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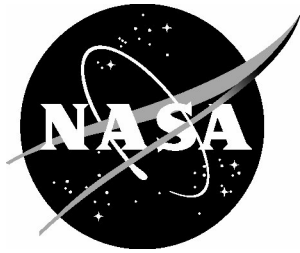
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We present a method to develop a physics-based, reduced-order model of a wind tunnel measurement system (including a sting, strain gage force balance, and test article) to facilitate preliminary wind tunnel test design by predicting structural modes. This reduced-order model is combined with a simple finite element beam model of a sting to create a hybrid reduced-order model which is used to estimate the dynamics of the full assembly. Comparisons between a full finite element model and the hybrid reduced-order model show that this hybrid reduced-order model is capable of predicting the first six natural frequencies to within 10% error for the geometry presented within the paper. The methodology presented within this paper can be used to develop reduced-order model parameters for a number of balances and test article to create hybrid reduced-order models to predict the dynamics of a number of measurement assemblies.

I. Introduction

Mechanical vibrations in wind tunnel force measurements are an important consideration in test configuration design. Characterization of the dynamics of these measurement systems, which typically consist of a test article, strain gage force balance, and sting, provides information that is useful in experimental design. For example, a dynamic characterization enables engineers to identify and change components to shift structural resonant frequencies out of the frequency regime of interest. Alternatively, test parameters can be selected to avoid known resonances all together. One of the prime motivations for a dynamic characterization is to inform force measurements. Force balances are commonly calibrated statically but the dynamic sensitivity can significantly deviate from the static sensitivity due to the influence of additional mechanical components [1–3]. A dynamic characterization provides an estimation of the bandwidth wherein the static sensitivity is valid; commonly this bandwidth is expressed as a certain percentage of the first natural frequency of the system (e.g., 60% - 70% is widely accepted for sensors such as accelerometers) [1, 4]. The

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following paper details an approach to modeling wind tunnel measurement systems using a reduced set of estimated parameters to simplify the process of wind tunnel test design.

Prior studies have experimentally characterized the dynamics of wind tunnel test assemblies via modal analysis or through in-situ dynamic calibration of the force balance [5–9]. Modal analysis requires instrumenting the assembly with accelerometers and striking various locations with an instrumented hammer. The natural frequencies and mode shapes are determined using modal parameter estimation routines. An in-situ calibration requires applying a known force and measuring the response of the force balance. One such method of calibration involves suspending a measurement configuration (including the test article, force balance, and sting) either horizontally or vertically, hanging a weight of known mass from the downstream end via a thin wire, cutting the wire supporting the mass, and then measuring the system step response [7, 10]. Additionally, the sum of weighted accelerations technique (SWAT) has been used to dynamically calibrate wind tunnel assemblies [9]. The SWAT process, in the context of dynamic calibration, is implemented by applying a known static force to get the static calibration and then applying a known dynamic force (e.g., step function) to get the dynamic calibration from discrete acceleration measurements [8, 11–13]. While these approaches characterize the realized assembly, they do not inform the design phase of the test configuration. Therefore, an approach for predicting natural frequencies and mode shapes is desirable.

Reduced-order models (ROMs) are one method to predict the lower-order dynamic behavior of systems. Since the advent of structural analysis tools such as finite-element (FE) software, devising methods to decrease computation time has become increasingly useful. For example, structural dynamics problems with large numbers of degrees of freedom (DOFs) can sometimes require an unwieldy amount of time and digital resources to create and solve. ROMs, by contrast, are intended to approximate system dynamics while also drastically minimizing the time and resources required to build and solve.

Many works have attempted to develop ROMs for wind tunnel measurement systems via a lumped-element approach. Specifically, several works have attempted to model a sting, which has cantilevered boundary conditions, using idealized masses and springs [14–18]. For example, Bernstein and Pankhurst successfully determined the first seven natural frequencies of the sting using a lumped-element approach, although only four mode shapes were reported [14]. Billingsley also utilized lumped elements to great effect, although only the first three sting natural frequencies and mode shapes were reported [15]. In each work, analyses were restricted to motion in the pitch plane. Buehrle added to the works of Billingsley by including both the pitch and the yaw planes in the analysis [17]. Parker applied these analyses in a de-tuning software tool designed to facilitate wind tunnel test design by determining natural frequencies and mode shapes for the composite wind tunnel measurement system [18]. These works were effective at predicting the dynamics in one or two planes of motion, but do not consider the coupled system consisting of three translations and three rotations.

The present work extends those of Bernstein and Pankhurst [14], Buehrle [17], and Parker [18], by developing a

fully three-dimensional hybrid-ROM for a wind tunnel measurement system that uses idealized lumped-elements for the test article and force balance, and a simple finite element model for the sting. In this hybrid-ROM, the force balance is modeled as a pure spring element (including linear coupling between planes of motion) with 3 translational and 3 rotational DOFs on each end, while the test article is modeled as a rigid body mass element with both translational and rotational inertia. Additionally, due to the regular geometry of the sting, beam finite elements are employed, the formulation for which include 3 translations and 3 rotations [19]. The advantage of this model is that it serves as a simple predictive tool to aid in the design of a wind tunnel test assembly. For example, engineers can use this model to quickly study effects of changing the test article mass or changing balance stiffness parameters without having to remesh an entire solid model geometry. Moreover, this technique could enable the construction of a digital library of pre-existing stings and force balances which can be used to estimate the natural frequencies of a generic wind tunnel test assembly.

The paper is organized as follows: The framework for the reduced-order model and its general assumptions are detailed in Section II, identification of system parameters and the design of a numerical experiment are described in Section III, and results of the numerical experiment are presented in Section IV. A comprehensive approach to reduced-order modeling of these systems benefits the aerospace community by facilitating preliminary design of wind tunnel tests based on a simplified set of parameters.

II. Reduced-Order Model

The goal in developing a ROM is to construct a set of ordinary differential equations that approximate the dynamics of a higher-order system using a reduced set of parameters. For linear, time-invariant mechanical systems, the governing differential equations are typically written in matrix form as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}. \quad (1)$$

In Equation (1), $\mathbf{M}$ is the global mass matrix, $\mathbf{C}$ is the global damping matrix, and $\mathbf{K}$ is the global stiffness matrix. The quantity $\mathbf{x}$ represents the global displacement vector, while $\mathbf{f}$ represents the global forcing vector.

In this work, a proportional damping model is included and is of the form

$$\mathbf{C} = \mu\mathbf{M} + \lambda\mathbf{K}, \quad (2)$$

where the damping matrix $\mathbf{C}$ is a linear combination of the mass and stiffness matrices, and μ and λ are proportionality constants [20].

Since three distinct components are to be included in the system model (i.e., the sting, force balance, and test article), consideration of the inertial and elastic properties of each component separately is necessary. Once the mass

and stiffness matrices for each component are determined, they are then assembled to create global mass and stiffness matrices for the composite system. The goal of this paper is to provide a systematic methodology to identify the global mass and stiffness matrices in Equation (1) for a wind tunnel measurement system such as the one shown in Figure 1a. Once these matrices have been identified, Equation (1) can be simulated in both the frequency- and time-domains to make predictions regarding the dynamic system response.

The ROM implemented in this work is intended to model wind tunnel test assemblies that obey a few assumptions. First, the sting is assumed to have a geometry such that its deflection can be modeled as an Euler-Bernoulli beam. Because of the regular geometry, the sting may be modeled using beam finite elements, the parameters for which are assigned in Section III.C. The test article is assumed to be a rigid body mass element which is valid provided that the sting and balance are more flexible than the test article. The test article rigidity assumption may only be valid at lower frequencies, since at higher frequencies, the elasticity of the test article may contribute to the overall dynamics of the system. The test article inertial properties are stored in a 6-by-6 matrix, denoted as M_A in Figure 1a, which corresponds to 3 translational and 3 rotational DOFs. The derivation for this rigid body mass matrix is included in Section III.A.

