

Risk-aware Data-driven Decision Making under Uncertainty



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Motivation

- Improving robustness of a given design
- “Fixing” a system that fails a Monte Carlo analysis test
- Carrying out performance vs. robustness tests

Outline

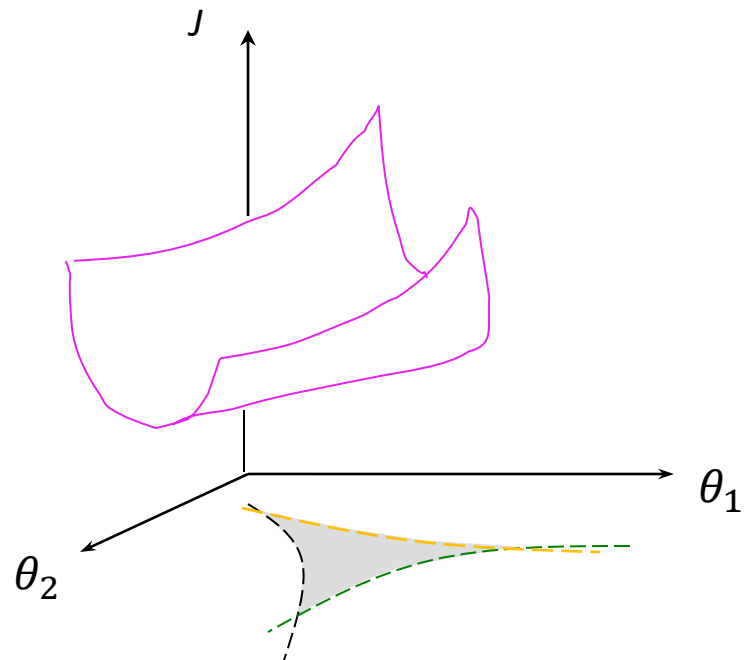
- Introductory concepts
- Solution strategies
- Scenario Theory

Optimization under Uncertainty

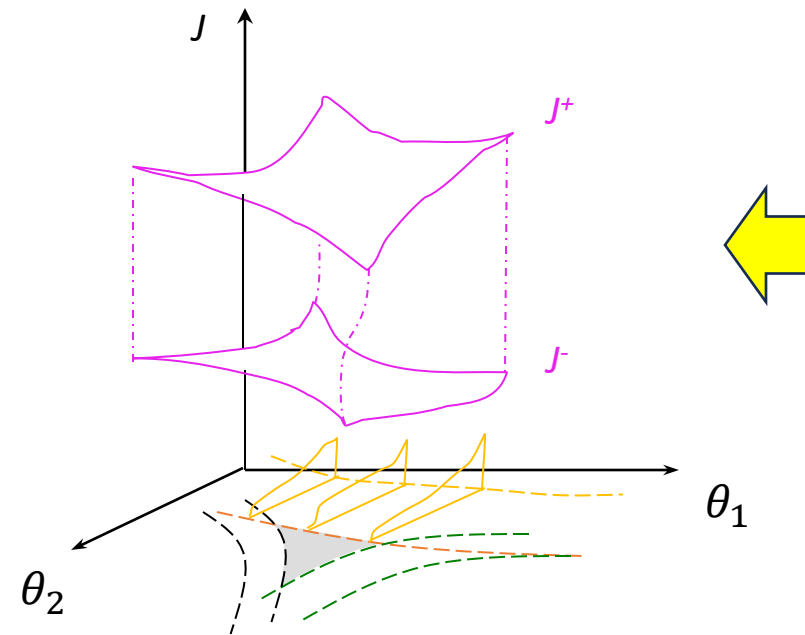
- Design processes are often performed by optimization
- Optimization is highly sensitive to what it is assumed to be known
- Applications of optimization under uncertainty: regression analysis, model calibration, controls, system identification, machine learning, systems theory, financial mathematics, decision making, etc.

Optimization under Uncertainty

Deterministic Setting

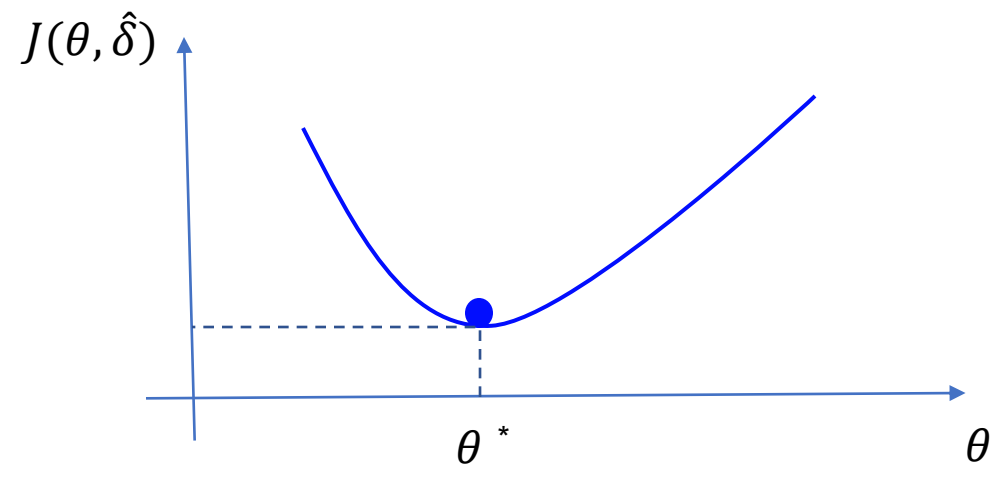
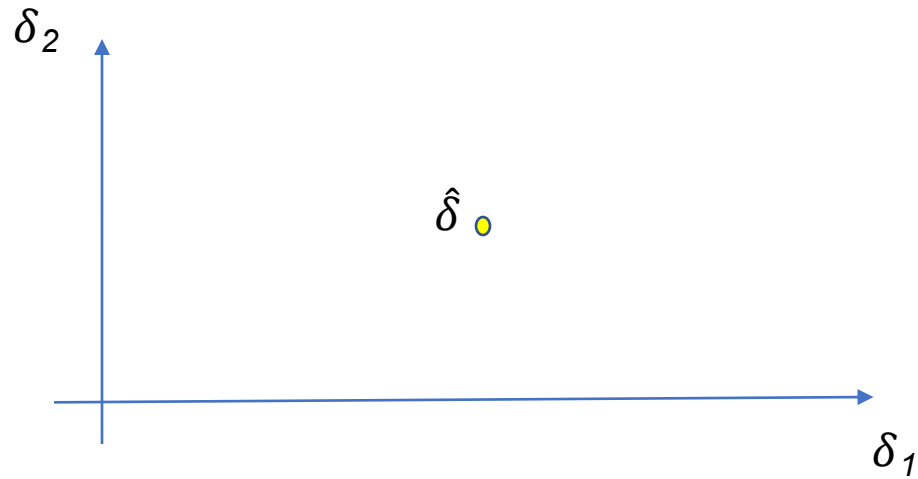


Probabilistic Setting



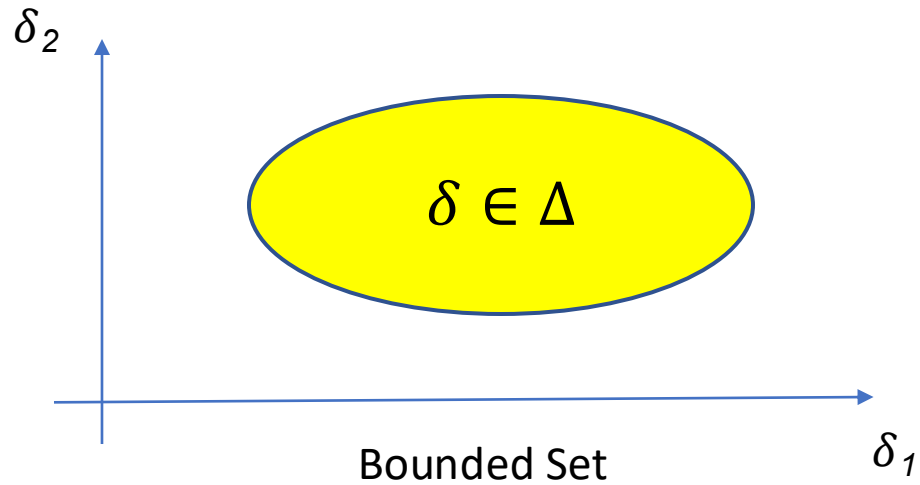
Optimization under Uncertainty: $\min_{\theta \in \Theta} J(\theta, \hat{\delta})$

- The value of δ is known



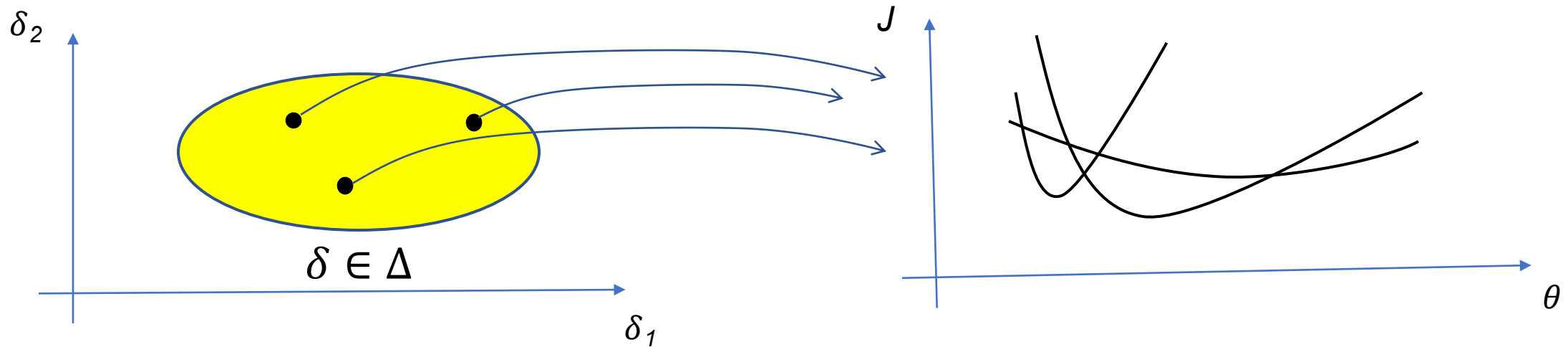
Optimization under Uncertainty

- The value of δ is known
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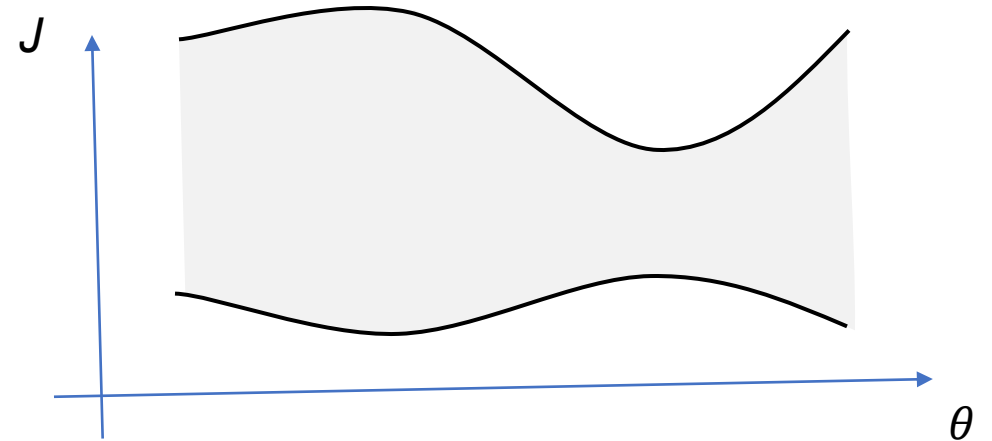
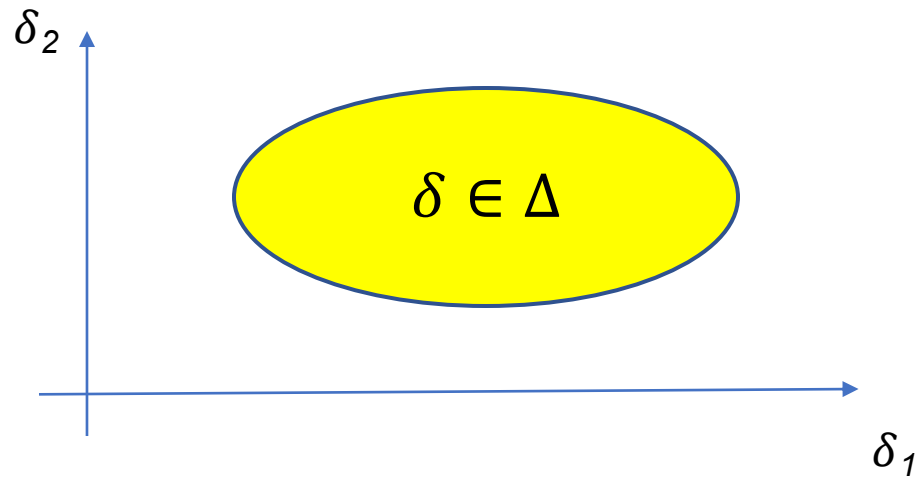
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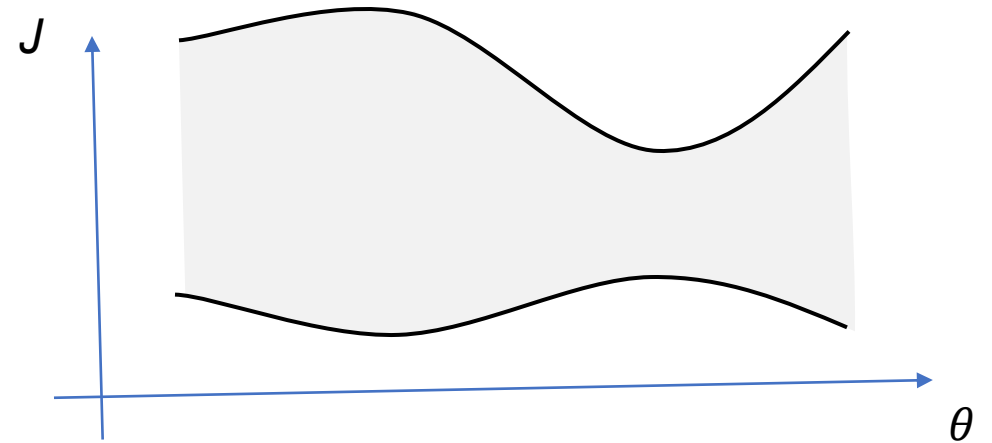
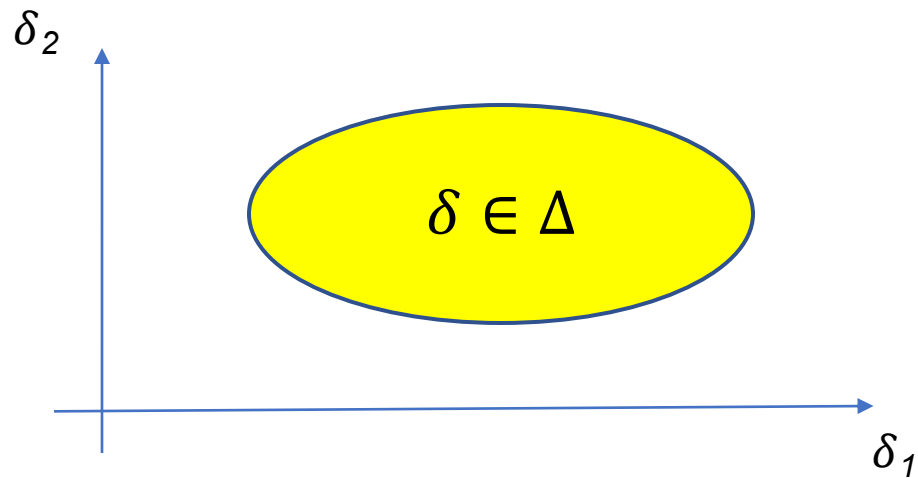
Optimization under Uncertainty

- The value of δ is known
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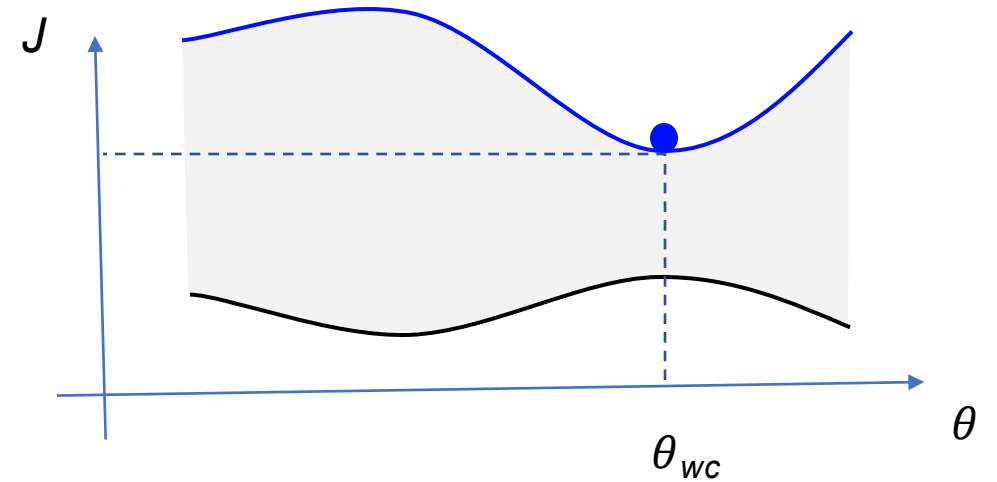
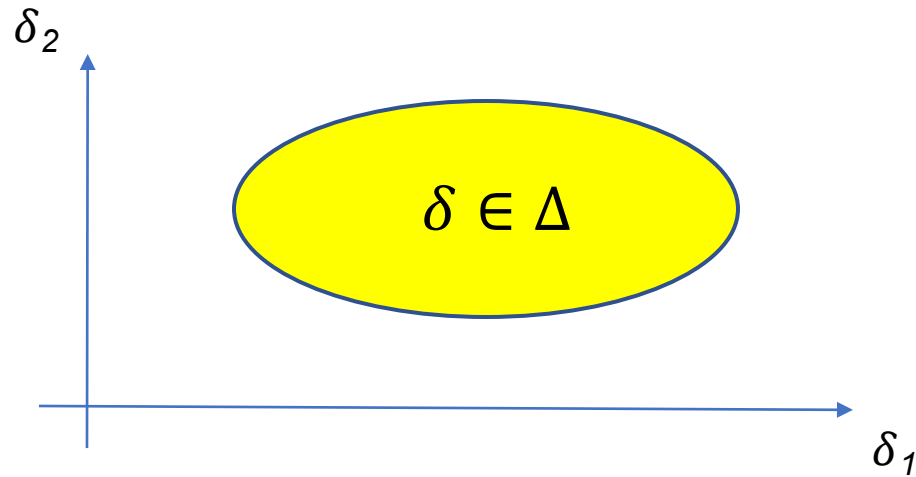


Optimization under Uncertainty

- The value of δ is known
- The value of δ is unknown: worst-case and chance-constrained



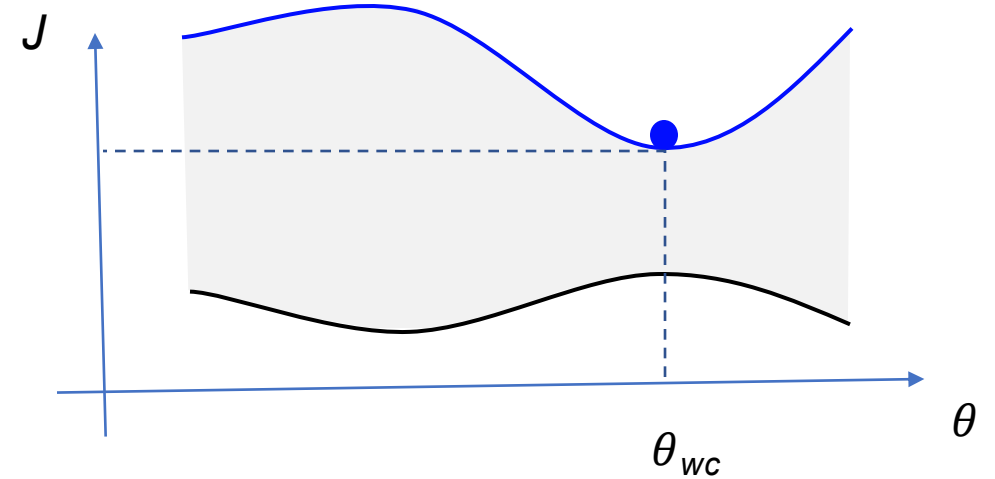
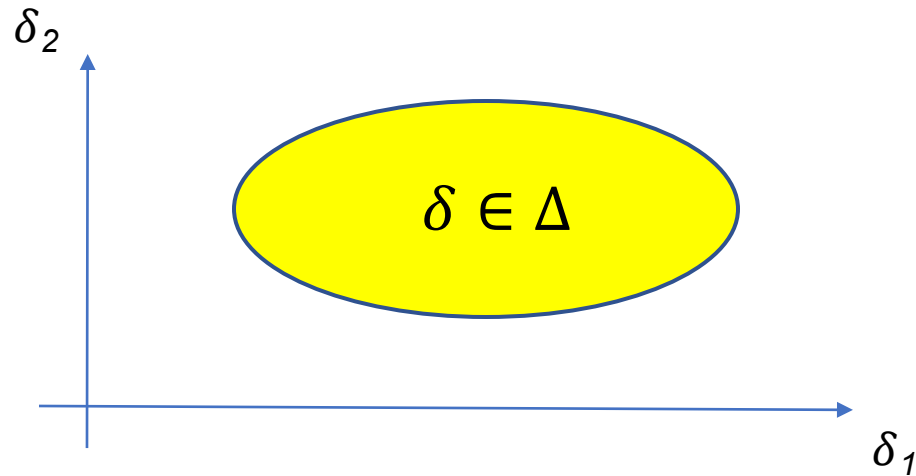
Worst-Case



