Frequency Domain and Control System Introduction, with application to Launch Vehicles
Part 2

April 3, 2014

Rob Hall
MSFC EV41
CRM Solutions, Inc.

Rev. 2

Summary

- **Charts provide an introduction to control system analysis/design approaches.**
- **Starts with fundamental concepts and advances to stability of a launch vehicle with unstable aerodynamics.**
- **Part 1**
  - Stability and Laplace Transforms
  - Eigenvalues
  - Spring-mass-damper system with Laplace Transforms and State Space
  - Closed-Loop vs. Open Loop
  - Root locus
  - Proportional-Integral-Derivative Control
  - Pole Placement
  - Step Response

- **Part 2**
  - Transfer function frequency response
  - Bode plots
  - Frequency response with Nyquist Plots
  - Bode vs. Nichols vs. Nyquist
  - Nyquist plots: full vs. half
  - Nyquist stability criterion
  - Stability margins
  - Stability results with an unstable plant: Launch vehicle example
  - Launch Vehicle Stability results with slosh
  - Launch vehicle frequency domain examples

Recall: First Order System Stability

- **Among uses for a Laplace Transform is to determine a solution to a linear differential equation, and examine its stability properties:**

Example :

$$\dot{x}(t) = ax(t) \quad \Rightarrow \quad \dot{x}(t) - ax(t) = 0$$

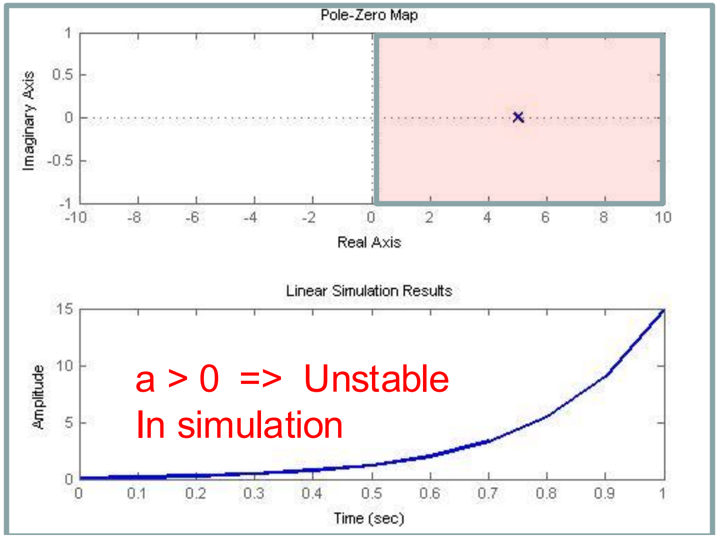
Take Laplace Transform :

$$sX(s) - x(0) - aX(s) = 0$$

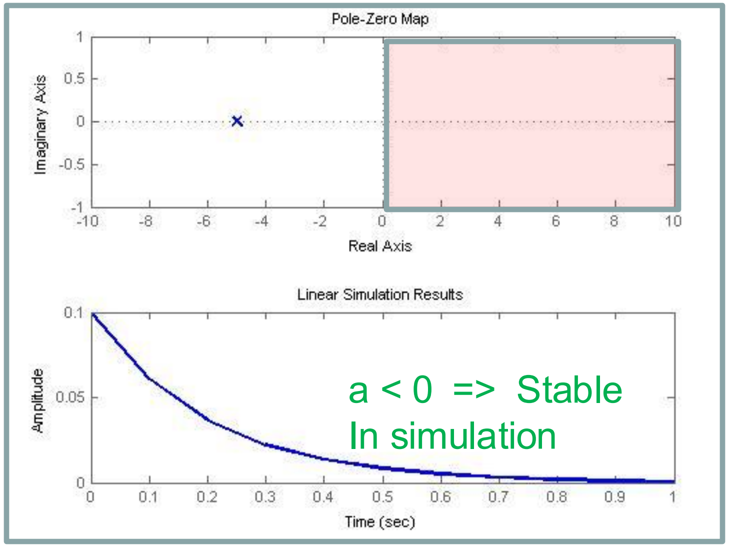
$$X(s) = \frac{x(0)}{s-a}$$

Take Inverse Laplace Transform :

$$x(t) = x(0)e^{at}$$



a > 0 => Unstable
In simulation



a < 0 => Stable
In simulation

- **The denominator of the transfer function set = 0 is the "characteristic equation", and for stability the characteristic eqn must not have positive real roots:**

$$s - a = 0$$

  - a > 0 => Unstable (roots in right half plane)
  - a < 0 => Stable (roots in left half plane)
  - Note: These roots of the characteristic equation are also called "poles".

Same result if A is matrix, and via Lyapunov Function

Given :

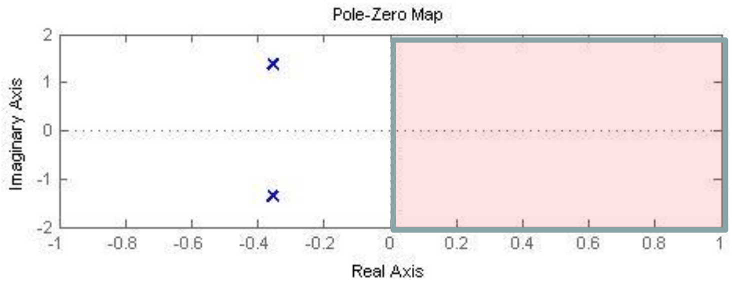
$$\dot{x}(t) = Ax(t)$$

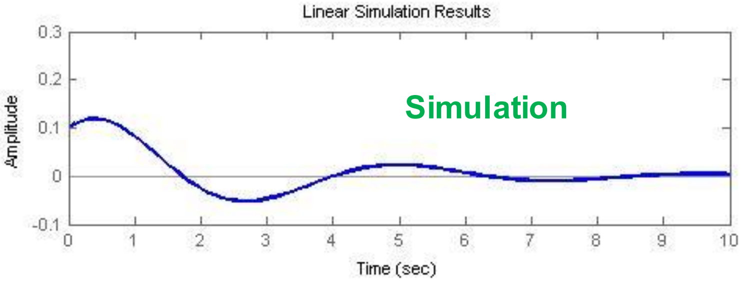
The solution is :

$$x(t) = x(0)e^{At}$$

◆ **For Asymptotical Stability:**

- A < 0
  - $\Rightarrow$ Roots in left hand plane (with Laplace transforms)
  - $\Rightarrow$ Eigenvalues are negative (with state space formulation)





◆ **Or if using Lyapunov Approach:**

Define Lyapunov Function $V(x)$:

$$V(x) = x^T Px$$

Then :

$$\dot{V}(x) = \dot{x}^T Px + x^T P\dot{x}$$

$$= (x^T A^T)Px + x^T P(Ax)$$

$$= x^T A^T Px + x^T PAx$$

$$= x^T (A^T P + PA)x$$

For Stability :

$$V(x) > 0 \qquad \Rightarrow \; P > 0$$

$$\dot{V}(x) < 0 \qquad \Rightarrow \; (A^T P + PA) < 0$$

Lyapunov Inequality
In matlab: lyap

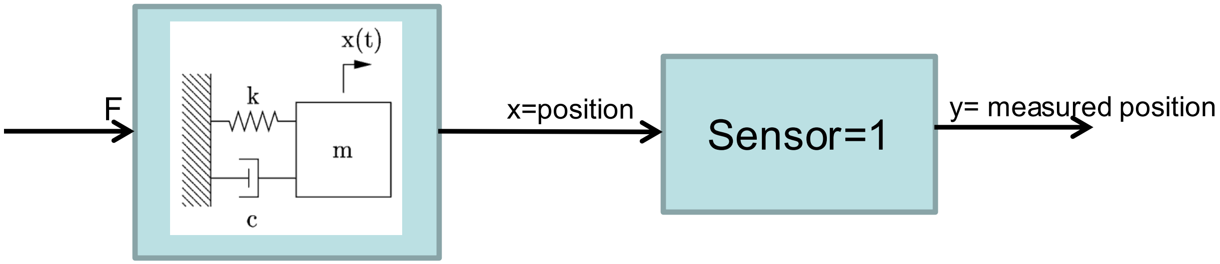
Example, let P = 1 :

$$(A^T(1) + (1)A) < 0$$

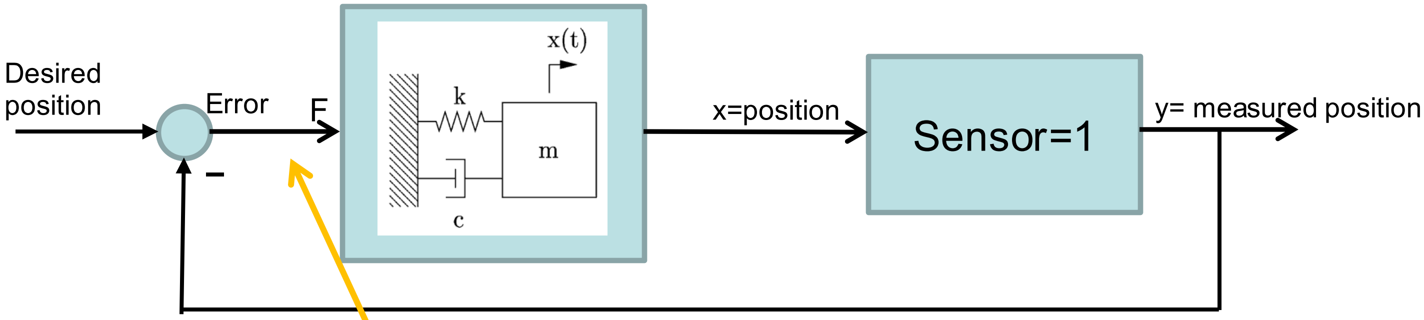
$A < 0$ for asymptotical stability ✓

Recall: Open Loop vs Closed Loop

- A system is “open loop” if the input has no information on how the system responds:

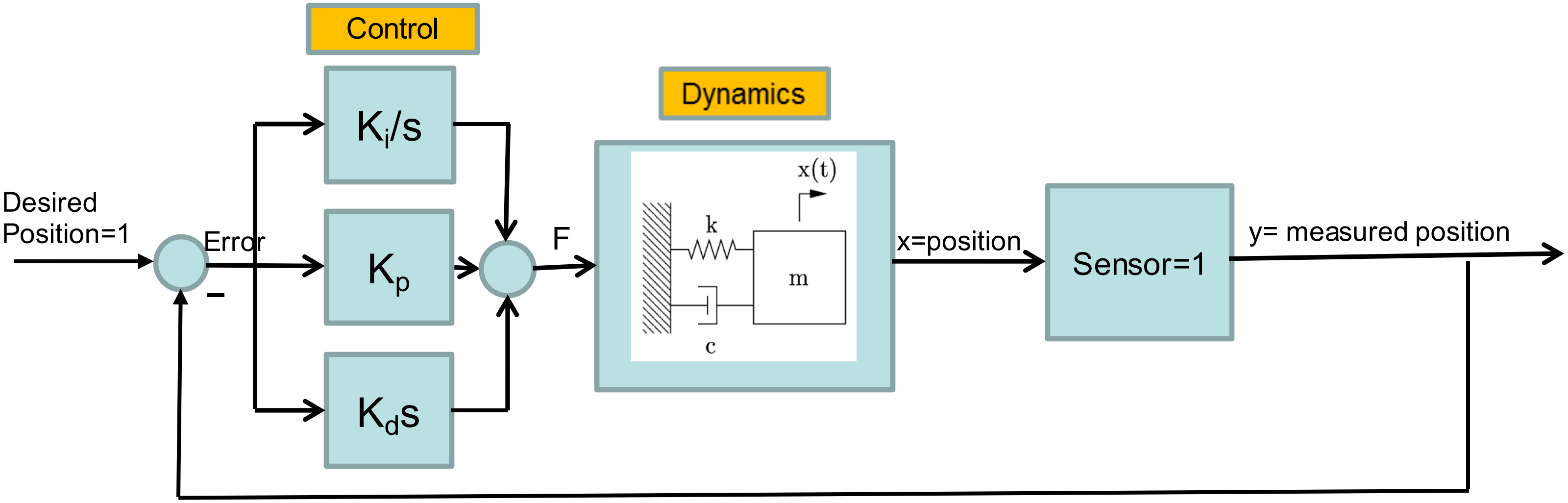


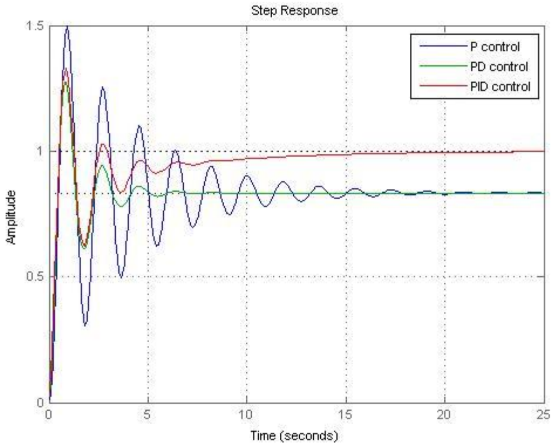
- We can “close the loop” by feeding back the output and use that to calculate the input:

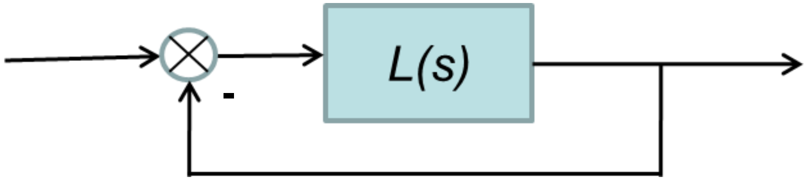


This is “proportional” control, because the force F is now proportional to the input Error: (Force=K_p*Error, K_p=1)

Recall: Proportional –Integral-Derivative (PID) Control







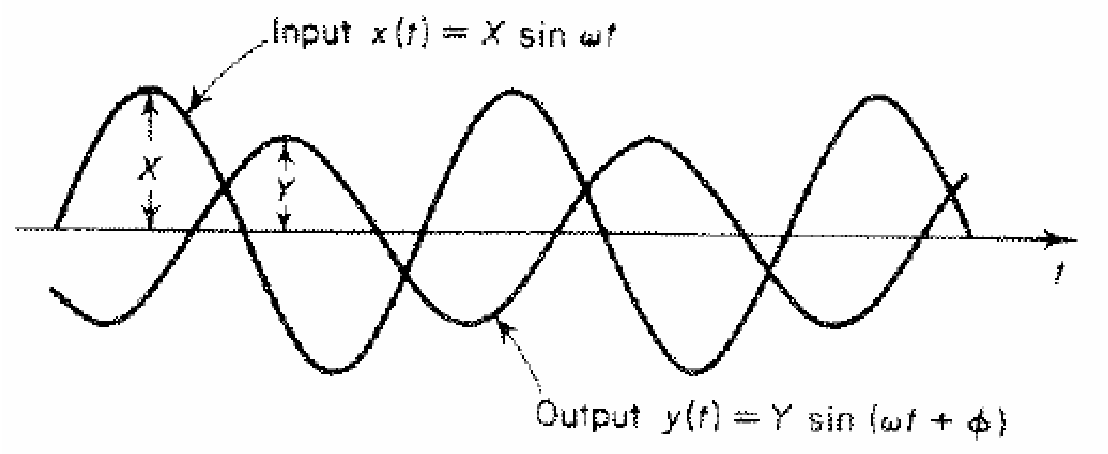
Closed Loop Transfer Function :

$$\frac{L(s)}{1+L(s)}$$

Closed Loop Characteristic Equation: *1+L(s)=0*
For stability, roots of characteristic equation should not be in the right half plane

Frequency Response of Transfer Function

- **For an input sine wave to a stable transfer function, the steady state output is a sine wave of the same frequency, however with a gain and phase shift.**



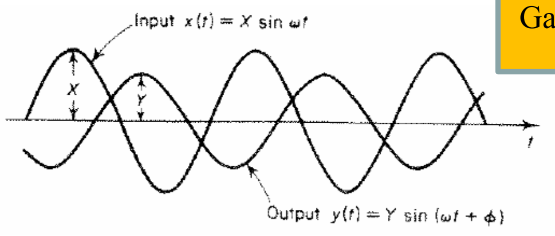
From Ogata, Ref [1]

Bode Plot

- **A Bode plot shows this steady state gain and phase shift as a function of input frequency.**

$x(t) = X\sin(\omega t)$

$y(t) = Y\sin(\omega t + \phi)$



$$\text{Gain} = \frac{Y}{X}$$

- **For example:**

$$G(s) = \frac{1}{s+5}$$

sys=tf(1, [1 5])

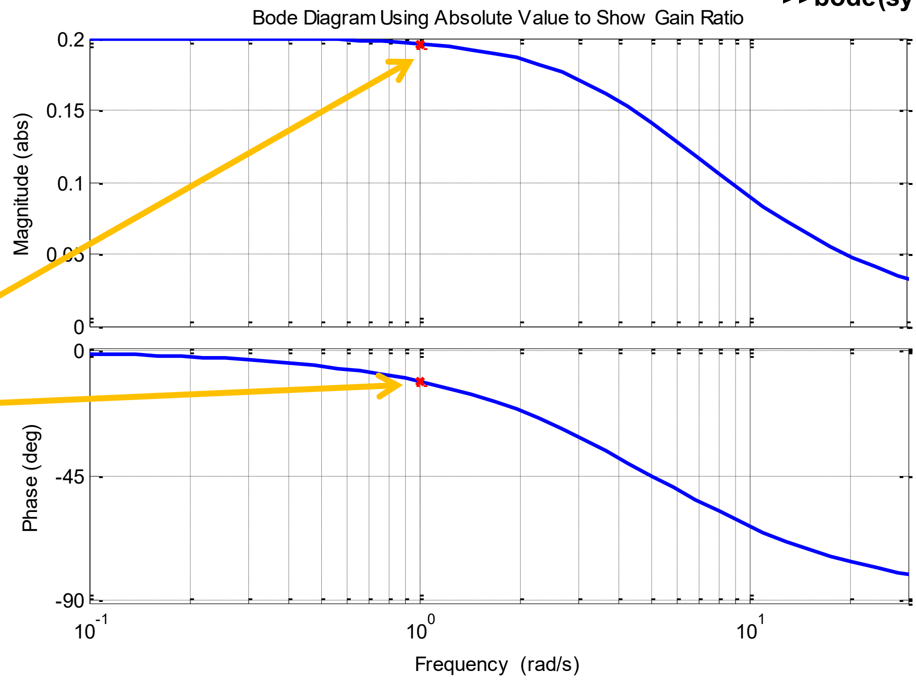
- **What is gain and phase shift at ω = 1 rad/sec?**

[mag, phase] = bode(sys, 1)

mag = 0.1961 phase = -11.3099 deg

Gain/Phase for all frequencies:

>>bode(sys)



Bode Plot, and Time Domain Sim, with Sine Wave Input of 1 rad/sec

- **A Bode plot shows steady state gain and phase shift as a function of input frequency.**
- **For example, given the system below, what is the gain and phase shift for a input sine wave with frequency of 1 rad/sec?:**

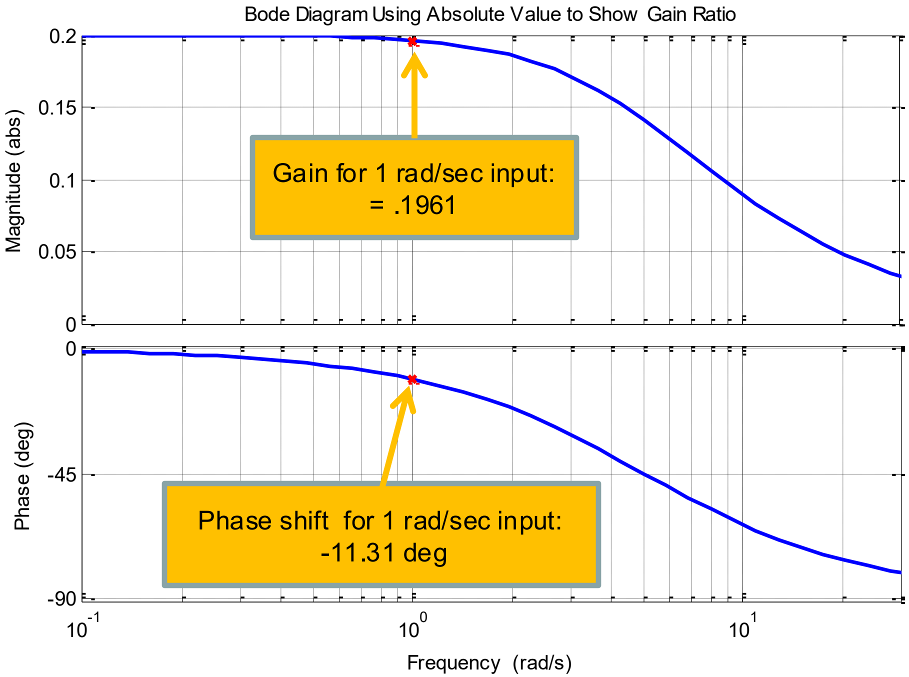
$$G(s) = \frac{1}{s+5}$$

```
sys=tf(1, [1 5])

bode(sys)
```

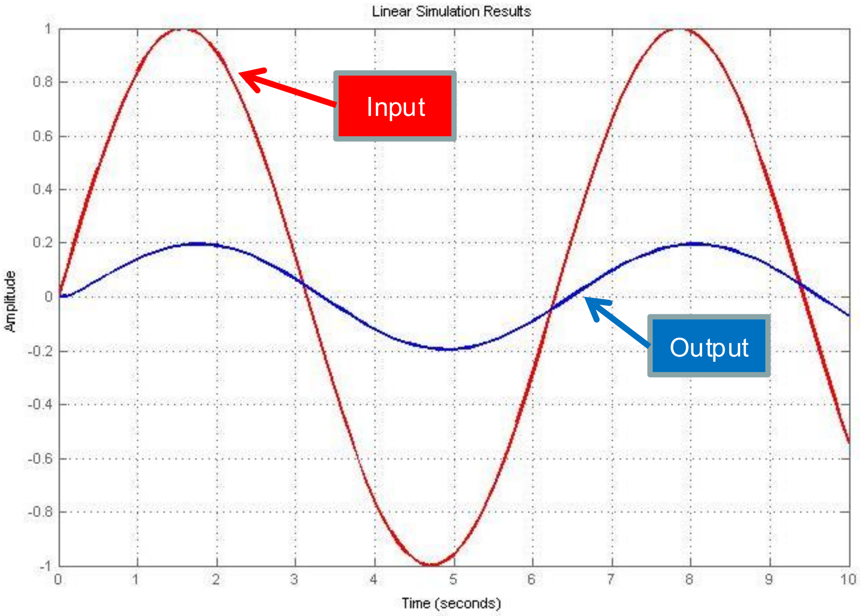
Matlab command ‘lsim’ is quick way to simulate a linear system

```
t=0:0.1:10;
u=sin(1*t);
lsim(sys, u, t)
```



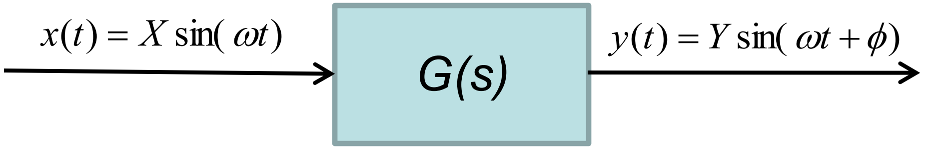
Gain for 1 rad/sec input: = .1961

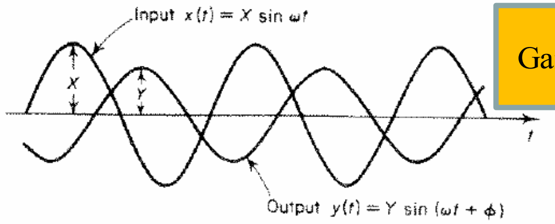
Phase shift for 1 rad/sec input: -11.31 deg



Note: One doesn’t need a Laplace transform to compute a frequency response, instead simply input sine waves and examine output.