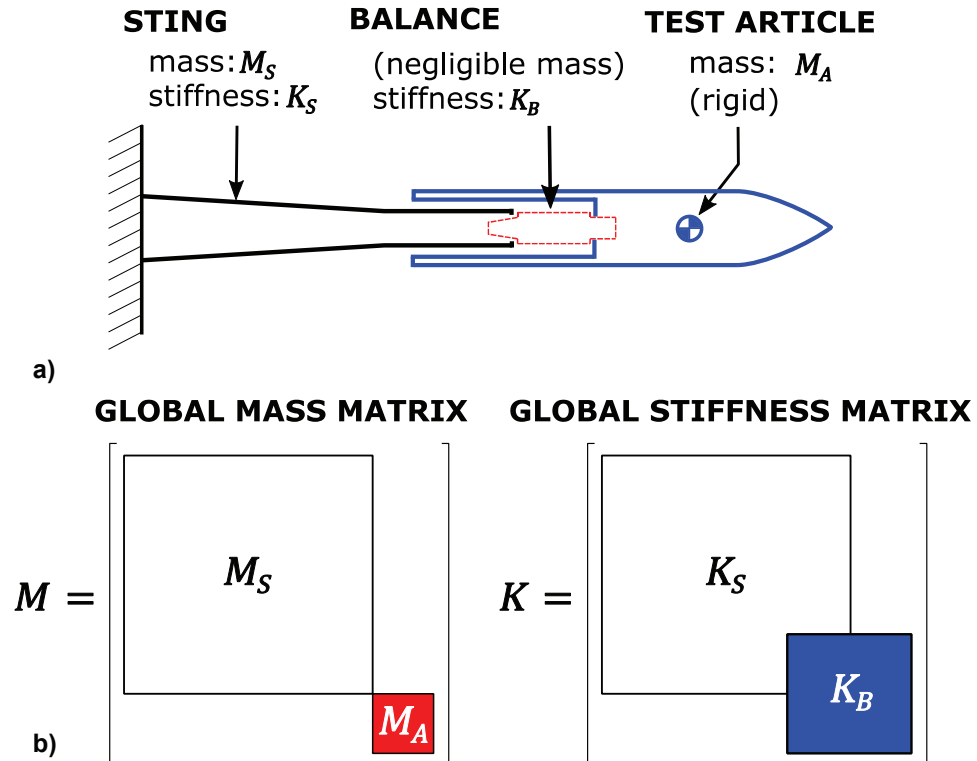


Fig. 1 a) Simplified illustration of a wind tunnel measurement system and its components. b) Visual representation for how to assemble global mass and stiffness matrices.

Lastly, the mass of the test article m_A is assumed to be significantly larger than the mass of the force balance m_B , or $m_A \gg m_B$, implying that the mass of the force balance may be neglected such that it may be modeled as a pure spring element. The stiffness matrix for the idealized balance, written as $\mathbf{K}_B$ in Figure 1, is a 12-by-12 matrix including 6 DOFs on either end of the balance. The method for identification of stiffness parameters is provided in Section III.B. A visual representation of the scheme implemented to assemble the global mass and stiffness matrices is shown in Figure 1b. Numerical parameters for the test article and force balance are assigned in Sections III.A and III.B, respectively.

For the purpose of in-depth study, two models will be developed in this work. The first model only aims to capture the dynamics of the force balance and test article using the aforementioned idealized lumped-elements; this model will be referred to hereafter as a ROM. The second model combines the force balance and test article ROM with a simple finite-element of the sting; this combined model is referred to as a hybrid-ROM.

III. System Parameter Identification Methodology & Numerical Experiment Description

A. Test Article Mass Matrix

Because the sting and force balance are typically more flexible than the test article, the test article is represented as a rigid body [1, 18]. The test article is initially assumed to be a point mass rigidly linked to the end of the force balance, such that the inertia of the test article is offset from the end of the balance at an angle. The kinematics of the balance are described using two coordinate systems in Figure 2. The O reference frame is a fixed inertial frame, while the B frame is attached to the end of the balance.

The mass matrix model for the test article is determined using a Lagrangian-based approach using 6 DOFs [21, 22].

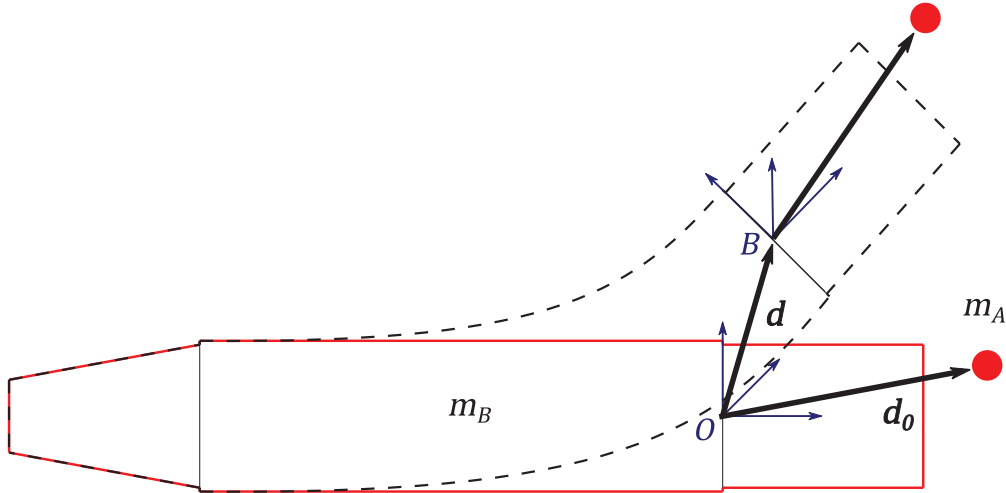


Fig. 2 Illustration of a force balance with a point mass rigidly fixed to the end of the balance. The inertial reference frame O is fixed in space to the initial location of the hanging end of the undeformed balance, while the B reference frame is free to translate and rotate as the balance deforms.

The Lagrangian is written as

$$L = T - V, \quad (3)$$

where T and V represent the test article kinetic and potential energies, respectively. The conservative form of Lagrange's Equation may be written as

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\mathbf{u}}} - \frac{\delta L}{\delta \mathbf{u}} = 0. \quad (4)$$

where $\mathbf{u}$ represents a set of generalized coordinates describing the motion of the test article or

$$\mathbf{u}^T = \begin{bmatrix} a & s & n & r & p & y \end{bmatrix}, \quad (5)$$

where a , s , n , r , p , and y represent the axial, side, normal, roll, pitch and yaw motions of the test article center of gravity (CG) in the O frame. In Equation (5), T represents the transpose operator. Since the test article is assumed to be rigid, the portions of Equations (3) and (4) dealing with the potential energy are ignored. Therefore, only the kinetic energy is necessary to obtain the mass matrix for the system, or

$$\frac{d}{dt} \frac{\delta T}{\delta \dot{\mathbf{u}}} = \mathcal{M}_A(\ddot{\mathbf{u}}, \dot{\mathbf{u}}, \mathbf{u}) \approx \mathbf{M}_A \ddot{\mathbf{u}}. \quad (6)$$

where $\mathcal{M}_A$ is a set of nonlinear equations. Linearizing $\mathcal{M}_A$ produces a constant mass matrix $\mathbf{M}_a$ that is multiplied by a generalized acceleration vector $\ddot{\mathbf{u}}$ as shown by the right-hand side of Equation (6).