- Formulation

$$\min_{\theta \in \Theta} \max_{\delta \in \Delta} J(\theta, \delta)$$

Worst-Case

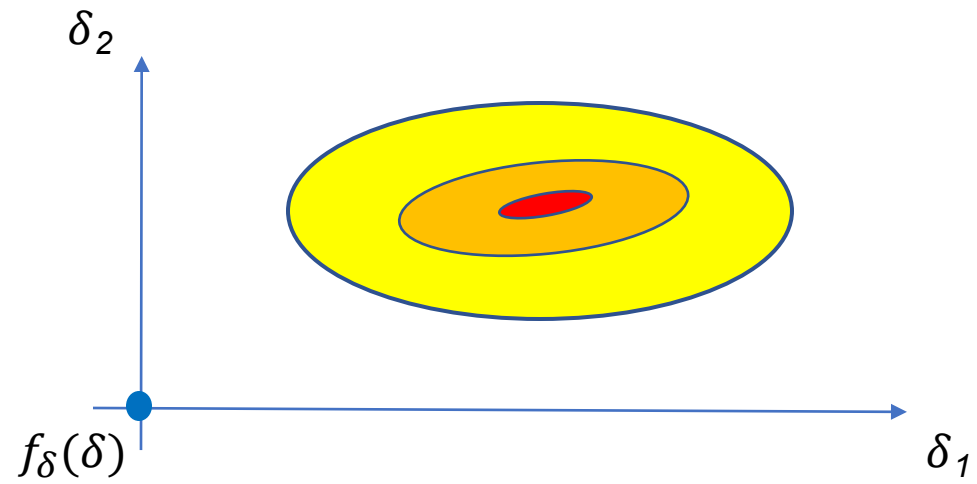


- Formulation

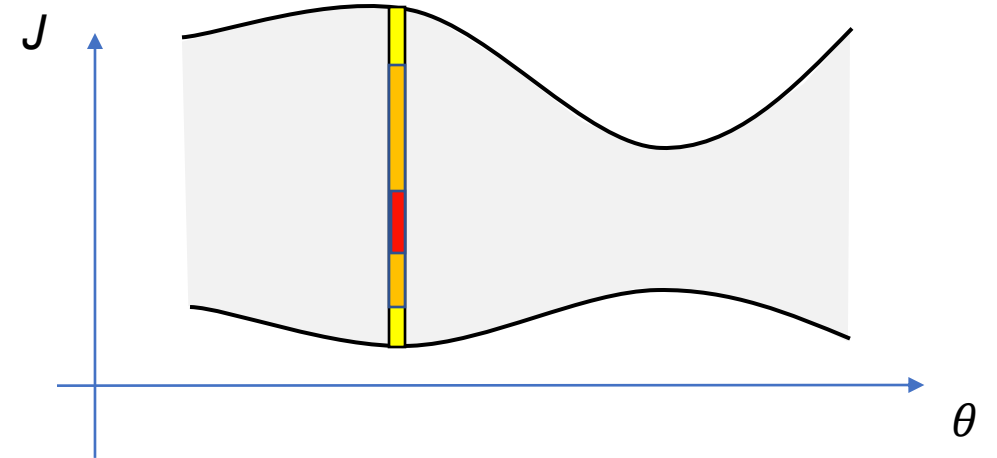
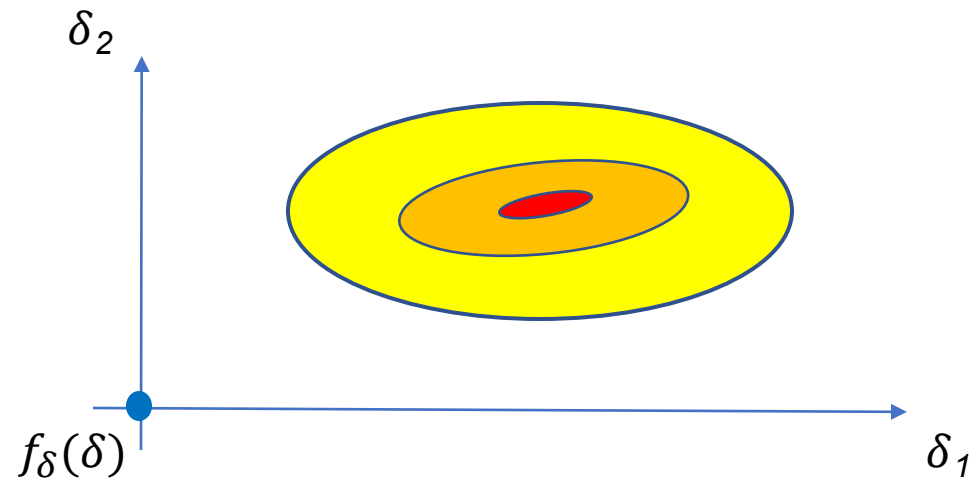
$$\min_{\theta \in \Theta} \max_{\delta \in \Delta} J(\theta, \delta)$$

- Intractability, performance guarantees, conservatism

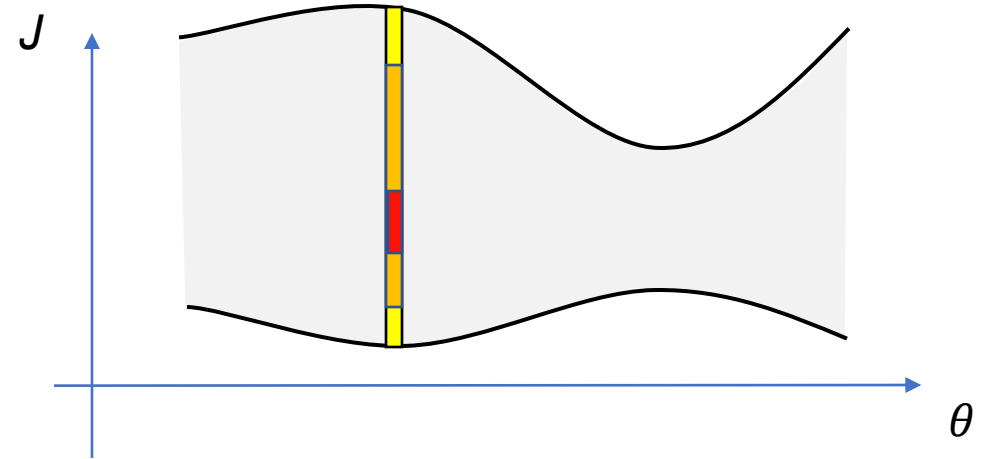
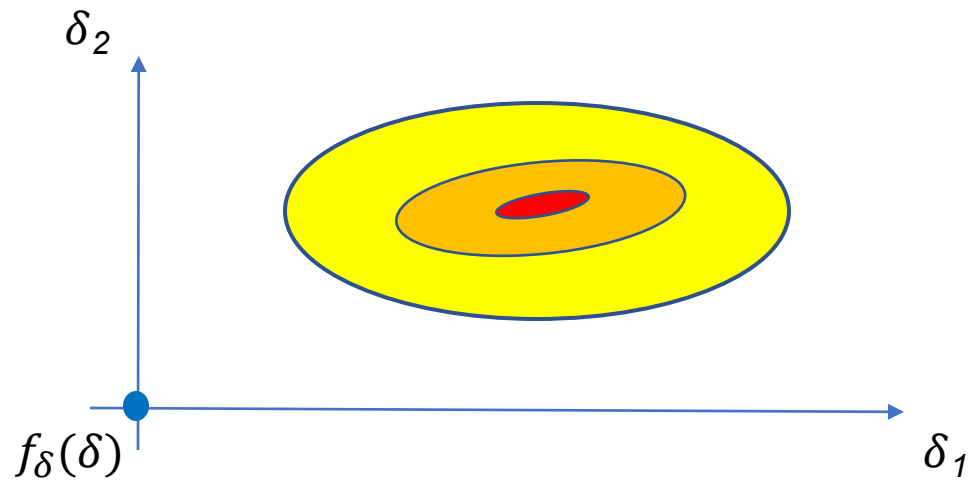
Probabilistic Uncertainty



Probabilistic Uncertainty

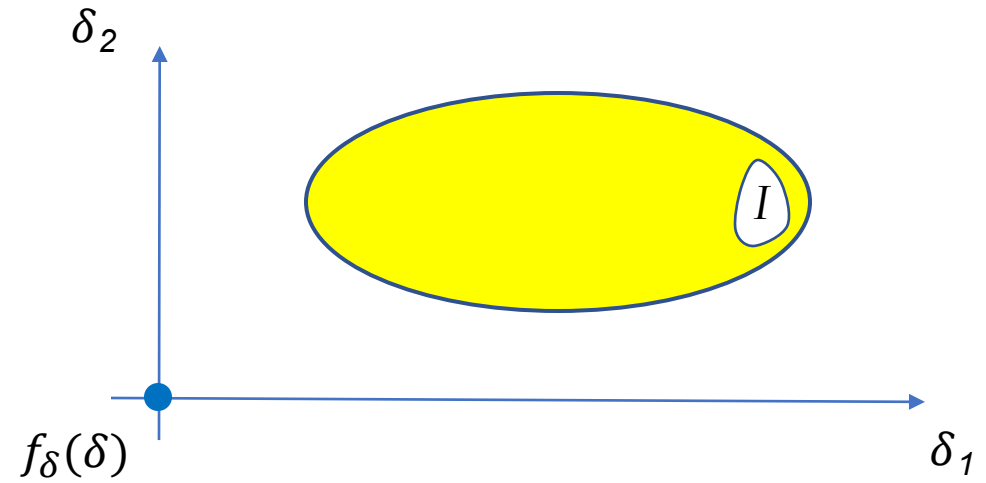
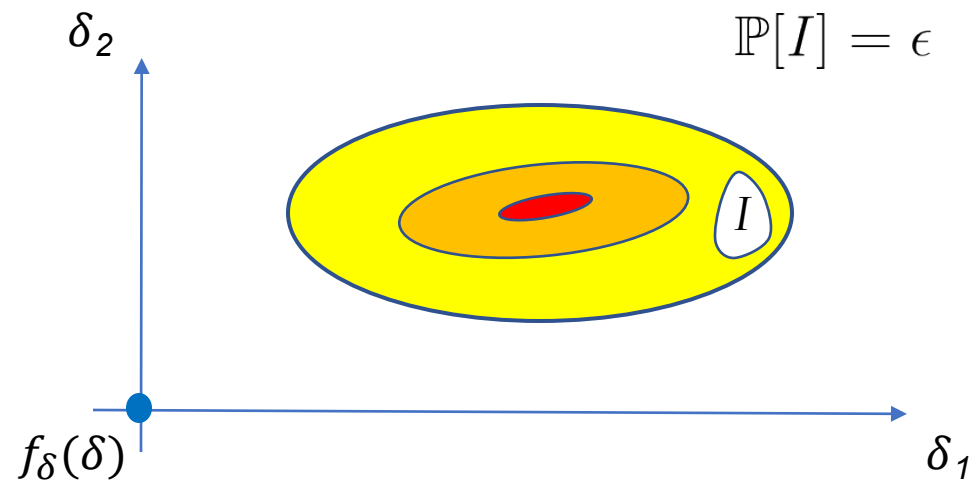


Chance-Constrained Optimization



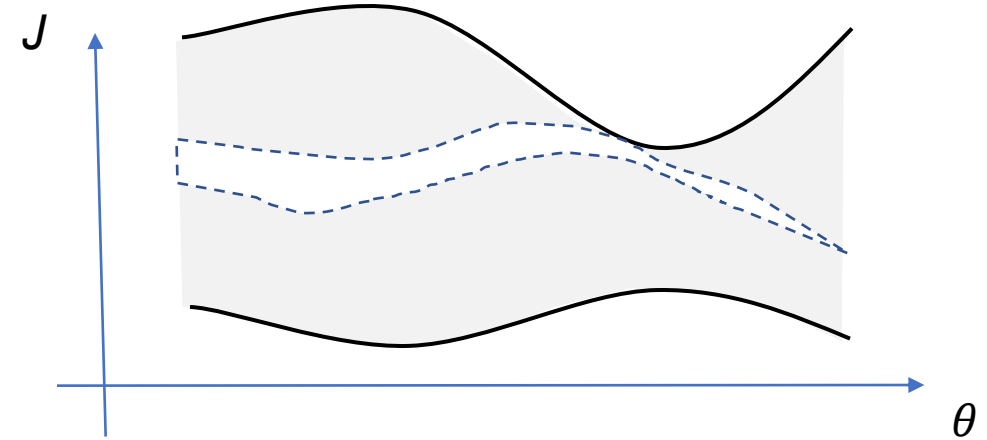
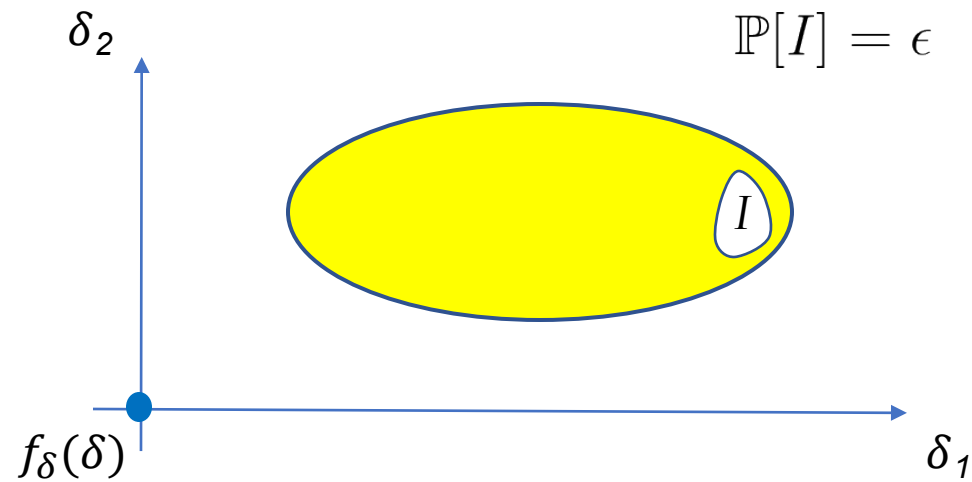
- Goal: improve performance by accepting some risk

Chance-Constrained Optimization



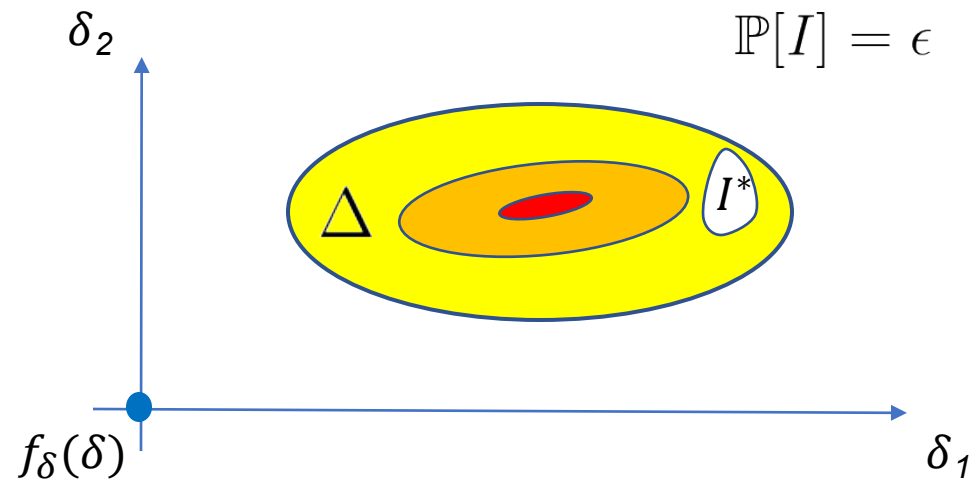
- Approach 1: find the WC solution after ignoring a low-probability region

Chance-Constrained Optimization



- Approach 1: find the WC solution after ignoring a low-probability region
 - The choice of I will likely be suboptimal, thus, the resulting design

Chance-Constrained Optimization

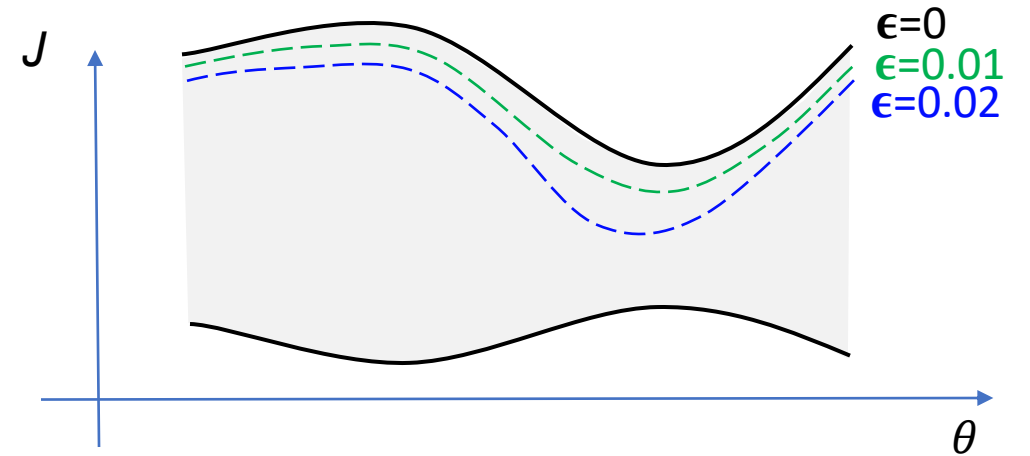
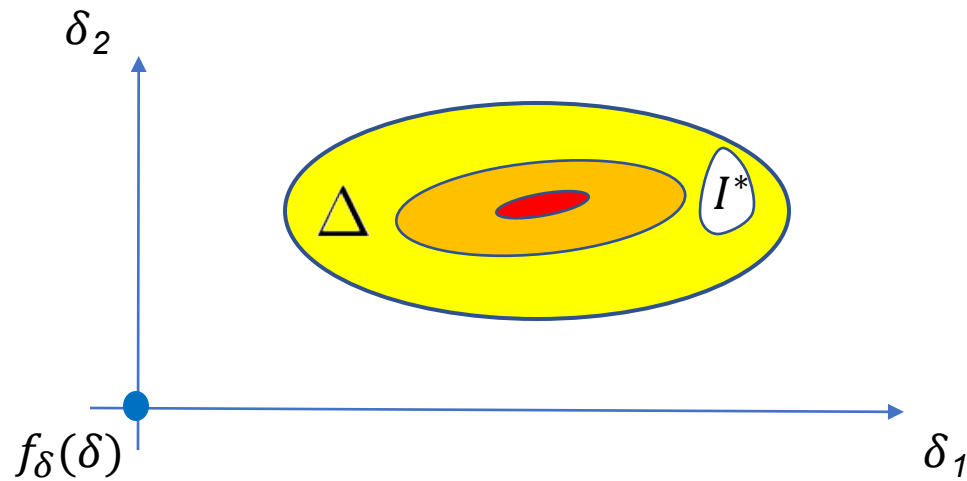


- Approach 2: search for both the optimal design and the optimal I

$$\min_{\theta \in \Theta, I} \left[\max_{\delta \in \Delta - I} J(\theta, \delta) : \mathbb{P}[I] = \epsilon \right]$$