Gain Shift is Typically Plotted in Decibels (dB)





$$\text{Gain} = \frac{Y}{X}$$

To get gain in decibels (dB):

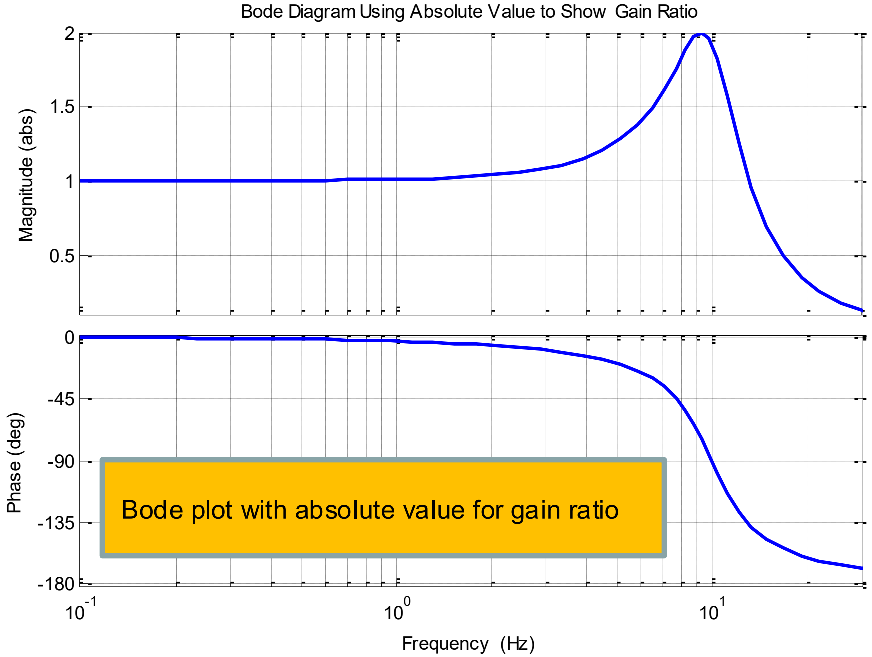
$\text{Gain}_{dB} = 20 * \log_{10}(\text{gain})$

Example, if the output gain doubles (factor of 2), this is ~ 6dB increase:

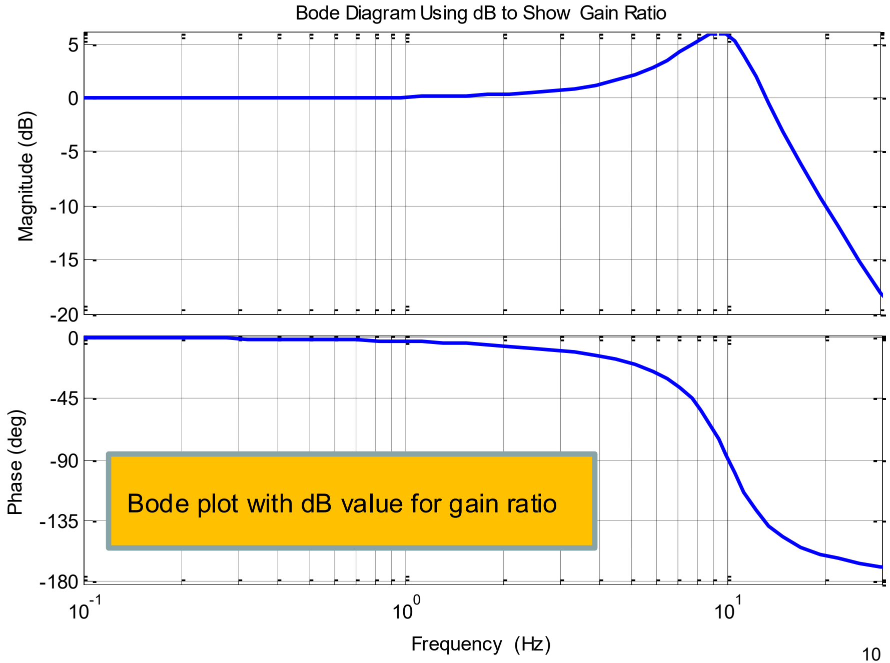
```
>>gain_db=20*log10(2)

gain_db =   6.0206
```

```
num = 10*2*pi;
den = [1 0.74*.7*num num^2];
lpsys = tf(num^2, den)
bode(lpsys)
```



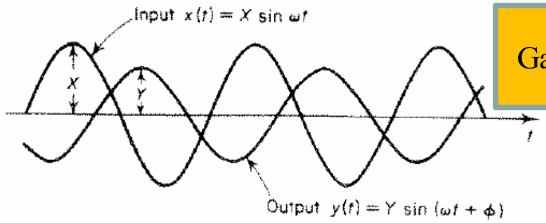
Bode plot with absolute value for gain ratio



Bode plot with dB value for gain ratio

Bode Gain ratio is Typically Shown in Decibels (dB)

$$x(t) = X\sin(\omega t) \rightarrow \boxed{G(s)} \rightarrow y(t) = Y\sin(\omega t + \phi)$$



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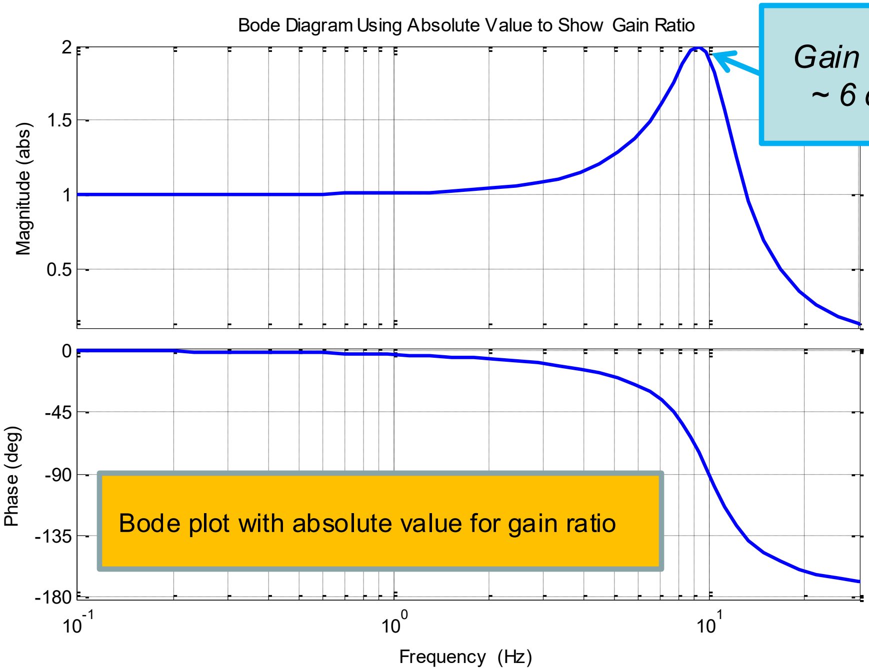
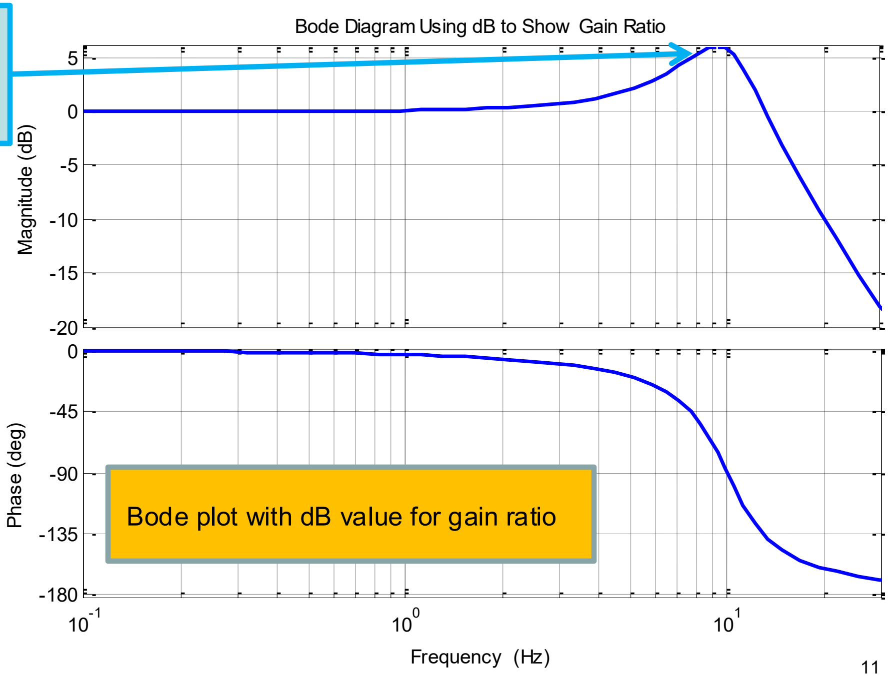


Bode Diagram Using Absolute Value to Show Gain Ratio

Bode plot with absolute value for gain ratio

Gain of 2
~ 6 dB



Bode Diagram Using dB to Show Gain Ratio

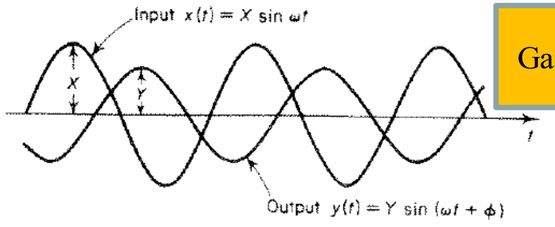
Bode plot with dB value for gain ratio

Bode Gain ratio is Typically Shown in Decibels (dB)

$x(t) = X\sin(\omega t)$

$G(s)$

$y(t) = Y\sin(\omega t + \phi)$



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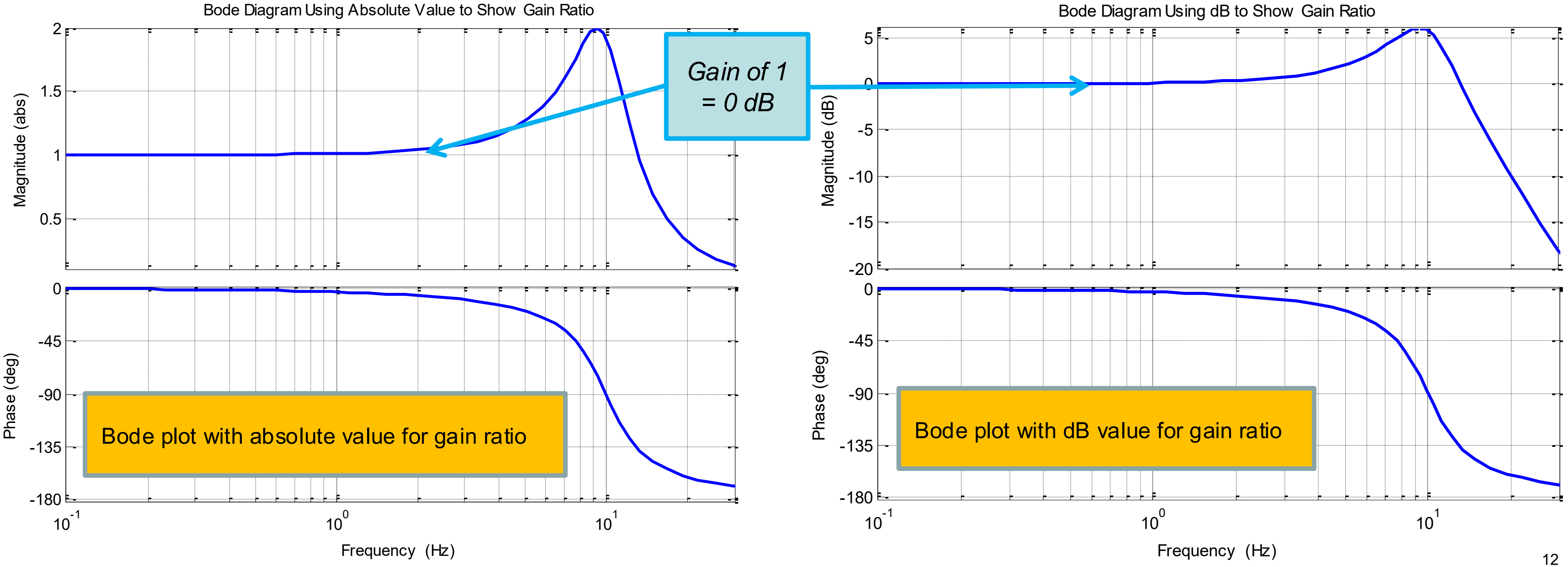


Bode Diagram Using Absolute Value to Show Gain Ratio

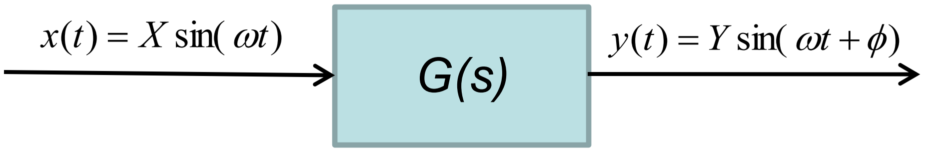
Bode Diagram Using dB to Show Gain Ratio

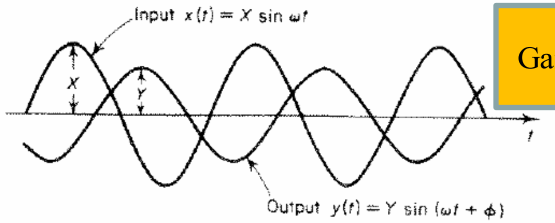
Gain of 1 = 0 dB

Bode plot with absolute value for gain ratio

Bode plot with dB value for gain ratio

Bode Gain ratio is Typically Shown in Decibels (dB)