The point-mass assumption initially ignores the rotational inertia due to the geometry of the test article to facilitate the derivation of the mass matrix, and will be generalized later to include finite volume effects. The axial, side, and normal coordinates for the initial position of the test article point mass can be written using the vector $\mathbf{d}_0$, or

$$\mathbf{d}_0^T = \begin{bmatrix} a_0 & s_0 & n_0 \end{bmatrix}, \quad (7)$$

Additionally, the DOFs listed in Equation (5) define the updated position vector $\bar{\mathbf{d}}$ for the test article CG. The updated position vector can be written as

$$\bar{\mathbf{d}} = \mathbf{d} + \mathbf{R}\mathbf{d}_0 \quad (8)$$

where $\mathbf{d}$ represents a vector of free translations in the axial, side, and normal DOFs, or

$$\mathbf{d}^T = \begin{bmatrix} a & s & n \end{bmatrix}. \quad (9)$$

The matrix $\mathbf{R}$ in Equation (8) represents the rotation of the B frame coordinate system established in Figure 2. The

matrix is decomposed into a set of Euler angles for each axis as

$$\mathbf{R} = \mathbf{R}_y \mathbf{R}_p \mathbf{R}_r, \quad (10)$$

where $\mathbf{R}_r$ represents a roll rotation, $\mathbf{R}_p$ represents a pitch rotation, and $\mathbf{R}_y$ represents a yaw rotation. The matrix multiplication in Equation (10) is performed and small angles are assumed (i.e. $r, p, y \ll 1$) such that the rotation matrix $\mathbf{R}$ is approximated as

$$\mathbf{R} \approx \begin{bmatrix} 1 & -y & p \\ y & 1 & -r \\ -p & r & 1 \end{bmatrix}. \quad (11)$$

Substituting Equations (7), (9), and (11) into Equation (8) results in the updated position vector $\bar{\mathbf{d}}$. The components of the updated position vector in Equation (8) may be written as

$$\bar{\mathbf{d}}^T = \begin{bmatrix} a + a_0 - s_0 y + n_0 p & s + a_0 y + s_0 - n_0 r & n - a_0 p + s_0 r + n_0 \end{bmatrix}. \quad (12)$$

With $\bar{\mathbf{d}}$ now known, the kinetic energy of the test article may now be obtained. The kinetic energy of the point-mass approximation for the test article is defined as

$$T = \frac{1}{2} m_A \dot{\bar{\mathbf{d}}} \cdot \dot{\bar{\mathbf{d}}}, \quad (13)$$

where $\dot{\bar{\mathbf{d}}}$ is the time derivative of the updated position vector $\bar{\mathbf{d}}$. Written out in terms of the components of the time derivative of Equation (12), Equation (13) becomes

$$T = \frac{1}{2} m_A [(\dot{a} - s_0 \dot{y} + n_0 \dot{p})^2 + (\dot{s} + a_0 \dot{y} - n_0 \dot{r})^2 + (\dot{n} - a_0 \dot{p} + s_0 \dot{r})^2]. \quad (14)$$

Applying Equation (14) to Equation (6) results in 6 equations from which the constant mass matrix can be derived as

$$\mathbf{M}_A = \begin{bmatrix} m_A & 0 & 0 & 0 & m_A n_0 & -m_A s_0 \\ 0 & m_A & 0 & -m_A n_0 & 0 & m_A a_0 \\ 0 & 0 & m_A & m_A s_0 & -m_A a_0 & 0 \\ 0 & -m_A n_0 & m_A s_0 & m_A (s_0^2 + n_0^2) & -m_A a_0 s_0 & -m_A a_0 n_0 \\ m_A n_0 & 0 & -m_A a_0 & -m_A a_0 s_0 & m_A (a_0^2 + s_0^2) & -m_A s_0 n_0 \\ -m_A s_0 & m_A a_0 & 0 & -m_A a_0 n_0 & -m_A s_0 n_0 & m_A (a_0^2 + n_0^2) \end{bmatrix}. \quad (15)$$

As previously stated, Equation (15) describes the inertial properties of a point mass in 6 DOFs. Generally speaking, a standard test article includes rotational inertia properties based on the geometry of the model. Therefore, the rotational inertia terms in Equation (15) may be generalized such that the matrix can be written as

$$\mathbf{M}_A = \begin{bmatrix} m_A & 0 & 0 & 0 & m_A n_0 & -m_A s_0 \\ 0 & m_A & 0 & -m_A n_0 & 0 & m_A a_0 \\ 0 & 0 & m_A & m_A s_0 & -m_A a_0 & 0 \\ 0 & -m_A n_0 & m_A s_0 & I_{aa} & -I_{as} & -I_{an} \\ m_A n_0 & 0 & -m_A a_0 & -I_{sa} & I_{ss} & -I_{sn} \\ -m_A s_0 & m_A a_0 & 0 & -I_{na} & -I_{ns} & I_{nn} \end{bmatrix}, \quad (16)$$

where I_{ij} in the bottom right quadrant of $\mathbf{M}_A$ represents the test article inertia tensor about the initial location of the B frame coordinate system. Furthermore, the inertia tensor is symmetric, meaning that $\mathbf{M}_a = \mathbf{M}_a^T$. In addition to using a Lagrangian-based approach, this mass matrix may also be derived using a Newtonian-based approach but is not shown here for the sake of brevity [23].

A solid model for a generic test article is shown in Figure 3a alongside its inertial and geometric properties in Figure 3b. The geometry for this test article is based on the AGARD-B standard wind tunnel model and has a fuselage diameter of 10.2 cm (4") [24]. The inertial and geometric properties listed in Figure 3b may be input into Equation (16) and subsequently lumped onto the end of the balance in the global mass matrix. For reference, the frequency corresponding to the first flexible mode of the test article in this work is 829.6 Hz.

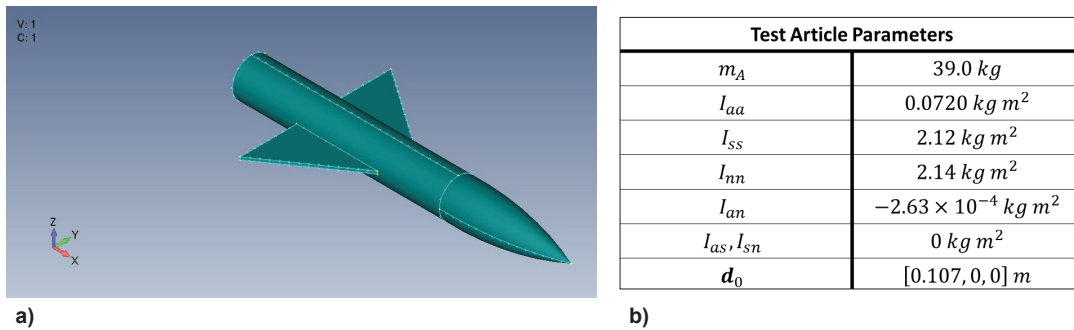


Fig. 3 a) Solid model of the test article, modeled after the AGARD-B standard wind tunnel test article, including inertial and geometric properties used to create the model. b) List of input parameters supplied to Equation (16) to generate rigid body mass matrix.

B. Force Balance Stiffness Matrix

Because the inertia of the force balance is much less than the inertia of the test article, the force balance can be approximated as a pure spring element with 3 translational and 3 rotational DOFs on each end. For this purpose, a solid model of a force balance was drawn based on dimensions provided by [25]. The force balance solid model, shown in Figure 4, was meshed so that the balance stiffness coefficients could be determined using the functionality of a commercial FE software package. The mesh can be seen in Figure 5, with several close-up views of key balance sections. The flex beams, T-strap of the balance axial section, and two cage sections at either balance end were meshed with hexahedral elements. Tetrahedral elements were used to mesh the remaining geometry.

