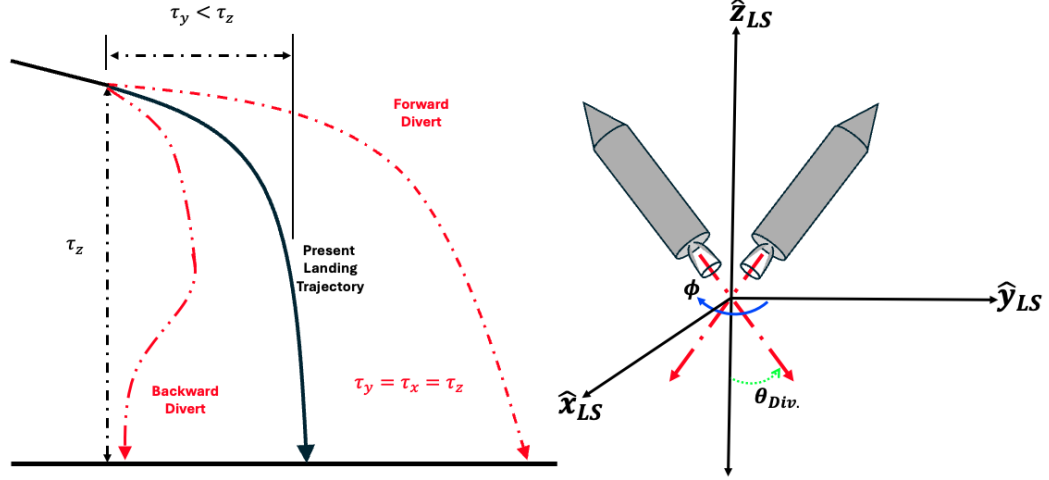


Figure 18 Divert-Space determination by applying the WE formulation. The total maneuvering time along the down-range and cross-range directions must be equal to the total flight (“airborne”) time τ_z , and the optimal attitude $\theta_{Div.}$ is constantly calculated while applying precise timing on the yaw control to guarantee effective energy dissipation and a safe landing.

The divert maneuver is initiated by setting the throttle to the selected maximum optimal value that satisfies both full energy dissipation and propellant constraints.

$$\frac{dm}{dt} = \dot{m} = \dot{m}_{max.} . \quad (34)$$

As shown in Fig. 18, a higher value of the throttle translates into a higher thrust-vector, and thus a higher optimal divert angle $\theta_{Div.}$. The higher the divert angle and the higher the thrust-vector, the higher the horizontal control-authority available to expand the divert space, but also translates into a higher propellant consumption. The optimal divert attitude angle is calculated using

$$\theta_{Div.} = \cos^{-1} \left[\frac{F_z}{F_{T_{max.}}} \right] , \quad (35)$$

where $F_{T_{max.}}$ is the maximum optimal thrust-force that corresponds to the maximum optimal mass-flow $\dot{m}_{max.}$.

Fig. 19 shows the WE approach cycle employed to determine the optimal divert attitude $\theta_{Div.}$, and the total flight time τ_z available to perform the divert maneuver.

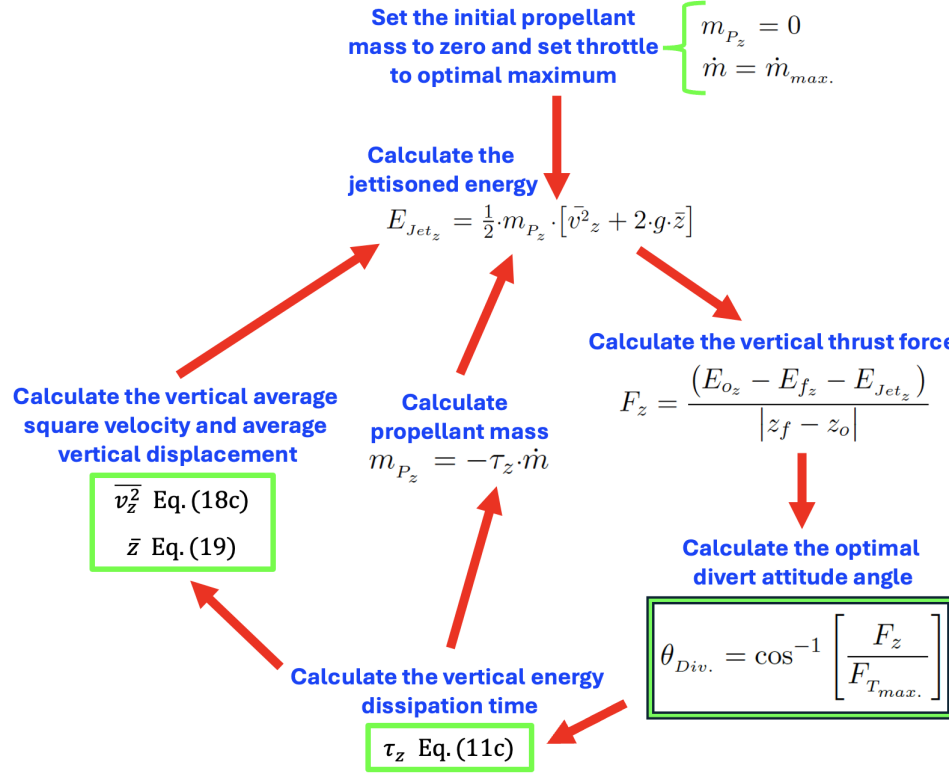


Figure 19 Divert maneuver WE calculation cycle to determine the optimal divert attitude angle $\theta_{Div.}$, and the total available flight time τ_z for optimal $\dot{m}_{max.}$. The calculation starts by setting the mass-flow to the maximum optimal mass-flow value, that combined with the calculated total flight time τ_z , is employed to estimate the total consumed propellant during the maneuver. This gives the flexibility of adjusting the optimal $\dot{m}_{max.}$ in order to satisfy propellant constraints and full energy dissipation at landing.