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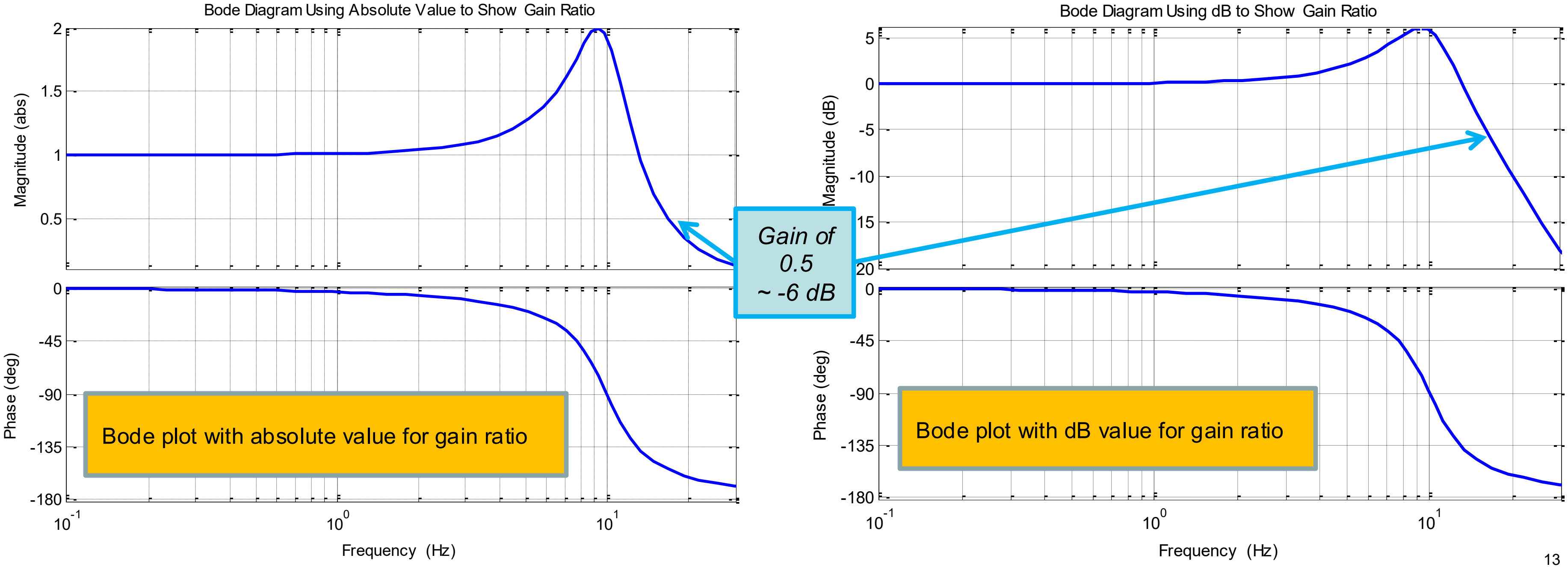
```

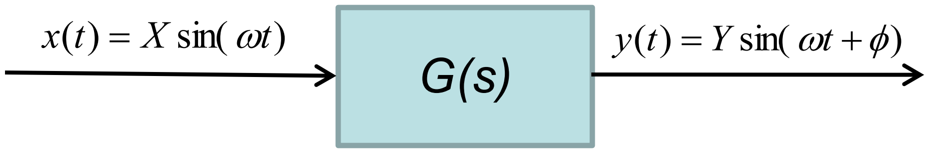


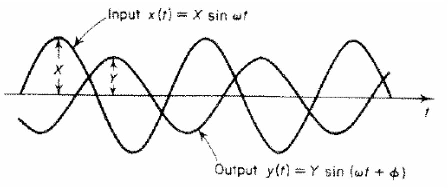
Gain of 0.5 ~ -6 dB

Bode plot with absolute value for gain ratio

Bode plot with dB value for gain ratio

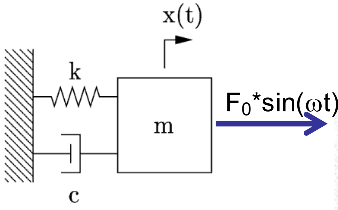
Frequency Response: Spring Mass Damped System



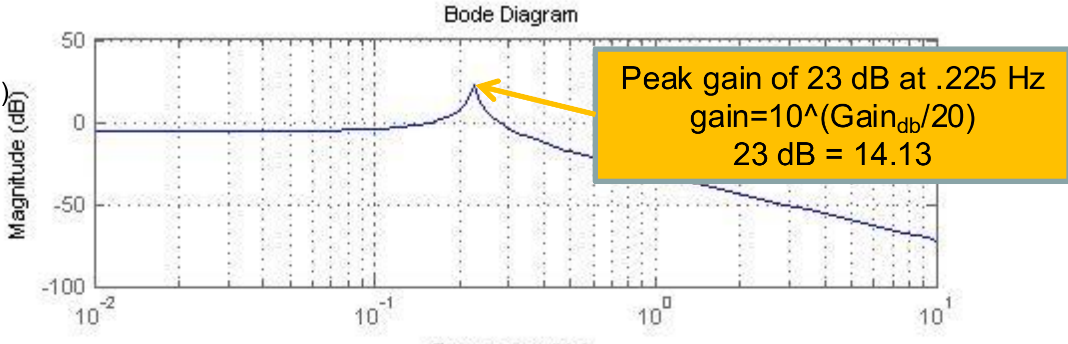


- **Another Example: Lightly Damped Spring-Mass System with resonance input:**

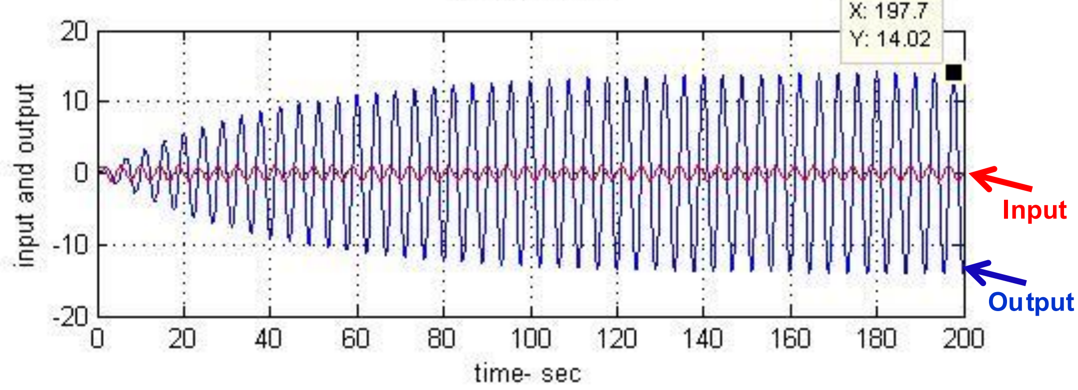
m=1, k=2, c=0.05, F_0=1



$$\omega = \sqrt{\frac{k}{m}} = 1.41 \text{ rad/sec} = 0.225 \text{ Hz}$$



Peak gain of 23 dB at .225 Hz
gain=10^(Gain$_{db}$/20)
23 dB = 14.13



Analytical Steady State Gain to Resonance (See Kreyszig):

$$X_{\max} = \frac{2mF_0}{c\sqrt{4m^2\omega^2 - c^2}} = 14.14$$

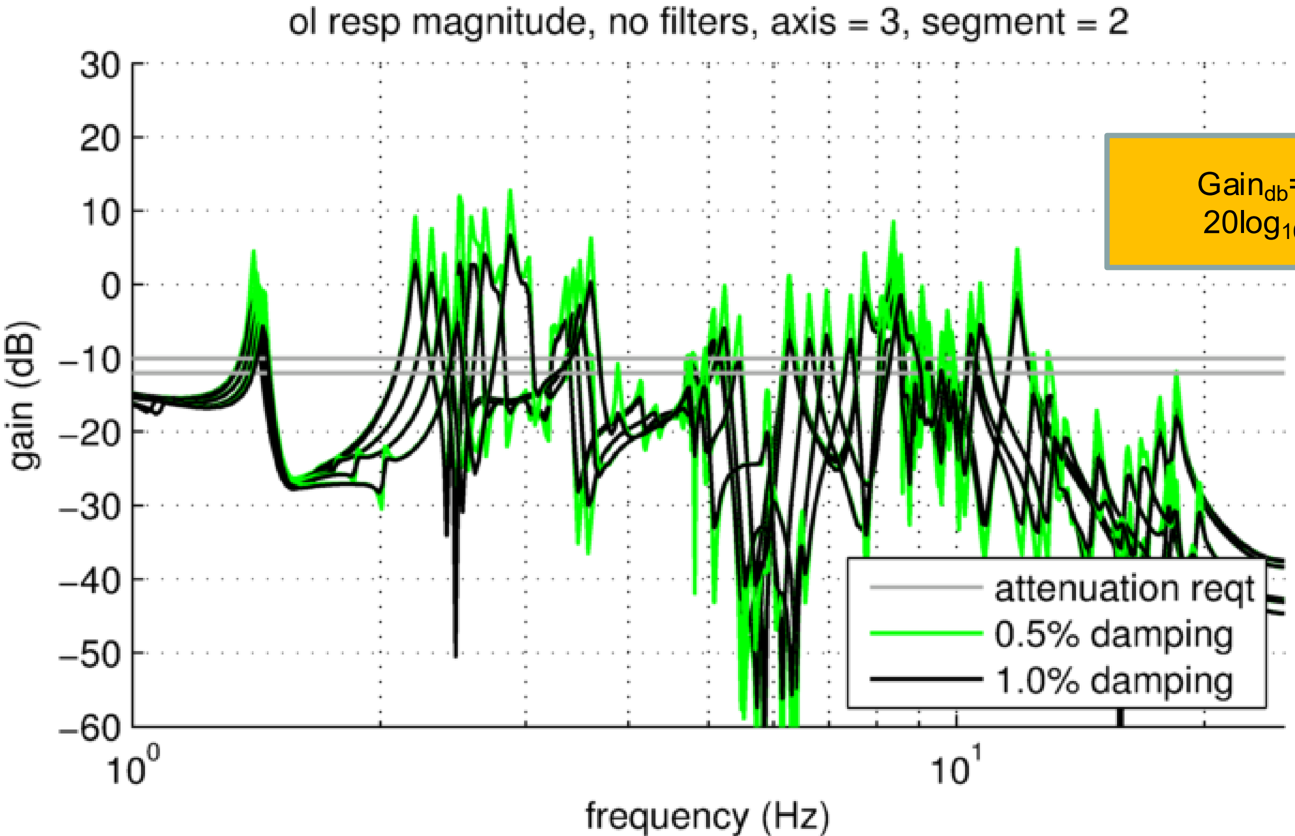
Important Result: For lightly damped flex, frequency response magnitude (steady state gain) is roughly proportional to ~ 1/c

From Ogata, Ref [1]

Example of flex gain ~ inversely proportional to damping

$$X_{\max} = \frac{2mF_0}{c\sqrt{4m^2\omega^2 - c^2}} \approx \frac{2mF_0}{c\sqrt{4m^2\omega^2}}$$

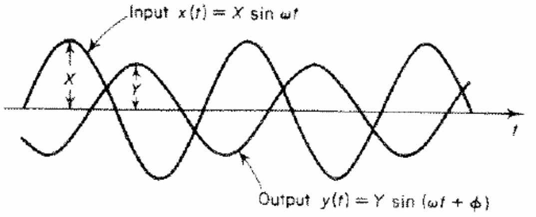
- Ascent Vehicle Yaw axis open loop response, no filters, core-only sensor blending (From John Wall/DCI)
- Doubling structural damping ~ halves flex gain



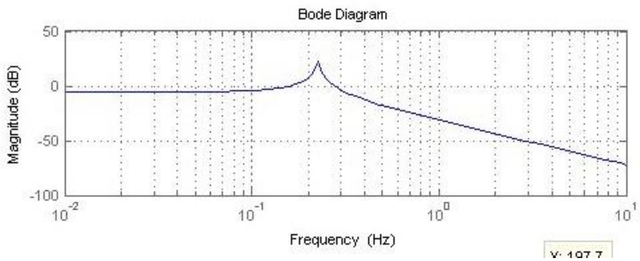
$\text{Gain}_{db} = 20\log_{10}(\text{gain})$
$20\log_{10}(0.5) = -6$ dB

Summary thus far

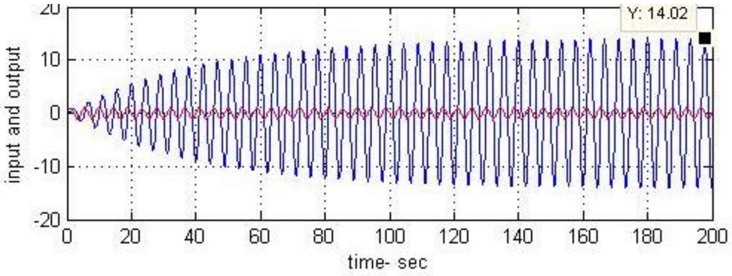
- For an input sine wave to a stable transfer function, the steady state output is a sine wave of the same frequency, however with a gain and phase shift.



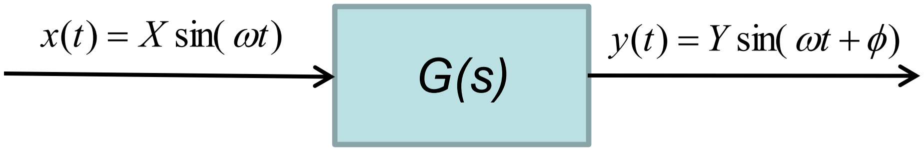
- A Bode plot shows the gain and phase shift as a function of frequency.

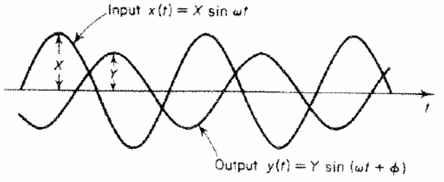


- For second order systems, the steady state amplitude is inversely proportional to the damping.



Frequency Response and Polar Plots





- **The output magnitude and phase change can be represented by a complex number (frequency response).**
- **Let *s=j*ω in transfer function, then solve. For example:**

Gain/Phase for ω = 1 rad/sec:

$$G(s) = \frac{1}{s+5}$$

$$\frac{1}{(j\omega)+5}$$

Evaluate gain/phase at $\omega = 1$ rad/sec:

$$\frac{1}{j+5} = \frac{1}{j+5} \times \frac{j-5}{j-5} = \frac{j-5}{-26} = .1923 - .0385j$$

Gain and Phase Shift easily calculated from complex number

$$|G(j\omega)| = \frac{Y}{X} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = 0.1961$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = -0.1976 \text{ rad} = -11.31 \text{ deg}$$

Gain/Phase in Matlab (as complex number):

```
sys=tf(1, [1 5])
 [re,im] = nyquist(sys, 1)

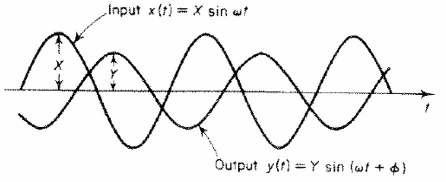
re =   0.1923   im =   -0.0385
```

Gain/Phase in Matlab:

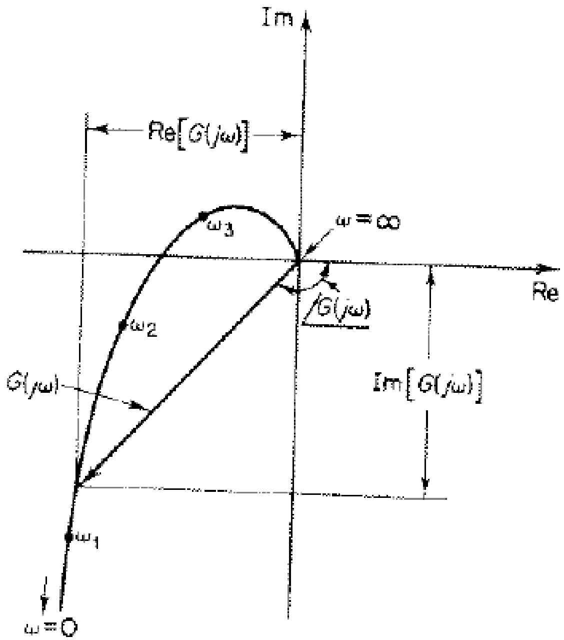
```
[mag, phase] = bode(sys, 1)