Balance stiffness matrix coefficients are assumed to be of the form obeying Hooke's law and can be written as

$$\mathbf{f}_B = \mathbf{K}_B \mathbf{u}_B, \quad (17)$$

where $\mathbf{K}_B$ can be broken up into four 6×6 sub-matrices, or

$$\mathbf{K}_B = \begin{bmatrix} \bar{\mathbf{K}}_{11} & \bar{\mathbf{K}}_{12} \\ \bar{\mathbf{K}}_{21} & \bar{\mathbf{K}}_{22} \end{bmatrix}. \quad (18)$$

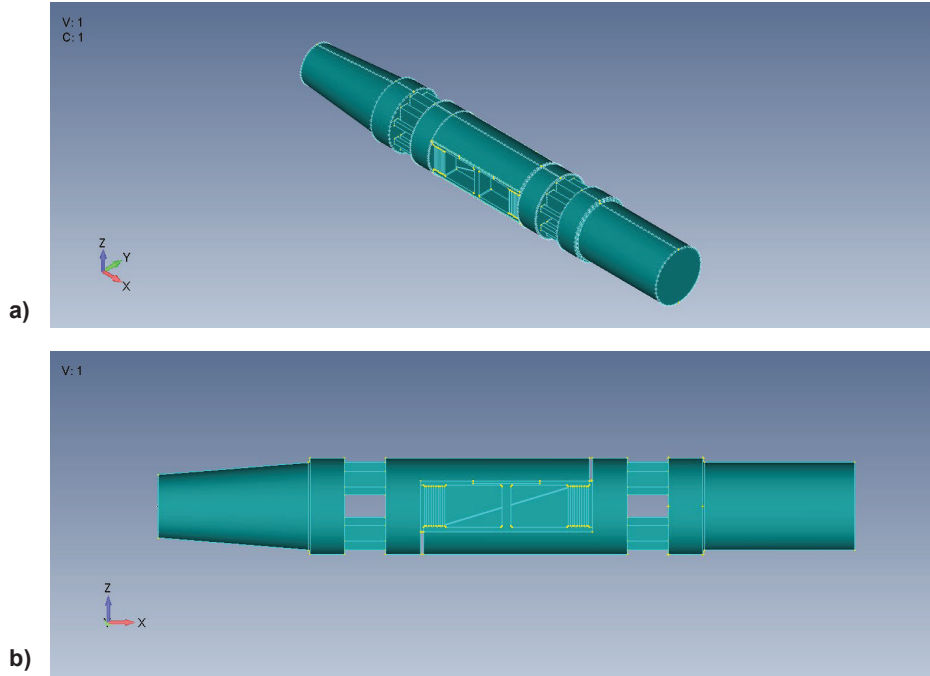


Fig. 4 Solid model of a force balance based on parameters from a NASA Langley technical memorandum [25]: a) Angled view; and b) frontal view.

In Equation (17), $\mathbf{u}_B$ is a 12-by-1 column vector of generalized translational and rotational DOFs representing each end of the balance. These DOFs may be written as

$$\mathbf{u}_B^T = \begin{bmatrix} a_1 & s_1 & n_1 & r_1 & p_1 & y_1 & a_2 & s_2 & n_2 & r_2 & p_2 & y_2 \end{bmatrix}, \quad (19)$$

where a_1, s_1, n_1, r_1, p_1 , and y_1 are the axial, side, normal, roll, pitch, and yaw DOFs at the sting end of the force balance; and a_2, s_2, n_2, r_2, p_2 , and y_2 are the axial, side, normal, roll, pitch, and yaw DOFs at the test article end of the force balance. All DOFs are depicted in Figure 6. The ends of the balance contribute little to the overall balance flexibility and have been removed to compute the stiffness matrix. The quantity $\mathbf{f}_B$ in Equation (17) represents a 12-by-1 column vector of reaction loads in the directions of each DOF, or

$$\mathbf{f}_B^T = \begin{bmatrix} A_1 & S_1 & N_1 & RM_1 & PM_1 & YM_1 & A_2 & S_2 & N_2 & RM_2 & PM_2 & YM_2 \end{bmatrix}. \quad (20)$$

where $A_1, S_1, N_1, RM_1, PM_1$, and YM_1 are the axial, side, normal, roll, pitch, and yaw reaction loads at the sting end of

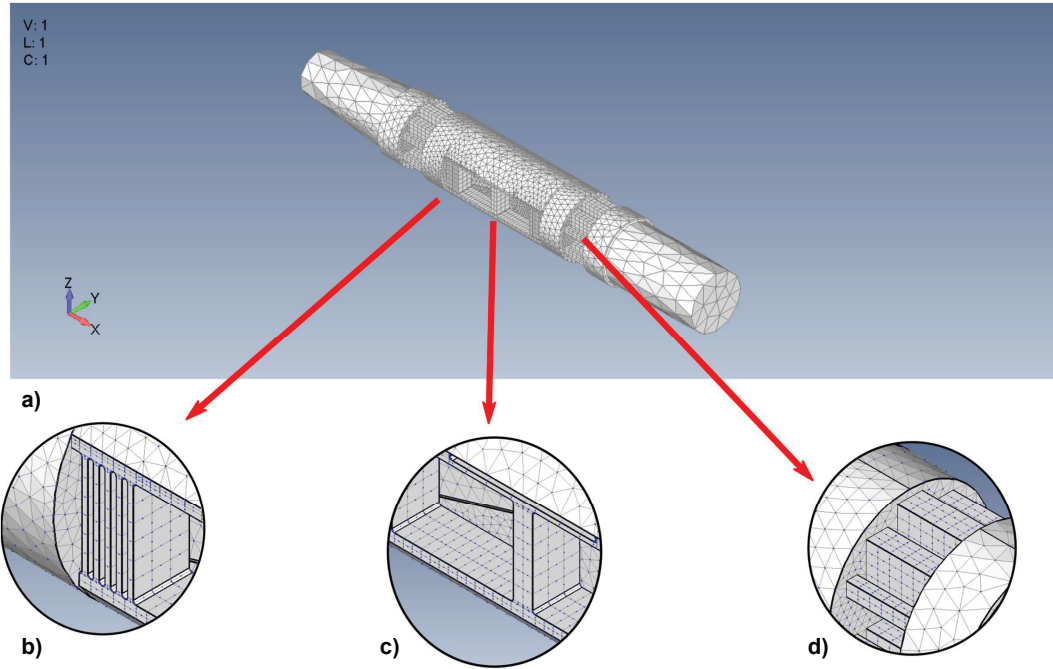


Fig. 5 a) Full view of the balance mesh. b) Close up view of force balance flexural beam mesh. c) Close up view of T-strap section mesh. d) Close up view of cage beam section mesh.

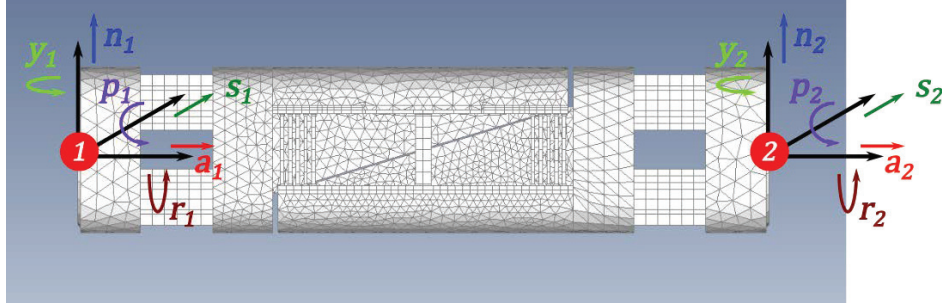


Fig. 6 A meshed model of the force balance without the metric and non-metric attachment points on either end. Labels for the DOFs associated with each end of the balance are provided.

the force balance; and A_2 , S_2 , N_2 , RM_2 , PM_2 , and YM_2 are the axial, side, normal, roll, pitch, and yaw reaction loads at the test article end of the force balance.

The balance stiffness matrix coefficients for the mesh in Figure 6 may be determined using the following approach implemented in FE software: (1) apply some non-zero displacement/angular constraint in the direction of a single generalized coordinate at either end of the balance while constraining the other DOFs to zero, (2) compute reaction loads in all 12 DOFs due to the applied constraint, (3) divide the column vector of reaction loads by the magnitude of the non-zero constraint to obtain a column of the force balance stiffness matrix, and (4) repeat steps (1)-(3) for each DOF until all stiffness coefficients are identified. The process was completed assuming a non-zero constraint of 1 mm and 0.001 rad for the translational and rotational DOFs respectively.