↓
Risk

Chance-Constrained Optimization

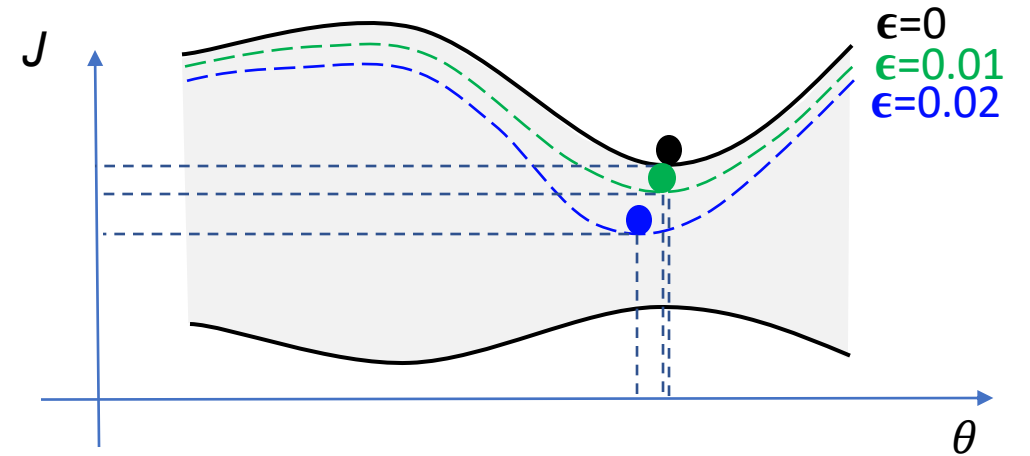
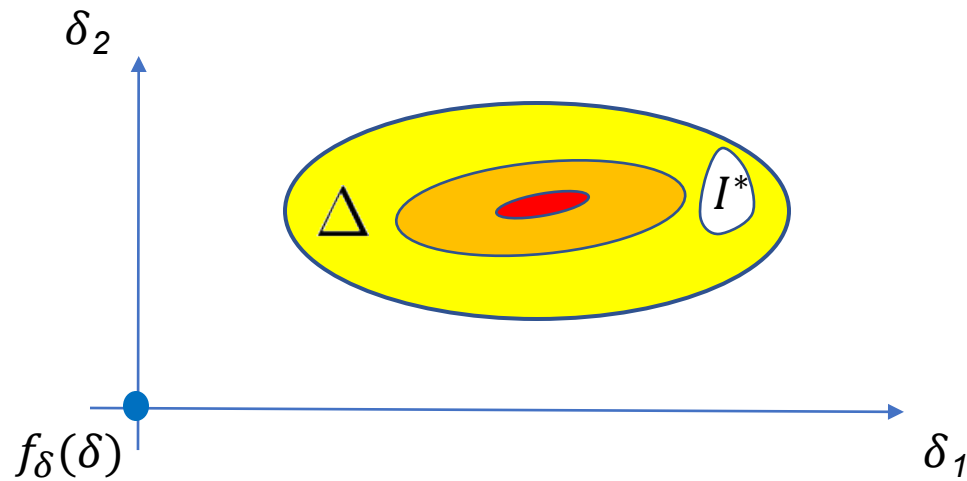


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\downarrow
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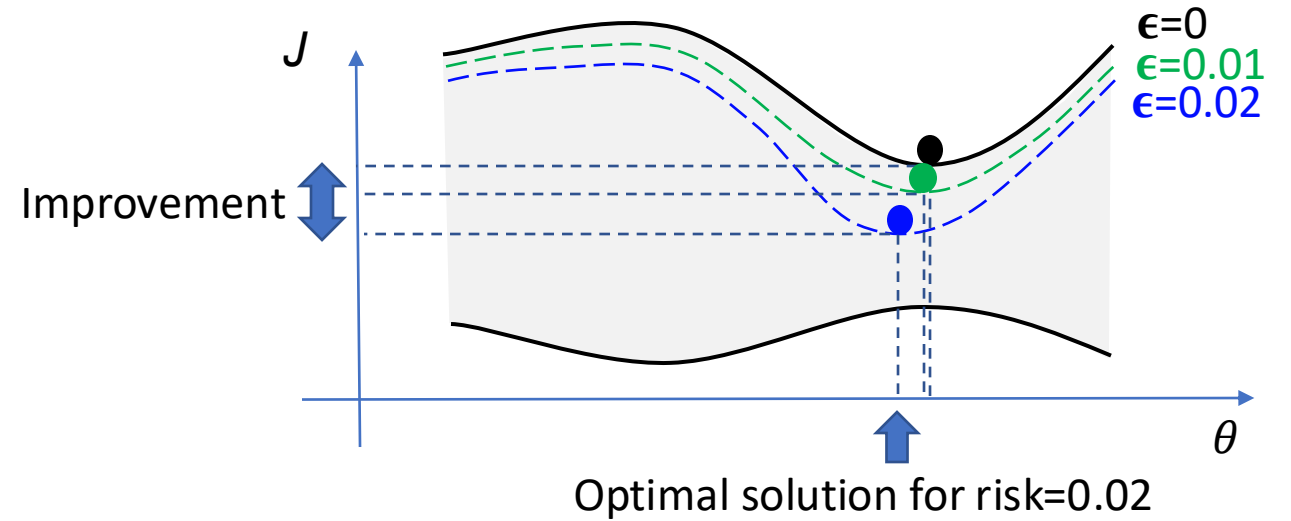
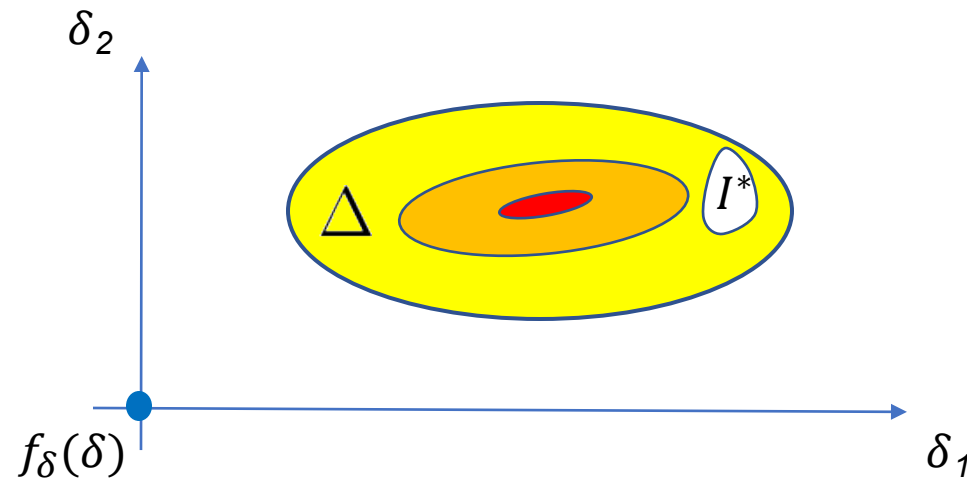


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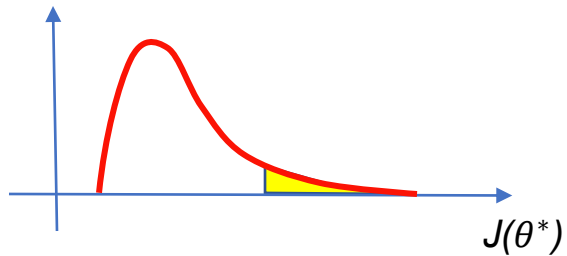
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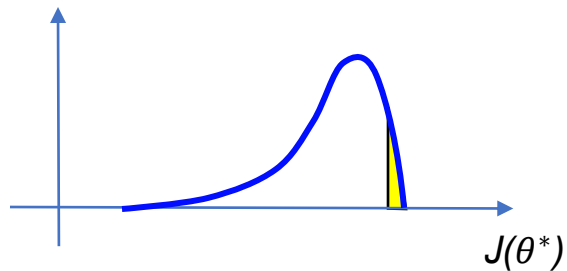
Chance-Constrained Optimization

- When does CCO pay off?

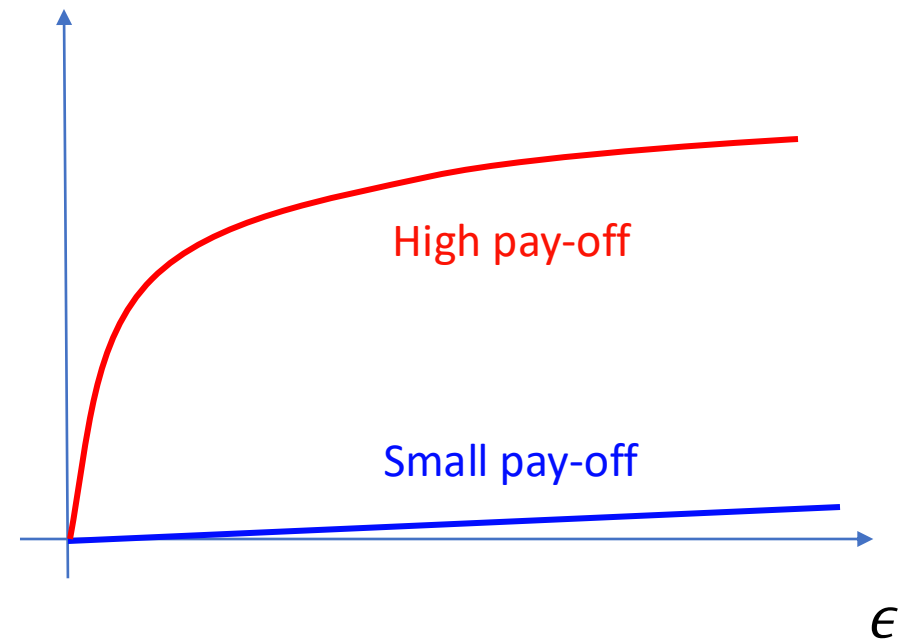
CASE 1



CASE 2

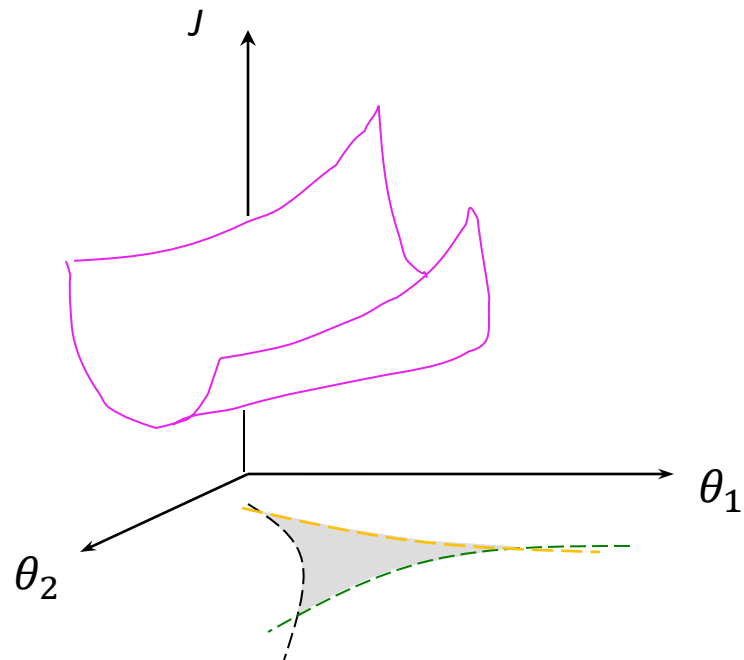


Cost
reduction

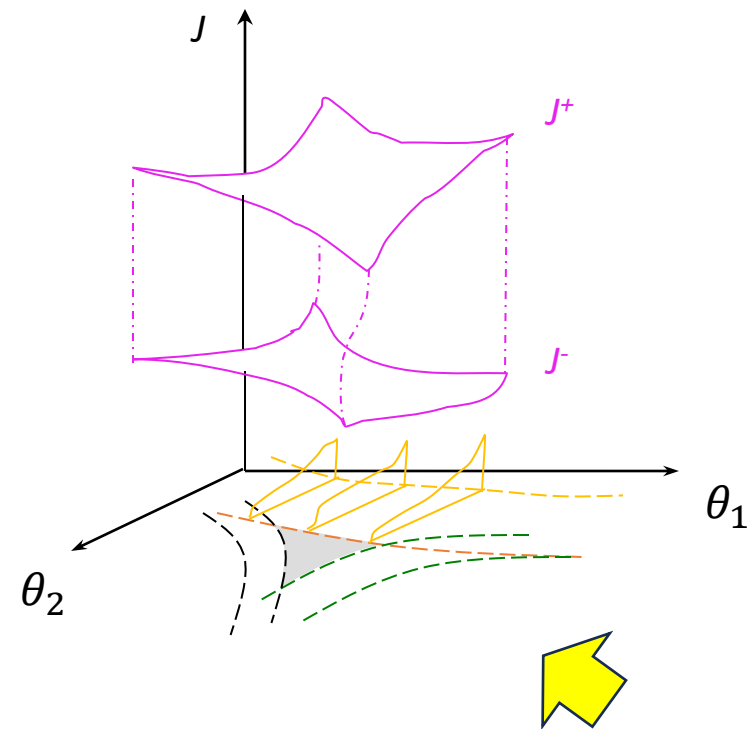


Optimization under Uncertainty

Deterministic Setting

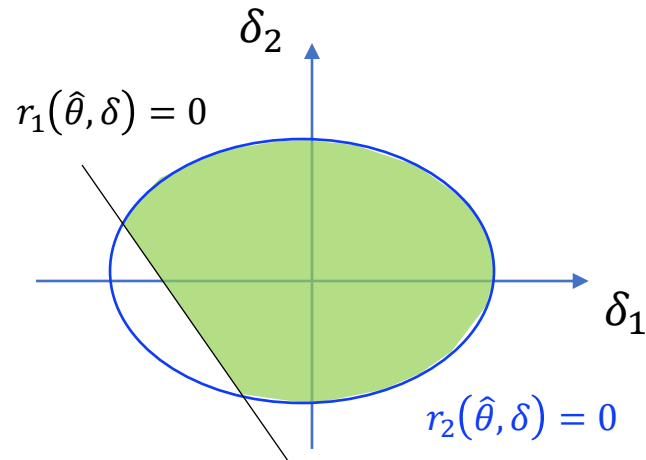


Probabilistic Setting



Failure Probability and Risk

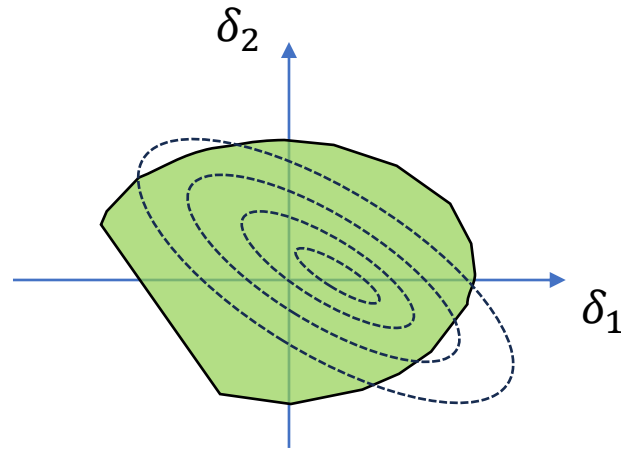
- Requirements: $r_k(\theta, \delta) \leq 0$, for $k = 1 \dots n_k$



Failure and **Success** domains (fixed θ)

Failure Probability and Risk

- Requirements: $r_k(\theta, \delta) \leq 0$, for $k = 1 \dots n_k$
- Probability of failure



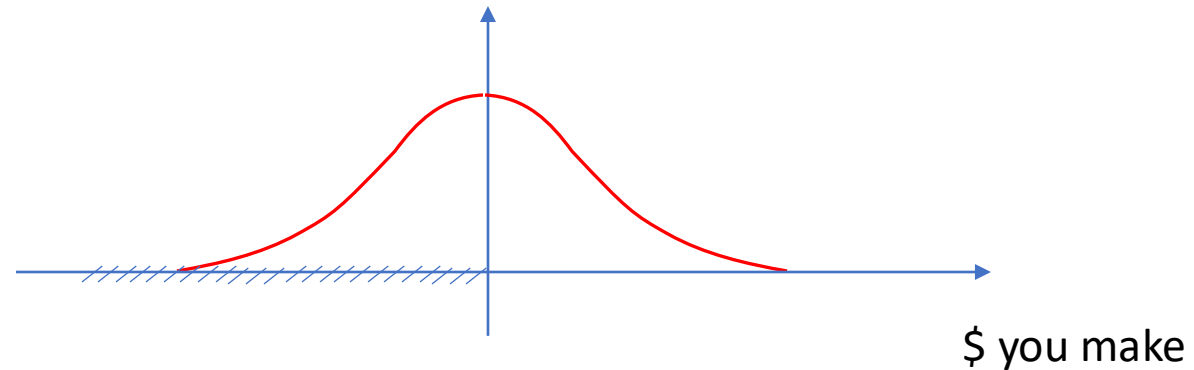
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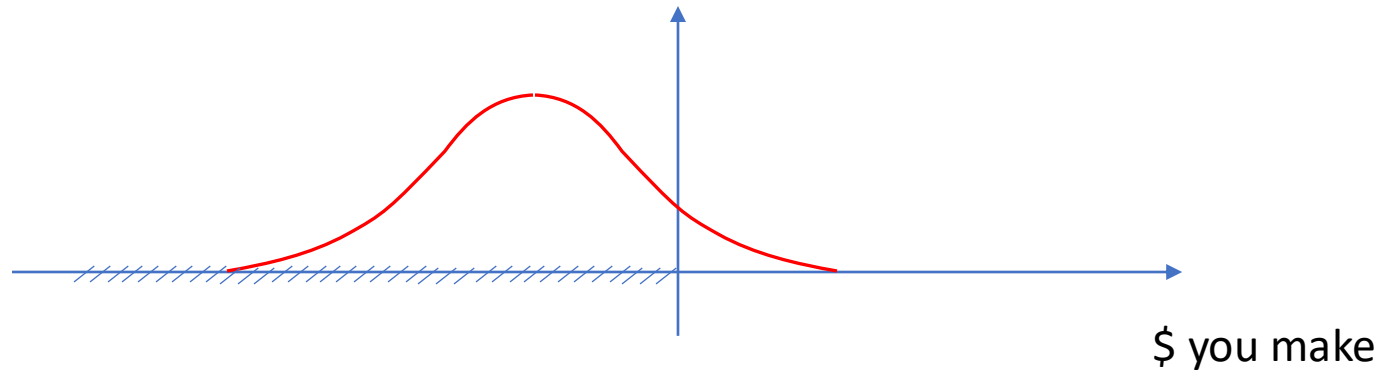
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- *Example: gambling*



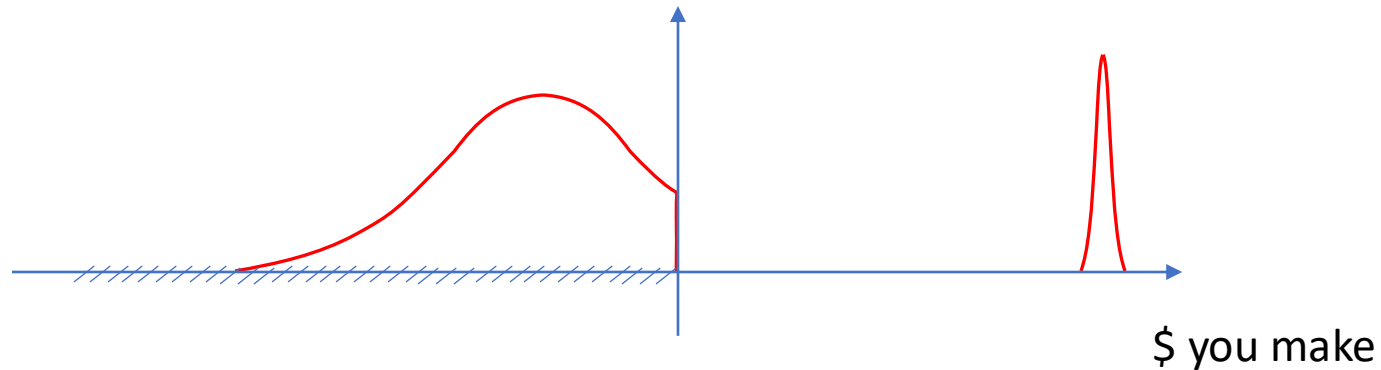
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Outline

- Introductory concepts
- Solution strategies
- Scenario Theory

Solution Strategies

$$\min_{\theta \in \Theta} J(\theta)$$

subject to: $r_k(\theta, \delta) \leq 0$ for $k = 1, \dots, n_r,$

- Decision variable: θ
- Uncertain parameters/changing operating conditions: δ

Solution Strategies

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- *Example: find the wing of minimal drag for which lift \geq weight for a given Mach number and a given mass*

Solution Strategies

$$\min_{\theta \in \Theta} J(\theta)$$

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- Decision variable: θ
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- δ characterized by a sequence of scenarios:

$$\mathcal{D} = \left\{ \delta^{(i)} \right\}_{i=1}^n$$

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Dataset could be obtained
experimentally or
synthetically

Solution Strategies

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- Uncertain parameters/changing operating conditions: δ
- δ characterized by a sequence of scenarios:

$$\mathcal{D} = \left\{ \delta^{(i)} \right\}_{i=1}^n$$

n does not have to be large

Solution Strategies

- Strict formulation

$$\begin{aligned} & \min_{\theta \in \Theta} J(\theta) \\ & \text{subject to: } r_k(\theta, \delta^{(i)}) \leq 0 \text{ for } k = 1, \dots, n_r, i = 1, \dots, n, \end{aligned}$$

- This program might be infeasible

Solution Strategies

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- This program might be infeasible
- *Example: find the wing of minimal drag for which lift \geq weight for a given set of Mach numbers and masses*

Solution Strategies: Relaxed Formulations

- Risk-averse vs. risk-agnostic

Solution Strategies

- Risk-averse formulation

$$\min_{\theta \in \Theta, \xi \geq 0} J(\theta) + \rho \sum_{i=1}^n \xi_i$$

$$\text{subject to: } r_k \left(\theta, \delta^{(i)} \right) \leq \xi_i \text{ for } k = 1, \dots, n_r, i = 1, \dots, n$$