Once the optimal divert attitude, the total flight time, and the optimal propellant mass-flow is obtained from the WE formulation cycle, the divert space boundary is calculated analytically by time management of the yaw direction as shown on the right of Fig. 18.

The divert space is symmetric with respect to the horizontal velocity vector at the moment the divert maneuver is initiated. Therefore, during the calculation of the boundary of the divert space, an alternate yaw angle ϕ' is defined with respect to the horizontal velocity vector, and then is transformed to the local LS coordinate reference. Therefore, during the divert boundary calculation, the cross-range direction is defined as to the right/left of the velocity vector, while the down-range is defined either along or against the initial horizontal velocity vector. To achieve zero horizontal velocity at the time of landing (full horizontal energy dissipation), precise time-management of the alternate yaw angle must be achieved. Appendix B contains the analytical solutions of the thrust-vector timing of the engine thrust-vector (right/left-forward/backward).

Fig. 20 shows examples of the divert space boundary calculated using the WE approach during equally spaced time intervals along the approach trajectory.

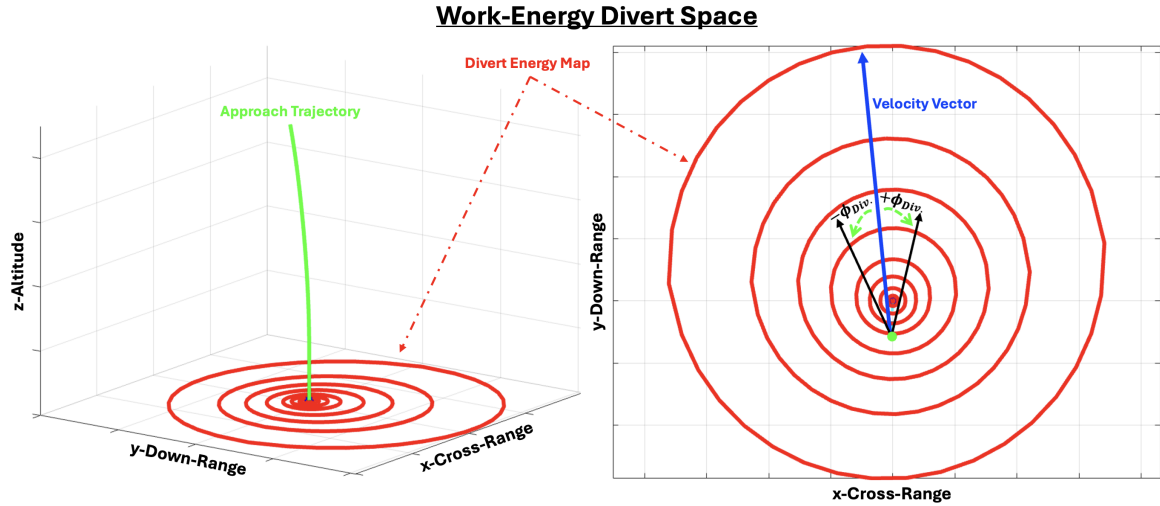


Figure 20 Divert space boundaries calculated using the WE formulation. The figure on the left shows the 3D nominal approach trajectory (green-line), while the ovals around the LS represent the divert boundaries initiated at equally spaced time intervals along the trajectory. The right plot shows a top view of the divert boundaries, and are symmetric with respect to the horizontal velocity vector present when the divert is initiated. The blue arrow represents the direction of the horizontal velocity vector direction at the beginning of the approach trajectory. The divert boundaries shrink around the LS as the divert maneuver is initiated later in time.

CONCLUSION

The WE approach applies the energy conservation principle to calculate instant guidance commands at any point during the trajectory in order to reach the desired final state. This approach simplifies the mathematics, and allows the calculation of real-time guidance commands to aid astronauts with manual control during different type of maneuvers (including landing, ascent, etc.), as well as providing them with real-time boundary limits of the available divert space during landing.

The physics-based WE formulation is more intuitive with respect to vehicle dynamics, and does not require linear algebra, neither the computational time and complexity to calculate, optimize, and propagate reference trajectories in order to calculate guidance commands. Instead, the approach seeks to satisfy the instant physical conditions that guarantee a safe landing, thus achieving similar performance and propellant consumption as convex-optimization methods with minimum computational time and cost. This approach is sensitive to the angular velocities employed to orient the vehicle into the optimal attitude and yaw, that may be used as tuning parameters specific to the vehicle. Possible numerical singularities during the WE computation are avoided by employing simple analytical expressions.

In summary, employing the WE formulation: Allows a more intuitive physical understanding of the trajectory and the consequences of changes due to perturbations and manual control, eliminates the need of constant calculation of reference trajectories, reduces the need of constant trajectory optimization, allows fast real-time calculation and recalculation of guidance commands at any point along the flight to achieve a safe landing, and allows fast real-time determination of the physically possible divert space for alternate landing maneuvers.

A similar WE formulation for real-time guidance in atmospheric conditions is in the process of being developed.

FUNDING SOURCES

The work presented here was funded under NASA's ESSCA Contract Number: 80MSFC18C0011.

ACKNOWLEDGMENTS

The author would like to acknowledge the team of Amentum Space Exploration Group (ASEG), as well as the NASA Human Landing System team, and the entire EV42 group at MSFC.

APPENDIX A: FINAL ROTATION MANEUVER FORMULATION

The transition region between the horizontal to the final vertical decent maneuver as shown in Fig. 4, is critical when trying to achieve complete horizontal energy dissipation. The goal of this final rotation maneuver is to successfully achieve vertical landing [attitude $\theta \approx 0$] with minimum horizontal velocity. From the WE perspective, this is achieved with a constant vertical force F_z obtained from Eq. (24b) as illustrated in Fig. 21.