The approach of applying a unit displacement to a single connection DOF while constraining all other connection DOFs and allowing the internal DOFs to deform is ideally identical to the Schur Complement, or Guyan Reduction as it is commonly called in structural analysis [26, 27]. Mathematically it is expressed as

$$\hat{\mathbf{K}} = \mathbf{K}_{cc} - \mathbf{K}_{ci}\mathbf{K}_{ii}^{-1}\mathbf{K}_{ic}, \quad (21)$$

where the subscript c denotes the connection DOFs, and the subscript i denotes the internal DOFs – those DOFs that are not connected to another structure. Given that stiffness matrices map displacements to forces, each column of the difference of matrices in Equation (21) represents the force required to enforce a unit displacement on the structure in a given direction while all other displacements are constrained to zero. A column of $\mathbf{K}_{cc}$ describes the force that needs to be applied to all connection DOFs in order for one of the connection DOFs to be displaced by unity while constraining all other DOFs on the structure – both connection and internal – to have zero displacement. A column of the product of matrices on the right hand side of Equation (21) describes the force required to enforce zero displacement at all connection DOFs, but allowing the internal DOFs to deform as if a unit displacement was applied to one of the connection DOFs. The superposition of these matrices gives a matrix $\hat{\mathbf{K}}$ whose columns represent the force required to

enforce a unit displacement at a single connection DOF while constraining all other connection DOFs and allowing all internal DOFs to deform.

The outcome is a unique stiffness matrix formed specifically for the force balance in Figure 4, but can then be used in place of the FEA model of the balance in Figure 5 for all future investigations. Although the force balance stiffness matrix computed via the unit displacement approach was used to calculate subsequent results in this paper, stiffness parameters were estimated through both methods, and both sets of stiffness parameters are included in Appendices A and B. Some numerical differences exist between the two sets of stiffness parameters; however, these differences resulted in a less than 1-percent difference from the natural frequencies reported in Section IV and were deemed insignificant.

C. Sting Parameters

Due to its regular geometry, the sting is to be modeled using beam elements. Each element has 2 nodes and 6 DOFs (i.e., 3 translations and 3 rotations) per node [19]. A beam of known length (such as the sting) can be divided into a number of beam elements. Mass and stiffness matrices can then be easily generated (denoted as $\mathbf{M}_S$ and $\mathbf{K}_S$ in Figure 1). Since the sting is fixed at one end, the first 6 rows and columns of each matrix are deleted to simulate cantilevered boundary conditions.

A solid model of the sting is displayed in Figure 7a, while a diagram with sting dimensions is included in Figure 7b. The sting designed for this work may be separated into two sections, namely, one linearly tapered section and a straight section. Included in both sections is a bored-out inner cavity with constant radius so that the tapered end of the force balance (see Figure 4) may be attached to the free end of the sting. Properties and dimensions for both sting sections (including the employed number of elements) are summarized in Table 1. The taper is modeled as a series of elements of constant length with diameters that decrease linearly from the element at the large end of the taper down to the element at the small end. With sub-models for all system components, the global ROM mass and stiffness matrices may be arranged, and a numerical experiment conducted. Equations for the sting, force balance, and test article are coupled in the physical coordinate system as illustrated in Figure 1.

D. Numerical Experiment Description

The dynamics of both the ROM (i.e., force balance and test article only) and hybrid-ROM (i.e., sting, force balance, and test article) were compared with FE models of the partial system assembly and the full system assembly, respectively. A description of this numerical experiment is provided here, while results for the ROM are compared with those of a partial assembly FE model in Section IV.A. Results comparing the hybrid-ROM with a full assembly FE model are provided in Section IV.B. Comparisons between the results of the ROM and those of the full FE model of the system were made using the natural frequencies and mode shapes, as well as frequency-domain and time-domain responses. An illustration of the full FE model is displayed in Figure 8.

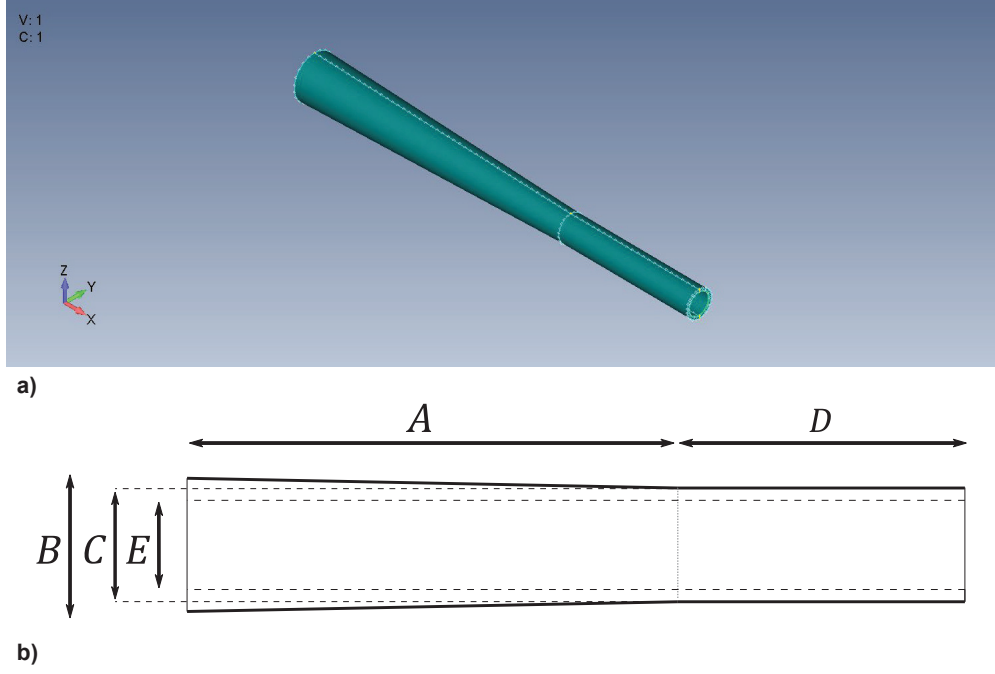


Fig. 7 a) Solid model of the sting. b) Diagram of the sting with dimensions. Letters in the diagram correspond to dimensions provided in Table 1.

Sting Properties		
Total Number of Elements		48
Elastic Modulus		200 GPa
Density		7840 kg/m ³
Tapered Section		
Number of Elements		32
Length	A	40.6 cm
Large-end Outer Diameter	B	6.40 cm
Small-end Outer Diameter	C	3.80 cm
Inner Diameter	E	3.30 cm
Straight Section		
Number of Elements		16
Length	D	20.3 cm
Outer Diameter	C	3.80 cm
Inner Diameter	E	3.30 cm

Table 1 List of sting properties and dimensions. Letters correspond to dimensions in Figure 7b.