mag =  0.1961  phase = -11.3099
```

Plotting Frequency Response



- **Let *s=jω* in transfer function, and the resulting complex number is the "frequency response"**
- **A nyquist plot is a plot this real vs imaginary complex number values for all frequencies.**
  - Results in a polar plot of gain vs. phase



$$|G(j\omega)| = \frac{Y}{X} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}$$

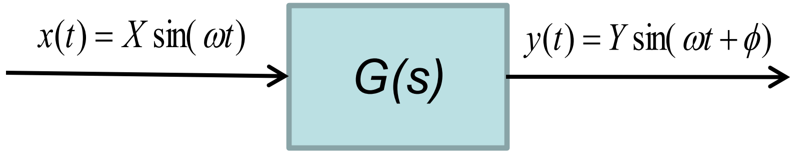
$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

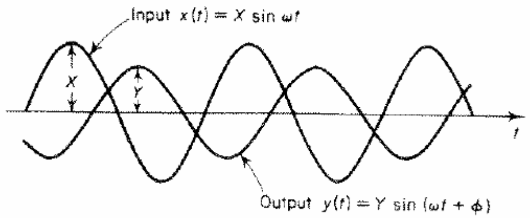
Plot of frequency response (real vs. imaginary values) with ω varied from 0 to ∞

Nyquist Plot

From Ogata, Ref [1]

Nyquist and Nichols Plots





$$G(s) = \frac{1}{s+5}$$

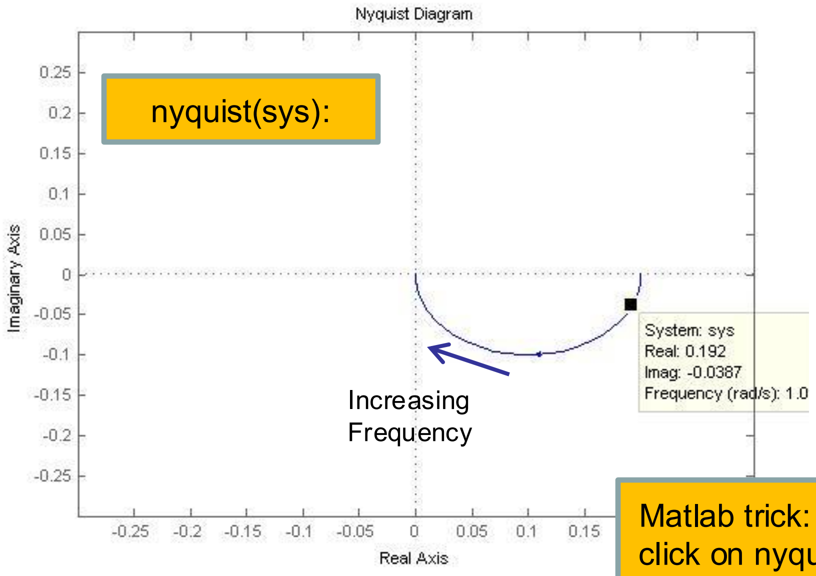
sys=tf(1, [1 5])

$$G(j\omega) = \frac{1}{j\omega + 5}$$

$$\left|G(j\omega)\right| = \frac{Y}{X} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = 0.1961 = -14.1\,\mathrm{dB}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = -0.1976\ \mathrm{rad} = -11.31\ \mathrm{deg}$$

Nyquist: Plots frequency response as complex number real vs. imaginary

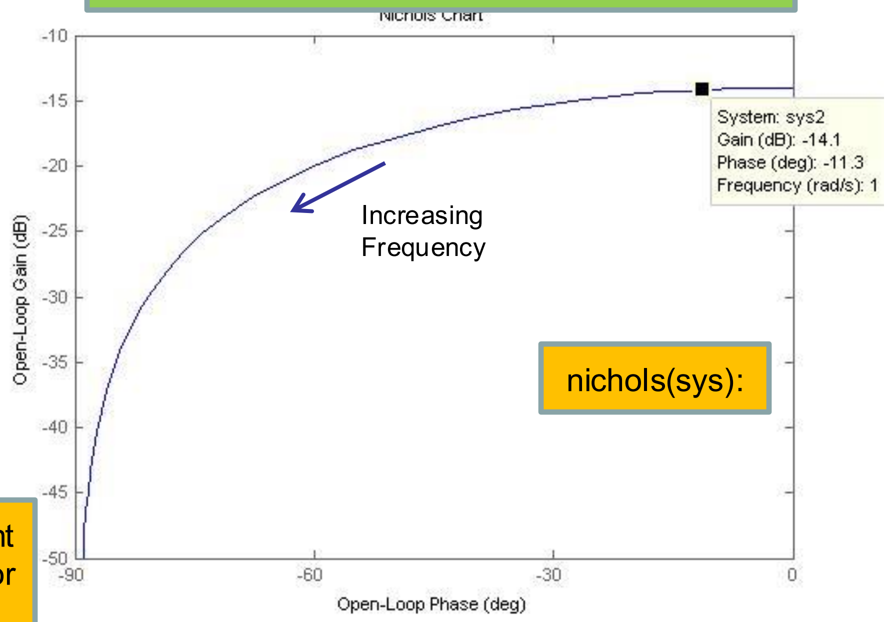


nyquist(sys):

System: sys
Real: 0.192
Imag: -0.0387
Frequency (rad/s): 1.0

Increasing Frequency

Nichols: Directly plots phase vs. gain



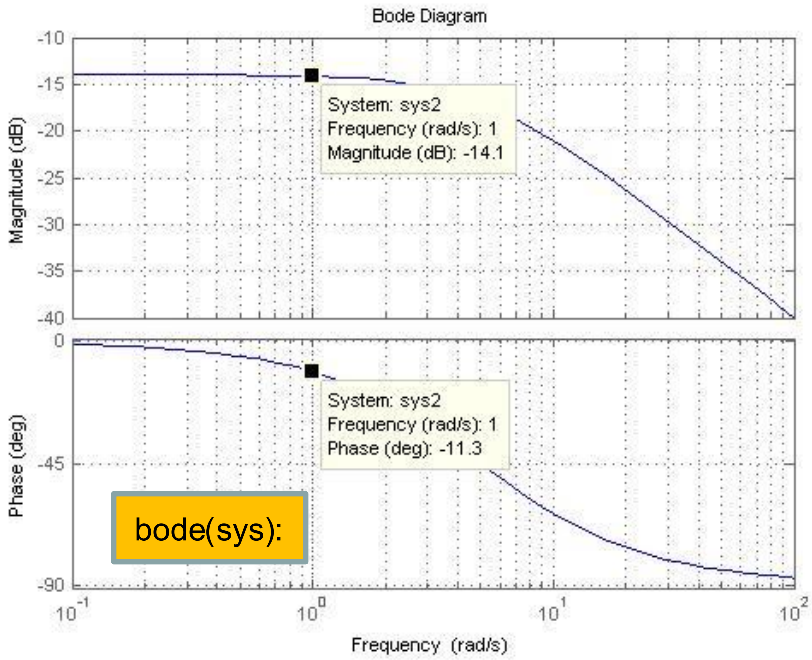
System: sys2
Gain (dB): -14.1
Phase (deg): -11.3
Frequency (rad/s): 1

Increasing Frequency

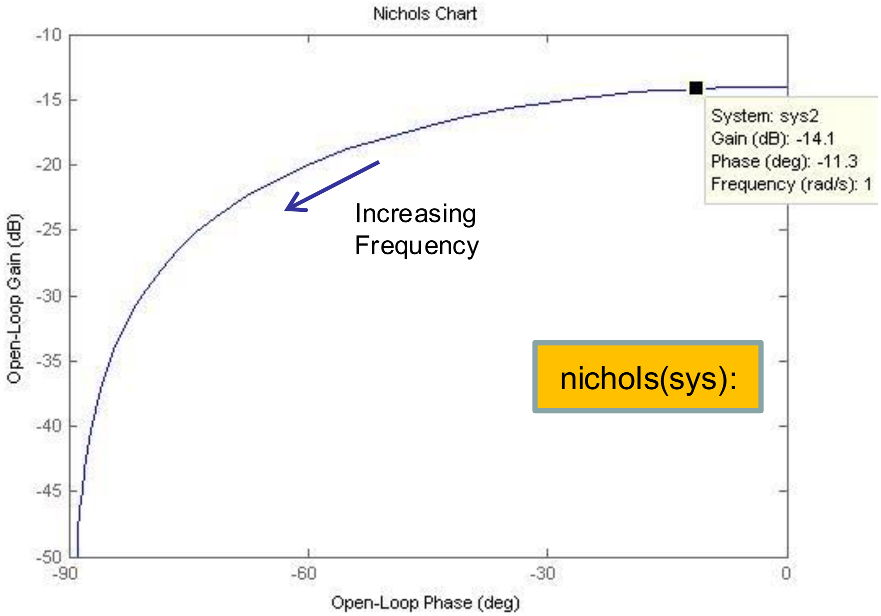
nichols(sys):

Matlab trick: right click on nyquist or nichols curve to show frequency

Summary: Bode, Nichols, and Nyquist Plots



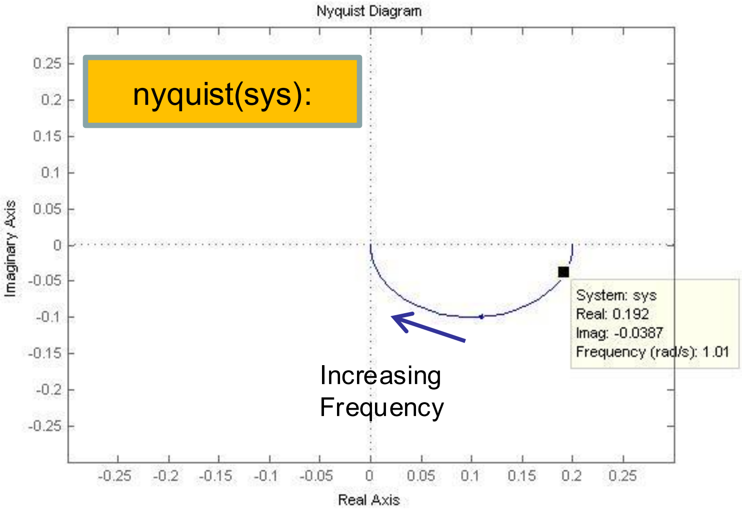
bode(sys):



Increasing Frequency

nichols(sys):

All three plot same data: The gain and phase shift as a function on input frequency



nyquist(sys):

Increasing Frequency

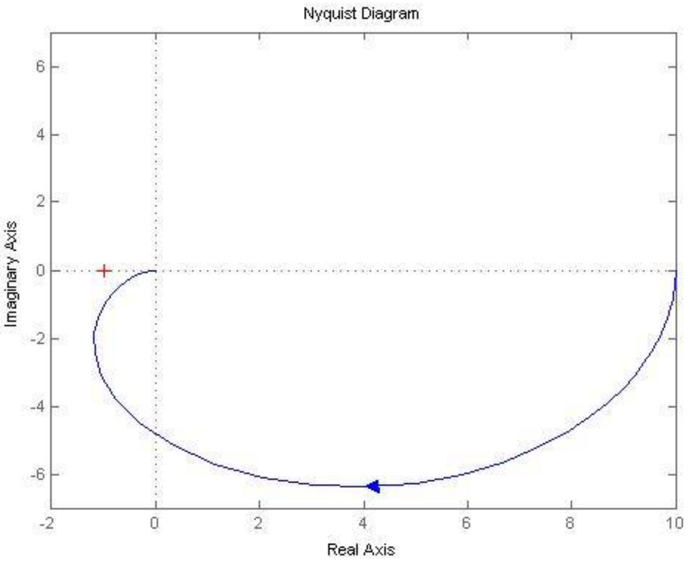
$$G(s) = \frac{1}{s+5}$$

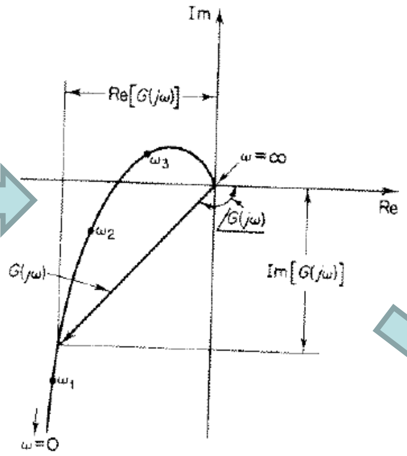
sys=tf(1, [1 5])

$$G(s) = \frac{1}{s+5} \qquad G(j\omega) = \frac{1}{j\omega+5}$$

Nyquist Plots with Gain and Phase

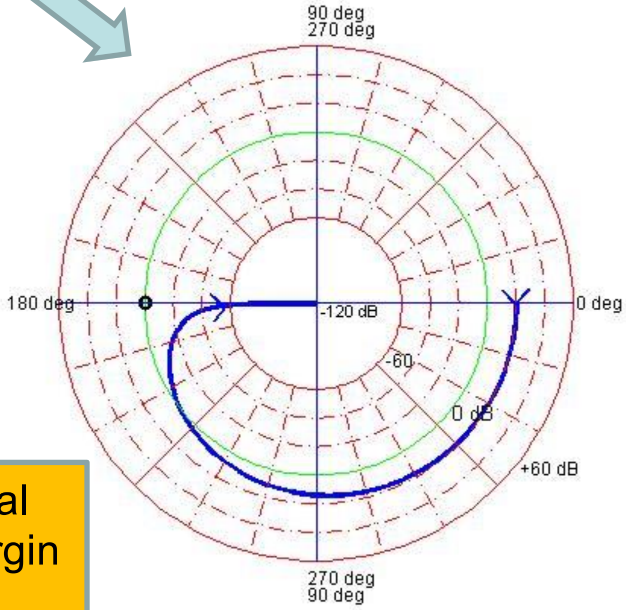
- Some Nyquist plots show gain and phase for the frequency response rather than a complex number.
- Result is a polar plot of gain and phase.





$$|G| = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}$$

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

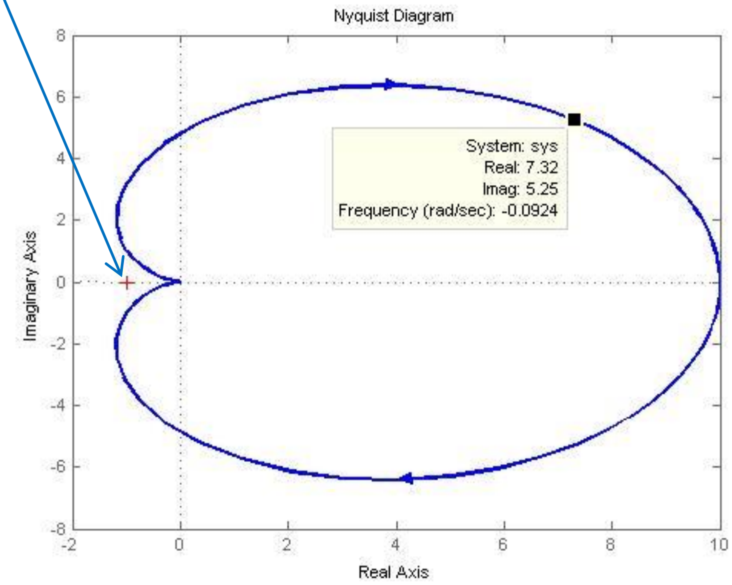


Nyquist Polar plot allows a visual inspection of gain and phase margin (more later).

Full Nyquist Plot vs. Half Nyquist Plot

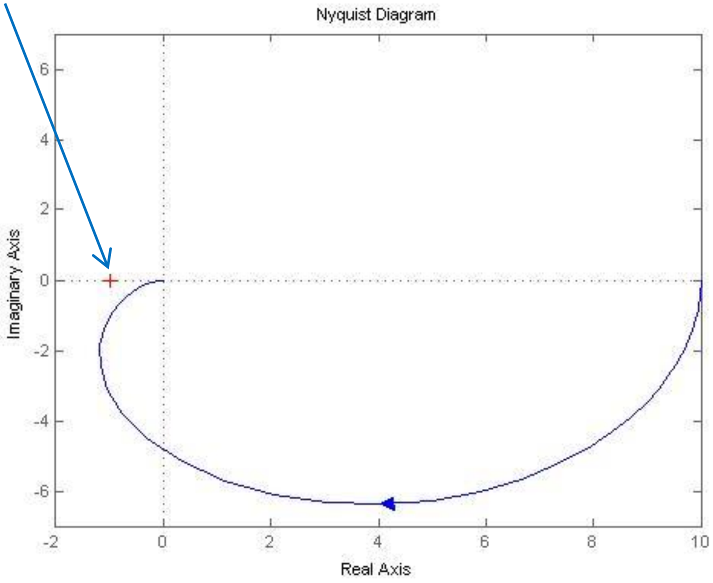
- **The frequency response drawn from $\omega = -\infty,\infty$ is a full Nyquist plot, while one drawn from $\omega = 0,\infty$ is a "half" Nyquist (or just Nyquist) plot.**

Critical point for Stability



Full Nyquist, drawn from $\omega = -\infty,\infty$

Critical point for Stability

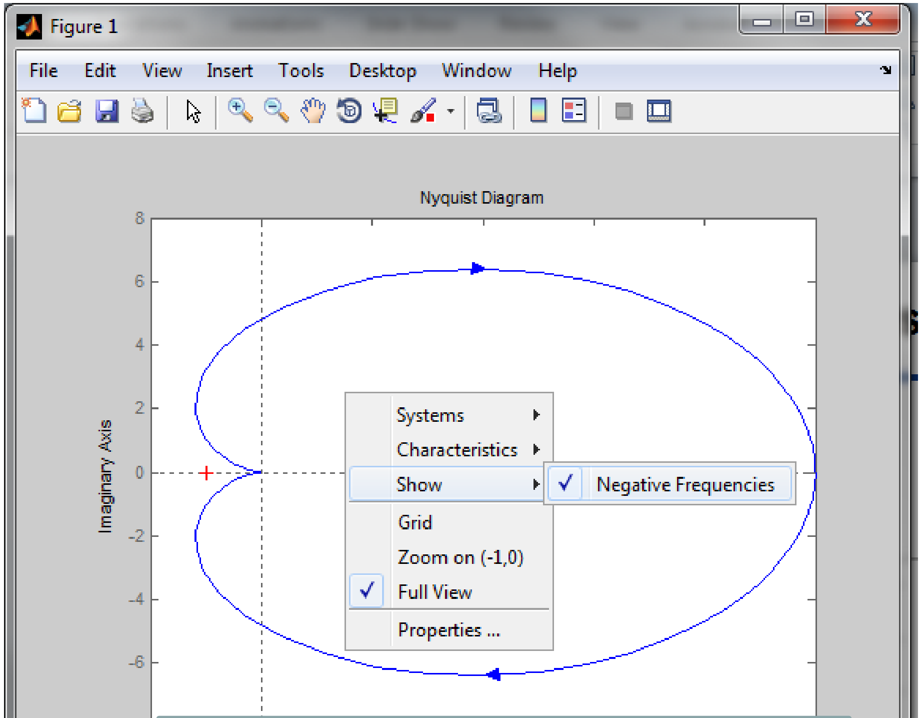


Nyquist, drawn from $\omega = 0,\infty$

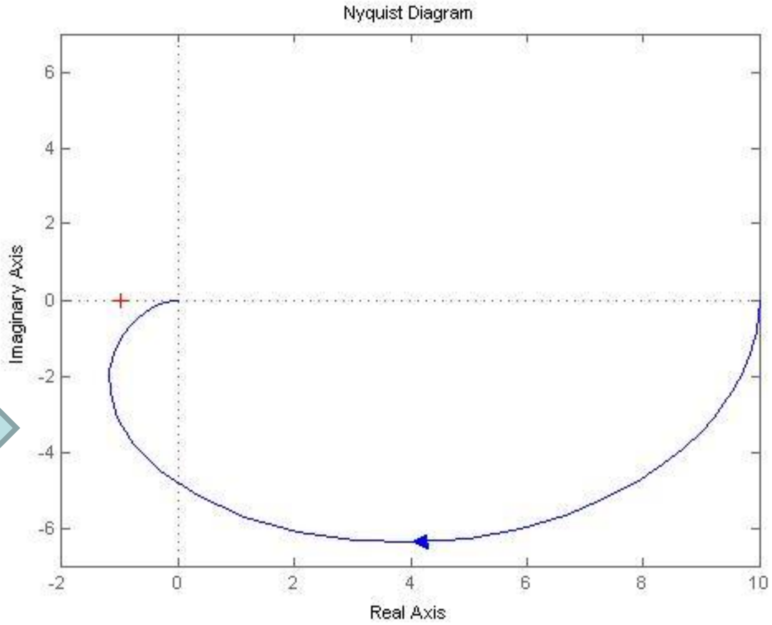
- **Note, in this example, the full Nyquist does not encircle the critical point.**
  - The 'critical point' for stability is discussed later.
- **This encirclement (or lack of) is apparent on a full Nyquist, but not as clear in a half Nyquist (or Bode or Nichols).**

Matlab Tip: Full Nyquist vs. Half Nyquist

- After Matlab draws a Full Nyquist plot, one can simply right click the plot border and deselect “Negative Frequencies” to go from a Full Nyquist to a Nyquist.



Full Nyquist, drawn from $\omega = -\infty,\infty$



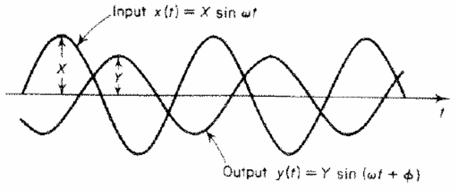
Nyquist, drawn from $\omega = 0,\infty$

Summary: Key Points

- **For an input sine wave to a stable transfer function, the steady state output is a sine wave of the same frequency, however with a gain and phase shift.**

$x(t) = X\sin(\omega t)$

$y(t) = Y\sin(\omega t + \phi)$



- **We can represent this gain and phase shift as a complex number by setting *s=jω*:**

$$G(s) = \frac{1}{s+5}$$

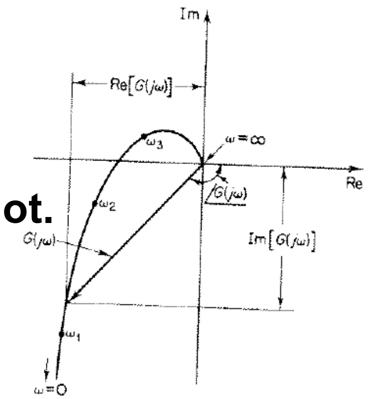
$$\frac{1}{(j\omega)+5}$$

Evaluate gain/phase at $\omega = 1$ rad/sec :

$$\frac{1}{j+5} = \frac{1}{j+5} \times \frac{j-5}{j-5} = \frac{j-5}{-26} = .1923 - .0385j$$

$$|G(j\omega)| = \frac{Y}{X} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = 0.1961$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = -0.1976 \text{ rad} = -11.31 \text{ deg}$$

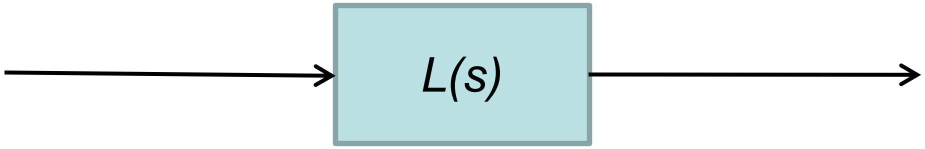


- **Plotting the complex number as a function of frequency is a Nyquist Plot.**
  - Plotting from $\omega = -\infty, \infty$ is a “full” Nyquist plot
  - Plotting from $\omega = 0, \infty$ is a Nyquist plot (or “half” Nyquist).

- **For a closed-loop system to be stable, the closed-loop characteristic equation, *1+L(s)=0*, must have no roots with positive real values (Z=0).**

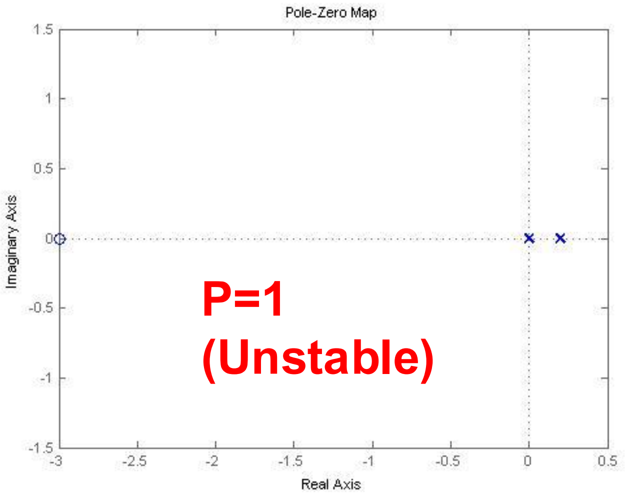
Recall Open Loop vs. Closed Loop Poles

Open Loop, can be unstable



Open Loop Transfer Function :

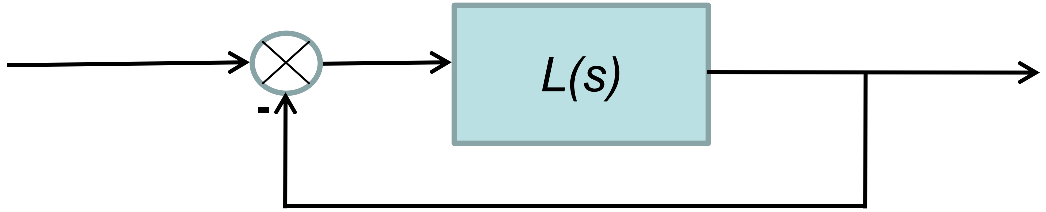
$$L(s) = \frac{2(s+3)}{s(5s-1)}$$



P=1
(Unstable)

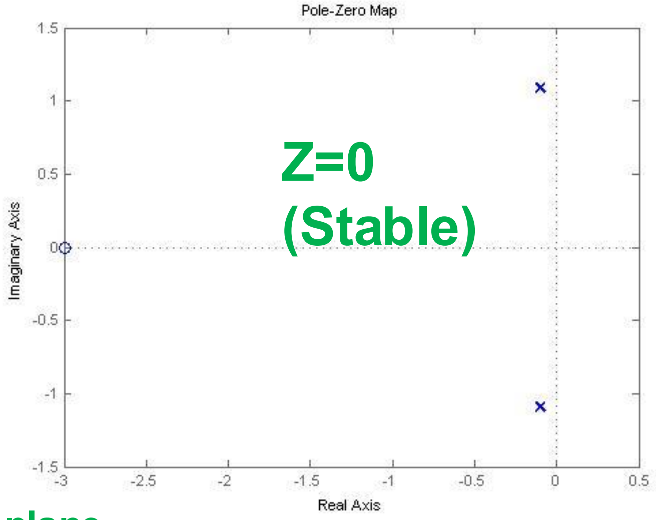
P=Number of poles in open loop transfer function in right hand plane

Closed Loop



Closed Loop Transfer Function :

$$\frac{L(s)}{1+L(s)} = \frac{\frac{2(s+3)}{s(5s-1)}}{1+\frac{2(s+3)}{s(5s-1)}} = \frac{2(s+3)}{s(5s-1)+2(s+3)} = \frac{2(s+3)}{5s^2+s+6}$$



Z=0
(Stable)

Z=Number of poles in closed loop transfer function in right hand plane

Nyquist Stability Criterion

- **Nyquist derived an approach to graphically determine the stability of a closed loop system from the open loop system:**

$$Z = N + P$$

where Z is the number of roots of $1 + L(s)$ (closed-loop characteristic equation) in right half plane ($= 0$ for stability)

P is the number of poles of the open loop transfer function $L(s)$ in the right hand plane

N is the number of clockwise encirclements that the open loop system makes around the $-1 + j0$ ("critical") point

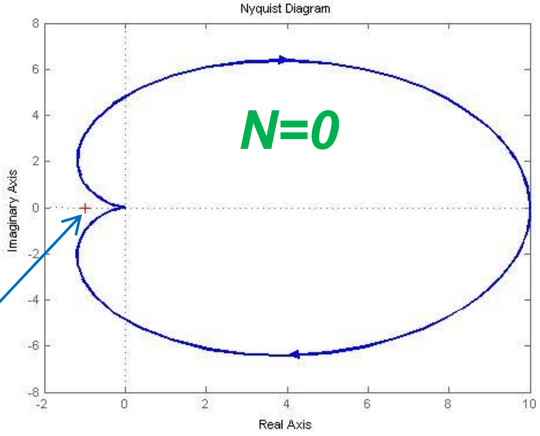
- **Notes:**
  - Z=0 for a stable closed-loop system (no poles in right hand plane).
  - N=1 for clockwise encirclement, N=-1 for counterclockwise encirclement.
  - If P=0 the open loop system (plant) is stable, and then N must = 0 as well.

- **Stable Plant Example:**

Open Loop Transfer Function :

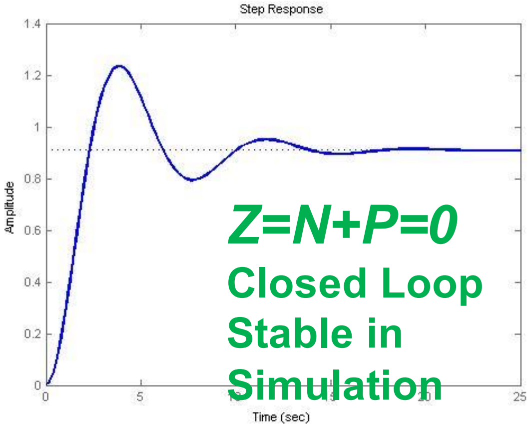
$$L(s) = \frac{10}{(3s+1)(5s+1)}$$

P=0



N=0

Critical point for Stability



Z=N+P=0

Closed Loop Stable in Simulation

Nyquist Stability Criterion

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$$Z = N + P$$

where Z is the number of roots of $1 + L(s)$ (closed-loop characteristic equation) in right half plane ($= 0$ for stability)

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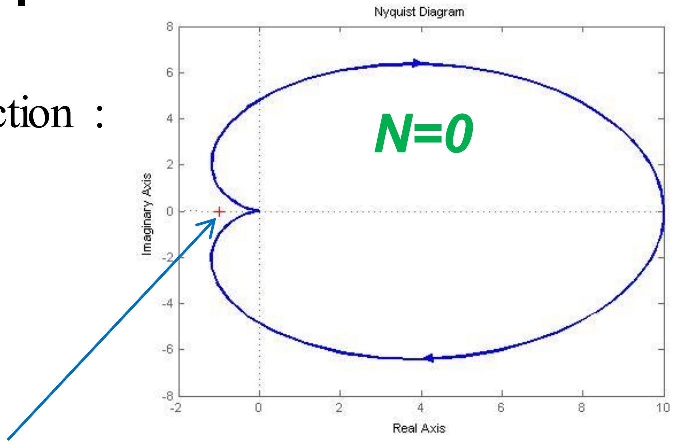


N=0

Critical point for Stability

Key Concept:

How close the plotted Nyquist Path come to the "critical" point is called "**margin**"

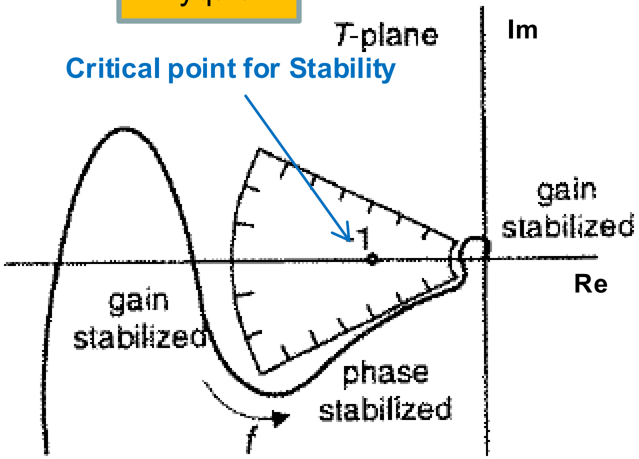
Hitting the critical point means you are at the point of unallowable "encirclements", hence failing the Nyquist Stability Criterion.

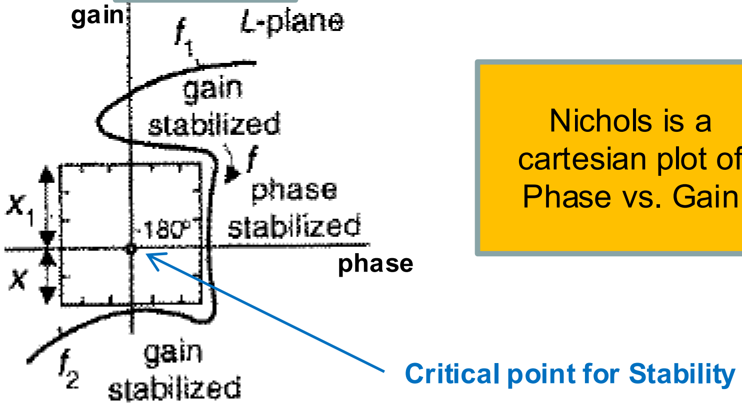
Gain and Phase Margins, Nyquist and Nichols

Nyquist

Nichols

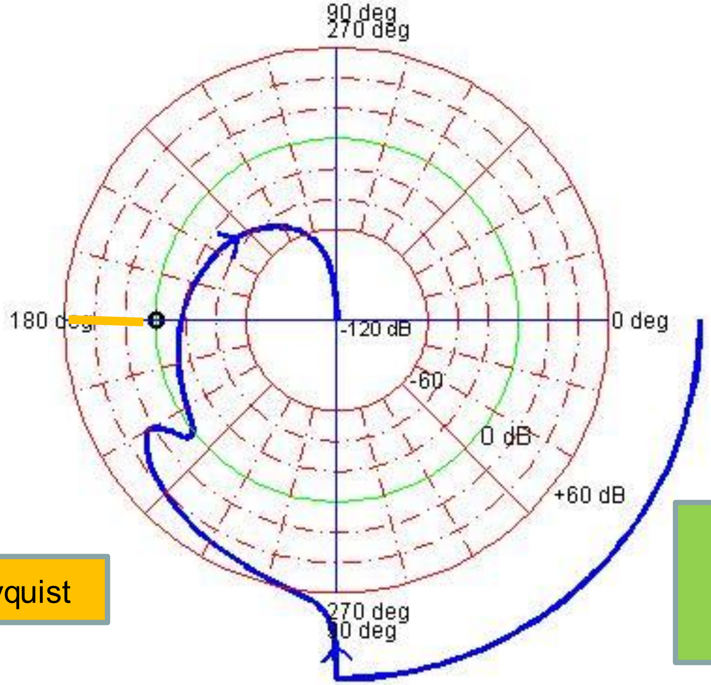
Nyquist is a polar plot of Phase vs Gain





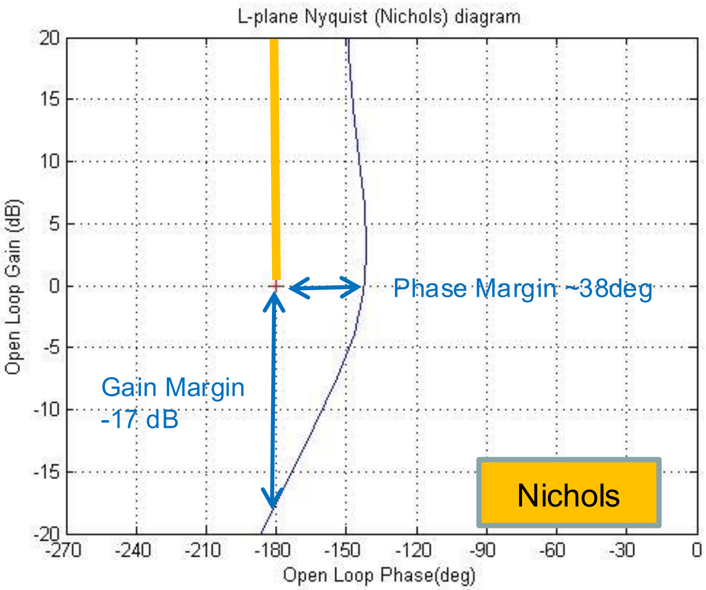
Nichols is a cartesian plot of Phase vs. Gain

From Lurie, Ref [2]



Nyquist and Nichols for Stable Plant

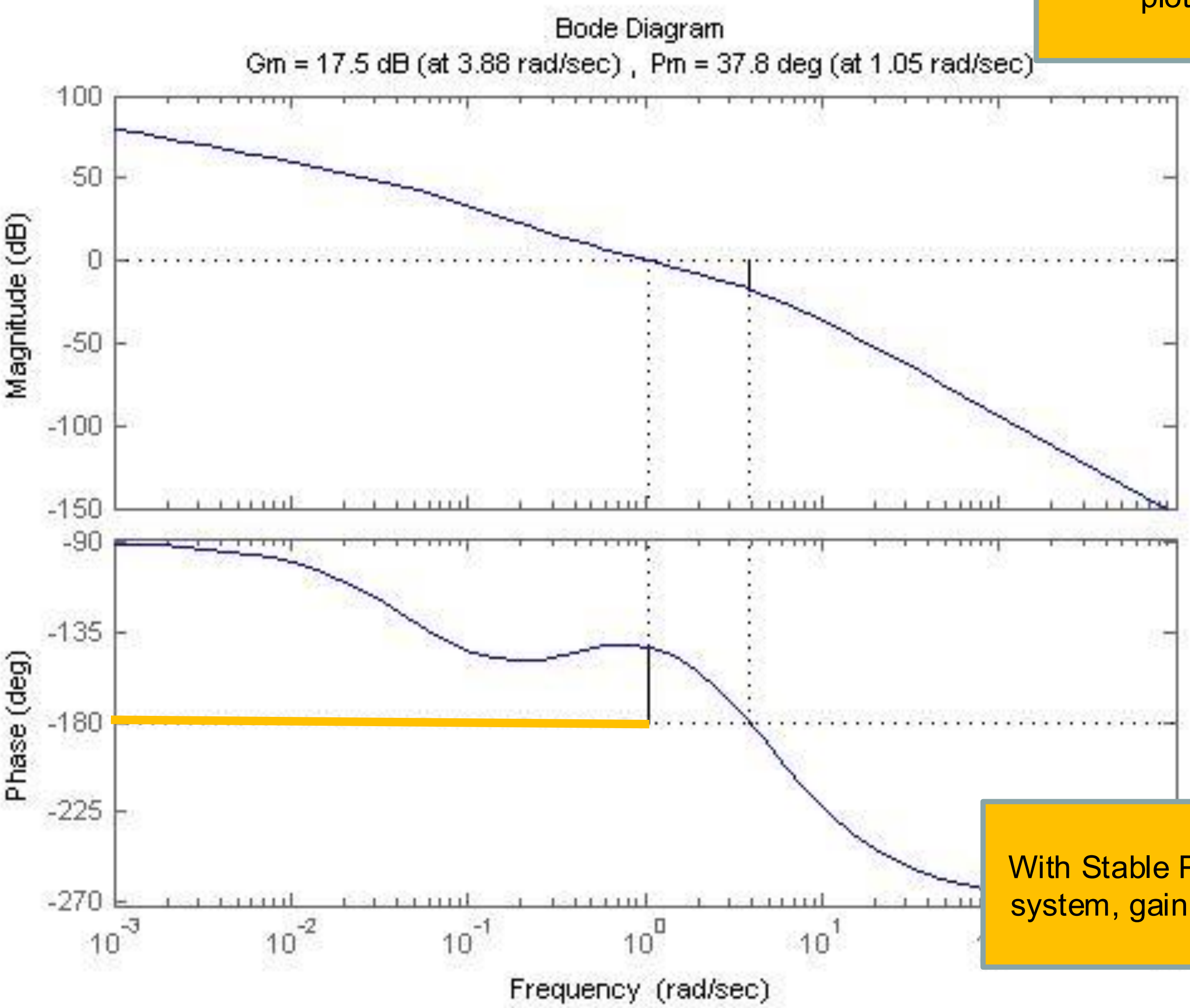
Nyquist



Nichols

Bode Plot Gain and Phase Margins, Stable Plant

Matlab command ‘margin’ gives a bode plot but shows margin as well



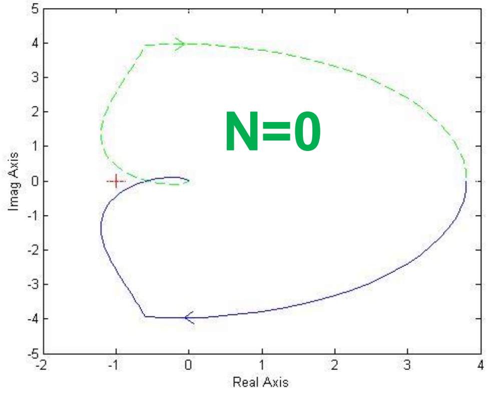
With Stable Plant, and stable closed loop system, gain < 0 dB at phase = -180 deg

Another Example: Gain and Phase Margins with Stable Plant

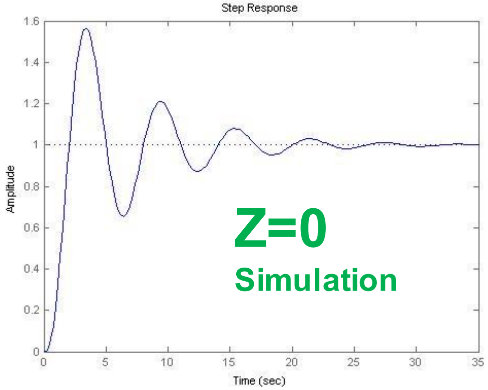
Open Loop Transfer Function :

$$L(s) = \frac{10}{s(s+1)(s+5)}$$

P=0



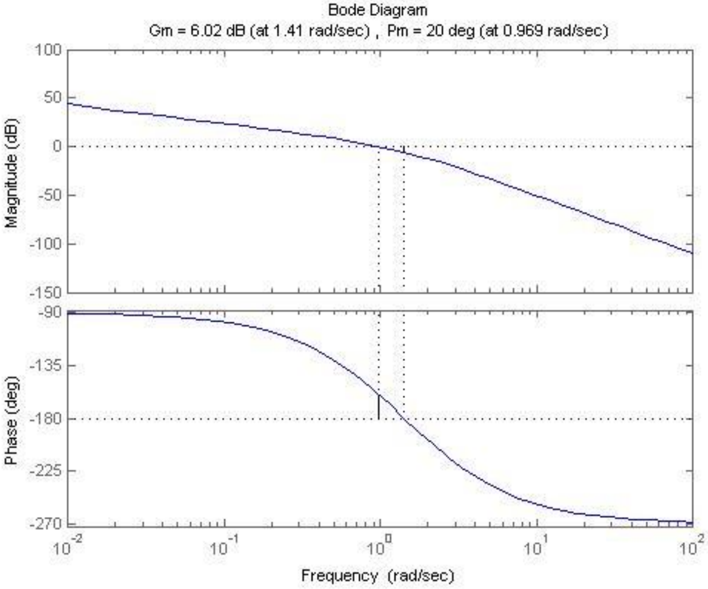
N=0

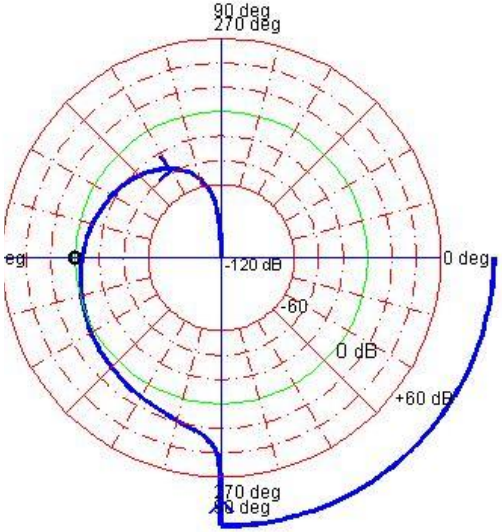


Z=0

Simulation

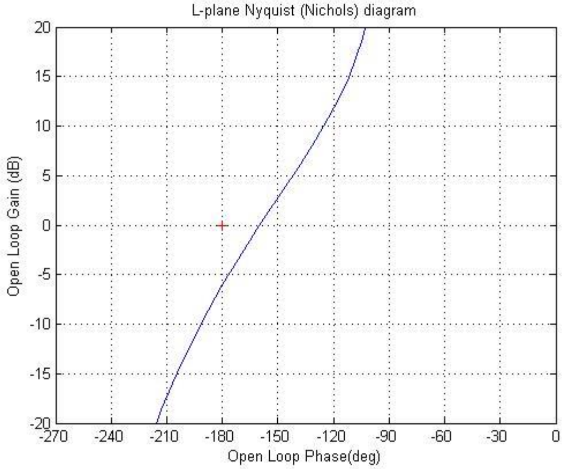
Bode





Nyquist

Nichols



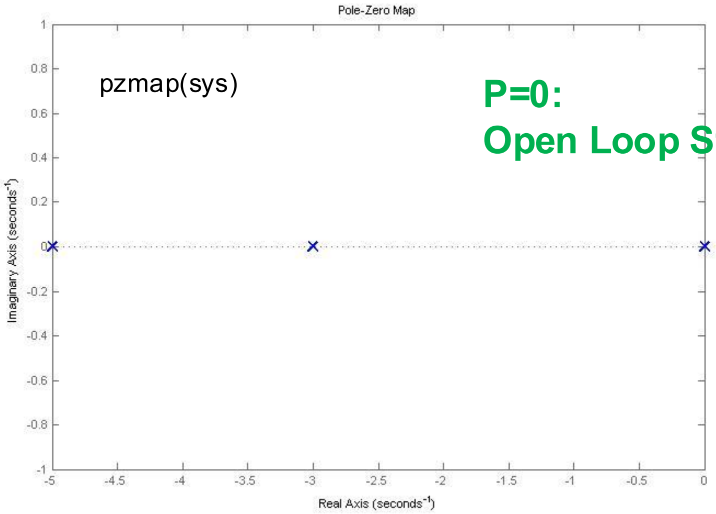
Stable Plant, Unstable with Feedback

Open Loop Transfer Function :

$$L(s) = \frac{200}{s(s+3)(s+5)} = \frac{200}{s^3 + 8s^2 + 15s}$$