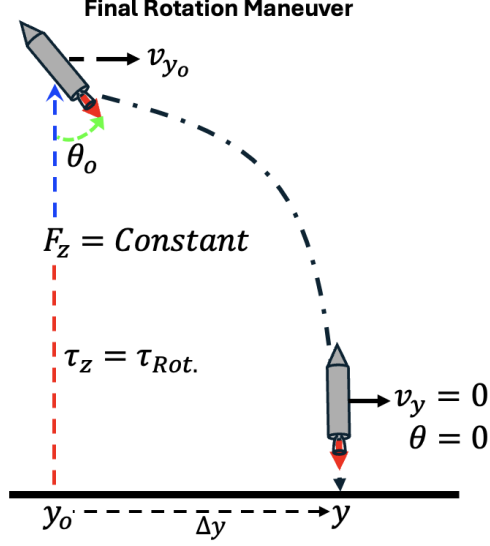


Figure 21 Final attitude rotation maneuver before landing. The maneuver starts with initial attitude θ_o and initial down-range velocity v_{y_o} . A constant vertical force F_z is necessary to dissipate the total vertical kinetic and potential energy, and thus to achieve a safe landing with final down-range velocity $v_y = 0$, total displacement Δy , and final attitude angle $\theta = 0$.

In order to formulate the solution with constant rotation, the down-range acceleration from Eq. (8b) is written as

$$\frac{dv_y}{dt} = \frac{\dot{m}}{m} \cdot v_e \cos(\phi) \sin(\theta) , \quad (36)$$

where by assuming a constant vertical force F_z , Eq. (5) is written in the form

$$F_T = \frac{F_z}{\cos(\theta)} = -v_e \cdot \dot{m} , \quad (37)$$

and the mass-flow rate is expressed as a function of the attitude angle θ as

$$\dot{m} = -\frac{F_z}{v_e \cdot \cos(\theta)} . \quad (38)$$

By plugging Eq. (38) into Eq. (36), and by assuming a constant angular attitude rate ω_θ

$$\frac{dv_y}{dt} = \frac{\partial v_y}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} = \omega_\theta \cdot \frac{dv_y}{d\theta} = -F_z \cdot \cos(\phi) \cdot \frac{\tan(\theta)}{m(\theta)} . \quad (39)$$

The mass as a function of the attitude angle $m(\theta)$ is obtained from Eq. (4) by setting the time $t = \frac{(\theta - \theta_o)}{\omega_\theta}$, and by employing Eq. (38), thus giving

$$m(\theta) = m_o \cdot \left[1 - \frac{F_z}{\omega_\theta m_o v_e} \cdot \frac{(\theta - \theta_o)}{\cos(\theta)} \right]. \quad (40)$$

The difference between down-range velocities is obtained by integrating and solving Eq. (39) with respect to the attitude angle in the form

$$v_y(\theta) - v_y(\theta_o) = -\frac{F_z}{\omega_\theta} \cdot \cos(\phi) \cdot \int_{\theta_o}^{\theta} d\theta' \cdot \frac{\tan(\theta')}{m(\theta')}. \quad (41)$$

By expanding to the first order, the reciprocal of the mass yields

$$\frac{1}{m(\theta')} \approx \frac{1}{m_o} \cdot \left[1 + \frac{F_z}{\omega_\theta m_o v_e} \cdot \frac{(\theta' - \theta_o)}{\cos(\theta')} \right], \quad (42)$$

the integral Eq. (41) is expressed as a simple trigonometric integral

$$v_y(\theta) - v_y(\theta_o) = -\frac{F_z}{\omega_\theta m_o} \cdot \cos(\phi) \cdot \int_{\theta_o}^{\theta} d\theta' \cdot \tan(\theta') \cdot \left[1 + \frac{F_z}{\omega_\theta m_o v_e} \cdot \frac{(\theta' - \theta_o)}{\cos(\theta')} \right], \quad (43)$$

with solution given by

$$v_y(\theta) - v_y(\theta_o) = \frac{F_z}{\omega_\theta m_o} \cdot \cos(\phi) \cdot \left\{ \ln \left[\frac{\cos(\theta)}{\cos(\theta_o)} \right] - \frac{F_z}{\omega_\theta m_o v_e} \cdot \left[\frac{(\theta - \theta_o)}{\cos(\theta)} - \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \cdot \frac{\cos(\theta)}{1 - \sin(\theta)} \right] \right] \right\}. \quad (44)$$

By setting $\theta = 0$ and $v_y(\theta) \xrightarrow{0} v_{y_o}$ [See Fig. 21], with $v_{y_o} = v_y(\theta_o)$, Eq. (44) is simplified to

$$v_{y_o} = \frac{F_z}{\omega_\theta m_o} \cdot \cos(\phi) \cdot \left\{ \ln [\cos(\theta_o)] - \frac{F_z}{\omega_\theta m_o v_e} \cdot \left[\theta_o + \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \right] \right\}. \quad (45)$$

Eq. (45) is written in the quadratic form with respect to ω_θ

$$\omega_\theta^2 - \frac{F_z}{m_o v_{y_o}} \cdot \cos(\phi) \cdot \ln [\cos(\theta_o)] \cdot \omega_\theta + \frac{F_z^2}{m_o^2 v_{y_o} v_e} \cdot \cos(\phi) \cdot \left\{ \theta_o + \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \right\}, \quad (46)$$

and by completing the square, the solution of the optimal constant attitude angular rate is given by

$$\omega_{\theta} = \frac{F_z}{2 \cdot m_o v_{y_o}} \cdot \left\{ \cos(\phi) \cdot \ln [\cos(\theta_o)] \pm \sqrt{\cos^2(\phi) \cdot \ln^2 [\cos(\theta_o)] - 4 \cdot F_z^2 \cdot \cos(\phi) \cdot \left(\frac{v_{y_o}}{v_e}\right) \cdot \left\{ \theta_o + \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \right\}} \right\} . \quad (47)$$