The dynamics of the full FE model are described using Equation (1), wherein the mass and stiffness matrices were exported via NASTRAN. Due to the large number of DOFs used in the full FE model, a modal coordinate system was used to fairly compare the FE and ROM models since the ROM is only meant to capture the lower-order dynamics of

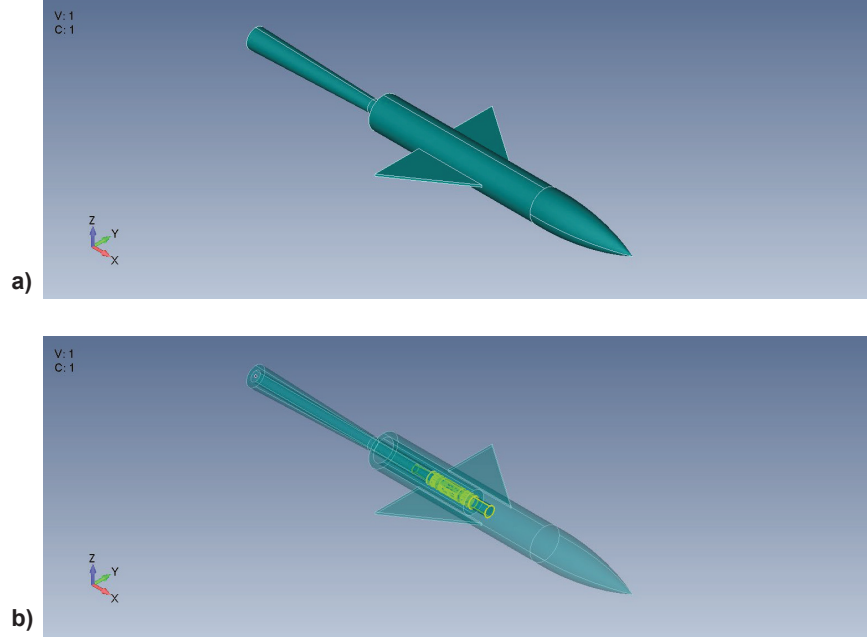


Fig. 8 a) Solid model of the full system. b) Model of the full system with the force balance visible.

the system up to the first 6 natural frequencies. The transformation from the physical coordinate system to the modal coordinate system is written as

$$\mathbf{x} = \mathbf{U}\mathbf{q}, \quad (22)$$

where $\mathbf{x}$ is the array of physical deflections, $\mathbf{U}$ is a matrix of mass normalized normal modes, and $\mathbf{q} = [q_1, \dots, q_n]^T$ is an array of modal contributions. Equation (22) may be substituted into Equation (1) and then multiplied by $\mathbf{U}^T$ to produce the following equation in modal coordinates:

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2q_n = g_n. \quad (23)$$

In this equation, n represents the mode number and ω_n , ζ_n , and g_n represent the natural frequency, damping ratio, and the modal force of the n^{th} mode. Damping ratios for each mode were individually set as 0.5% with the assumption that μ and λ from Equation (2) are equal to zero and $2\zeta_n/\omega_n$, respectively.

Frequency domain analyses were performed using a direct-solve approach on the matrix form of Equation (23) which can be written as

$$\ddot{\mathbf{q}} + \bar{\mathbf{C}}\dot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} = \mathbf{g}, \quad (24)$$

where $\bar{\mathbf{C}}$ is the diagonal modal damping matrix, $\mathbf{\Omega}$ is the diagonal matrix of eigenvalues ω_n^2 , and $\mathbf{g} = \mathbf{U}^T\mathbf{f}$ is the modal forcing vector. Assuming harmonic forcing of the form $\mathbf{g} = \mathbf{G}(\omega)e^{j\omega t}$ where $\mathbf{G}$ is a vector of complex modal force

amplitudes as a function of frequency, the time harmonic solution in modal coordinates for Equation (24) is written in the form of $\mathbf{q} = \mathbf{Q}(\omega)e^{J\omega t}$ where $\mathbf{Q}$ is the vector of complex modal amplitudes as a function of frequency. The assumed solution is then substituted into Equation (24) to obtain

$$\mathbf{Z}\mathbf{Q} = \mathbf{G}, \quad (25)$$

where

$$\mathbf{Z} = \mathbf{\Omega} + J\omega\bar{\mathbf{C}} - \omega^2\mathbf{I}. \quad (26)$$

where $\mathbf{I}$ is the identity matrix and $J = \sqrt{-1}$. The complex modal amplitudes $\mathbf{Q}$ are found by pre-multiplying Equation (25) by the inverse of $\mathbf{Z}$. Time-domain analyses were conducted by taking the scaled inverse-Fast Fourier Transform (IFFT) of $\mathbf{Q}$. To obtain a real time-series from a complex spectrum using the IFFT algorithm, the input spectrum must be two-sided and conjugate symmetric about the central frequency bin.

IV. Results & Discussion

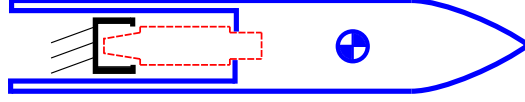
Results are presented for two different simulations derived from the full wind tunnel measurement system depicted in Figure 8, illustrations of which are provided in Figure 9. In the first case as per Figure 9a, the sting is not included, and only the force balance and test article are considered. A comparison of the natural frequencies computed using both the FE model and the ROM is provided in Section IV.A. Further provided in Section IV.A are modal complex spectra resulting from applying a 2000 N infinitesimal width impulse applied in the lift direction of the test article CG for both the FE and reduced-order models. In the second case shown in Figure 9b, the cantilevered sting is included with the force balance and test article attached to the free end. In this case, comparisons are made between the natural frequencies and normal modes computed for the hybrid-ROM and the full assembly FE model. Additionally, modal spectra and a physical time-domain simulation of the test article CG in response to a 2000 N infinitesimal width impulse are computed and presented for both the hybrid-ROM and the full assembly FE model. Because the test article is treated as an idealized mass with distributed rotational inertia for both the ROM and hybrid-ROM, the impulse is technically applied to the node at the end of the force balance, rather than at the test article CG as is otherwise possible with the FE model included in the comparison.

A. Case 1: Force Balance & Test Article

The first 6 natural frequencies computed for both the FE model of the partial assembly and the ROM are provided, in addition to the percent difference between the natural frequencies of the two models, in Table 2.

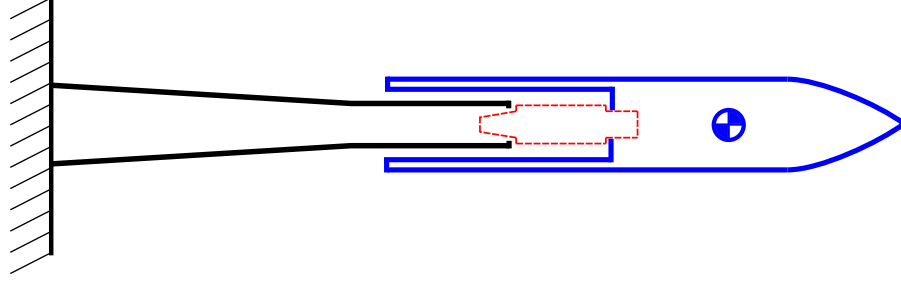
The predicted natural frequencies for both models are within 7% of one another. We note that these predicted natural frequencies serve as an upper bound for the full assembly including the sting, force balance, and test article. In other

Configuration #1: Balance and Test Article



a)

Configuration #2: Sting, Balance, and Test Article



b)

Fig. 9 Illustrations of two simulated configurations for both the full FE model and the hybrid-ROM: a) Only the balance and test article, assuming a rigid boundary condition on one end of the balance; b) the full wind tunnel measurement system including sting, balance, and test article, with the rigid boundary condition applied to the sting.

Mode	FE, Hz	ROM, Hz	$\Delta\%$	Type
1	12.1	11.8	-2.83	Yaw
2	14.9	14.6	-2.07	Pitch
3	59.2	58.7	-0.89	Roll
4	84.8	85.1	0.27	Axial
5	110.0	115.3	4.86	Yaw
6	138.1	147.2	6.61	Pitch

Table 2 Comparison of the natural frequencies computed for the FE model of the balance and test article with those computed by the ROM for the first 6 modes.

words, the natural frequencies will be lower when the sting is considered. Additionally, the fundamental frequency of the test article is well above the frequencies in Table 2, meaning that representing the test article as a rigid body is valid in the scope of this work.