```
num2 = 200
den2 = [1 8 15 0]

sys=tf(num2, den2)
```



pzmap(sys)

P=0: Open Loop Stable

Clsoed - Loop Transfer Function :

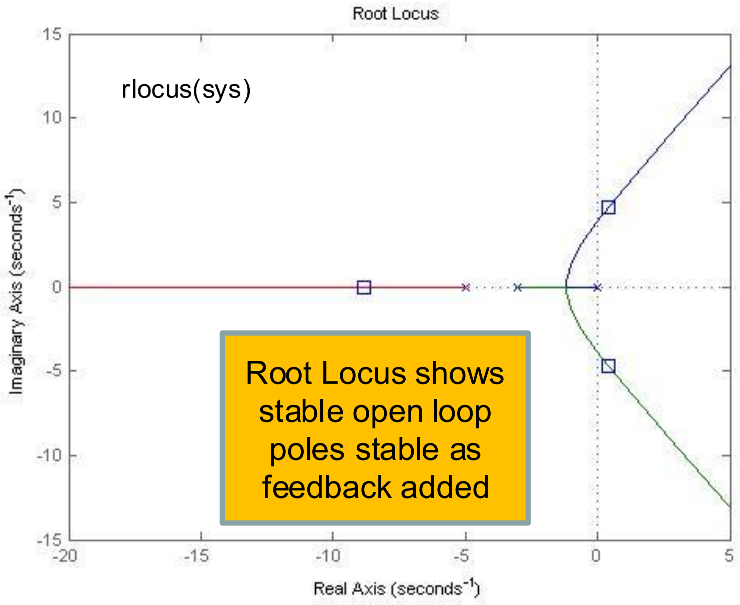
$$L_{CL}(s) = \frac{L(s)}{1 + L(s)} = \frac{\dfrac{200}{s(s+3)(s+5)}}{1 + \dfrac{200}{s(s+3)(s+5)}}$$

$$= \frac{200}{s(s+3)(s+5) + 200} = \frac{200}{s^3 + 8s^2 + 15s + 200}$$