A further approximation for the angular rate may be obtained by noticing that the vertical force $F_z \sim m_o \cdot g$, where g is the local value of the lunar gravity acceleration. Therefore $\frac{F_z}{m_o v_e} \sim \frac{g}{v_e} \lll 1$, and the second term on the RHS of Eq. (45) may be ignored. The approximated angular rate is thus given by

$$\omega_{\theta} \approx \frac{F_z}{m_o v_{y_o}} \cdot \cos(\phi) \cdot \ln [\cos(\theta_o)] . \quad (48)$$

The estimated time for the final rotation maneuver is simply calculated using

$$\tau_{Rot.} = -\frac{\theta_o}{\omega_{\theta}} . \quad (49)$$

The down-range total displacement as a function of the angle is formulated from Eq. (44) in the form

$$\frac{dy}{dt} = \frac{\partial y}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} = \omega_{\theta} \cdot \frac{dy}{d\theta} = v_y(\theta) , \quad (50)$$

that is integrated and expressed as

$$y(\theta) - y(\theta_o) = \frac{(\theta - \theta_o)}{\omega_{\theta}} \cdot v_{y_o} + \frac{F_z^2}{\omega_{\theta}^3 m_o^2 v_e} \cdot \cos(\phi) \cdot (\theta - \theta_o) \cdot \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \cdot \frac{\cos(\theta)}{1 + \sin(\theta)} \right] + \frac{F_z}{\omega_{\theta}^2 m_o} \cdot \cos(\phi) \cdot \int_{\theta_o}^{\theta} d\theta' \cdot \ln \left[\frac{\cos(\theta')}{\cos(\theta_o)} \right] . \quad (51)$$

The logarithmic integral with trigonometric terms can be solved using a Taylor expansion with respect to θ_o using

$$\begin{aligned} f(\theta') &= \ln \left[\frac{\cos(\theta')}{\cos(\theta_o)} \right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(\theta_o)}{n!} \cdot (\theta' - \theta_o)^n \\ &\approx -\frac{\sin(\theta_o)}{\cos(\theta_o)} \cdot (\theta' - \theta_o) - \frac{1}{2} \cdot \frac{1}{\cos^2(\theta_o)} \cdot (\theta' - \theta_o)^2 - \frac{1}{3} \cdot \frac{\sin(\theta_o)}{\cos^2(\theta_o)} \cdot (\theta' - \theta_o)^3 \\ &\quad - \frac{1}{12} \cdot \frac{[1 + \sin^2(\theta_o)]}{\cos^3(\theta_o)} \cdot (\theta' - \theta_o)^4 - \frac{1}{60} \cdot \frac{\sin(\theta_o) \cdot [5 + \sin^2(\theta_o)]}{\cos^4(\theta_o)} \cdot (\theta' - \theta_o)^5 \\ &\quad - \frac{1}{360} \cdot \frac{[5 + 18 \cdot \sin^2(\theta_o) + \sin^4(\theta_o)]}{\cos^5(\theta_o)} \cdot (\theta' - \theta_o)^6 , \end{aligned} \quad (52)$$

where in this work, the expansion is expressed as a 6th order polynomial. By plugging the truncated expansion from Eq. (52) into Eq. (51), and by integrating with respect to θ' , the following solution is obtained

$$\begin{aligned}
y(\theta) - y(\theta_o) \approx & \frac{(\theta - \theta_o)}{\omega_\theta} \cdot v_{y_o} + \frac{F_z^2}{\omega_\theta^3 m_o^2 v_e} \cdot \cos(\phi) \cdot (\theta - \theta_o) \cdot \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \cdot \frac{\cos(\theta)}{1 + \sin(\theta)} \right] \\
& - \frac{F_z}{2 \cdot \omega_\theta^2 m_o} \cdot \cos(\phi) \cdot \left\{ \frac{\sin(\theta_o)}{\cos(\theta_o)} \cdot (\theta - \theta_o)^2 + \frac{1}{3} \cdot \frac{1}{\cos^2(\theta_o)} \cdot (\theta - \theta_o)^3 \right. \\
& + \frac{1}{6} \cdot \frac{\sin(\theta_o)}{\cos^2(\theta_o)} \cdot (\theta - \theta_o)^4 + \frac{1}{30} \cdot \frac{[1 + \sin^2(\theta_o)]}{\cos^3(\theta_o)} \cdot (\theta - \theta_o)^5 \\
& + \frac{1}{180} \cdot \frac{\sin(\theta_o) \cdot [5 + \sin^2(\theta_o)]}{\cos^4(\theta_o)} \cdot (\theta - \theta_o)^6 \\
& \left. + \frac{1}{1260} \cdot \frac{[5 + 18 \cdot \sin^2(\theta_o) + \sin^4(\theta_o)]}{\cos^5(\theta_o)} \cdot (\theta - \theta_o)^7 \right\}.
\end{aligned} \tag{53}$$