The modal frequency-domain results from applying the previously described numerical impulse to the ROM and the FE model of the partial assembly are shown in Figure 10. The resonance peaks computed by both the ROM and the FE model of the partial assembly occur near the same frequencies which is expected from the agreement of the eigen-analysis study. Additionally, the low-order approximations for both the components included in the ROM perform reasonably well in predicting modal amplitude in at least the first 6 modes. The largest disparities occur in the first two modes where the ROM under-predicts the displacement amplitudes of the first 2 modes by a factor of approximately

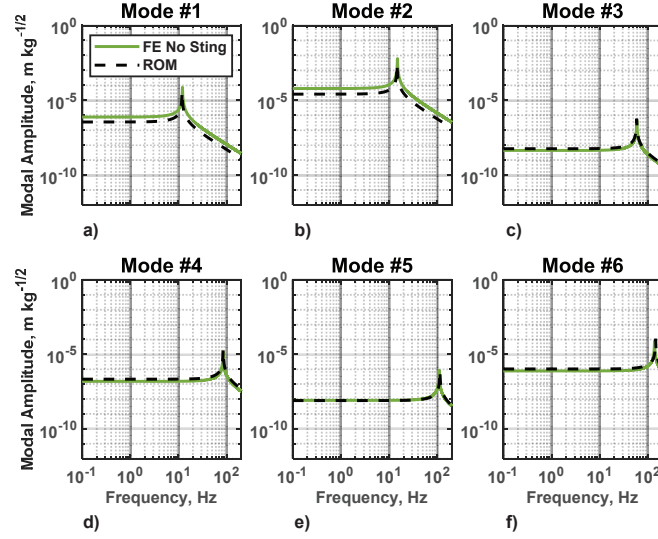


Fig. 10 Modal spectra for the first 6 modes computed in response to a 2000 N impulse for the partial assembly excluding the sting where a-f) correspond to modes #1-6, respectively.

2. These results suggest the ROM can also provide low-order amplitude approximations provided that the modeling assumptions described in Sections II and III are satisfied.

B. Case 2: Full System

In this section, comparisons are made between the hybrid-ROM and full assembly FE model, wherein both models include the test article, force balance, and sting. The first 6 natural frequencies computed by both models are provided in Table 3. Similar to Case 1, the natural frequencies computed by either model agree for each mode within 5%, suggesting that adding the sting model to the ROM of the force balance and test article is a viable method to estimate the low-order dynamics of the full assembly.

The first 3 mode shapes predicted by both the hybrid-ROM and full assembly FE model are shown in Figure 11.

Mode	FE, Hz	Hyb.-ROM, Hz	$\Delta\%$	Type
1	7.55	7.30	-3.24	Yaw
2	8.07	7.79	-3.42	Pitch
3	30.4	31.6	3.94	Yaw
4	34.6	36.1	4.43	Pitch
5	50.2	49.3	-1.74	Roll
6	82.7	82.6	-0.06	Axial

Table 3 Comparison of the natural frequencies computed for the full FE model with those computed by the hybrid-ROM for the first 6 modes.

Mode shapes predicted using the FE model are shown in the left column, while the mode shapes predicted by the hybrid-ROM are shown in the right column. The first mode, computed and illustrated in Figures 11a and b for both models, demonstrates that FE model is dominated by yaw motion with some side deflection. In consonance with FE prediction, the hybrid-ROM also predicts large yaw deflections with some side motion. A similar comparison can be made for the second mode shape illustrated in Figures 11c and d, as well as the third mode shape in Figures 11e and f. These are each dominated by pitch and yaw motion, respectively.

Modal frequency-domain results in response to a 2000 N impulse applied to the normal translation DOF of the test article CG for both the full assembly FE model and the hybrid-ROM are provided in Figure 12. As with Case 1, the full FE model and hybrid-ROM provide comparable complex modal spectra. The amplitudes are reasonably represented by the hybrid-ROM, with the exception of modes 5 and 6 which appear to over-predict the amplitude of the FE model by a factor of 4 or 5. For illustrative purposes, the spectra computed for Case 1 are also included in Figure 12 to demonstrate the reduction in natural frequencies for each mode as a result of including the sting in the model.

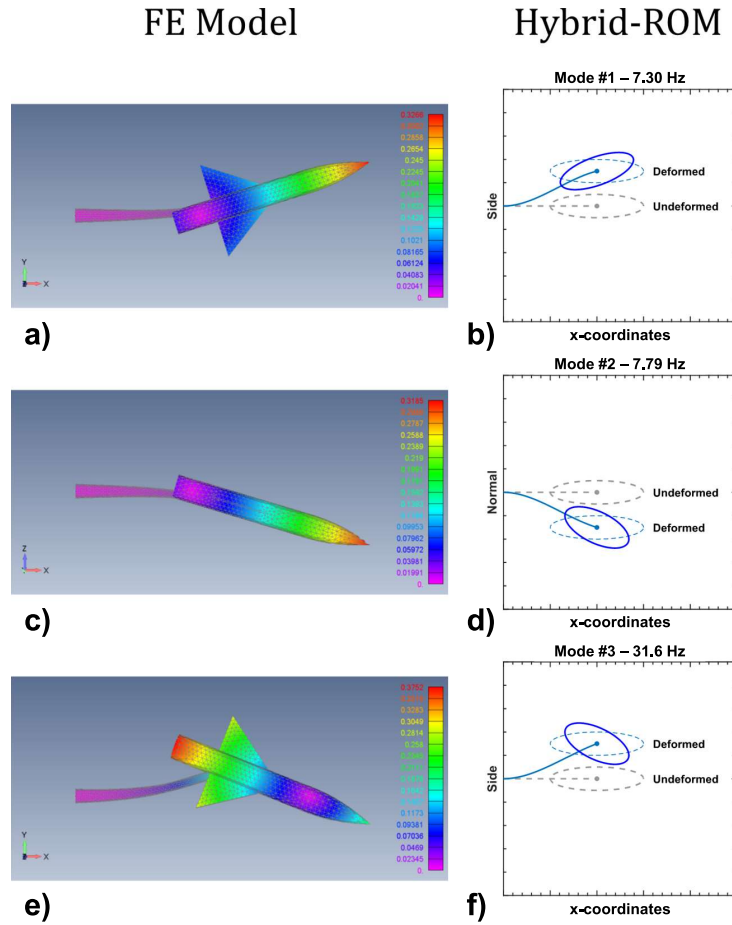


Fig. 11 Comparison of the first three mode shapes computed by the full FE and hybrid-ROM models: a-b) Mode #1; c-d) Mode #2; and e-f) Mode #3.

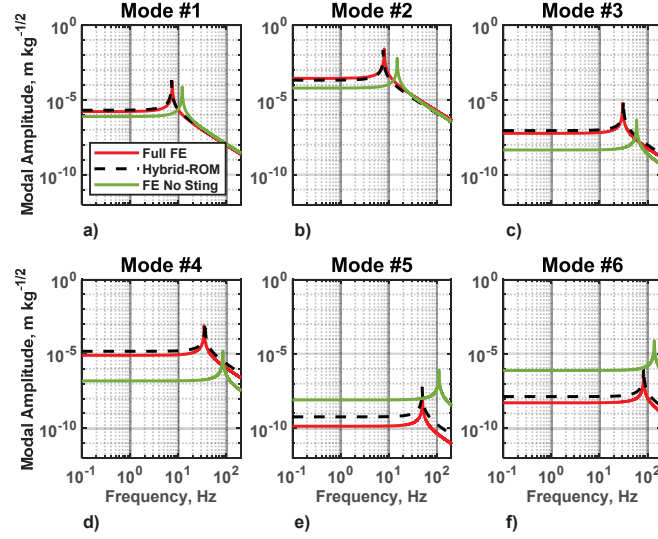


Fig. 12 Modal spectra for the first 6 modes computed in response to a 2000 N impulse for the full wind tunnel measurement system for the full assembly FE model and hybrid-ROM where a-f) correspond to modes #1-6, respectively. The FE and Hybrid-ROM models represent the same components, while the FE no sting is only included for illustration purposes.