```
cl_sys=feedback(sys,1)
pole(cl_sys)
```

-8.8562
0.4281 + 4.7328i
0.4281 - 4.7328i

Z=2: Two Unstable Closed Loop Poles



rlocus(sys)

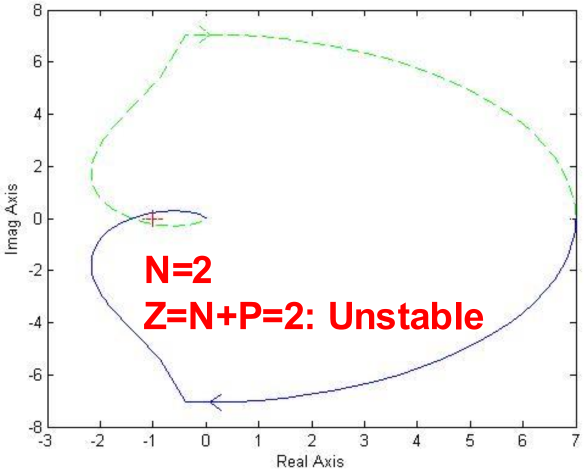
Root Locus shows stable open loop poles stable as feedback added

Continued Stable Plant, Unstable with Feedback: Nyquist, Bode, Nichols

Open Loop Transfer Function

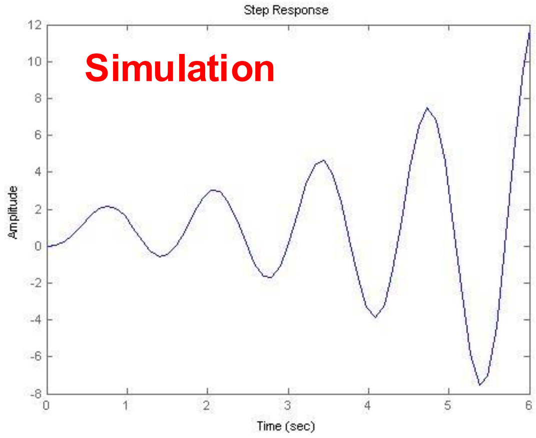
$$L(s)=\frac{200}{s(s+3)(s+5)}$$

P=0

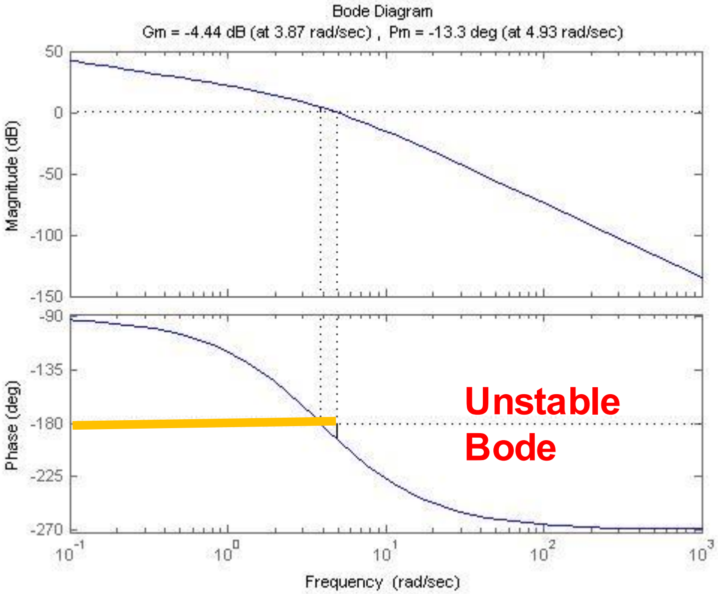


N=2

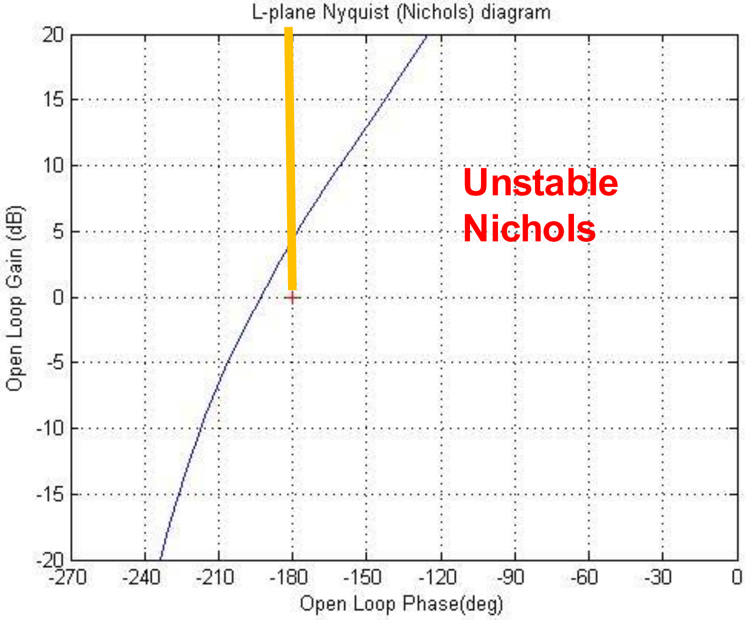
Z=N+P=2: Unstable



Simulation



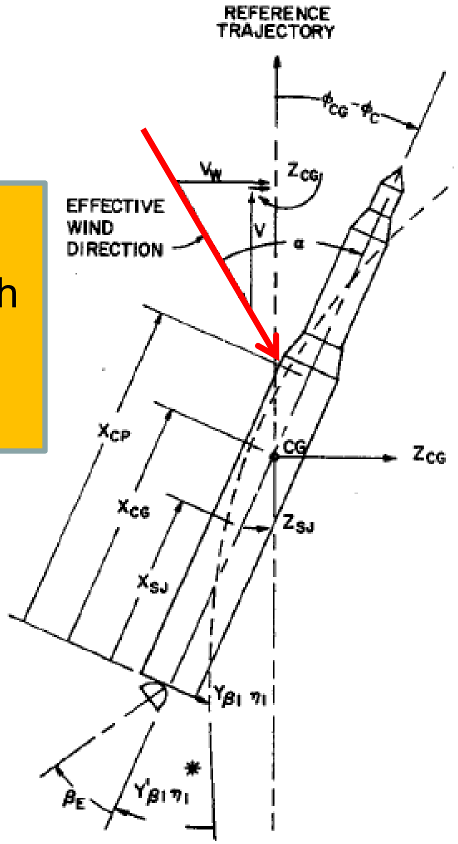
Unstable Bode



Unstable Nichols

Aerodynamic Instability in Launch Vehicles

- **Launch vehicles are aerodynamically unstable when the center of pressure (aero force application point) is above the center of mass.**
  - Generally the case.



Center of Pressure is above mass center, which drives vehicle attitude away from equilibrium

Adding fins moves center of pressure down, which helps mitigate aero instability

From Frosch, Vallely, Ref [4]

- **Hence when one derives the equations of motion and corresponding Laplace transform, one expects to see at lease one unstable pole.**

Launch Vehicle Rigid Body Equations of Motion, and Roots

$\phi = \text{rotation}$

$Z = \text{translation}$

$c_1 = C_{Z_\alpha} qA(X_{cg} - X_{cp}) / I$

$c_2 = \dfrac{FX_{cg}}{I}$

$k_4 = \dfrac{F}{M}$

$k_3 = \dfrac{F-D}{M}$

$k_7 = (C_{Z_\alpha} qA) / M$

$\alpha = \phi - \dfrac{Zs}{V}$

$$\phi s^2 = -c_1\alpha - c_2\beta$$

$$Zs^2 = k_7\alpha + k_3\phi + k_4\beta$$

PD Control :

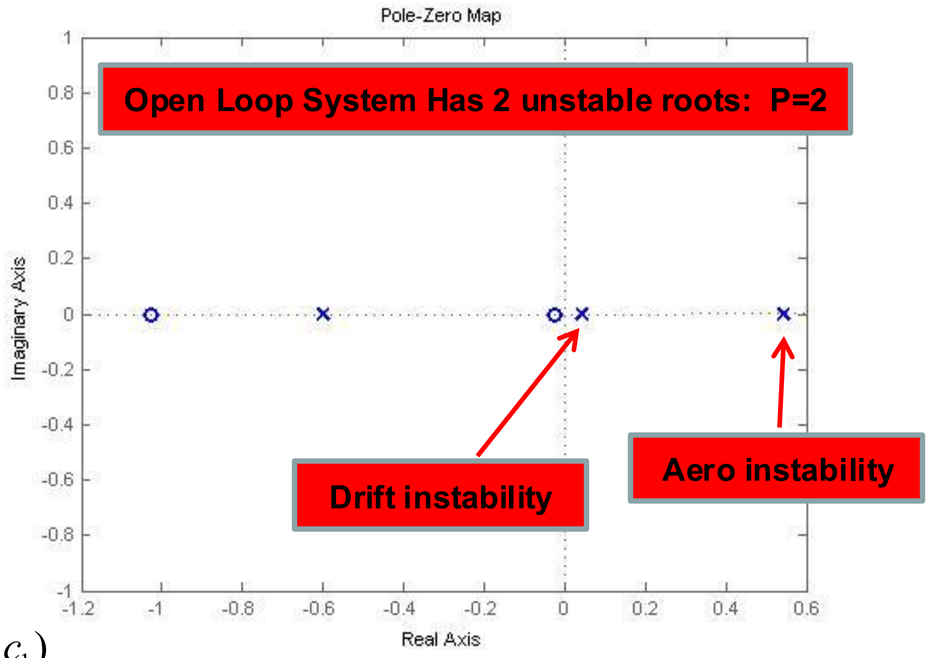
$$\beta = (K_d s + K_p)\phi_c$$

Open Loop Transfer Function, L(s):

$$\frac{\phi}{\phi_c} = \frac{c_2K_ds^2 + \dfrac{K_d}{V}(k_7c_2 - k_4c_1)s + c_2K_ps + \dfrac{K_p}{V}(k_7c_2 - k_4c_1)}{s^3 + \dfrac{k_7}{V}s^2 + c_1s - \dfrac{c_1k_3}{V}}$$

L(s) at t=60 sec, Ares:

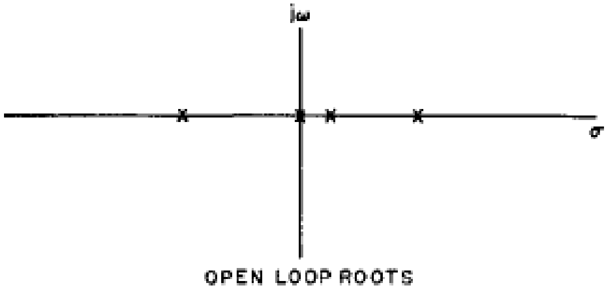
```
1.235 s^2 + 1.299 s + 0.03025
----------------------------------
s^3 + 0.01371 s^2 - 0.3277 s + 0.01371
```



Open Loop System Has 2 unstable roots: P=2

Drift instability

Aero instability



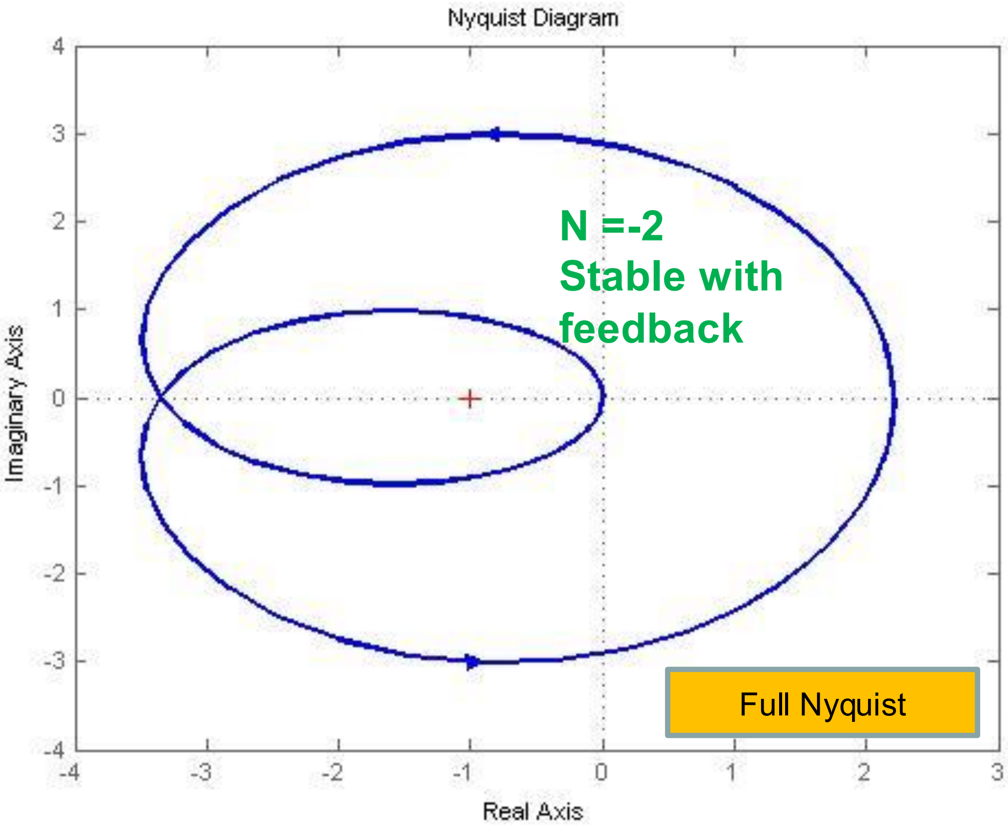
OPEN LOOP ROOTS

Saturn Data

From Frosch, Vallely, Ref [4]

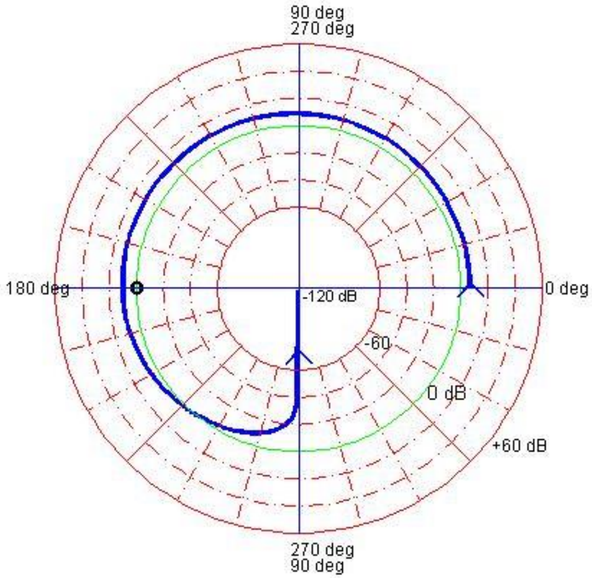
Is Closed Loop Design Stable?

- To meet Nyquist Criterion, open loop system must have two counterclockwise encirclements (Z=N+P, since P=2 N must equal -2)
- Create Nyquist Plot of Open Loop Transfer Function:



N =-2
Stable with feedback

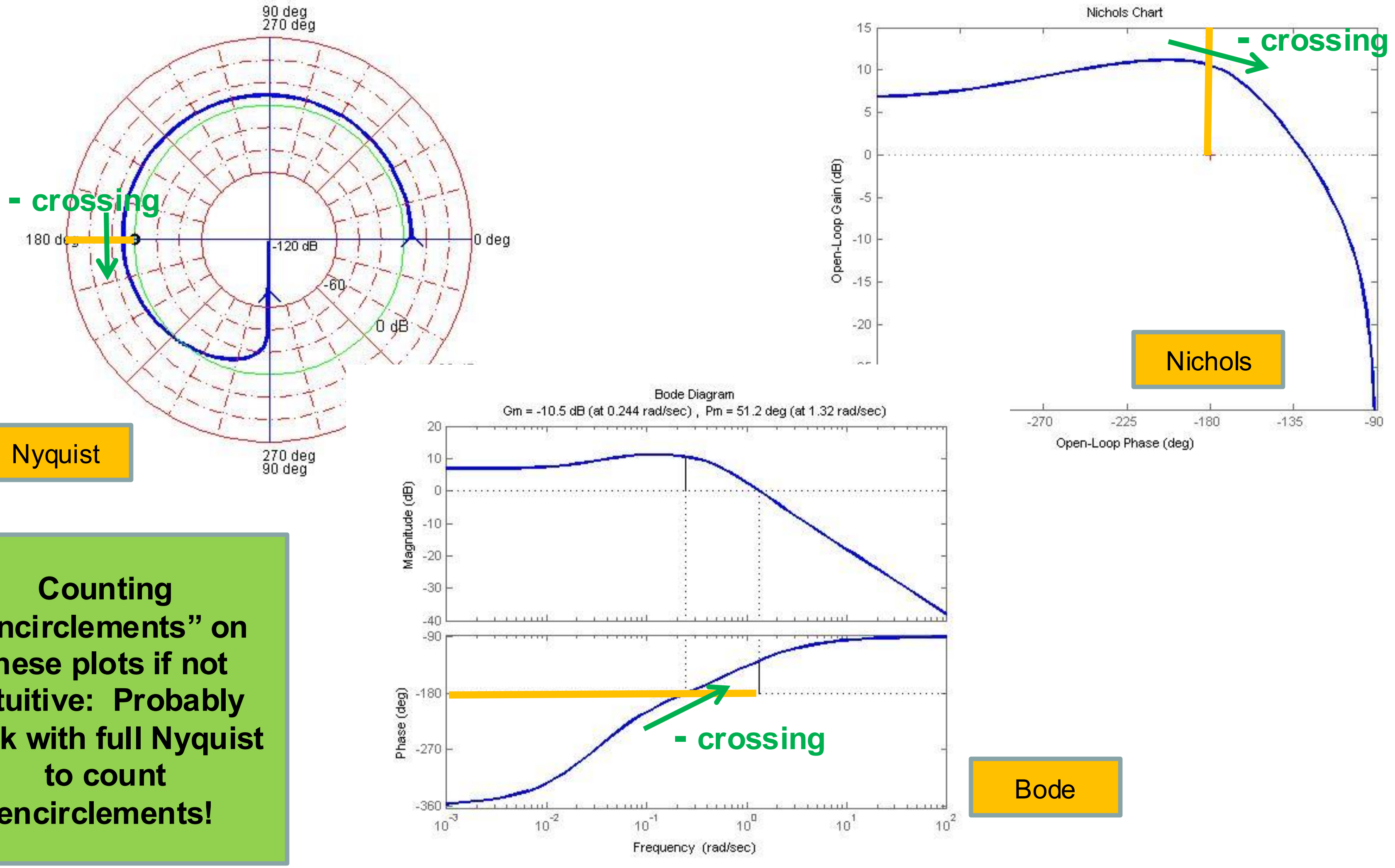
Full Nyquist



Nyquist

Nichols, Bode, and Nyquist Plots

- **Should have a negative crossing of stability line to be stable.**
  - One negative crossing on bode, nichols, and nyquist plots maps to two counterclockwise encirclements on full nyquist.



Counting “encirclements” on these plots if not intuitive: Probably stick with full Nyquist to count encirclements!

Launch Vehicle Rigid Body Equations of Motion: Add slosh

$\sigma = \text{slosh displacement}$

$Z = \text{translation}$

$\phi = \text{rotation}$

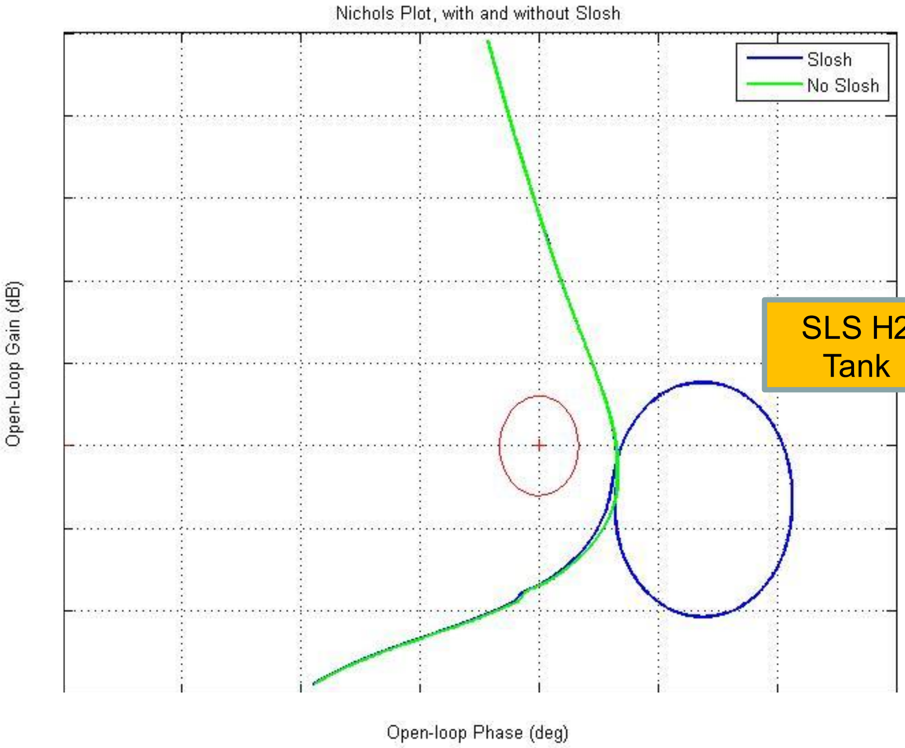
$$Zs^2 + \frac{1}{M} m_s \sigma s^2 - k_4 \phi = k_7 \alpha + k_3 \phi + k_4 \beta$$

$$\phi s^2 - \frac{m_s}{I} l \sigma s^2 - \frac{m_s}{I} k_3 \sigma = -c_1 \alpha - c_2 \beta$$

$$\sigma s^2 + 2\varsigma_s \omega_s \sigma s + \omega_s{}^2 \sigma - l \phi s^2 - k_3 \phi + Zs^2 = 0$$

PD Control:

$$\beta = (K_d s + K_p)\phi_c$$



SLS H2 Tank

$$c_1 = C_{Z_\alpha} q A (X_{cg} - X_{cp}) / I$$

$$c_2 = \frac{F X_{cg}}{I}$$

$$k_4 = \frac{F}{M}$$

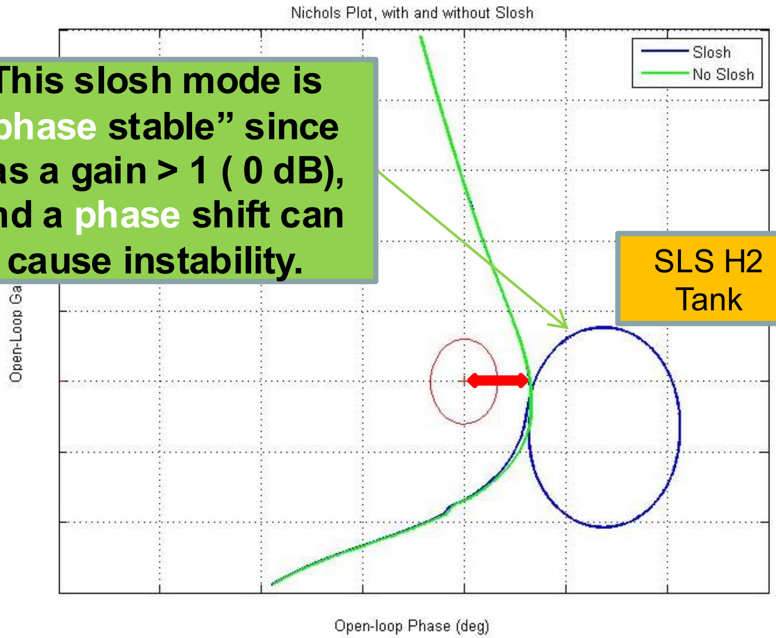
$$k_3 = \frac{F - D}{M}$$

$$k_7 = (C_{Z_\alpha} q A) / M$$

$$\alpha = \phi - \frac{Zs}{V}$$

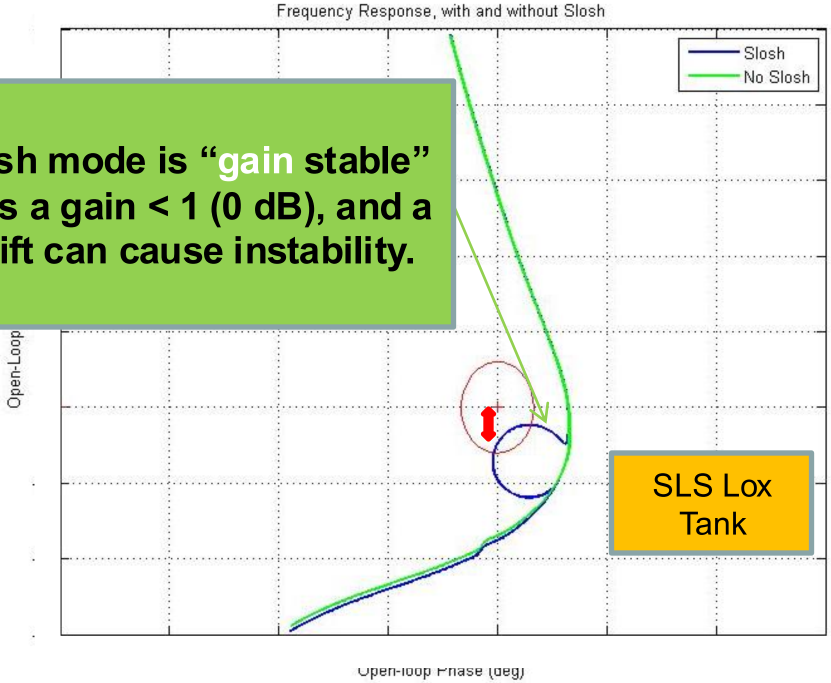
From Frosch, Vallely, Ref [4]