Solution Strategies

- Risk-averse formulation

$$\min_{\theta \in \Theta, \xi \geq 0} J(\theta) + \rho \sum_{i=1}^n \xi_i$$

$$\text{subject to: } r_k \left(\theta, \delta^{(i)} \right) \leq \xi_i \text{ for } k = 1, \dots, n_r, i = 1, \dots, n$$

- *Example: find the wing of minimal drag for which lift \geq weight for a fraction of the Mach numbers and masses*

Solution Strategies



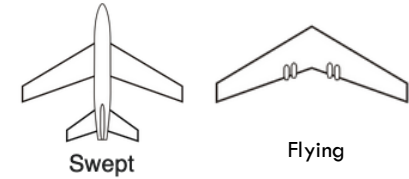
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- We can identify the most robust design of a given architecture
- *Example: find the backward swept wing of minimal drag for which lift \geq weight for as many Mach - mass combinations as possible*

Solution Strategies



- Risk-averse formulation

$$\min_{\theta \in \Theta, \xi \geq 0} J(\theta) + \rho \sum_{i=1}^n \xi_i$$

$$\text{subject to: } r_k \left(\theta, \delta^{(i)} \right) \leq \xi_i \text{ for } k = 1, \dots, n_r, i = 1, \dots, n$$

- We can identify the most robust design of a given architecture
- *Example: how does the most robust backward swept wing compares to the most robust flying wing*

Solution Strategies

- Risk-averse formulation

$$\min_{\theta \in \Theta, \xi \geq 0} J(\theta) + \rho \sum_{i=1}^n \xi_i$$

$$\text{subject to: } r_k \left(\theta, \delta^{(i)} \right) \leq \xi_i \text{ for } k = 1, \dots, n_r, i = 1, \dots, n$$

- The scenarios that violate the requirements will be called outliers
- The outliers are chosen optimally

Solution Strategies

- Risk-averse formulation

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$$\text{subject to: } r_k \left(\theta, \delta^{(i)} \right) \leq \xi_i \text{ for } k = 1, \dots, n_r, i = 1, \dots, n$$

- The trade-off between performance and robustness can be studied by changing ρ , i.e., by changing the number of outliers

Solution Strategies

- Risk-averse formulation

$$\min_{\theta \in \Theta, \xi \geq 0} J(\theta) + \rho \sum_{i=1}^n \xi_i$$

$$\text{subject to: } r_k \left(\theta, \delta^{(i)} \right) \leq \xi_i \text{ for } k = 1, \dots, n_r, i = 1, \dots, n$$

- *Example: how much can the drag be reduced if we let lift < 0.9×weight be for a few additional Mach - mass combinations*

Solution Strategies

- Easy Risk-agnostic approach
 1. Eliminate “bad” data upfront
 2. Solve for θ using a worst-case approach
- Drawbacks
 1. The best scenarios to remove depend on θ and we don't know θ^*
 2. Performance will likely be suboptimal

Solution Strategies

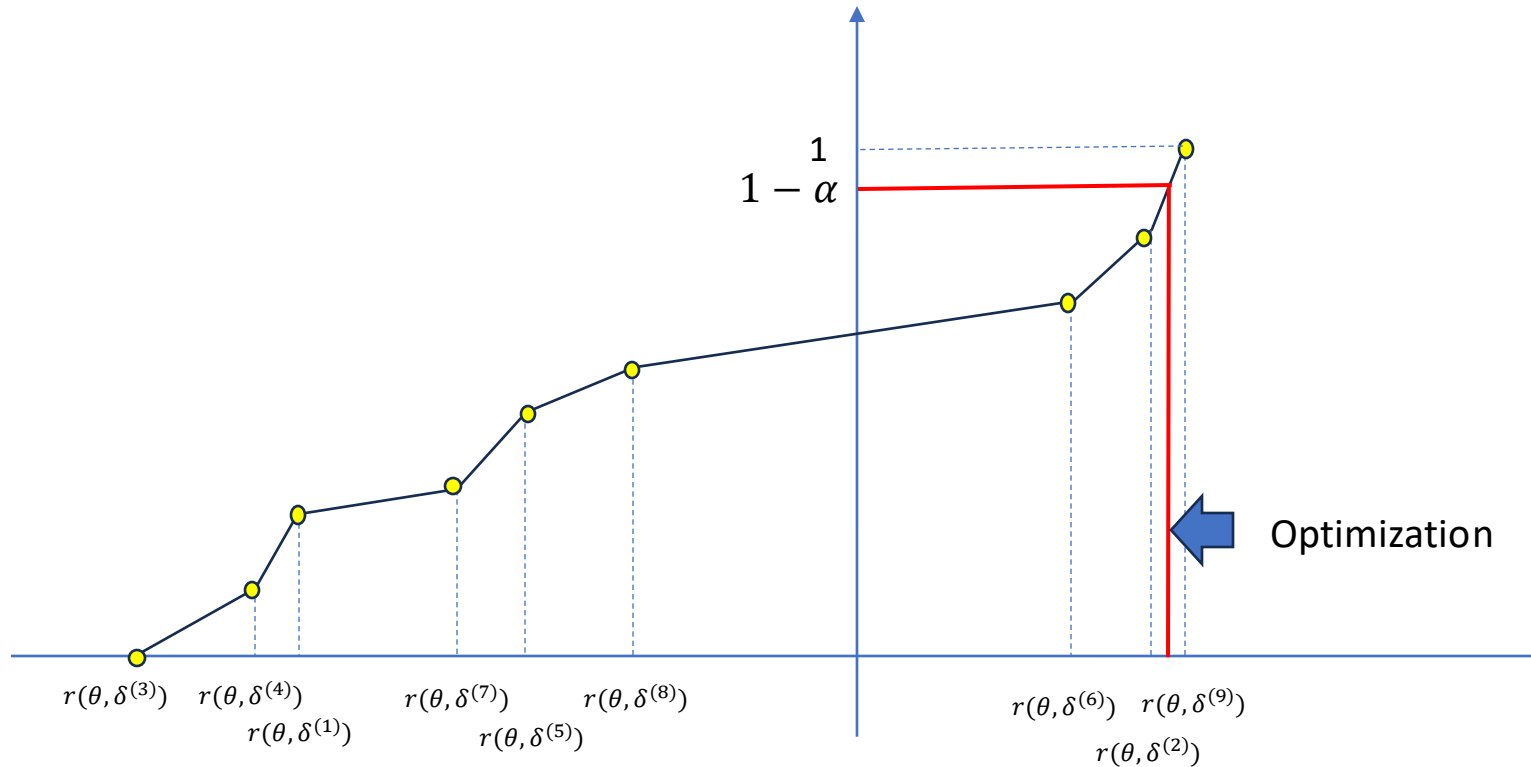
- Risk-agnostic formulation

$$\min_{\theta \in \Theta} J(\theta)$$

$$\text{subject to: } F_{r_k(\theta, \mathcal{D})}^{-1}(1 - \alpha_k) \leq 0, \quad k = 1, \dots, n_r$$

Solution Strategies

- Risk-agnostic formulation



Solution Strategies

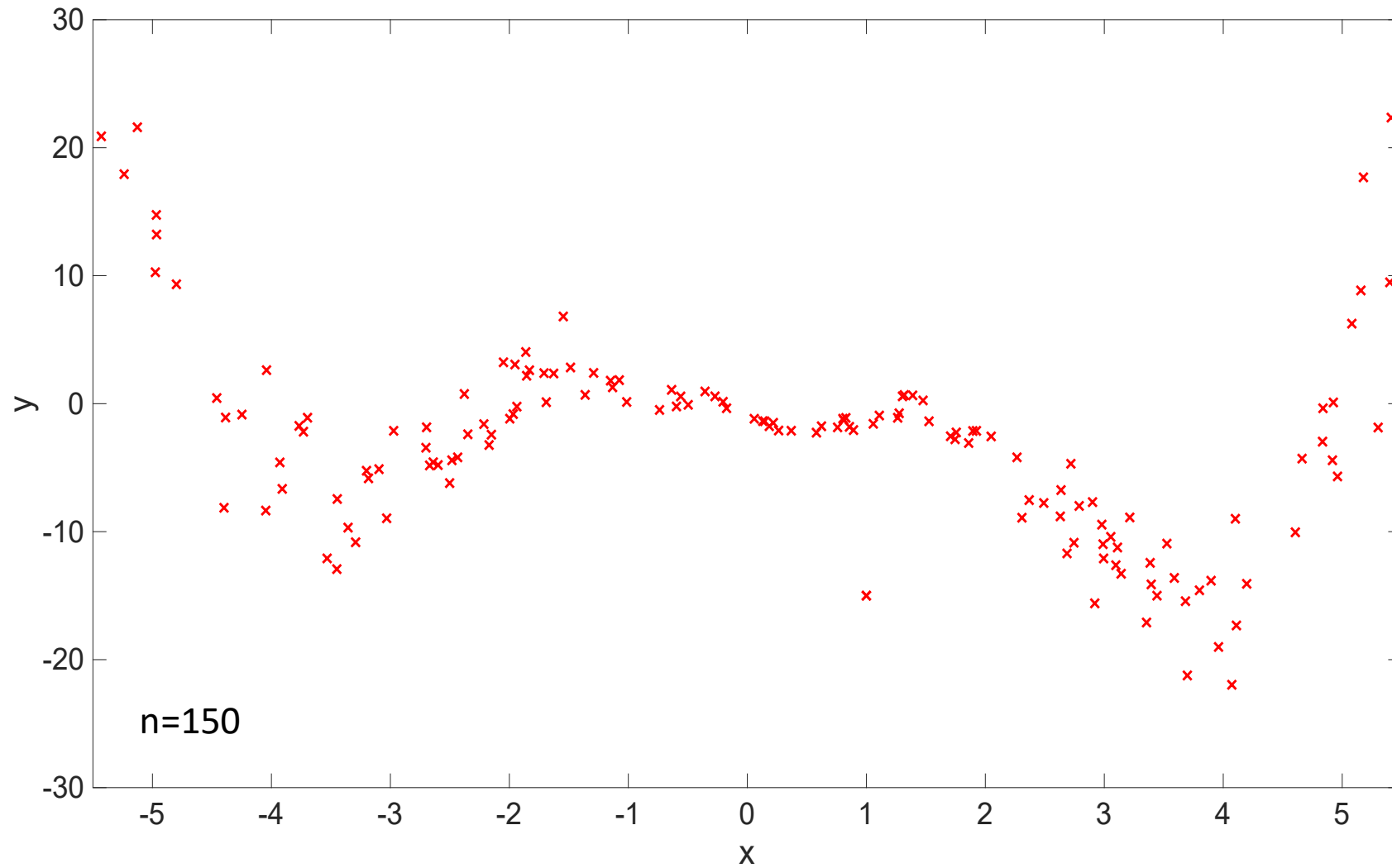
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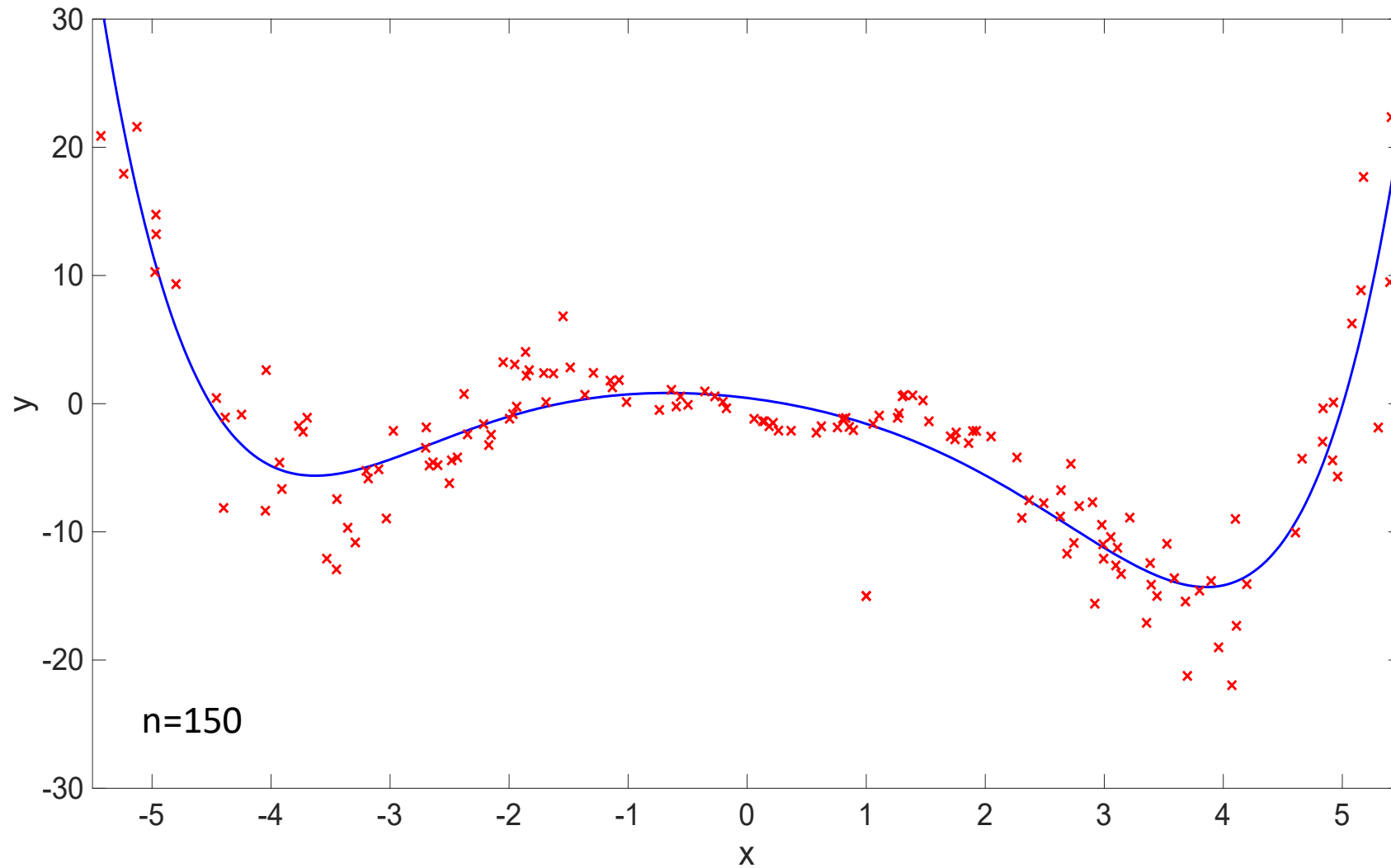
$$\text{subject to: } F_{r_k(\theta, \mathcal{D})}^{-1}(1 - \alpha_k) \leq 0, \quad k = 1, \dots, n_r$$

- *Example: how much can the drag be reduced if we let lift < weight be for a few Mach - mass combinations*