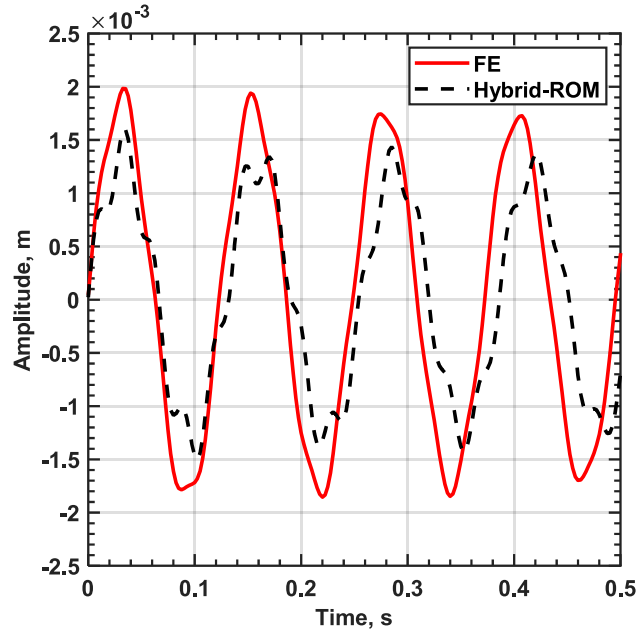


Fig. 13 A time-domain representation of the physical motion of the normal DOF of the test article CG in response to a 2000 N impulse for the full assembly FE model and hybrid-ROM.

The displacement time history in physical coordinates for the normal translation of the test article CG subjected to the 2000 N impulse is shown in Figure 13. Results were computed in modal space for computational speed and then transformed to physical coordinates. A total of 25 modes were used in the computation of the full FE model in

order to adequately capture the dynamics of the assembly. For the hybrid-ROM, only the first 6 modes were used to predict dynamics. In Figure 13, the first two cycles of the low-frequency oscillation for the two models are in phase, but start to diverge out of phase between the second and third cycles. The change in phase between the two models is due to the slight difference in natural frequencies of the second mode of oscillation which is dominated by motion in the normal and pitch DOFs when subjected to an impulse in the normal translation DOF. The hybrid-ROM does predict large amplitude higher-frequency content that is visible in Figure 13 due to the pitch-dominated fourth hybrid-ROM mode. The amplitude of fourth hybrid-ROM mode has nearly twice the amplitude of the fourth FE mode, while the second hybrid-ROM mode has roughly half the amplitude of the second FE mode as is visible in Figure 12.

V. Summary & Conclusions

We present a process to develop a reduced-order model to approximate the lower-order dynamics of a wind tunnel measurement system. A ROM was constructed by treating the force balance as a pure spring element and the test article as a lumped rigid body mass element. A reduced-order stiffness matrix representing the force balance was determined by applying static constraints to a finite-element model and calculating the reaction loads. The test article mass matrix was analytically derived using a Lagrangian-based approach. Moreover, an extension of this ROM was presented in the form of a hybrid-ROM, wherein the ROM of the force balance and test article was combined with a simple numerical model of the sting. Due to its regular geometry, the sting was modeled using beam finite elements, which require significantly fewer DOFs than 3D solid elements to capture the appropriate structural dynamics. Comparisons were drawn between the ROM and an FE model of the partial assembly, as well as that of the hybrid-ROM and a FE model of the full test assembly consisting of a sting, force balance, and test article. The hybrid-ROM appears to provide reasonable predictions for the natural frequencies of the system – matching the higher fidelity FE models to within $\pm 6\%$. Some limitations of the hybrid-ROM become visible when performing dynamic frequency-domain and time-domain analyses, such as inaccurately predicting the amplitude of oscillation. The hybrid-ROM may benefit the aerospace community as it can assist in wind tunnel test design by setting important system dynamic benchmarks that can then be adjusted using a small set of parameters. Notably, the hybrid-ROM is useful when trying to approximate the dynamic sensitivity, usable measurement bandwidth, or response for a wind tunnel measurement system without needing the resources of a new finite-element model for each iteration of the design phase.

Appendix

A. Stiffness Parameters – Unit Displacement Method

$$\bar{\mathbf{K}}_{11} = \begin{bmatrix} 11232894 & 520 & -1155868 & 57 & 89945 & -13 \\ 52011426236 & 40358 & -2657 & -4334 & 860514 & \\ -1155868 & 40358 & 18471080 & -812 & -1387147 & 4681 \\ 57 & -2657 & -812 & 9785 & 66 & -2806 \\ 89945 & -4334 & -1387147 & 66 & 129712 & -512 \\ -13 & 860514 & 4681 & -2806 & -512 & 82374 \end{bmatrix} \quad (\text{A.1})$$

$$\bar{\mathbf{K}}_{22} = \begin{bmatrix} 11232894 & 520 & -1155868 & 57 & -85746 & -92 \\ 520 & 11426236 & 40358 & -2657 & 1801 & -876269 \\ -1155868 & 40358 & 18471080 & -812 & 1420450 & -1453 \\ 57 & -2657 & -812 & 9785 & -58 & -2402 \\ -85746 & 1801 & 1420450 & -58 & 134774 & -75 \\ -92 & -876269 & -1453 & -2402 & -75 & 84769 \end{bmatrix}. \quad (\text{A.2})$$

$$\bar{\mathbf{K}}_{12} = \begin{bmatrix} -11232894 & -520 & 1155868 & -57 & -89945 & 13 \\ -520 & -11426236 & -40358 & 2657 & 4334 & -860514 \\ 1155868 & -40358 & -18471080 & 812 & 1387147 & -4681 \\ -57 & 2657 & 812 & -9785 & -66 & 2806 \\ 85746 & -1801 & -1420450 & 58 & 81134 & -199 \\ 92 & 876269 & 1453 & 2402 & -146 & 48424 \end{bmatrix} \quad (\text{A.3})$$

$$\bar{\mathbf{K}}_{21} = \bar{\mathbf{K}}_{12}^T \quad (\text{A.4})$$

B. Stiffness Parameters – Schur Complement Method

$$\bar{\mathbf{K}}_{11} = \begin{bmatrix} 11240456 & 847 & -1175509 & 59 & 91436 & 3 \\ 847 & 11567612 & 42076 & -2704 & -4449 & 871164 \\ -1175509 & 42076 & 18770708 & -822 & -1409438 & 4818 \\ 59 & -2704 & -822 & 9804 & 67 & -2825 \\ 91436 & -4449 & -1409438 & 67 & 131510 & -523 \\ 3 & 871164 & 4818 & -2825 & -523 & 83250 \end{bmatrix} \quad (\text{B.1})$$

$$\bar{\mathbf{K}}_{22} = \begin{bmatrix} 11240456 & 847 & -1175509 & 59 & -87241 & -125 \\ 847 & 11567612 & 42076 & -2704 & 1946 & -887108 \\ -1175509 & 42076 & 18770708 & -822 & 1443702 & -1577 \\ 59 & -2704 & -822 & 9804 & -58 & -2414 \\ -87241 & 1946 & 1443702 & -58 & 136718 & -86 \\ -125 & -887108 & -1577 & -2414 & -86 & 85674 \end{bmatrix} \quad (\text{B.2})$$

$$\bar{\mathbf{K}}_{12} = \begin{bmatrix} -11240456 & -847 & 1175509 & -59 & -91436 & -3 \\ -847 & -11567612 & -42076 & 2704 & 4449 & -871164 \\ 1175509 & -42076 & -18770708 & 822 & 1409438 & -4818 \\ -59 & 2704 & 822 & -9804 & -67 & 2825 \\ 87241 & -1946 & -1443702 & 58 & 82724 & -210 \\ 125 & 887108 & 1577 & 2414 & -153 & 49167 \end{bmatrix}. \quad (\text{B.3})$$

$$\bar{\mathbf{K}}_{21} = \bar{\mathbf{K}}_{12}^T \quad (\text{B.4})$$

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