The total displacement during the final rotation maneuver is obtained when the attitude angle reaches zero [$\theta = 0$], thus giving

$$\begin{aligned}
y(0) - y(\theta_o) \approx & -\frac{\theta_o}{\omega_\theta} \cdot v_{y_o} - \frac{F_z^2}{\omega_\theta^3 m_o^2 v_e} \cdot \cos(\phi) \cdot \theta_o \cdot \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \\
& - \frac{F_z}{2 \cdot \omega_\theta^2 m_o} \cdot \cos(\phi) \cdot \left\{ \frac{\sin(\theta_o)}{\cos(\theta_o)} \cdot \theta_o^2 - \frac{1}{3} \cdot \frac{1}{\cos^2(\theta_o)} \cdot \theta_o^3 + \frac{1}{6} \cdot \frac{\sin(\theta_o)}{\cos^2(\theta_o)} \cdot \theta_o^4 \right. \\
& - \frac{1}{30} \cdot \frac{[1 + \sin^2(\theta_o)]}{\cos^3(\theta_o)} \cdot \theta_o^5 + \frac{1}{180} \cdot \frac{\sin(\theta_o) \cdot [5 + \sin^2(\theta_o)]}{\cos^4(\theta_o)} \cdot \theta_o^6 \\
& \left. - \frac{1}{1260} \cdot \frac{[5 + 18 \cdot \sin^2(\theta_o) + \sin^4(\theta_o)]}{\cos^5(\theta_o)} \cdot \theta_o^7 \right\}.
\end{aligned} \tag{54}$$

Just before landing, the WE formulation may approach a singularity when the denominator in Eq. (24a) approaches zero [$(y_f - y_o) \rightarrow 0$] along the down-range. For the vertical component, this problem may be avoided by just keeping the force F_z at its constant previous value. In these type of conditions, it is no longer suitable to calculate the optimal approach angle using Eq. (25). Therefore, in order to drive the horizontal velocity to its final desired value, the attitude angle is directly solved from Eq. (9b) inside the same calculation cycle illustrated in Fig. 6 using

$$\theta_{opt.} = \sin^{-1} \left\{ \frac{[v_y(\tau_z) - v_{y_o}]}{v_e \cos(\phi) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right]} \right\}. \tag{55}$$

Inside the same cycle, the optimal yaw angle is solved from Eq. (9a) in the form

$$\phi_{opt.} = \sin^{-1} \left\{ \frac{[v_x(\tau_x) - v_{x_o}]}{v_e \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right]} \right\}, \tag{56}$$

with $\tau_x = \delta \cdot \tau_z$, where the positive constant $\delta \leq 1$, and its value is adjusted heuristically as a tuning parameter. In order not to induce oscillations into the small horizontal component of the velocity, the optimal attitude angle obtained from Eq. (55) is truncated to a maximum angle $\theta_{Max.} = \gamma \cdot |v_{y_o}| \text{ rad.}$, with the positive constant γ obtained heuristically.

APPENDIX B: TIME-MANAGEMENT FOR DIVERT SPACE CALCULATION

The outer boundaries of the divert space are limited to the total vertical descent (“airborne”) time τ calculated using Eq. (11c), and the optimal attitude angle θ obtained from Eq. (35). When the divert maneuver is initiated, the divert space is symmetric with respect to the latest horizontal velocity vector. Therefore, for divert calculations purposes, the cross-range position is defined as to the right/left of the current velocity vector (not with respect to the LS) defined with a yaw angle ϕ' , initial cross-range velocity $v_{x_o} = 0$, and initial down-range velocity v_{y_o} . Once the divert space is calculated with respect to the velocity vector, it is then transformed into the LS coordinates.

In order to maneuver the lander to the maximum boundaries, the yaw angle is changed according to the desired divert direction, and the change of the yaw angle orientation must be timed precisely to guarantee a zero horizontal velocity at the time of landing. Therefore, Eq. (9a) is broken into two maneuvers with different yaw angles (pointing right/left), and times for the cross-range in the form

$$v_{x_1} = \cancel{v_{x_o}}^0 + v_e \sin(\phi'_1) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_{x_1} \right], \quad (57a)$$

$$\cancel{v_{x_2}}^0 = v_{x_1} + v_e \sin(\phi'_2) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_1} \cdot \tau_{x_2} \right], \quad (57b)$$

where $\phi'_1 = \phi'$ and $\phi'_2 = -\phi'$ [or $\phi'_2 = \pm\phi' + \pi$], and $m_1 = m_o + \dot{m} \cdot \tau_{x_1}$. By noticing that $\tau = \tau_{x_1} + \tau_{x_2}$ [and thus $\tau_{x_2} = \tau - \tau_{x_1}$], Eq. (57a) is plugged into Eq. (57b) and solved for τ_{x_1} thus giving

$$\tau_{x_1} = -\frac{m_o}{\dot{m}} \cdot \left[1 - \sqrt{1 + \frac{\dot{m}}{m_o} \cdot \tau} \right]. \quad (58)$$

In a similar manner, Eq. (9b) is broken into two maneuvers with two different yaw angles (forward/backward) for the down-range (forward/backward) component in the form

$$v_{y_1} = v_{y_o} + v_e \cos(\phi'_1) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_{y_1} \right], \quad (59a)$$

$$\cancel{v_{y_2}}^0 = v_{y_1} + v_e \cos(\phi'_2) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_1} \cdot \tau_{y_2} \right], \quad (59b)$$

with $\phi'_1 = \phi'$ and $\phi'_2 = -\phi'$ [or $\phi'_2 = \pm\phi' + \pi$], $m_1 = m_o + \dot{m} \cdot \tau_{y_1}$, and $\tau = \tau_{y_1} + \tau_{y_2}$, Eq. (59a) is plugged into Eq. (59b) to solved for τ_{y_1}

$$\tau_{y_1} = -\frac{m_o}{\dot{m}} \cdot \left[1 - \sqrt{1 + \frac{m_o}{\dot{m}} \cdot \tau} \cdot e^{\frac{v_{y_o}}{2v_e \cos(\phi') \sin(\theta)}} \right], \quad (60)$$

and $\tau_{y_2} = \tau - \tau_{y_1}$.