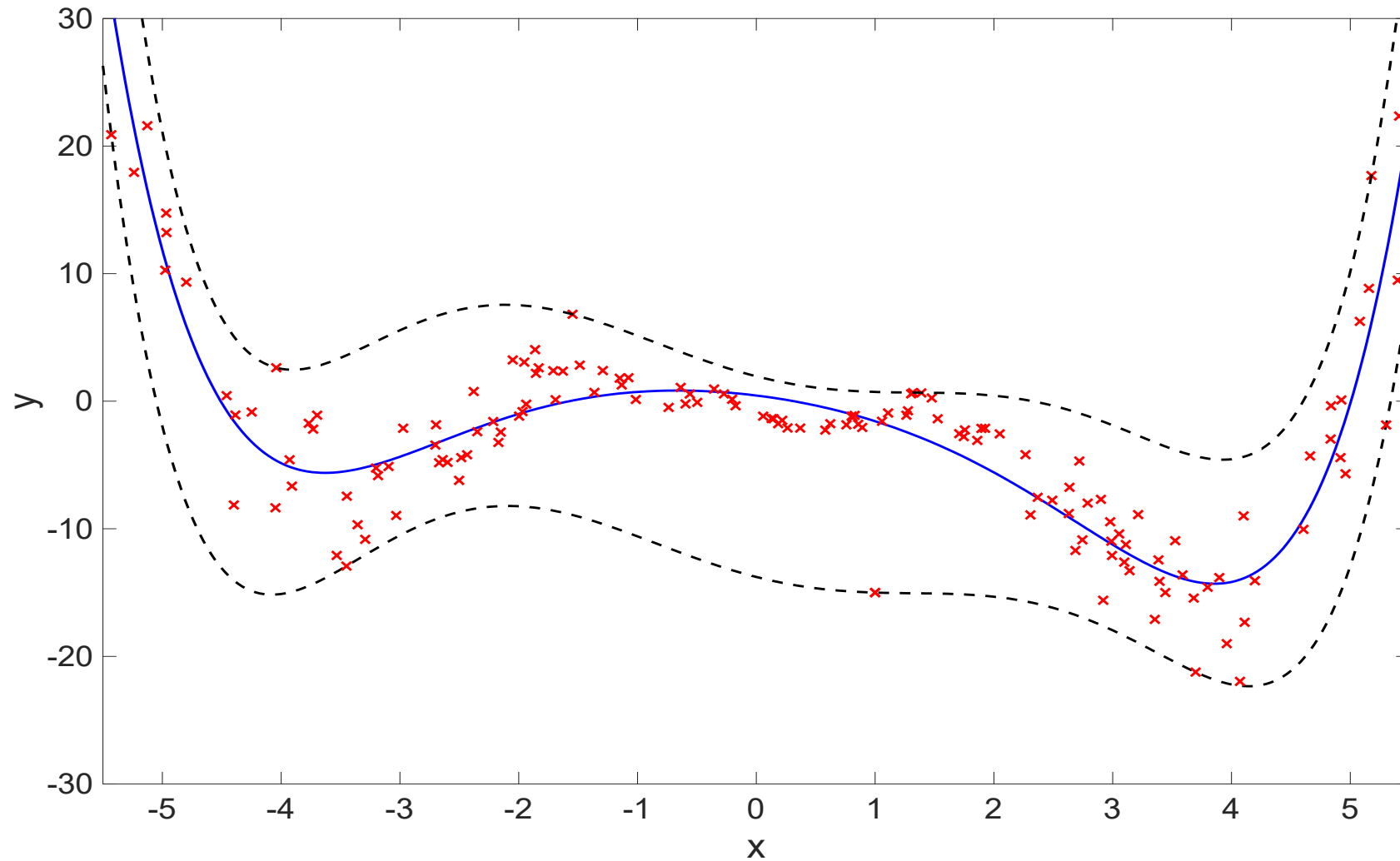
Example: Interval Predictor Models



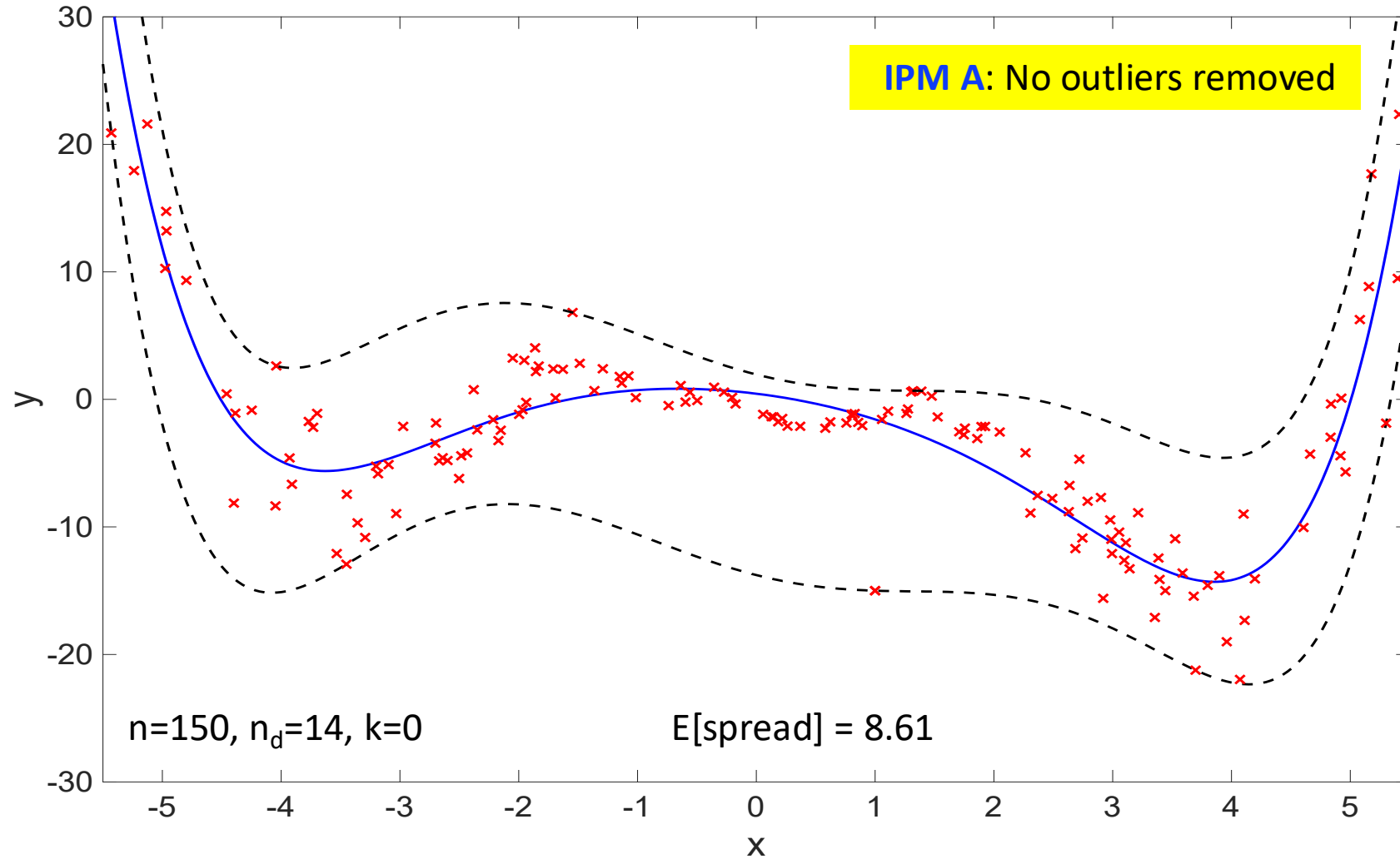
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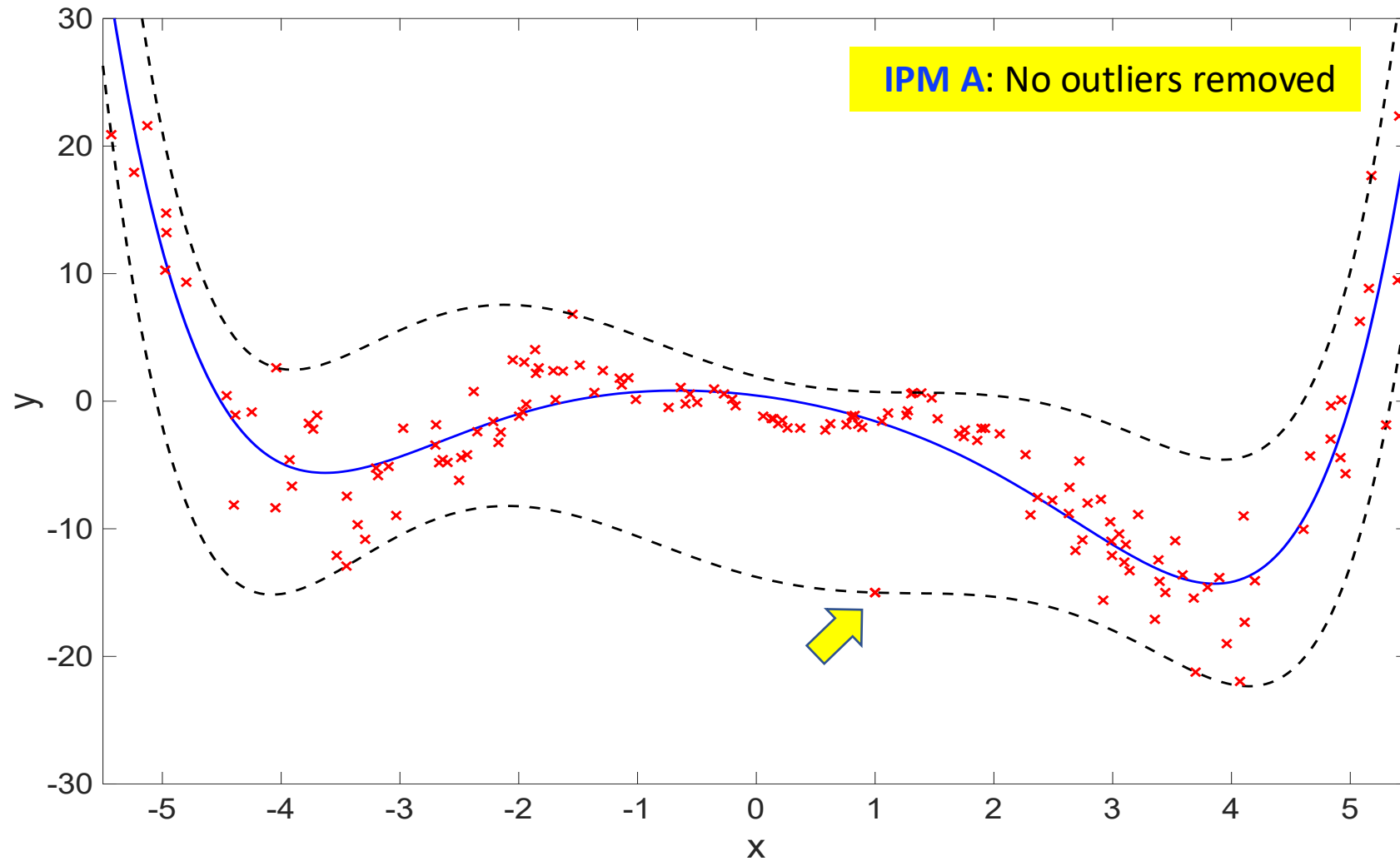
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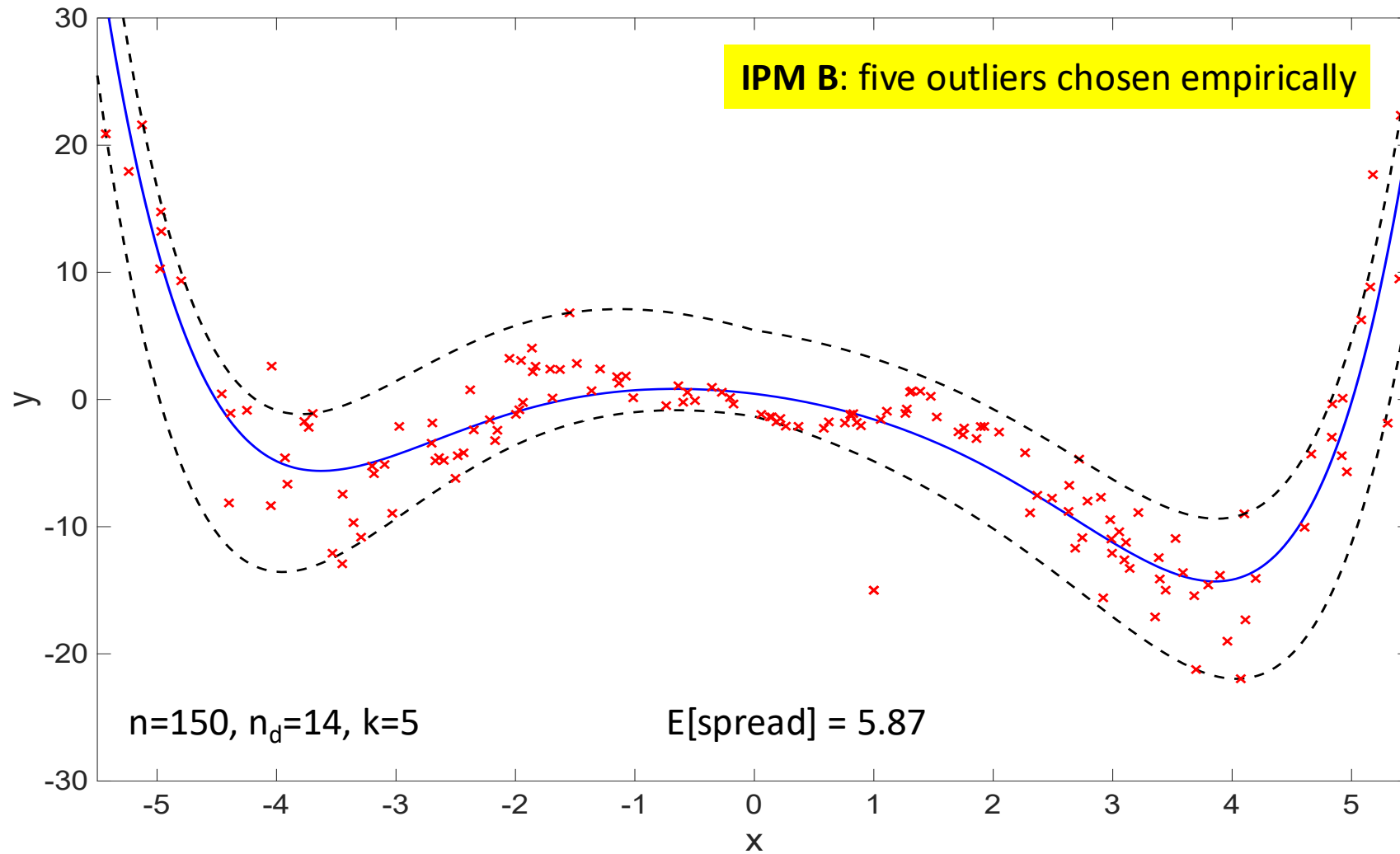
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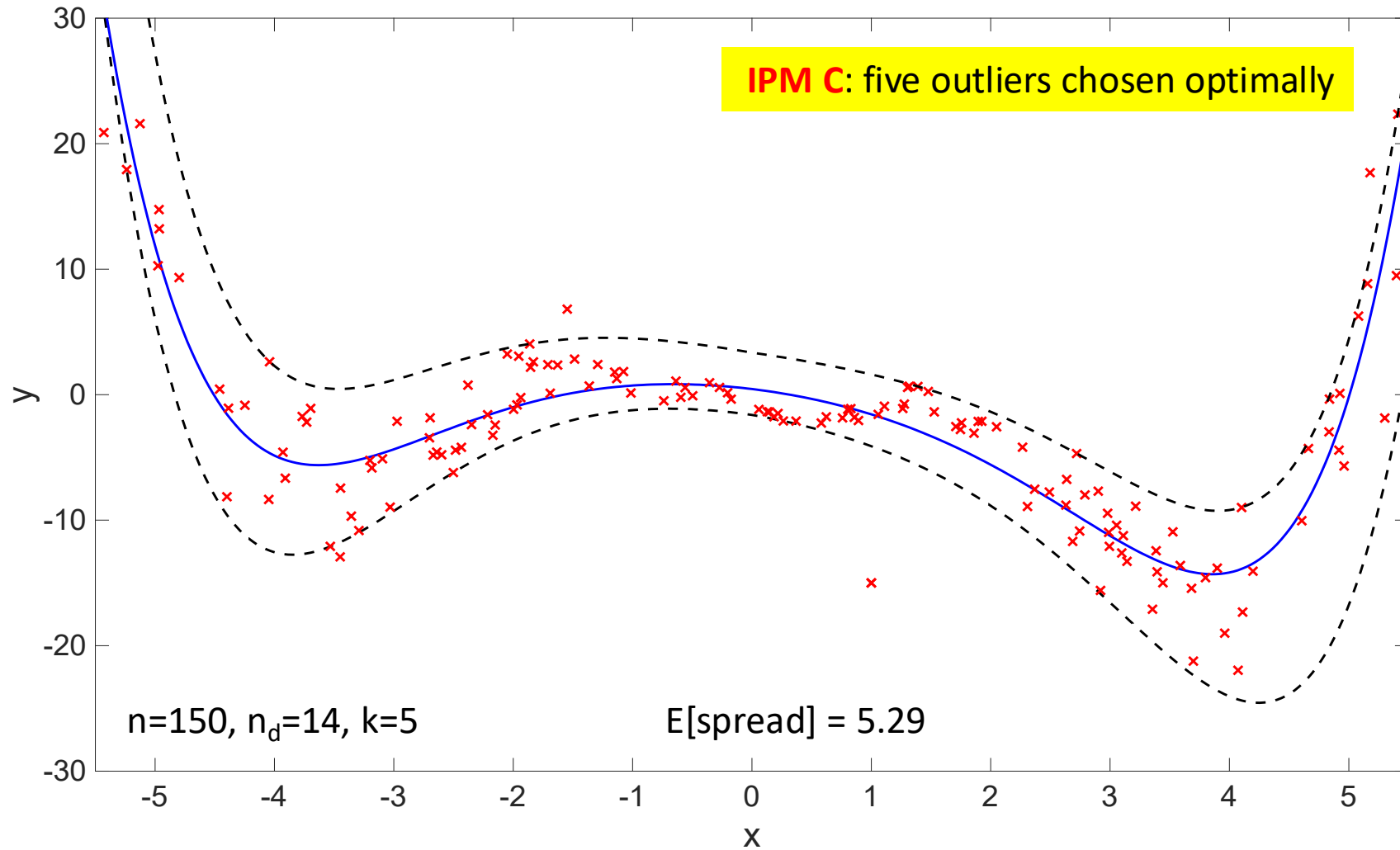
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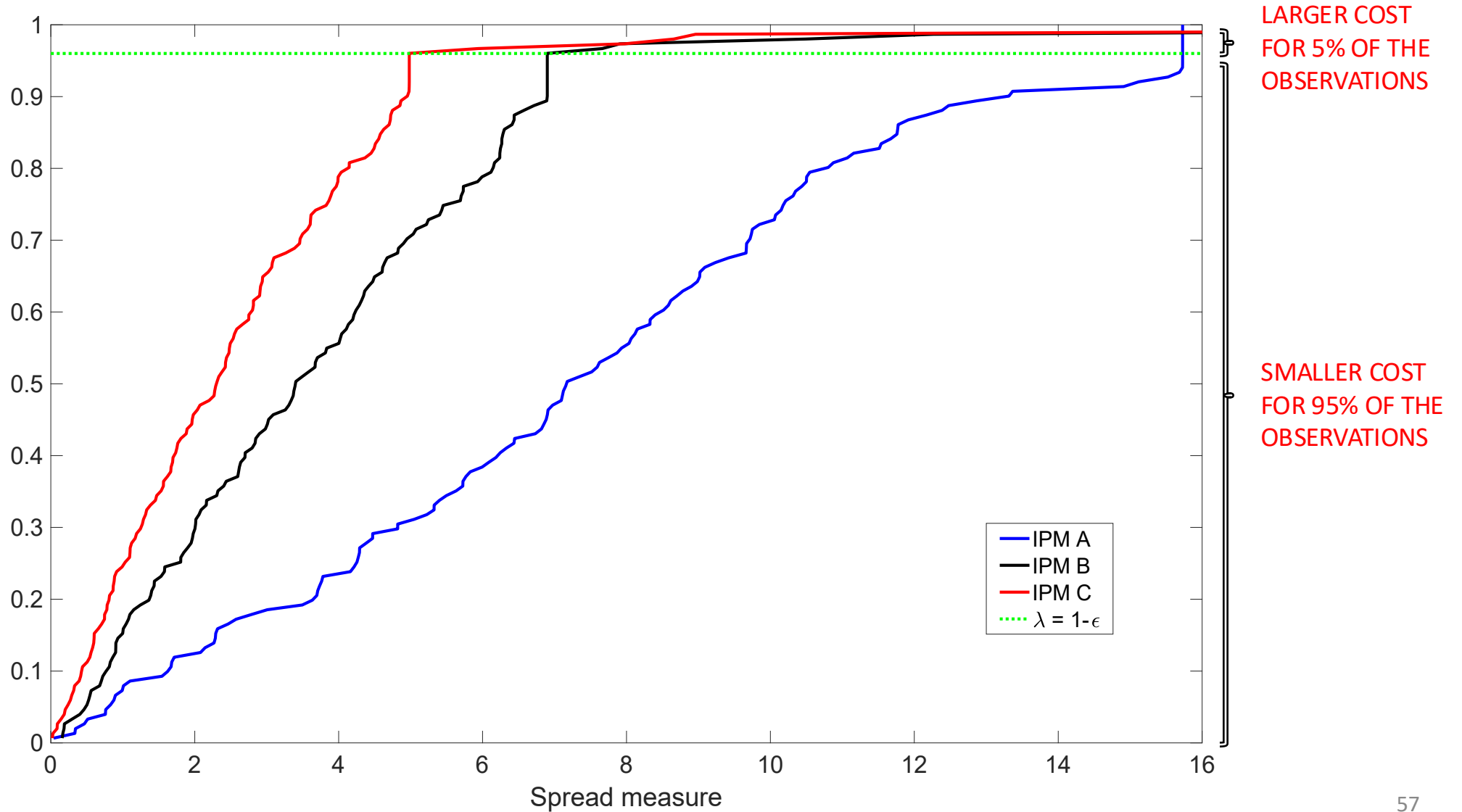
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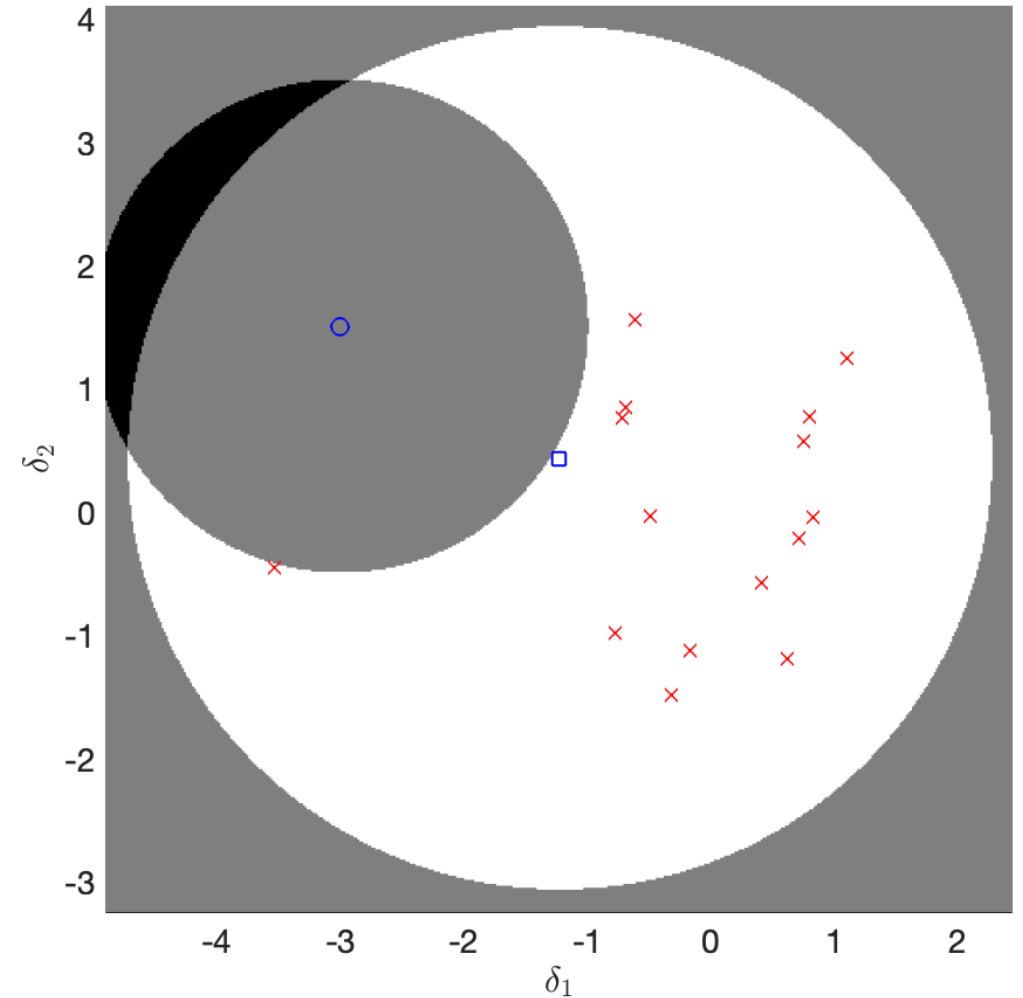


Solution Strategies: Preventing Overfitting

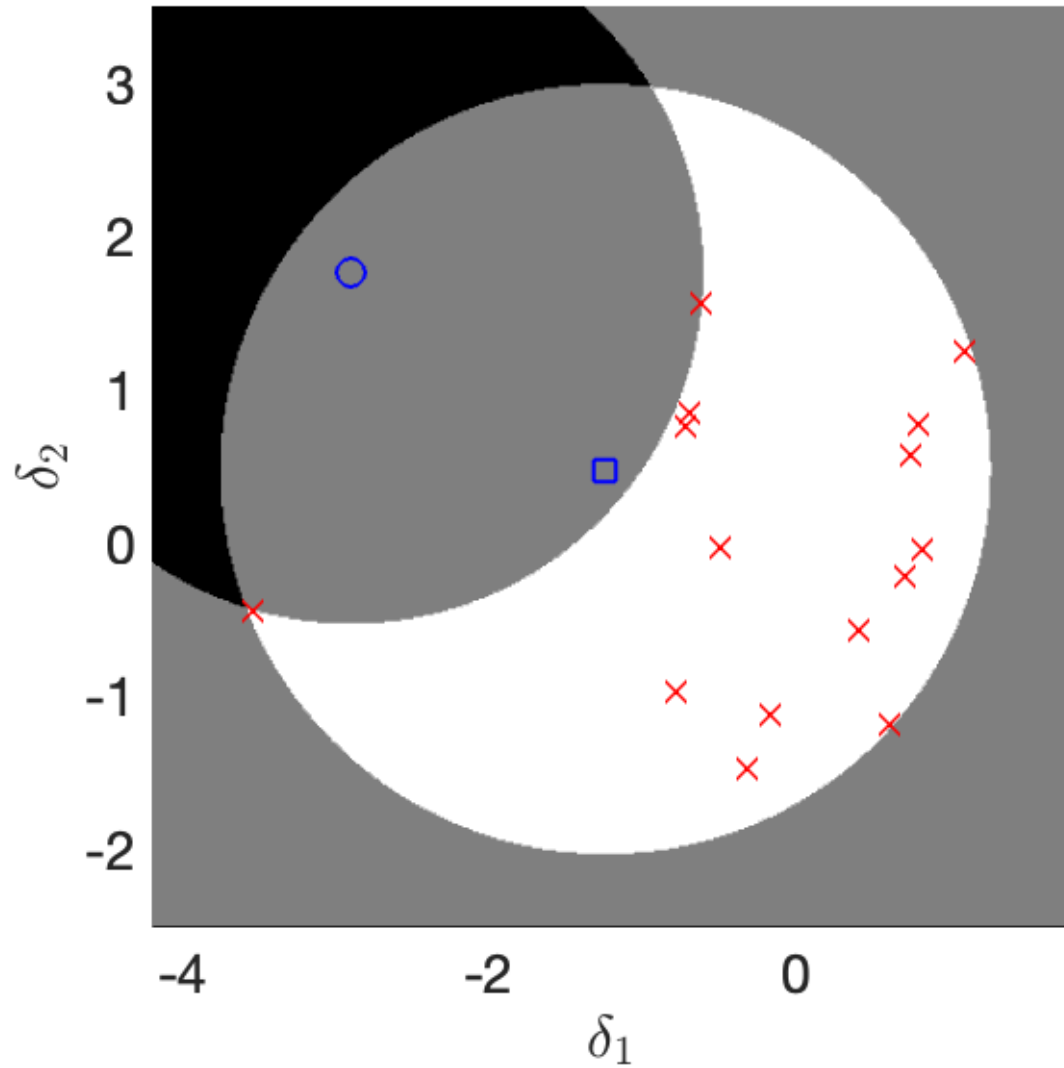
- Data-driven designs overfit the training data
- Overfitted models generalize poorly
- We seek designs that satisfy the requirements at the vicinity of the data
- Nominal data vs. perturbed data
- These strategies are suitable when
 - The dataset is small: insufficient measurements or expensive models
 - There is uncertainty in the data: biological & environmental applications