Launch Vehicle Stability: Gain Stable and Phase Stable



This slosh mode is "phase stable" since has a gain > 1 (0 dB), and a phase shift can cause instability.

SLS H2 Tank



This slosh mode is "gain stable" since has a gain < 1 (0 dB), and a gain shift can cause instability.

SLS Lox Tank

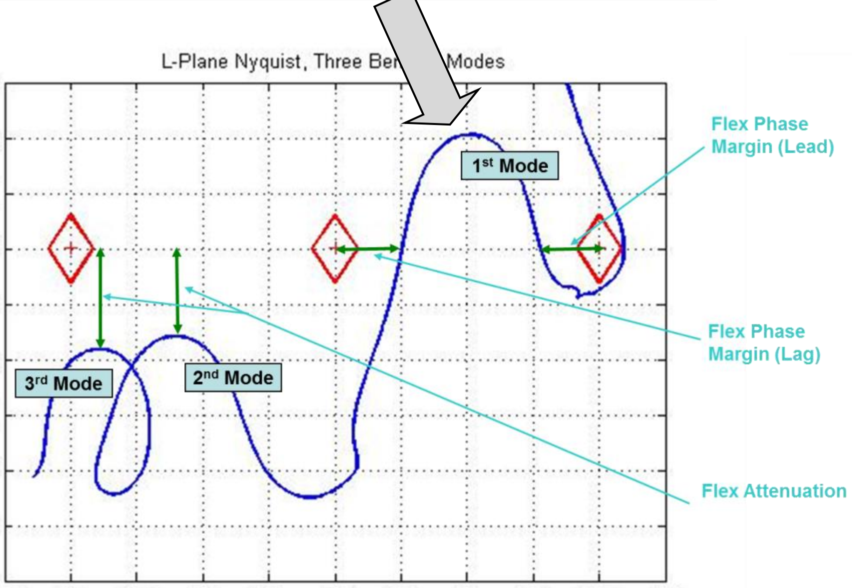
Generally like to gain stabilize rather than phase stabilize, but if the mode is too low frequency, then gain stabilization may not be feasible.

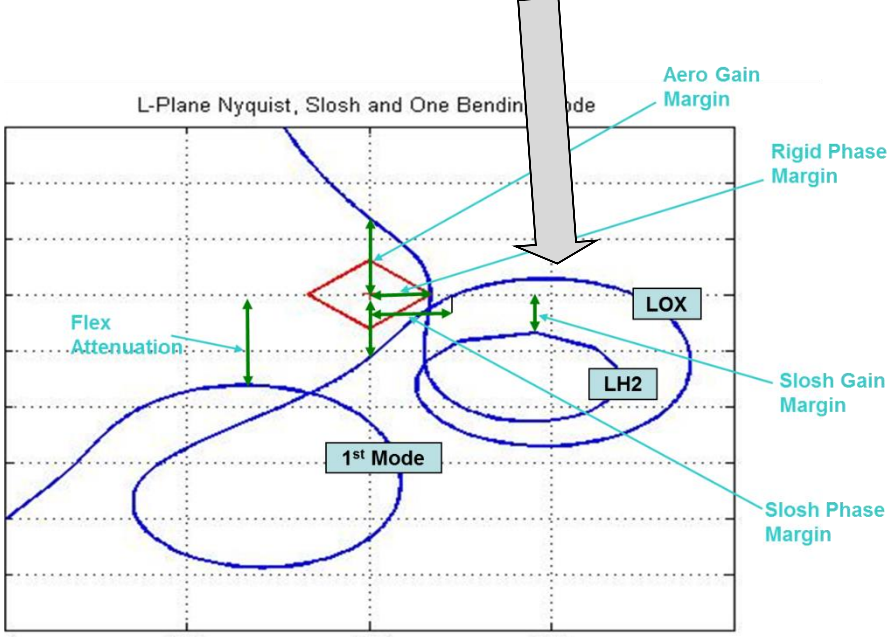
From Frosch, Vallely, Ref [4]

Ares I Stability Margin Definitions

First Stage Fundamental Bending Mode Phase Stabilized, As with Saturn, Atlas, Delta, Titan, etc.

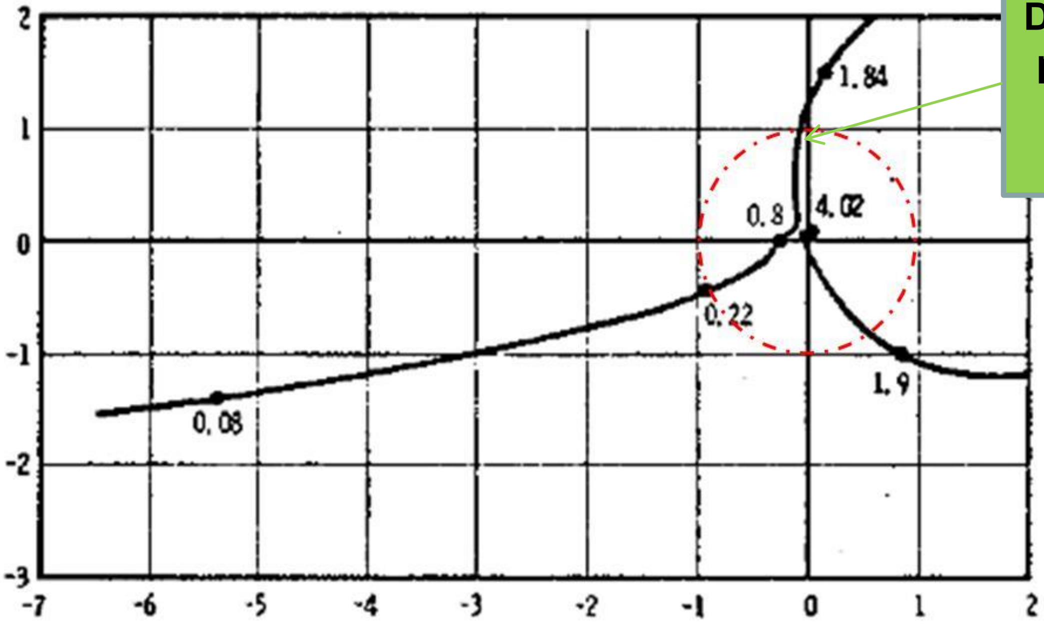
Upper Stage LOX Mode Phase Stabilized, As with Shuttle





	Rigid Body, Aero, Slosh		Flex	
	Gain Margin (dB)	Phase Margin (deg)	Phase Stabilized Flex Modes	Gain Stabilized Flex Modes
			Phase Margin (deg)	Flex Atten (dB)
Nominal	>6	>30	>45	>10
Dispersed	>3	>20	>30	>6

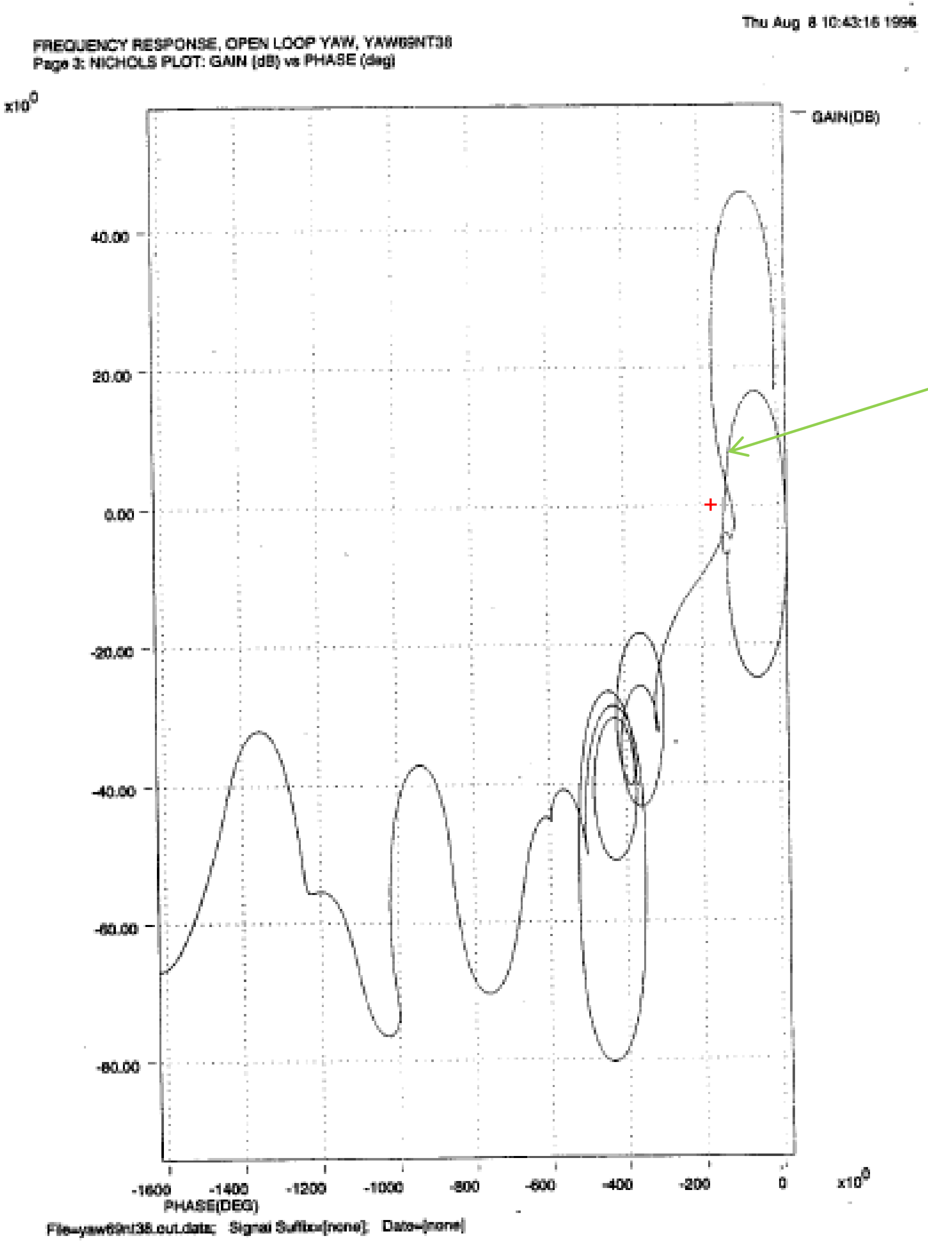
Delta 2 Frequency Response (Nyquist)



Delta 2 Flex mode was phase stabilized with ~88 degrees phase margin

Figure 5: Delta Lift-Off Nyquist Results From Ref [9], Nyquist Circle Added

Shuttle Frequency Response



Thu Aug 8 10:43:16 1996

FREQUENCY RESPONSE, OPEN LOOP YAW, YAW69NT38
Page 3: NICHOLS PLOT: GAIN (dB) vs PHASE (deg)

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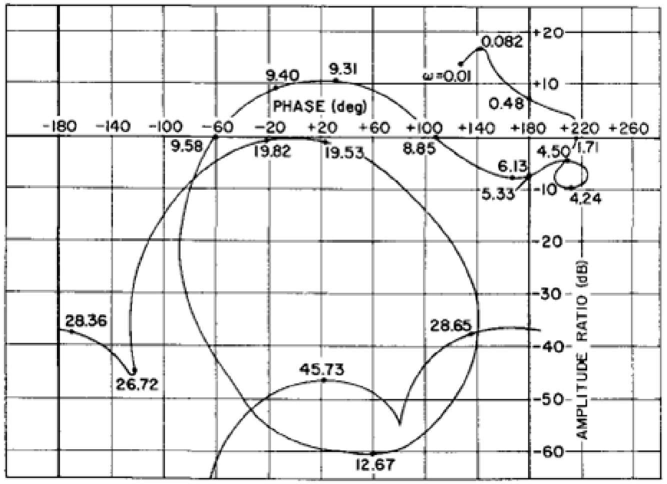
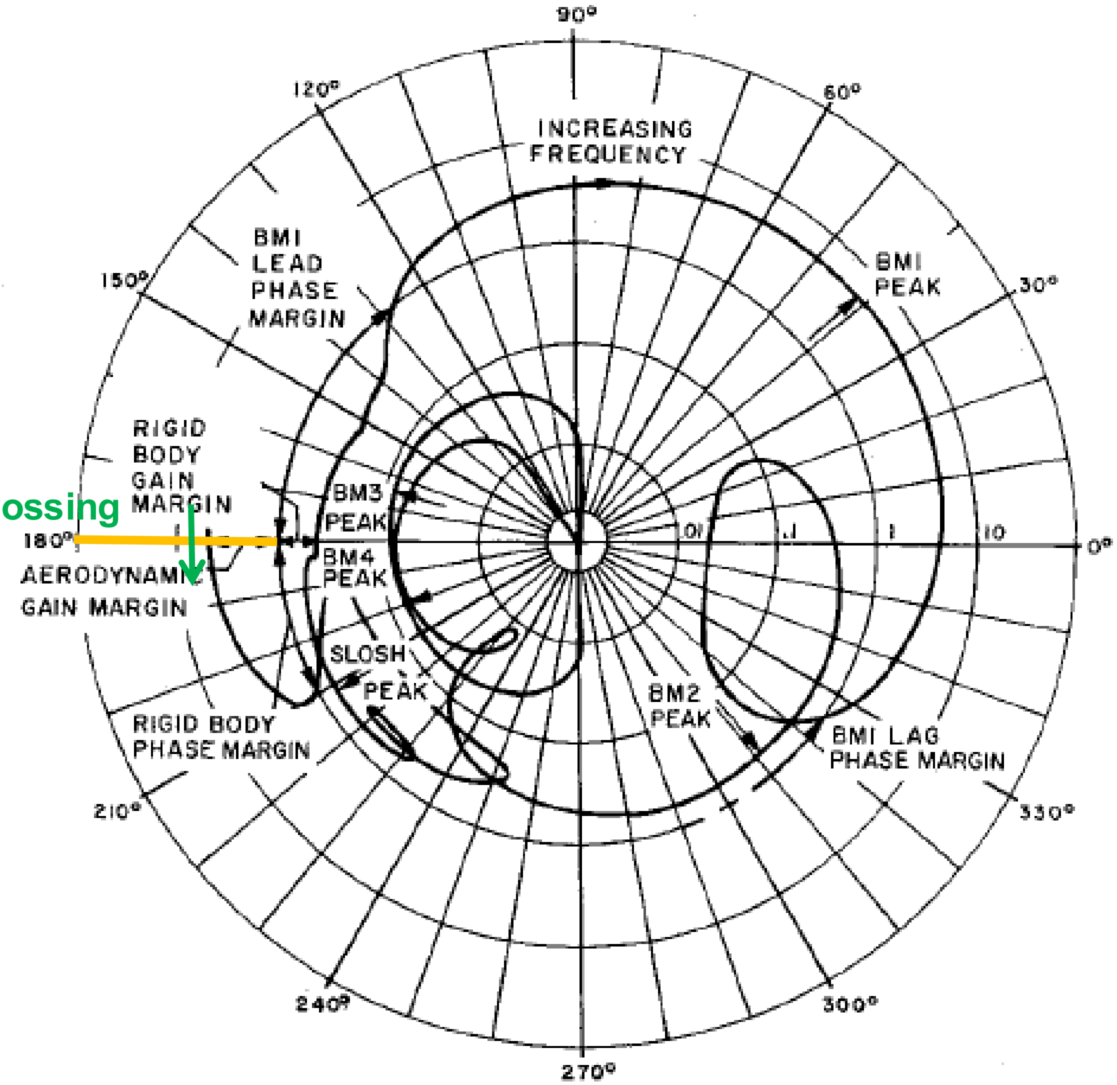
Shuttle Lox tank was phase stabilized. Lox Phase Margin was a challenge!

Saturn 5 Nyquist and Nichols





Negative crossing of Stability Line with gain > 1 (0 dB) reflects required counterclockwise encirclements from Nyquist Criteria

From Frosch, Vallely, Ref [4]

Conclusions

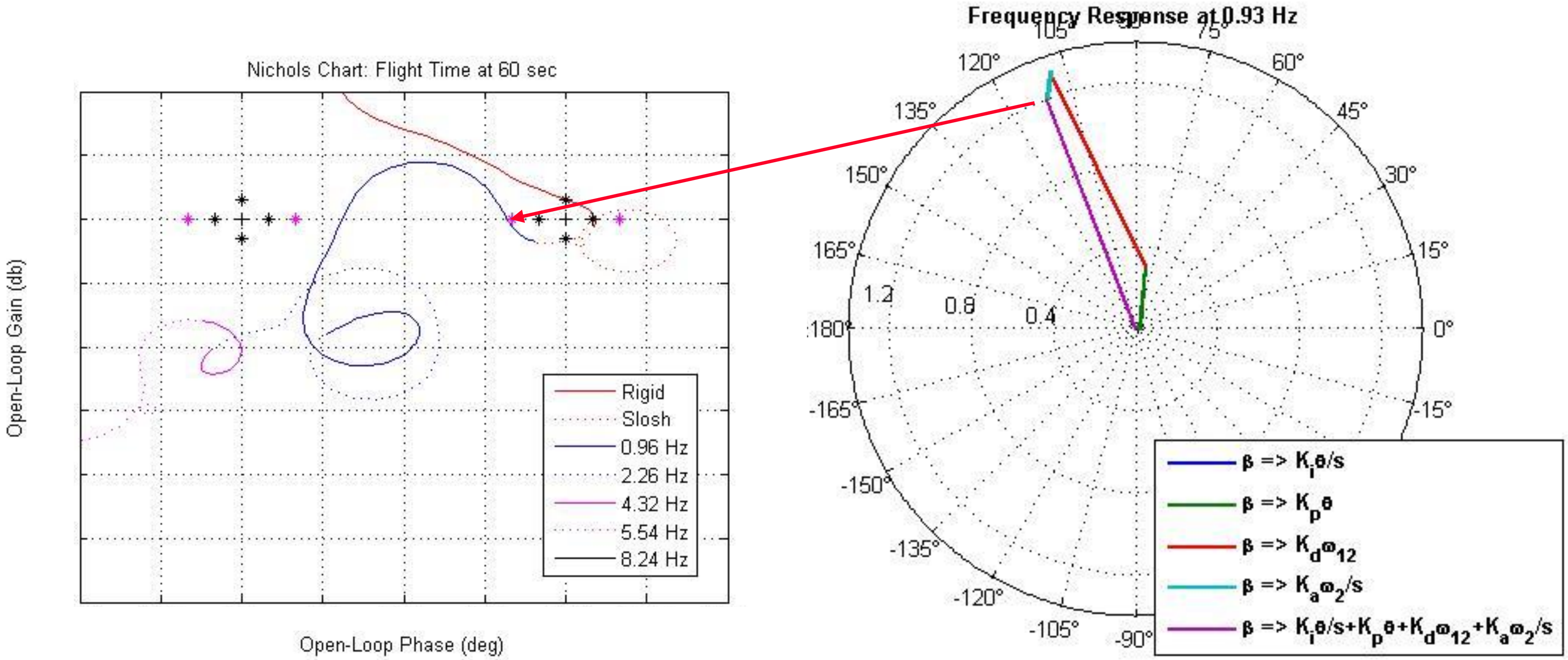
- **Nyquist, Bode, and Nichols plots are all useful in graphically examining the frequency response of linear systems.**
  - All three capture the gain and phase shift of a sine wave into the system.
- **Full Nyquist plots are useful to graphically determine the closed-loop stability, and the number of unstable poles, without actually computing the closed-loop transfer function.**
  - Computing the closed-loop transfer function for complex systems can be significant effort when done by hand.
  - Easier to visualize stability with full Nyquist plot because it plots the entire frequency range (-∞,∞) rather than half (0, ∞) with Nichols, Bode, and “half” Nyquist.
- **Stability margins are perhaps more easily recognized with Nichols and Bode plots rather than Nyquist.**
  - Nyquist is a polar plot of gain and phase, and many are more acquainted with cartesian coordinates.
  - However gain and phase margins are available, and consistent, on all three.
- **Gain and/or phase stabilization is acceptable with slosh and flex dynamics.**
  - Gain stabilization is generally preferred because it is more robust to time lags (latencies) in the system.
  - Flex/slosh stability margins are easily viewed on frequency response plots.

References

1. Ogata, K., *Modern Control Engineering*, 2nd Edition, Prentice Hall, New Jersey, 1990.
2. Lurie, B., Enright, P., *Classical Feedback Control with Matlab*, Marcel Dekker, New York, 2000.
3. Garcia-Sanz, M., “Stability Criteria in Non-Polar Diagrams”, Int. J. Elect. Engin. Educ., Vol 36, pp 65-72, Manchester U.P, 1999.
4. Frosch, J., Vallely, D., “Saturn AS-501/S-IC Flight Control System Design”, J. Spacecraft, Vol. 4, No. 8, August 1967.
5. Kreyszig, E., *Advanced Engineering Mathematics*, John Wiley & Sons, Inc., 2006.

Backup

Ares DAC 2 results, Nichols vs. Nyquist, PID control

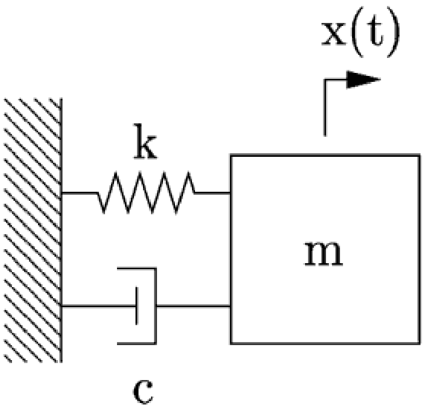


The Nyquist Plot allows user to breakdown the frequency response of each component of a PID controller. This allows visual quantification of how each controller state is impacting margin.