The optimal divert yaw angle is the angle that naturally makes the down-range time equals to the airborne time (vertical time of flight). By setting $\tau_y = \tau_z$, and the final down-range velocity to zero [$y(\tau_y) = 0$] in Eq. (11b), the following expression is obtained

$$-\tau_z \cdot v_e \cos(\phi) \sin(\theta) = \frac{m_o}{\dot{m}} \cdot v_{y_o} + [y(\tau_z) - y_o] . \quad (61)$$

By solving Eq. (61) for the yaw angle yields

$$\phi_{Div.} = \pm \cos^{-1} \left[\frac{-v_{y_o}}{v_e \sin(\theta_{Div.}) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right]} \right] , \quad (62)$$

where this angle may be taken with respect to the local velocity vector when performing the divert maneuver. Therefore, the spacecraft may be stirred either to the right or to the left, preferable towards the same direction where the engine is already pointing in order to reduce the maneuver workload and to maximize the divert range.

APPENDIX C: LS-TO-BODY TRANSFORMATION OF ANGULAR-RATES

The LS attitude and yaw angles [θ and ϕ] defined in Fig. 3, greatly simplify the mathematics employed to derive the angular-ratio commands. However, these angular ratios are given with respect to the local LS coordinate frame, while the lander needs to perform these rotations with respect to its body-reference (or navigation), as shown on the right of Fig. 3.

In this derivation, the angular-rates in the navigation-frame are separated into two simple rotations to derive the frame transformation summarized in Fig. 22.

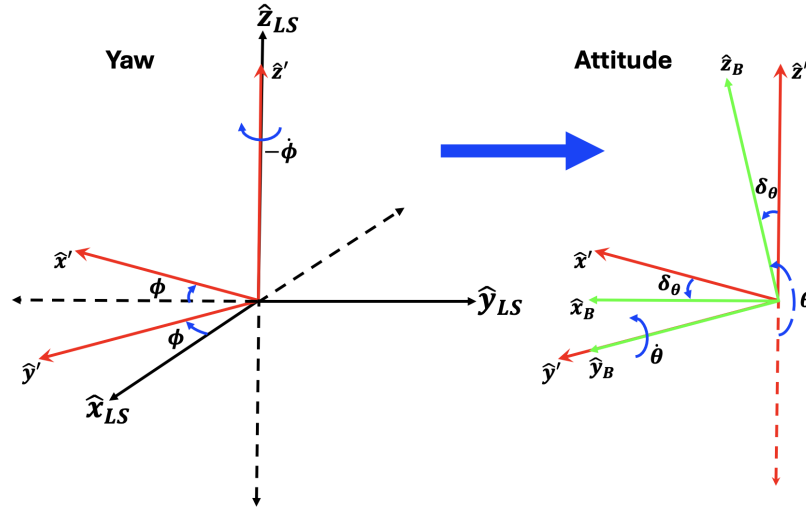


Figure 22 Yaw and attitude angular rotation-rate transformation from LS-coordinates into the body frame. First rotation (Yaw) is performed from the LS frame into the intermediate prime frame, while the second rotation (Attitude) is performed from the intermediate prime frame into the body (navigation) frame of the lander.

As shown on the left of Fig. 22, the Yaw rotation consists on a left-hand (negative) angular rotation of magnitude ϕ with respect to the $\hat{\mathbf{z}}_{\text{LS}}$ axis, and that coincides with the intermediate prime reference $\hat{\mathbf{z}}'$ expressed in the form

$$\omega_{\text{LS}}^{\mathbf{r}'} = -\dot{\phi} \cdot \hat{\mathbf{z}}_{\text{LS}} = -\dot{\phi} \cdot \hat{\mathbf{z}}', \quad (63)$$

with the frame-rotation from the LS frame into the intermediate prime frame represented by the matrix operation

$$\begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix} = \begin{bmatrix} -\sin(\phi) & -\cos(\phi) & 0 \\ \cos(\phi) & -\sin(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{x}}_{\text{LS}} \\ \hat{\mathbf{y}}_{\text{LS}} \\ \hat{\mathbf{z}}_{\text{LS}} \end{bmatrix}. \quad (64)$$

The attitude rotation shown on the right of Fig. 22, consists on a right-hand angular rotation of magnitude δ_θ along the intermediate prime $\hat{\mathbf{y}}'$ axis that coincides with the body $\hat{\mathbf{y}}_{\text{B}}$ axis, and that is expressed in the form

$$\omega_{\mathbf{r}'}^{\text{B}} = \dot{\theta} \cdot \hat{\mathbf{y}}' = \dot{\theta} \cdot \hat{\mathbf{y}}_{\text{B}}, \quad (65)$$

with the frame-rotation from the intermediate prime frame into the body frame represented by

$$\begin{bmatrix} \hat{\mathbf{x}}_{\text{B}} \\ \hat{\mathbf{y}}_{\text{B}} \\ \hat{\mathbf{z}}_{\text{B}} \end{bmatrix} = \begin{bmatrix} \cos(\delta_\theta) & 0 & -\sin(\delta_\theta) \\ 0 & 1 & 0 \\ \sin(\delta_\theta) & 0 & \cos(\delta_\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix}. \quad (66)$$

Therefore, the total angular-rate transformation between the LS and body is given by

$$\omega_{\text{LS}}^{\text{B}} = \omega_{\text{LS}}^{\mathbf{r}'} + \omega_{\mathbf{r}'}^{\text{B}} = -\dot{\phi} \cdot \hat{\mathbf{z}}' + \dot{\theta} \cdot \hat{\mathbf{y}}_{\text{B}}, \quad (67)$$

where the prime intermediate frame is expressed in terms of the body frame by inverting the matrix from Eq. (66)

$$\begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix} = \begin{bmatrix} \cos(\delta_\theta) & 0 & \sin(\delta_\theta) \\ 0 & 1 & 0 \\ -\sin(\delta_\theta) & 0 & \cos(\delta_\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{x}}_{\text{B}} \\ \hat{\mathbf{y}}_{\text{B}} \\ \hat{\mathbf{z}}_{\text{B}} \end{bmatrix}, \quad (68)$$

and the $\hat{\mathbf{z}}'$ axis written in terms of the body frame yields

$$\hat{\mathbf{z}}' = -\sin(\delta_\theta) \cdot \hat{\mathbf{x}}_{\text{B}} + 0 \cdot \hat{\mathbf{y}}_{\text{B}} + \cos(\delta_\theta) \cdot \hat{\mathbf{z}}_{\text{B}}. \quad (69)$$