Example: Data Enclosure

- Goal: choose a set of minimum volume that encloses a set of random points
- The set is the intersection between circle 1 and the complement of circle 2
- The center of circle 2 should be inside circle 1

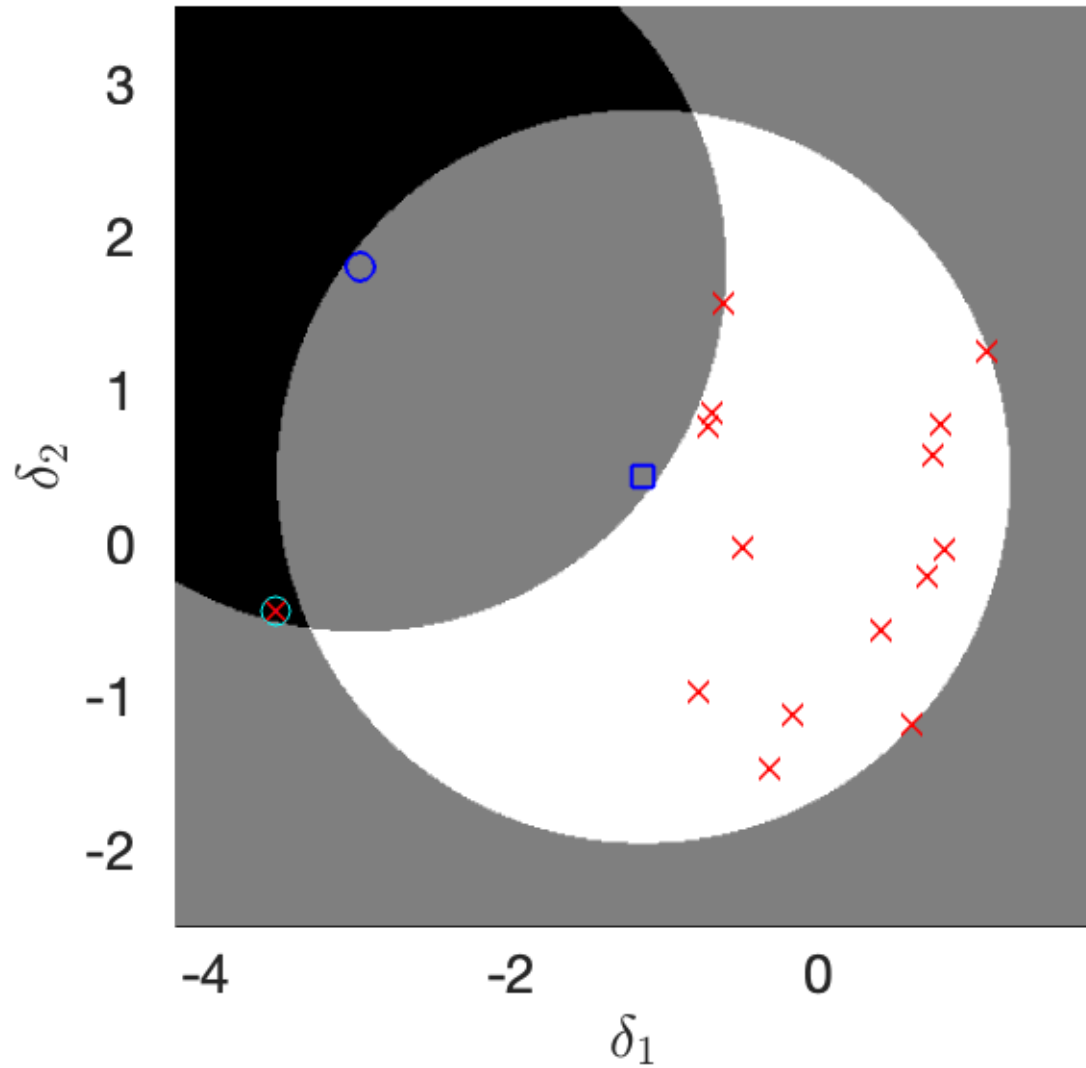


Design 1



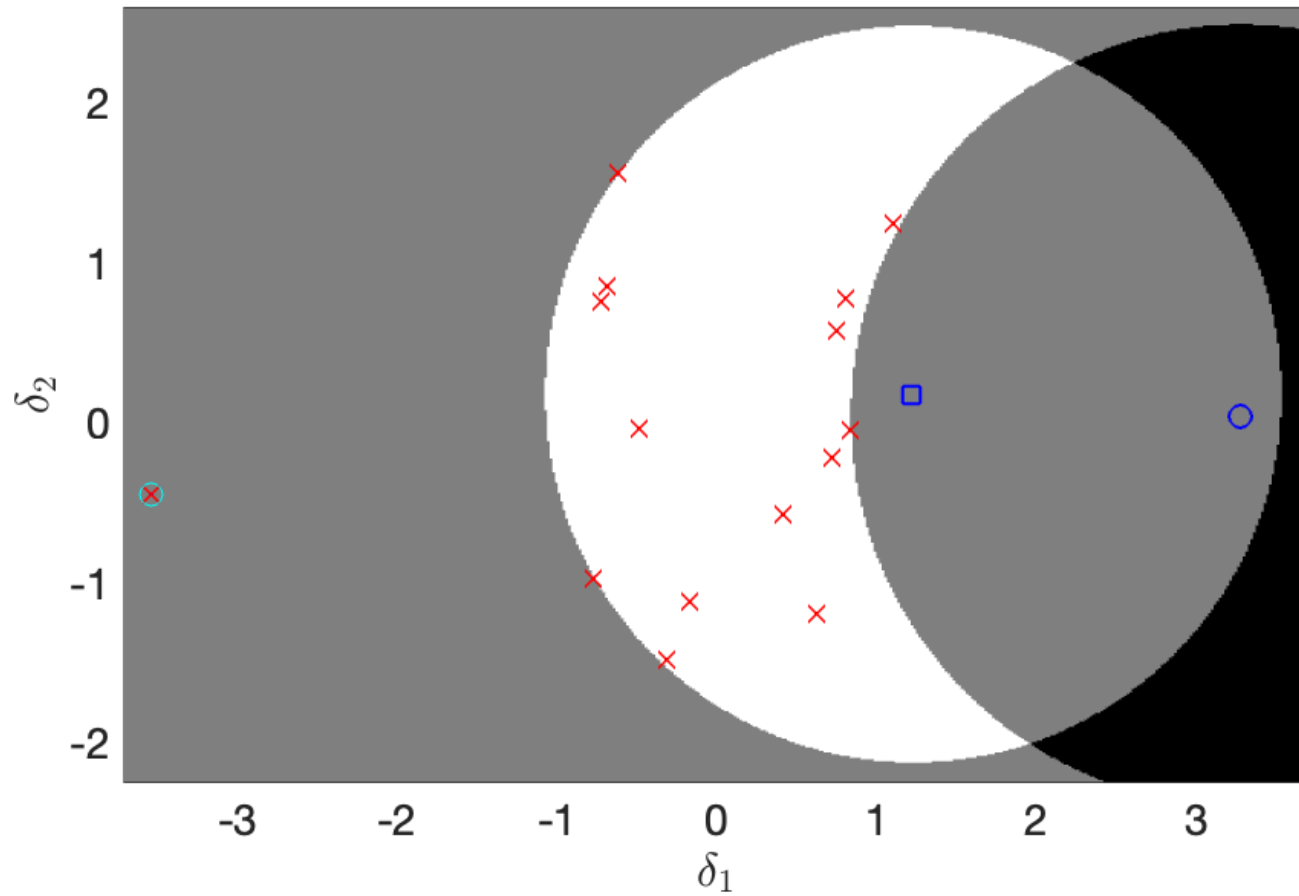
	Ove rfit	Risk- averse	OP	no	J*
θ_1	Y	Y	WC	0	11.2
θ_2	Y	Y	WC	1	10.5
θ_3	Y	N	CC	1	8.3
θ_4	N	Y	WC	0	15.1
θ_5	N	Y	CC	1	13.4
θ_6	N	N	CC	1	9.6
θ_7	N	N	CC	2	6.5

Design 2



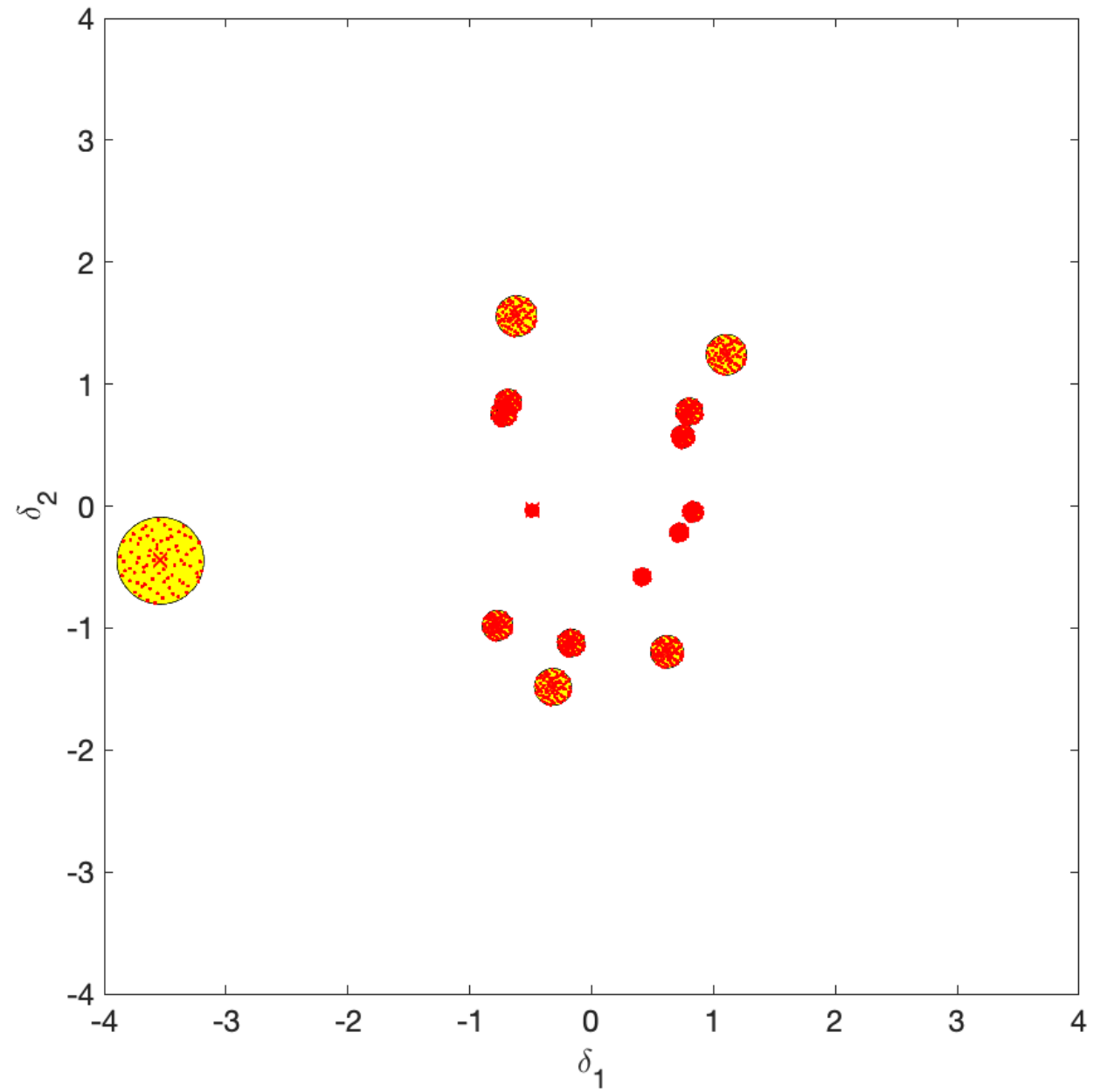
	Ove rfit	Risk- averse	OP	no	J*
θ_1	Y	Y	WC	0	11.2
θ_2	Y	Y	WC	1	10.5
θ_3	Y	N	CC	1	8.3
θ_4	N	Y	WC	0	15.1
θ_5	N	Y	CC	1	13.4
θ_6	N	N	CC	1	9.6
θ_7	N	N	CC	2	6.5

Design 3

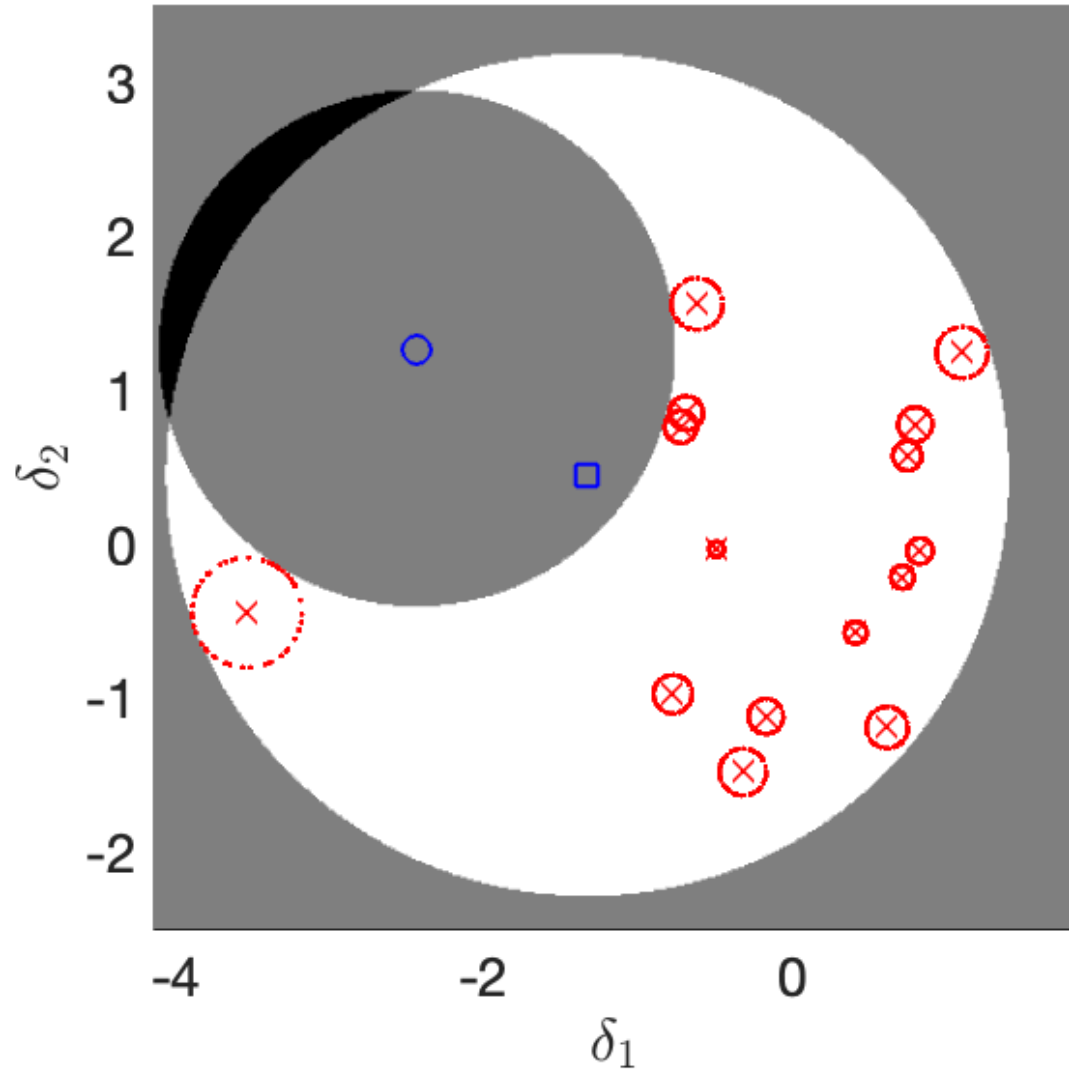


	Ove rfit	Risk- averse	OP	no	J*
θ_1	Y	Y	WC	0	11.2
θ_2	Y	Y	WC	1	10.5
θ_3	Y	N	CC	1	8.3
θ_4	N	Y	WC	0	15.1
θ_5	N	Y	CC	1	13.4
θ_6	N	N	CC	1	9.6
θ_7	N	N	CC	2	6.5

Non-adversarial perturbations

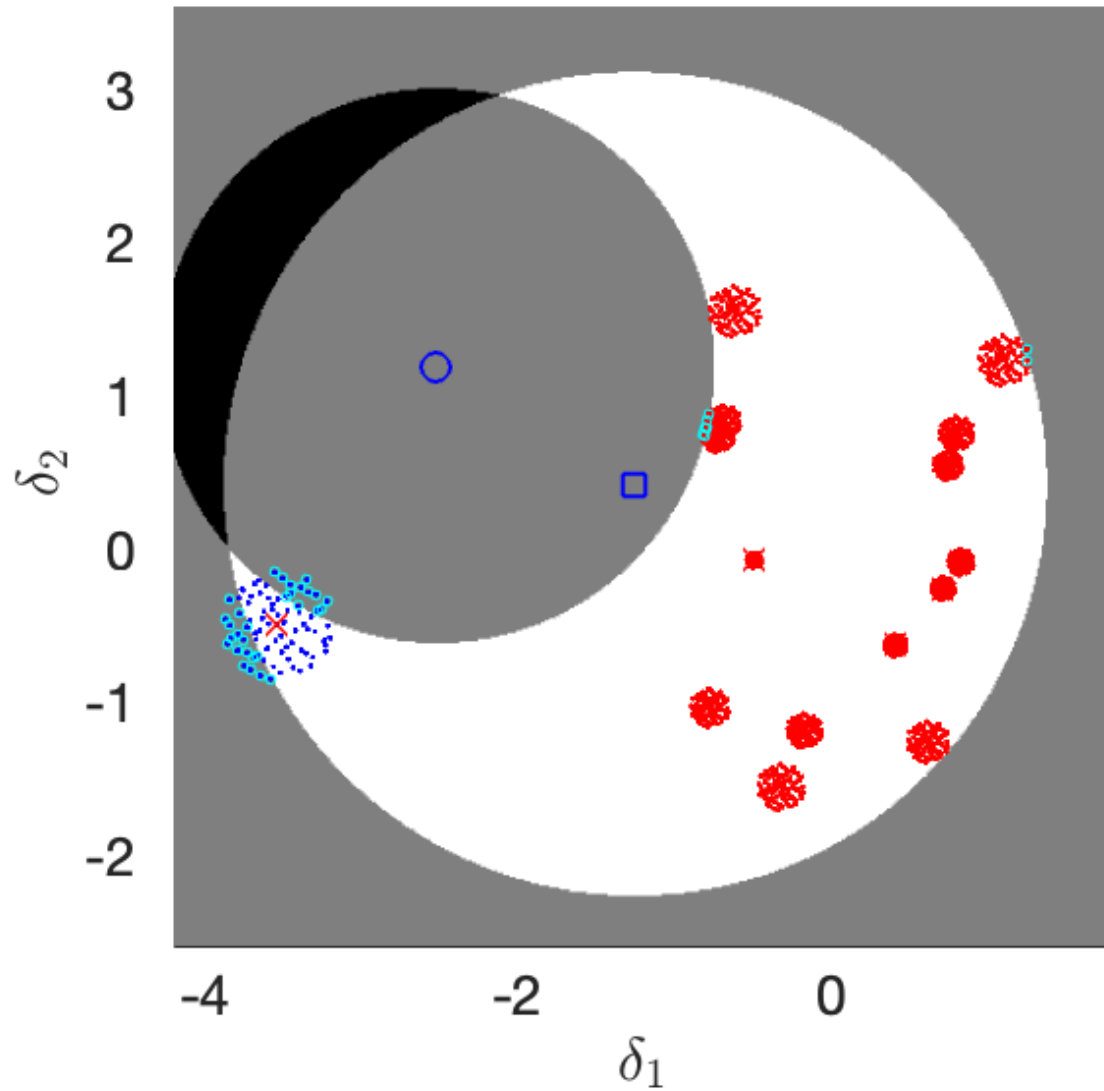


Design 4



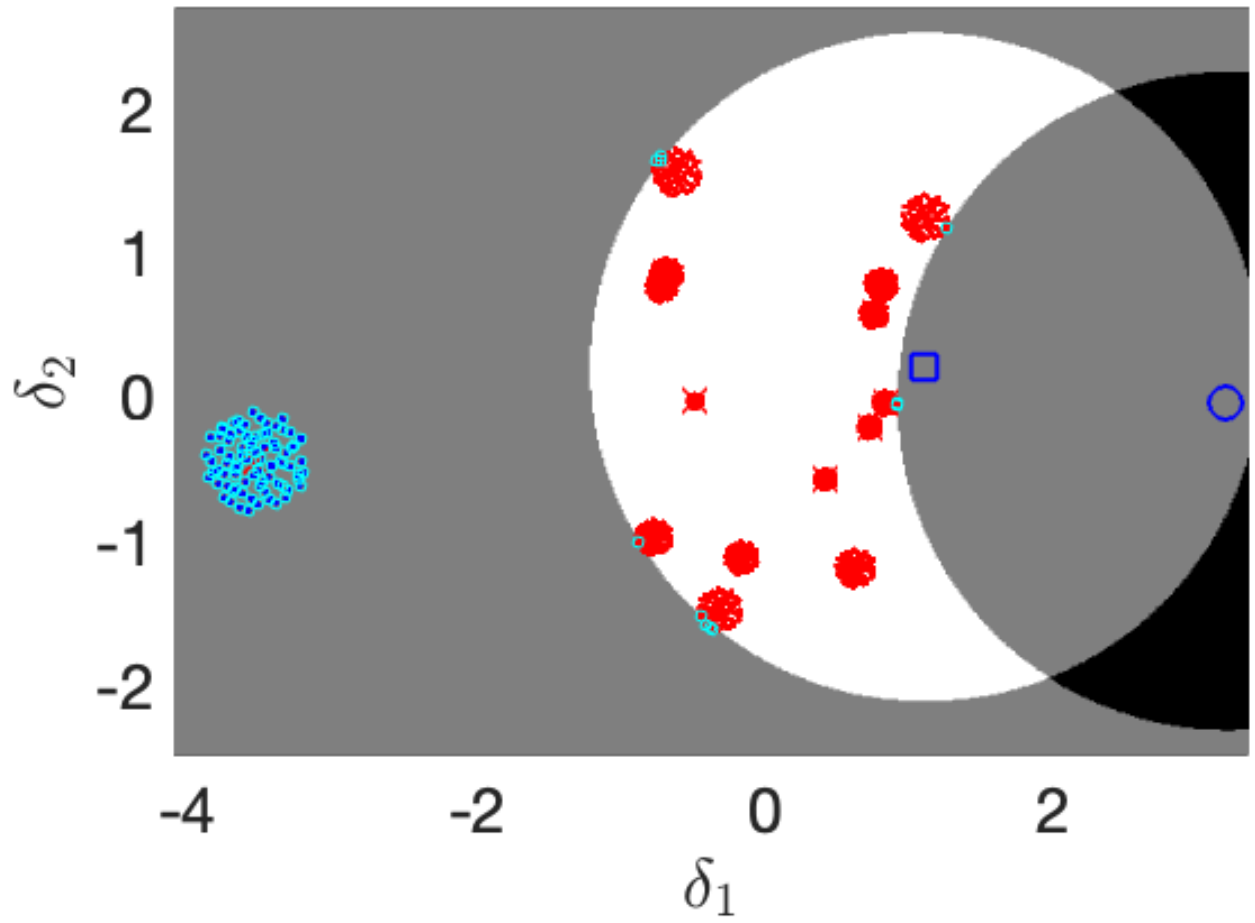
	Ove rfit	Risk- averse	OP	no	J*
θ_1	Y	Y	WC	0	11.2
θ_2	Y	Y	WC	1	10.5
θ_3	Y	N	CC	1	8.3
θ_4	N	Y	WC	0	15.1
θ_5	N	Y	CC	1	13.4
θ_6	N	N	CC	1	9.6
θ_7	N	N	CC	2	6.5

Design 5



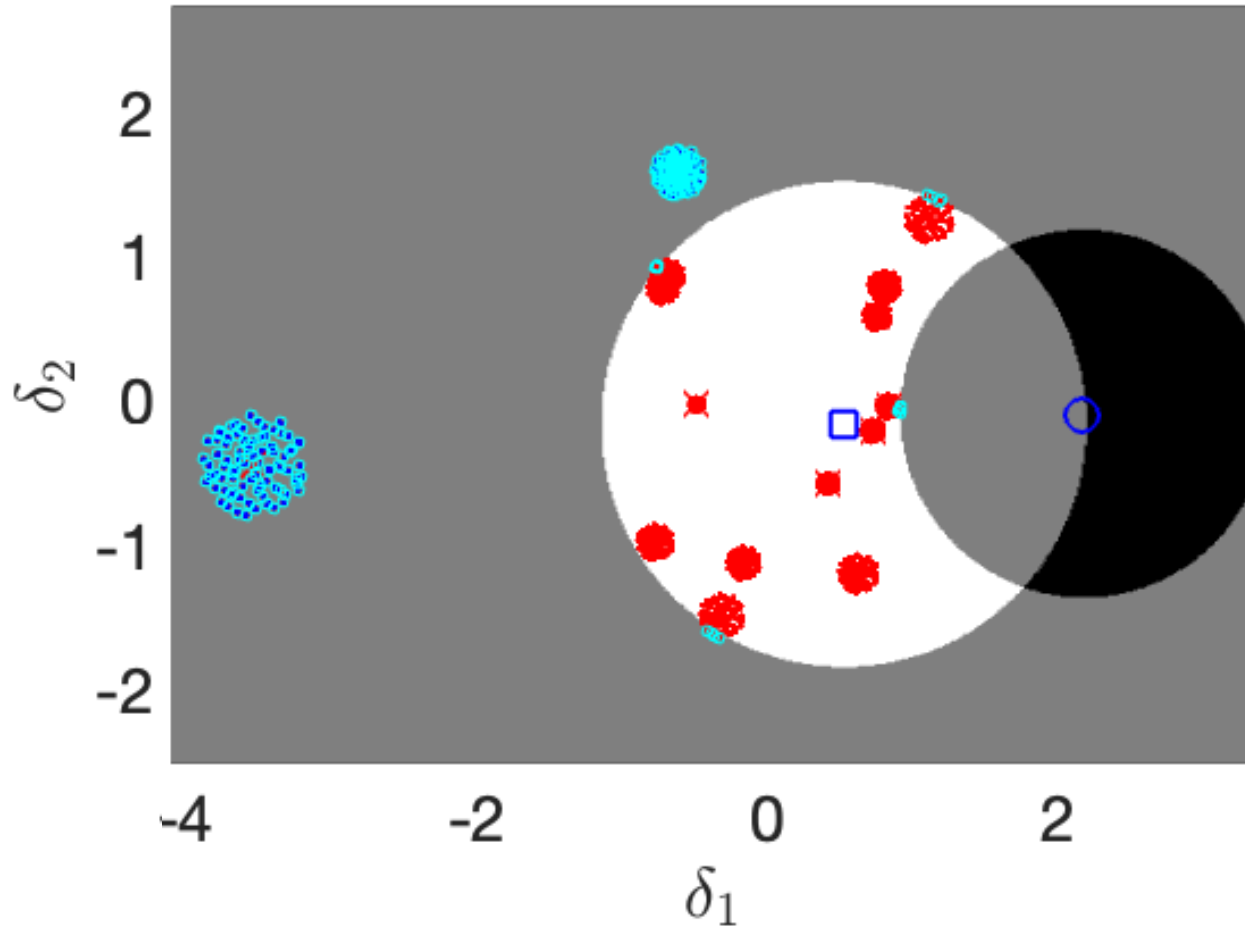
	Ove rfit	Risk- averse	OP	no	J^*
θ_1	Y	Y	WC	0	11.2
θ_2	Y	Y	WC	1	10.5
θ_3	Y	N	CC	1	8.3
θ_4	N	Y	WC	0	15.1
θ_5	N	Y	CC	1	13.4
θ_6	N	N	CC	1	9.6
θ_7	N	N	CC	2	6.5

Design 6



	Ove rfit	Risk- averse	OP	no	J*
θ_1	Y	Y	WC	0	11.2
θ_2	Y	Y	WC	1	10.5
θ_3	Y	N	CC	1	8.3
θ_4	N	Y	WC	0	15.1
θ_5	N	Y	CC	1	13.4
θ_6	N	N	CC	1	9.6
θ_7	N	N	CC	2	6.5

Design 7



	Ove rfit	Risk- averse	OP	no	J*
θ_1	Y	Y	WC	0	11.2
θ_2	Y	Y	WC	1	10.5
θ_3	Y	N	CC	1	8.3
θ_4	N	Y	WC	0	15.1
θ_5	N	Y	CC	1	13.4
θ_6	N	N	CC	1	9.6
θ_7	N	N	CC	2	6.5

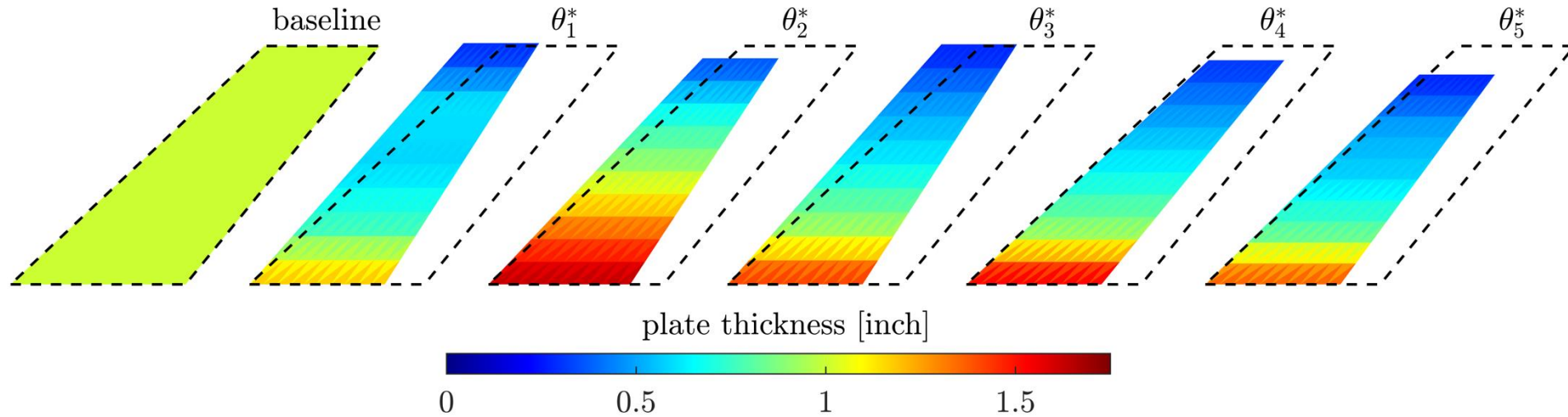


Performance metric

Example: Aeroelastic Wing Design

- Design of an aero-elastic wing subject to flutter constraints
- δ (6) : Mach number, the mass- and stiffness-proportional Rayleigh damping coefficients, and the dynamic pressure, viscosity and target lift coefficient at the steady load case
- θ (9) : the root chord, the tip chord, the semispan, the wing sweep, & the plate thickness of the five zones
- J : the mean of a linear combination of mass & drag
- Requirements: flutter stability and acceptably low stress at static loading
- Dataset: $n=50$ scenarios (high cost, avoid overfitting)
- Goal: explore the trade off between robustness & performance

Example: Aeroelastic Wing Design

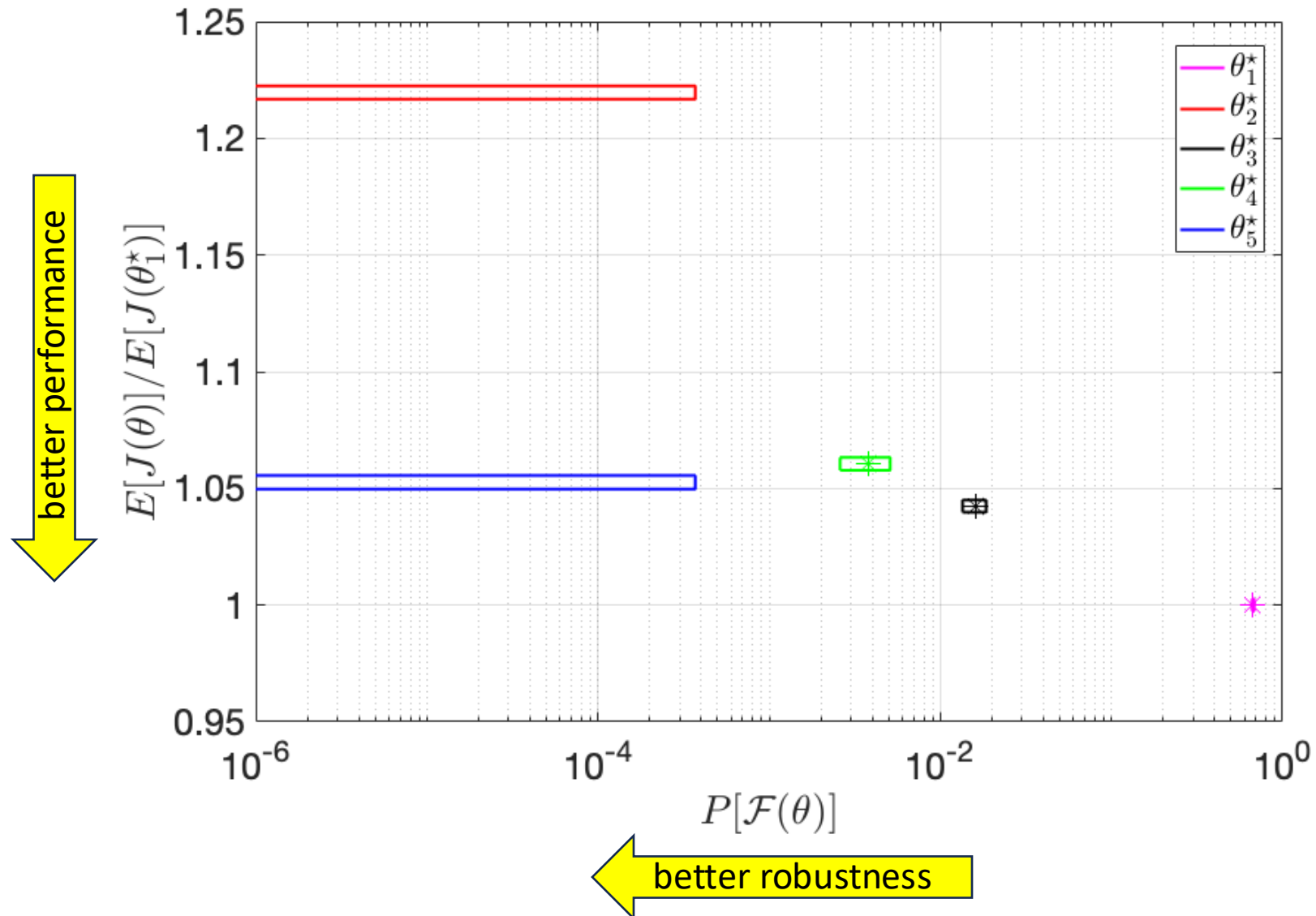


Training dataset					Testing dataset, $n=1e4$					
	n	m	n_o	$\mathbb{E}_\delta[f I]$	$\mathbb{E}_\delta[f]$	$P_\delta[\mathcal{F}_1]$	$P_\delta[\mathcal{F}_2]$	$P_\delta[\mathcal{F}]$	ℓ_1	ℓ_2
θ_1^*	1	1	0	3.08	3.12	3.75×10^{-1}	4.81×10^{-1}	6.73×10^{-1}	0.072	0.127
θ_2^*	50	1	0	3.80	3.81	0	0	0	0	0
θ_3^*	50	1	0	3.25	3.25	9.20×10^{-3}	6.90×10^{-3}	1.60×10^{-2}	0.011	0.012
θ_4^*	50	[1, 2]	0	3.31	3.31	3.70×10^{-3}	1.00×10^{-4}	3.80×10^{-3}	0.088	0.001
θ_5^*	67	1	0	3.33	3.28	0	0	0	0	0

Performance metric

Robustness metric

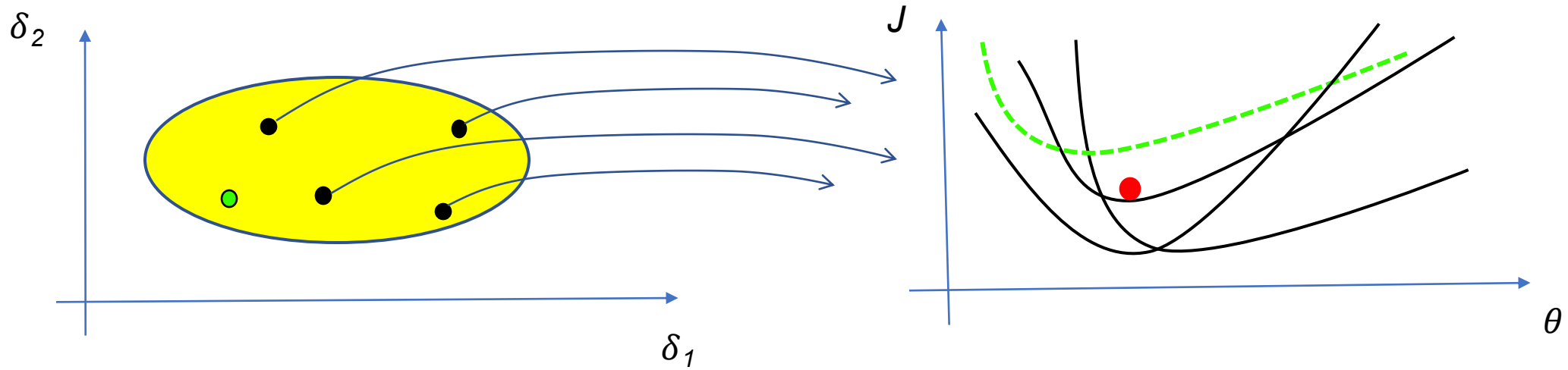
Example: Aeroelastic Wing Design



Outline

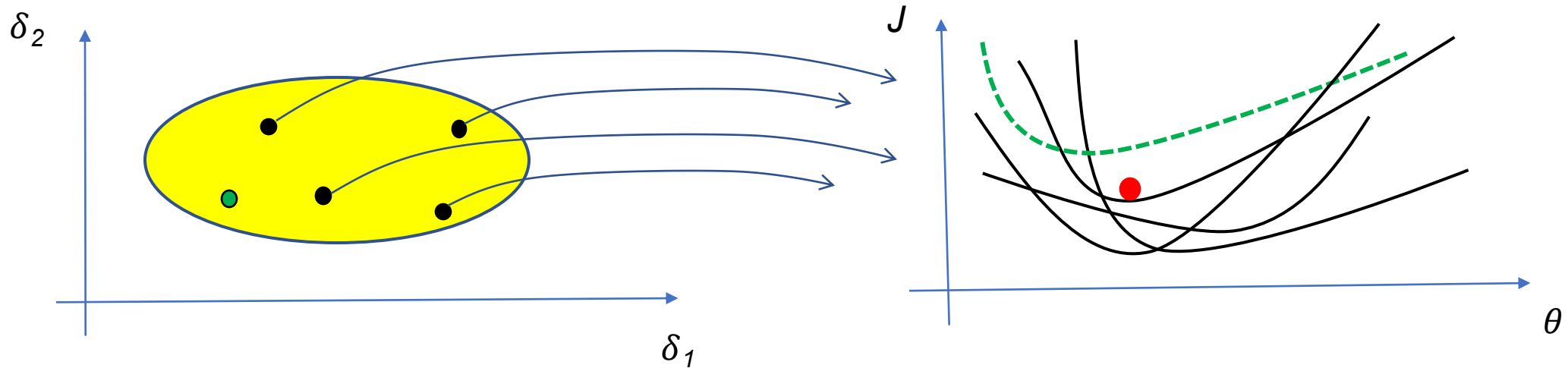
- Problem statement
- Solution strategies
- **Scenario Theory**
 - Nominal data and perturbed data

Scenario Optimization: Introduction



Q: What is the probability of a future scenario increasing the optimum cost?