By plugging the solution from Eq. (69) into Eq. (67), the angular-rates transformation is expressed in the form

$$\omega_{\text{LS}}^{\text{B}} = \dot{\phi} \cdot \sin(\delta_\theta) \cdot \hat{\mathbf{x}}_{\text{B}} + \dot{\theta} \cdot \hat{\mathbf{y}}_{\text{B}} - \dot{\phi} \cdot \cos(\delta_\theta) \cdot \hat{\mathbf{z}}_{\text{B}}. \quad (70)$$

Notice from Fig. 22, that $\delta_\theta = \theta - \frac{\pi}{2}$. Therefore, the final form of the frame transformation of the angular-rates from LS into body is given by

$$\omega_{\text{LS}}^{\text{B}} = -\dot{\phi} \cdot \cos(\theta) \cdot \hat{\mathbf{x}}_{\text{B}} + \dot{\theta} \cdot \hat{\mathbf{y}}_{\text{B}} - \dot{\phi} \cdot \sin(\theta) \cdot \hat{\mathbf{z}}_{\text{B}}. \quad (71)$$

APPENDIX D: FORMULATION OF A PHYSICS-BASED LUNAR ALTIMETER

In general during the final descent phase, a lander relies on the fusion of both Inertial Navigation System (INS), and Radar Altimeter (through Kalman filtering) to accurately estimate the altitude and vertical velocity needed to achieve a safe landing. In case of radar blackout or failure when performing this critical maneuver, the INS integration becomes less reliable with respect to time due to the accumulation of process noise during the forward state propagation process.

In an effort to reduce error estimation, a physics-based altimeter to help correct the INS during final descent is suggested. While the INS is based on the integration and double integration of accelerometer data fused with gyroscopic data to estimate velocity and position vectors, the physics-based altimeter is based on the expected exerted acceleration from the operating and well characterized engines fused with the gyroscopic data instead of relying on accelerometers.

For an interval Δt with constant net engine mass-flow $\dot{m}$, the vertical states are calculated using Eqs. (9c) and (10c) in the form

$$v_z = v_{z_o} - g \cdot \Delta t - v_e \cos(\theta_i) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right], \quad (72a)$$

$$z = z_o + \left[v_{z_o} + v_e \cos(\theta_i) \right] \cdot \Delta t - \frac{1}{2} \cdot g \cdot \Delta t^2 - v_e \cos(\theta_i) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right], \quad (72b)$$

where v_{z_o} , z_o , θ_o , m_o , and ω_θ are the latest vertical velocity, altitude, attitude, estimated mass, effective exhaust velocity, and attitude-rate states of the lander with standard deviations: $\sigma_{v_{z_o}}$, σ_{z_o} , σ_{θ_o} , σ_{m_o} , σ_{v_e} , σ_{ω_θ} , and σ_{ω_ϕ} .

The average attitude and yaw angles during the integration interval Δt are estimated with: $\theta_i = \theta_o + \frac{1}{2} \cdot \omega_\theta \cdot \Delta t$ and $\phi_i = \phi_o + \frac{1}{2} \cdot \omega_\phi \cdot \Delta t$, and the variances for the estimated states are calculated using

$$\sigma_{v_z}^2 = \sigma_{v_{z_o}}^2 + \sigma_g^2 \cdot \Delta t^2 + \sigma_{v_e}^2 \cdot \cos^2(\theta_i) \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] + \frac{v_e^2 \cos^2(\theta_i) \cdot \Delta t^2}{[m_o + \dot{m} \cdot \Delta t]^2} \cdot \left[\sigma_{\dot{m}}^2 + \frac{\dot{m}^2}{m_o^2} \cdot \sigma_{m_o}^2 \right] + v_e^2 \sin^2(\theta_i) \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right], \quad (73a)$$

$$\sigma_z^2 = \sigma_{z_o}^2 + \sigma_{v_{z_o}}^2 \cdot \Delta t^2 + \frac{1}{4} \cdot \sigma_g^2 \cdot \Delta t^4 + \sigma_{v_e}^2 \cdot \cos^2(\theta_i) \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 + v_e^2 \sin^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 + v_e^2 \cos^2(\theta_i) \cdot \left[\frac{\sigma_{\dot{m}}^2}{\dot{m}^2} + \frac{\sigma_{m_o}^2}{m_o^2} \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2, \quad (73b)$$

with the standard deviations of the modeled gravity and mass-flow represented by σ_g and $\sigma_{\dot{m}}$.

In general, the same physics-base navigation formulation may be employed in both cross-range and down-range horizontal position and horizontal velocity estimations by using Eqs. (9a) and (9b) together with Eqs. (10a) and (10b) in the form

$$v_x = v_{x_o} + v_e \sin(\phi_i) \sin(\theta_i) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right], \quad (74a)$$

$$v_y = v_{y_o} + v_e \cos(\phi_i) \sin(\theta_i) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right], \quad (74b)$$

$$x = x_o + \left[v_{x_o} - v_e \sin(\phi_i) \sin(\theta_i) \right] \cdot \Delta t + v_e \sin(\phi_i) \sin(\theta_i) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right], \quad (74c)$$

$$y = y_o + \left[v_{y_o} - v_e \cos(\phi_i) \sin(\theta_i) \right] \cdot \Delta t + v_e \cos(\phi_i) \sin(\theta_i) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right], \quad (74d)$$

with variances calculated using

$$\begin{aligned} \sigma_{v_x}^2 = \sigma_{v_{x_o}}^2 + & \left\{ \sigma_{v_e}^2 \cdot \sin^2(\phi_i) \sin^2(\theta_i) + v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \sin^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \\ & + \frac{v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \Delta t^2}{[m_o + \dot{m} \cdot \Delta t]^2} \cdot \left[\sigma_{\dot{m}}^2 + \frac{\dot{m}^2}{m_o^2} \cdot \sigma_{m_o}^2 \right], \end{aligned} \quad (75a)$$

$$\begin{aligned} \sigma_{v_y}^2 = \sigma_{v_{y_o}}^2 + & \left\{ \sigma_{v_e}^2 \cdot \cos^2(\phi_i) \sin^2(\theta_i) + v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \cos^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \\ & + \frac{v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \Delta t^2}{[m_o + \dot{m} \cdot \Delta t]^2} \cdot \left[\sigma_{\dot{m}}^2 + \frac{\dot{m}^2}{m_o^2} \cdot \sigma_{m_o}^2 \right], \end{aligned} \quad (75b)$$

$$\begin{aligned} \sigma_x^2 = \sigma_{x_o}^2 + \sigma_{v_{x_o}}^2 \cdot \Delta t^2 + & \left\{ \sigma_{v_e}^2 \cdot \sin^2(\phi_i) \sin^2(\theta_i) + v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \sin^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \\ & \times \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 \\ & + v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \left[\frac{\sigma_{\dot{m}}^2}{\dot{m}^2} + \frac{\sigma_{m_o}^2}{m_o^2} \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2, \end{aligned} \quad (75c)$$

$$\begin{aligned} \sigma_y^2 = \sigma_{y_o}^2 + \sigma_{v_{y_o}}^2 \cdot \Delta t^2 + & \left\{ \sigma_{v_e}^2 \cdot \cos^2(\phi_i) \sin^2(\theta_i) + v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \cos^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \\ & \times \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 \\ & + v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \left[\frac{\sigma_{\dot{m}}^2}{\dot{m}^2} + \frac{\sigma_{m_o}^2}{m_o^2} \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2. \end{aligned} \quad (75d)$$

REFERENCES

- [1] S. Boyd and L. Vandenberghe, "Convex Optimization," *Cambridge University Press*, Cambridge UK, 2004.
<https://doi.org/10.1017/CBO9780511804441>
- [2] A. Miele, "The calculus of variations in applied aerodynamics and flight mechanics, In: Optimization Techniques with Applications to Aerospace Systems," *Academic Press*, 1962, pp. 99-170.
[https://doi.org/10.1016/S0076-5392\(08\)62092-5](https://doi.org/10.1016/S0076-5392(08)62092-5)
- [3] J. Meditch, "On the Problem of Optimal Thrust Programming for a Lunar Soft Landing," *IEEE Transactions on Automatic Control*, Vol. **9**, No. 4, 1964, pp. 477-484.
<https://doi.org/10.1109/TAC.1964.1105758>
- [4] F. Gazzola and E. M. Marchini, "The Moon Lander Optimal Control Problem Revisited," *Mathematics in Engineering*, Vol. **3**, Iss. 5, 2021, pp. 1-14.
<https://doi.org/10.3934/mine.2021040>
- [5] A. W. Wilhite, J. Wagner, R. Tolson, and M. M. Moen, "Lunar Module Descent Mission Design," *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, 2012.
<https://doi.org/10.2514/6.2008-6939>
- [6] D. Malyuta, T. P. Reynolds, M. Szmuk, B. Açıkmeşe, and M. Mesbahi, "Fast Trajectory Optimization via Successive Convexification for Spacecraft Rendezvous with Integer Constraints," *AIAA Scitech 2020 Forum*, 2020.
<https://doi.org/10.2514/6.2020-0616>
- [7] M. Szmuk and B. Açıkmeşe, "Successive Convexification for 6-DoF Mars Rocket Powered Landing with Free-Final-Time," *AIAA Scitech Forum*, Session: Entry, Descent and Landing GN&C Technology I, 2018.
<https://doi.org/10.2514/6.2018-0617>
- [8] T. Reynolds, D. Malyuta, M. Mesbahi, B. Açıkmeşe, and J. M. Carson, "A Real-Time Algorithm for Non-Convex Powered Descent Guidance," *AIAA Scitech Forum*, Session: Entry, Descent and Landing GN&C Technology III, 2020.
<https://doi.org/10.2514/6.2020-0844>
- [9] R. Mu, Y. Deng, and P. Wu, "Adaptive Convex Optimization Guidance for Lunar Landing," *Aerospace*, **10**(7), 634, 2023.
<https://doi.org/10.3390/aerospace10070634>
- [10] P. Lu, "Augmented Apollo Powered Descent Guidance," *Journal of Guidance, Control, and Dynamics*, Vol. **42**, No. 3, 2019, pp. 447-457.
<https://doi.org/10.2514/1.G004048>
- [11] P. Lu, "Theory of Fractional-Polynomial Powered Descent Guidance," *Journal of Guidance, Control, and Dynamics*, Vol. **43**, No. 3, 2020, pp. 398-409.
<https://doi.org/10.2514/1.G004556>
- [12] J. Orphee, N. Olson, K. Clements, M. Hawkins, A. Summers, W. Wu, M. Fritzinger, and K. Miller, "Evaluation of Precision Landing Performance Using a Generalized Aerospace Simulation in SIMULINK Framework," *AAS 23-094*, 2023.
<https://ntrs.nasa.gov/citations/20230000011>