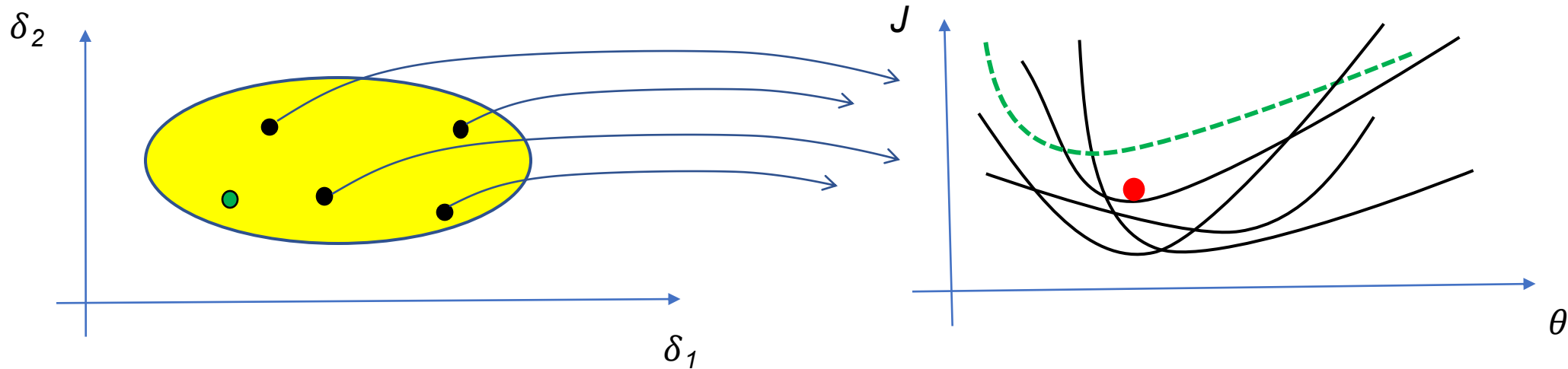
Scenario Optimization: Introduction



Q: What is the probability of a future scenario increasing the optimum cost?

A1: Start by modeling the distribution of the data....

Scenario Optimization: Introduction



Q: What is the probability of a future scenario increasing the optimum cost?

A1: Start by modeling the distribution of the data...

A2: Use Scenario Theory and avoid modeling the uncertainty

Scenario Theory

- Scenario theory provides an upper bound on the probability that an **unseen** datum will violate the requirements imposed upon a data-based optimum
- In general,

$P[\text{Violation}] < \text{Bound (amount of data, models' complexity, number of outliers)}$

Scenario Theory

- Scenario theory provides an upper bound on the probability that an **unseen** datum will violate the requirements imposed upon a data-based optimum
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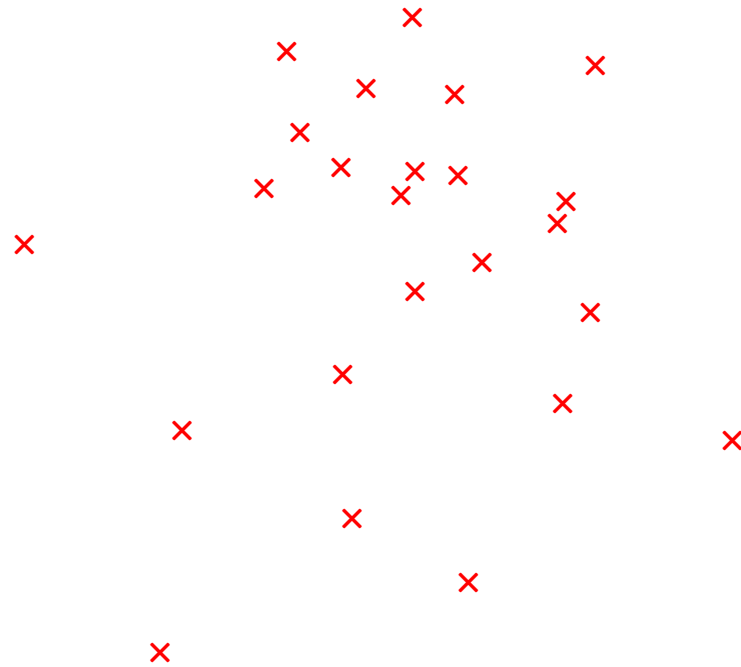
$$P[\text{Violation}] < \text{Bound (amount of data, models' complexity, number of outliers)}$$

- Features
 - Bound is non-asymptotic and distribution-free
 - Does not require modeling the uncertainty!
 - Bound is often tight
 - The only requirement is for the scenarios to be IID
 - Arbitrary dependency on the uncertain parameters & design variable
 - Convexity in the design variables helps

Scenario Theory

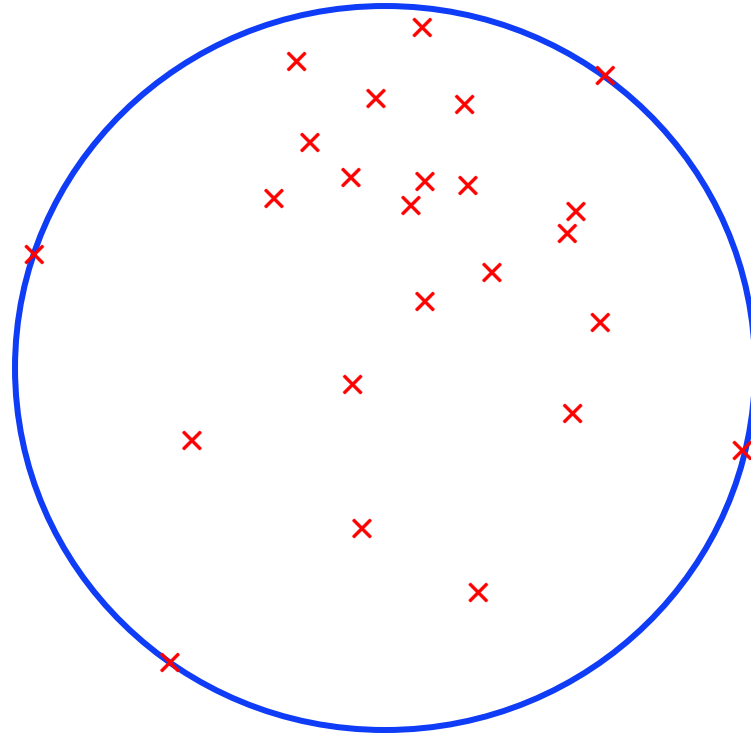
- Scenario theory yields a rigorous certificate of robustness

Example: Data-enclosing Set



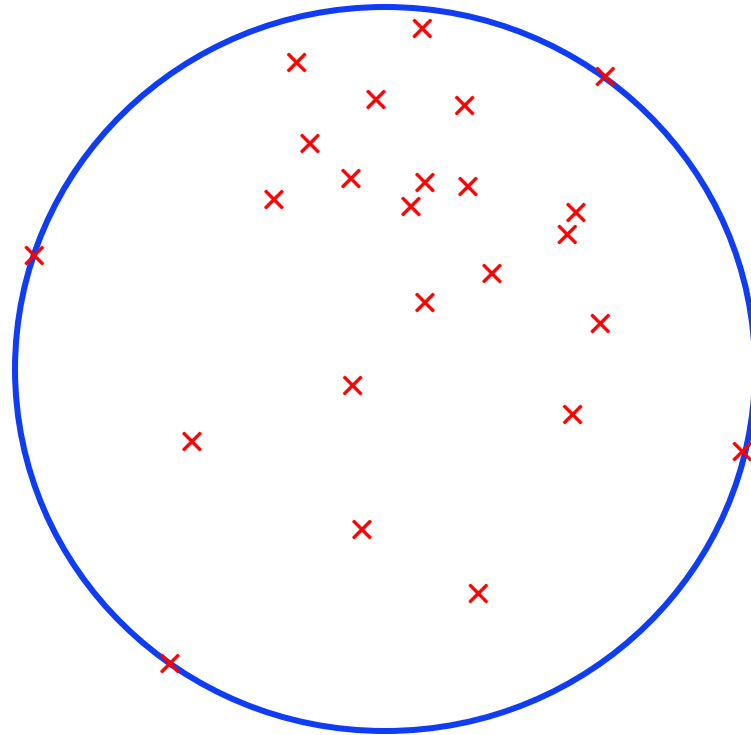
Compute the data-enclosing circle of minimum volume?

Example: Data-enclosing Set



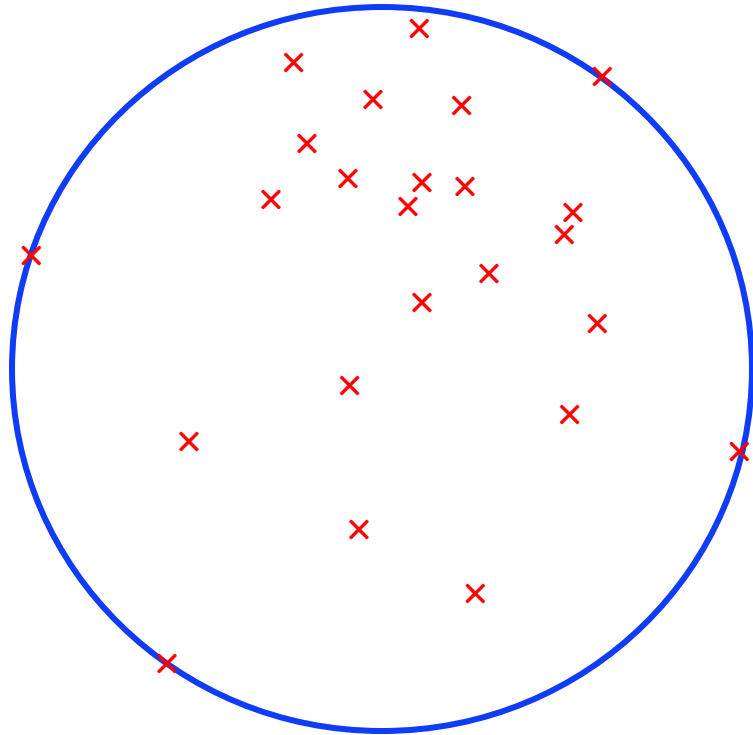
Compute the data-enclosing circle of minimum volume?

Example: Data-enclosing Set



What is the probability of a new datum falling outside the circle?

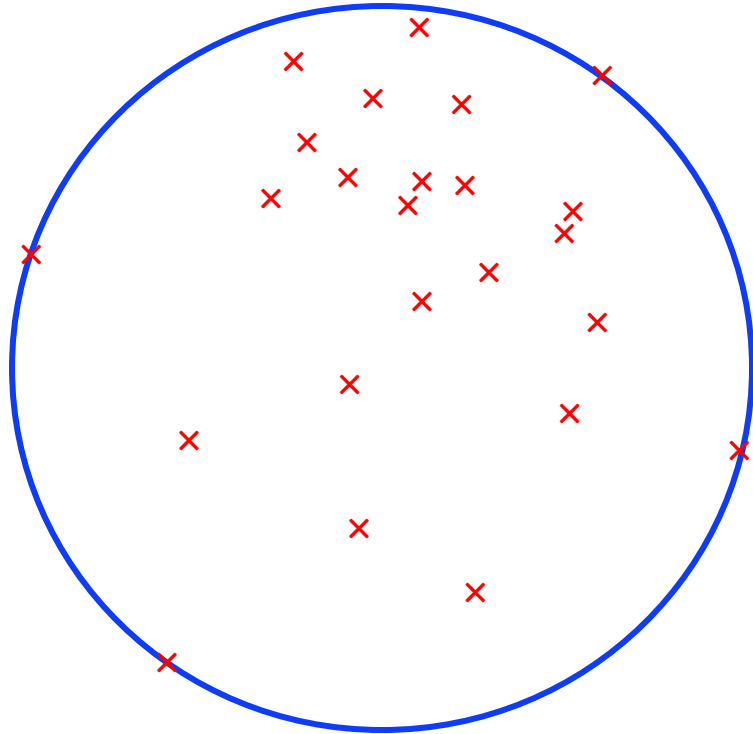
Example: Data-enclosing Set



Bound only requires evaluating
an algebraic expression

$$n=24, n_d=3, k=0, \beta=1e-3: P[x \notin C] \leq 0.38$$

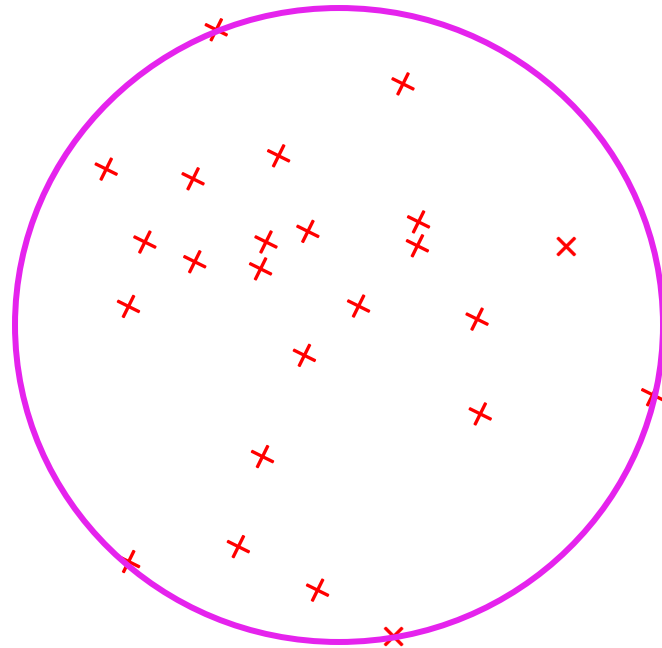
Example: Data-enclosing Set



No need to model the distribution of the data!!

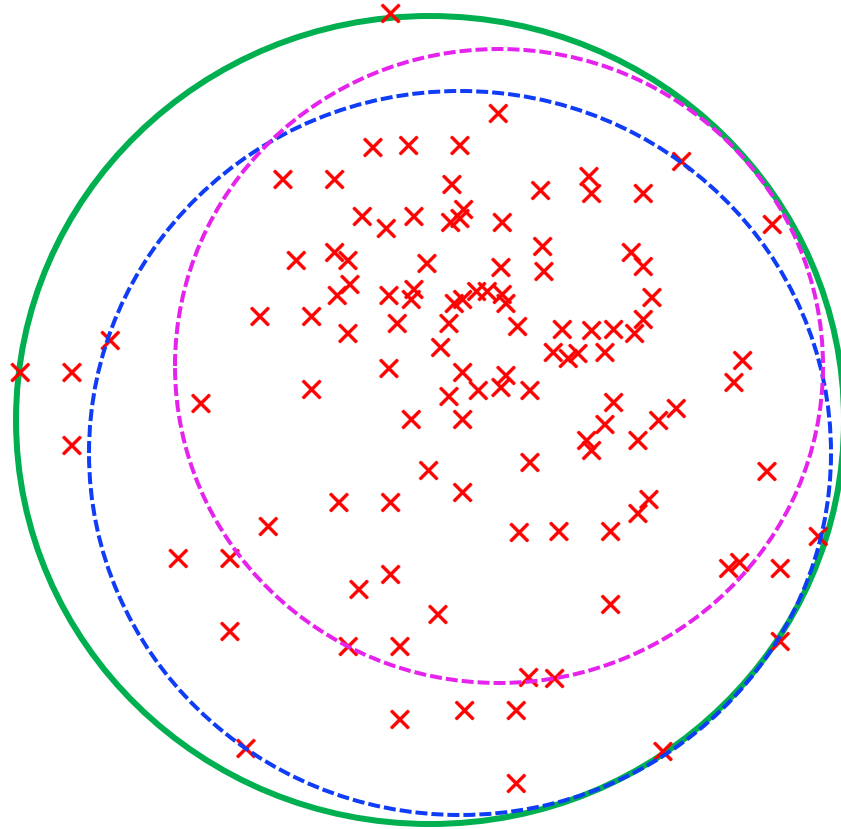
$$n=24, n_d=3, k=0, \beta=1e-3: P[x \notin C] \leq 0.38$$

Example: Data-enclosing Set



$$n=24, n_d=3, k=0, \beta=1e-3: P[x \notin C] \leq 0.38$$

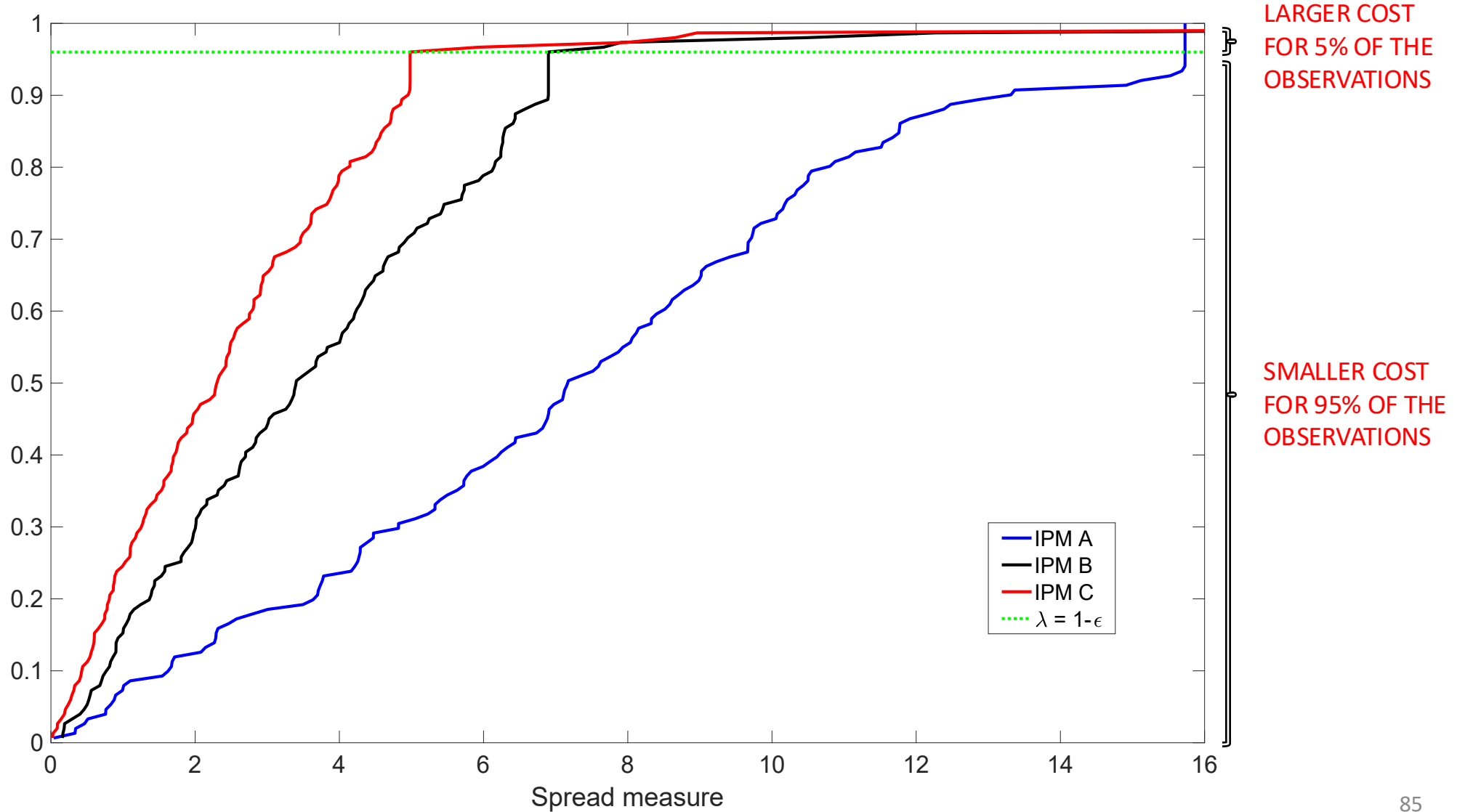
Example: Data-enclosing Set



- Bound is tight
- Tighter bound can be computed by making additional calculations

$$n=120, n_d=3, k=0, \beta=1e-3: P[x \notin \mathbf{C}] \leq 0.129$$

Example: Interval Predictor Models



Example: Interval Predictor Model Revisited

	PERFORMANCE	BOUND ($\beta=0.01$)
IPM A (n=150, d=14, k=0)	8.61	0.155
IPM B (n=150, d=14, k=5)	5.87	0.285
IPM C (n=150, d=14, k=5)	5.29	0.285

- A performance improvement of 32% increased the bound 0.13

- Bound is validated by using more data (we might not have it in practice)
- Dividing the data set into a training set & a validation set is not needed

Conclusions

- Strategies to data-driven optimization were presented
- Applicability
 - The requirements are continuous functions of θ : simulation-friendly setting
 - The computational complexity does not depend on n_δ
 - Strategy is parallelizable, but a large n might not be required
 - Standard, gradient-based algorithms are applicable
- Relaxation is used to prevent infeasibility
- The most robust solution out of any given design class can be pursued
- The trade-off between robustness and performance can be explored
- This framework can be integrated into the MC analyses used for verification: trial and error approaches might not converge

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- Crespo, L. G, Kenny S. P., Giesy D. P.; Interval predictor models for linear parametric models, ASME Journal of Verification, Validation and Uncertainty Quantification, 2016.

Risk-aware Data-driven Decision Making under Uncertainty



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