Recall: Transfer Function for spring-mass system:

- **Unforced spring-mass with small damping:**

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=0$$



Take Laplace Transform :

$$m\left[s^2X(s)-sx(0)-\dot{x}(0)\right]+c\left[sX(s)-x(0)\right]+kX(s)=0$$

$$X(s)=\frac{mx(0)s+m\dot{x}(0)+cx(0)}{s^2+\frac{c}{m}s+\frac{k}{m}}$$

Characteristic Equation :

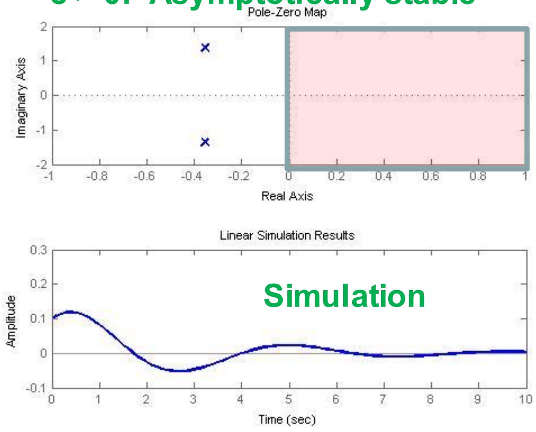
Real

Imaginary

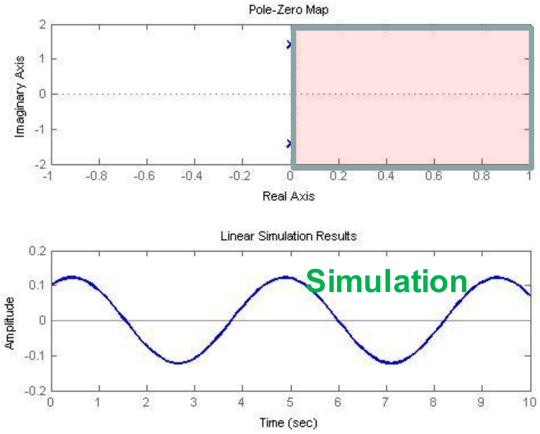
$$s^2+\frac{c}{m}s+\frac{k}{m}=0 \quad\Rightarrow\quad \text{Roots}=\ \lambda_{1,2}=s_{1,2}=\frac{1}{2}\left[-\frac{c}{m}\pm\sqrt{\left(\frac{c}{m}\right)^2-4\frac{k}{m}}\right]$$

For stability, real part of roots (eigenvalues) cannot be > 0

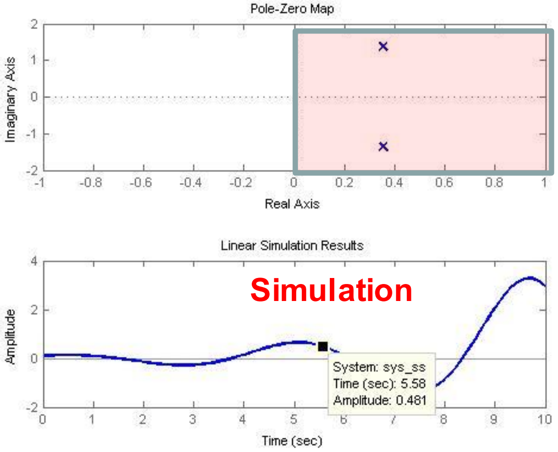
c > 0: Asymptotically stable



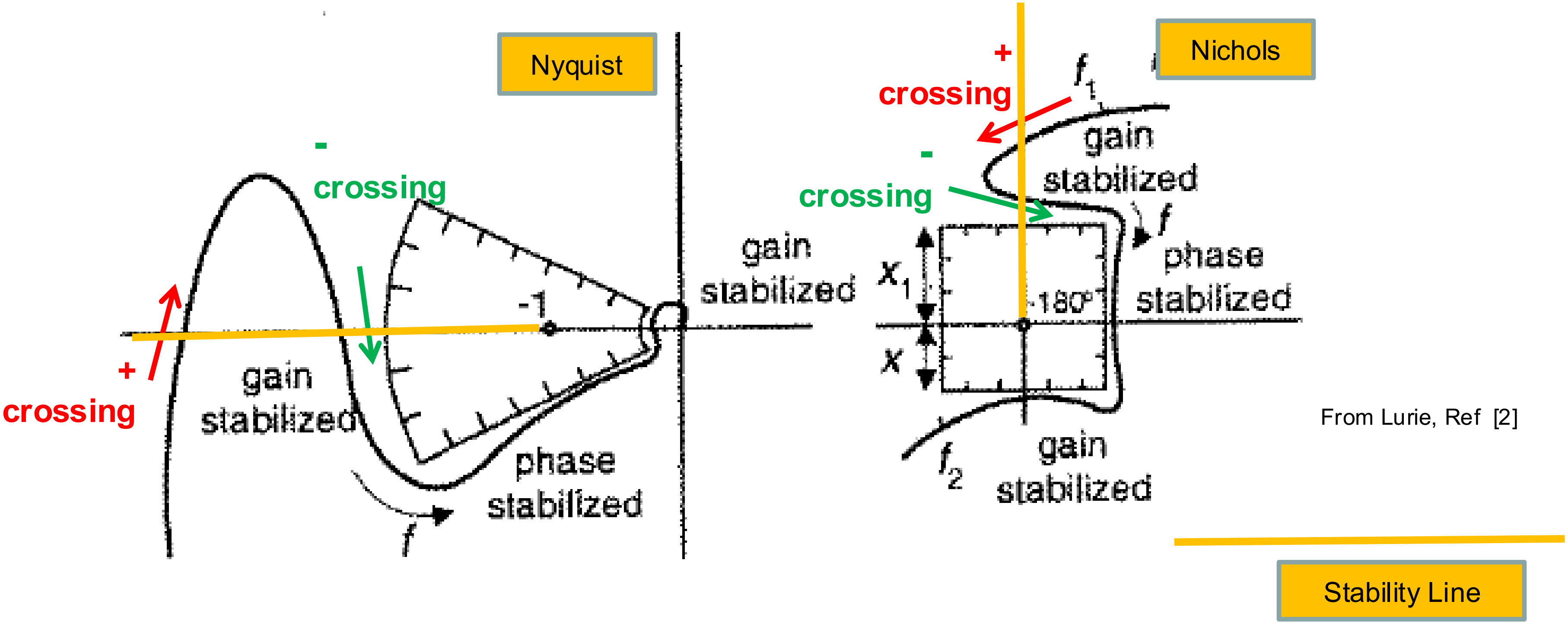
c = 0: Stable



c < 0: Unstable



Crossing with Nyquist and Nichols

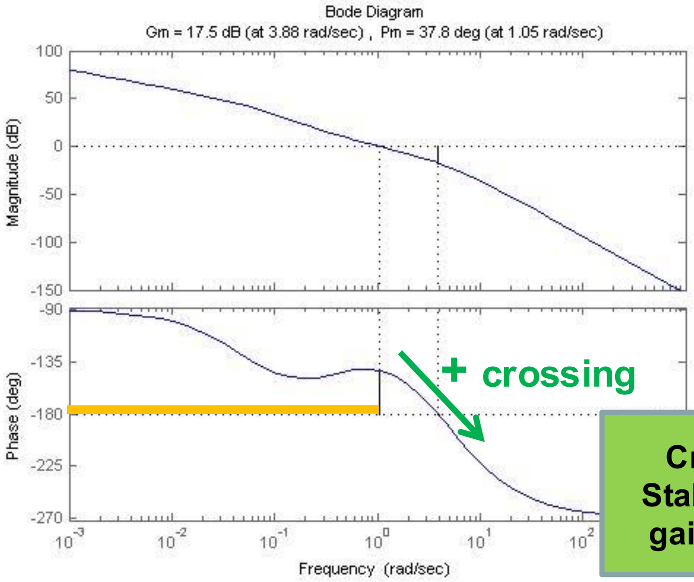


From Lurie, Ref [2]

- ♦ **A crossing of the 180 deg stability line with gain > 1 (0 dB) on a Nichols and Nyquist maps to encirclements on a full Nyquist.**
  - A negative crossing maps to a negative encirclement.
  - Positive crossings generally indicate instability.
  - An unstable plant must have a negative crossing for closed loop stability
  - For a stable plant there should be no crossing of the stability line, or at least the number of crossings should sum to zero.
    - Stable plant can have no encirclements.

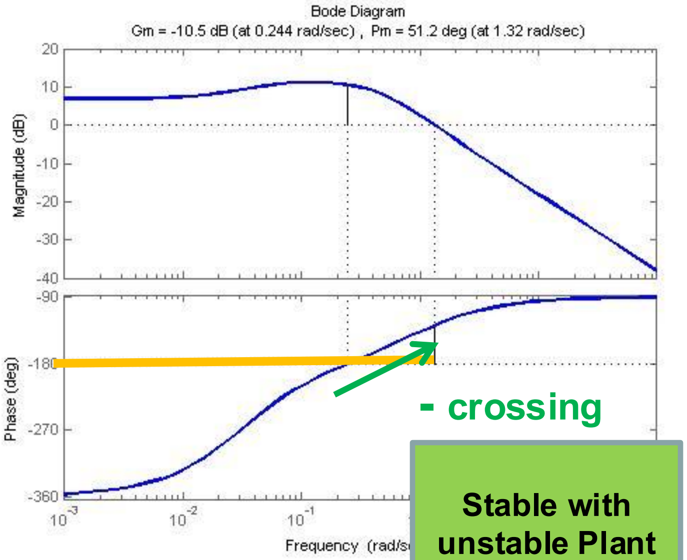
Crossings With Bode Plot

- **The direction that the response curve crosses the -180 deg line on Bode for gain > 0 dB also maps to encirclements (negative crossing denotes negative encirclement).**



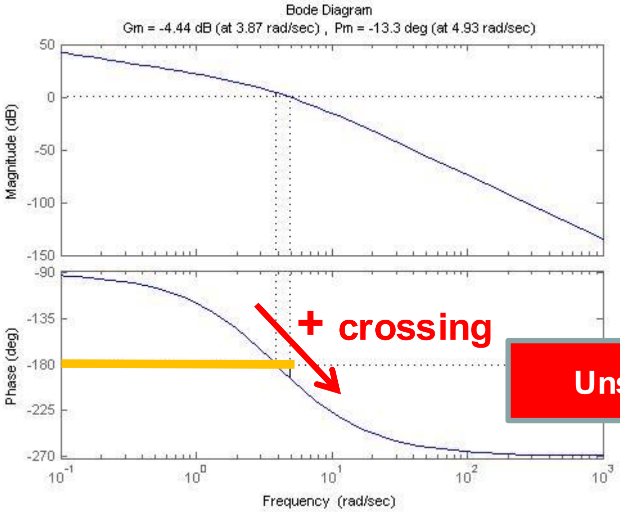
+ crossing

Crossing Stable since gain < 0 dB



- crossing

Stable with unstable Plant



+ crossing

Unstable

See, Garcia-Sanz Ref [3]

Unstable Plant Example, encirclements and crossings

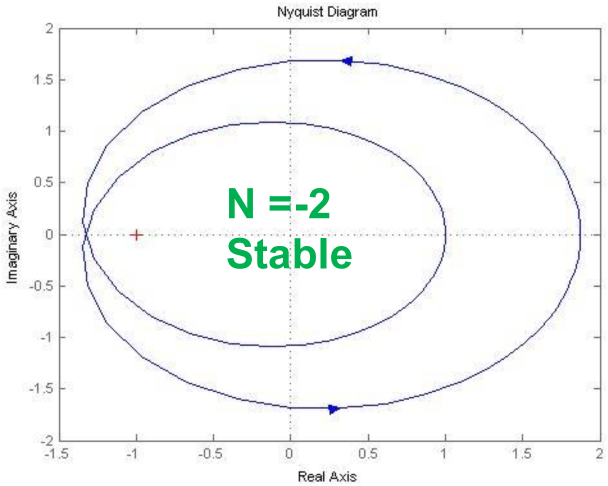
Open Loop Transfer Function :

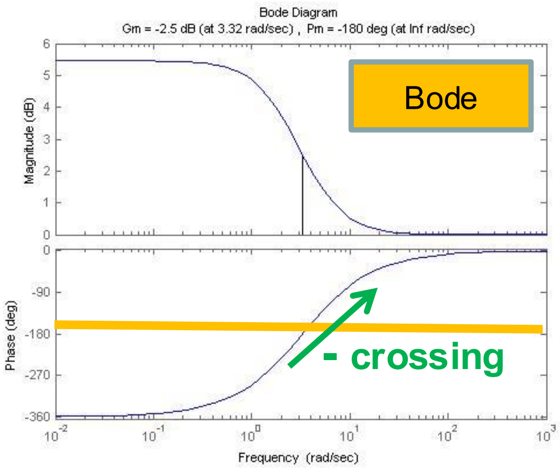
$$L(s) = \frac{(s+3)(s+5)}{(s-4)(s-2)}$$

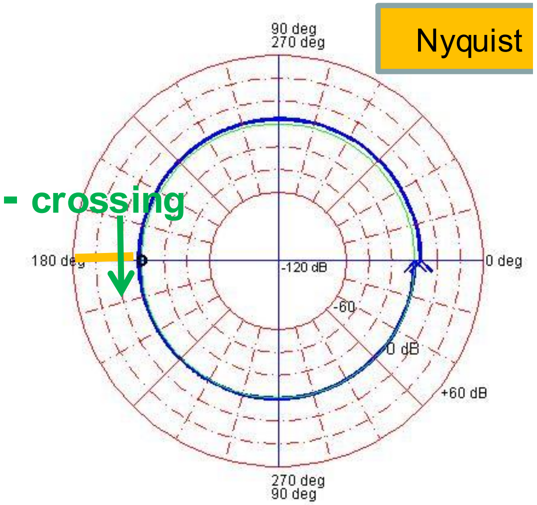
P=2
Two Unstable
Open loop Poles

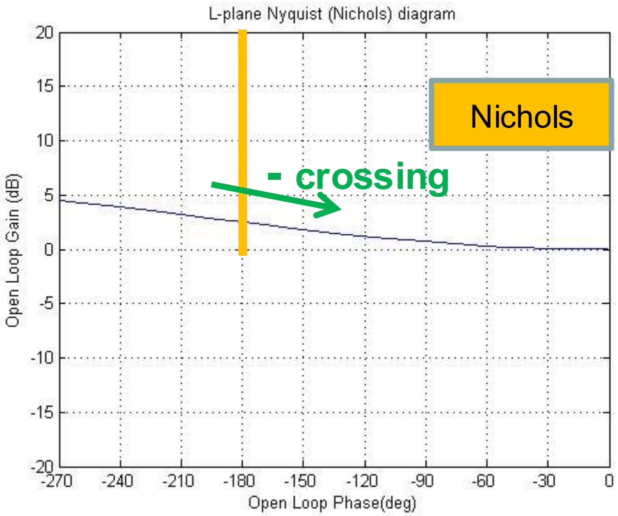
Nyquist Criteria: Z=N+P

For closed loop Stability (Z=0)
Hence N must =-2: two
Counterclockwise encirclements
on full Nyquist







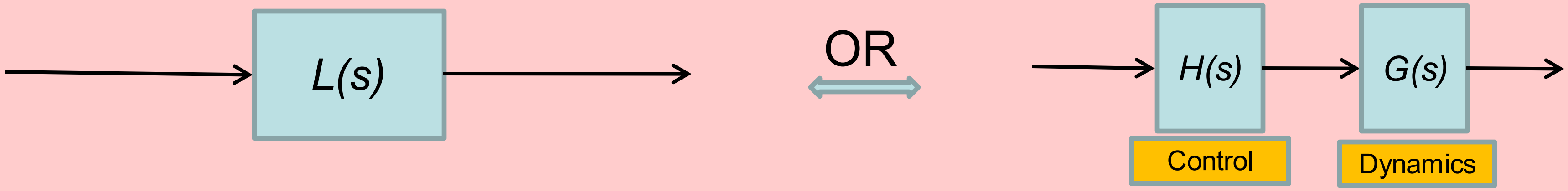


- **Important Point:**
  - Since the Nichols, Bode, and Nyquist are not “full” frequency plots, they will only show 1 crossing of the stability line for each two encirclements.

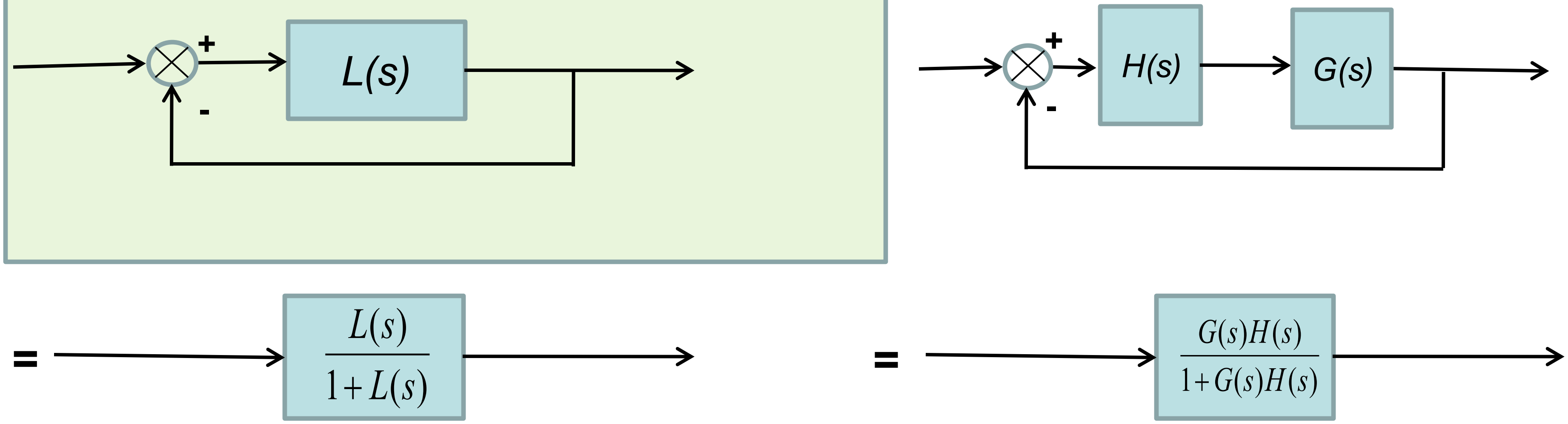
Recall: Open Loop vs. Closed Loop Transfer Functions

◆ **Open loop**

Open Loop System may be Stable or Unstable



◆ **Closed Loop**

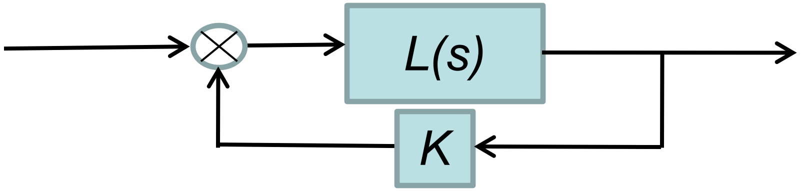


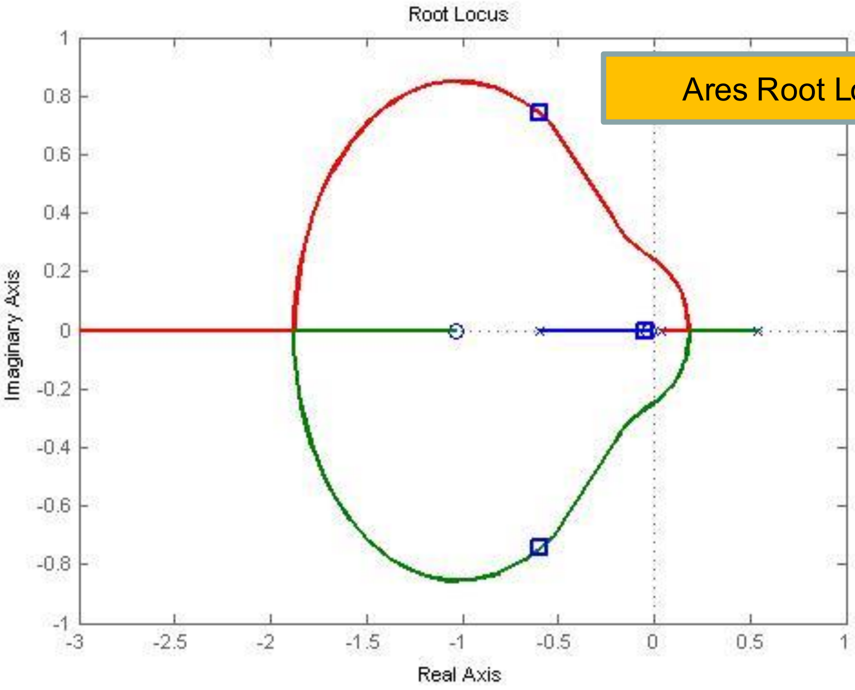
$$\frac{L(s)}{1+L(s)}$$

$$\frac{G(s)H(s)}{1+G(s)H(s)}$$

Closed Loop Characteristic Equation: $1+L(s)=0$ ($1+G(s)H(s)=0$)
For stability, roots of characteristic equation should not be in the right half plane

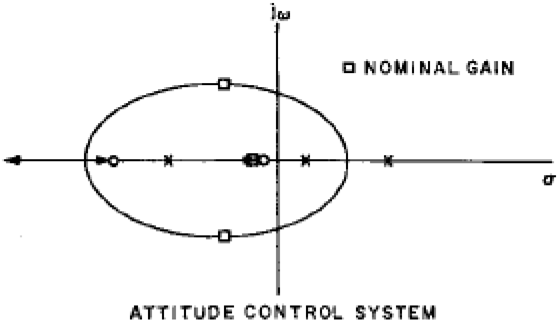
Root Locus for Ares and Saturn





Ares Root Locus

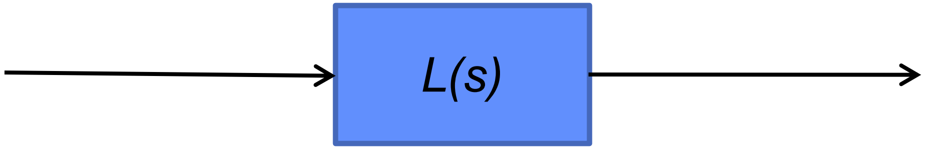
□ **Final Position of Pole in Design**

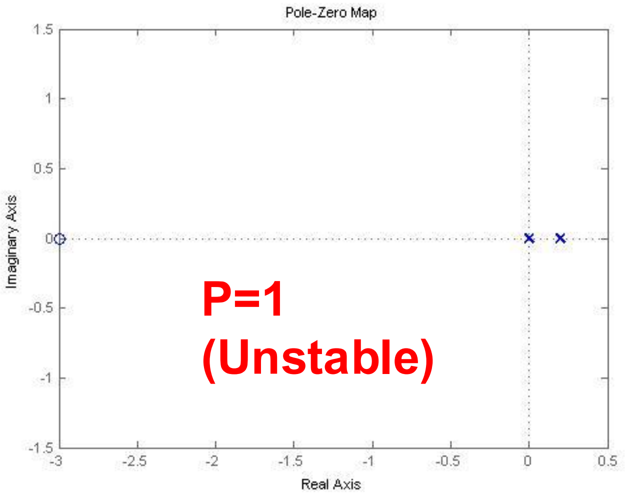


Saturn Root Locus

Recall: Open Loop vs. Closed Loop Example

Open Loop, can be unstable





P=1
(Unstable)

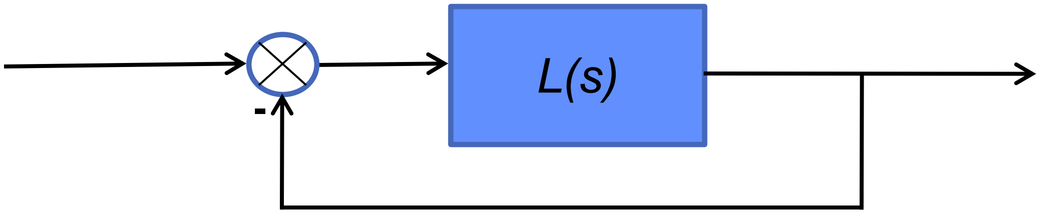
Ex : Open Loop Transfer Function :

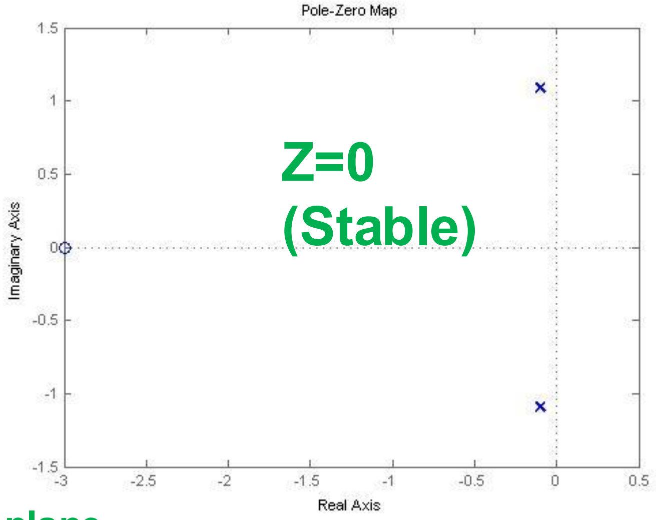
$$L(s)=\frac{2(s+3)}{s(5s-1)}$$

P=Number of poles in open loop transfer function in right hand plane

---

Closed Loop





Z=0
(Stable)

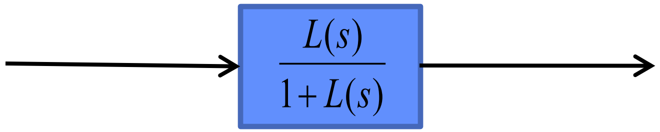
Closed Loop Transfer Function :

$$\frac{L(s)}{1+L(s)}=\frac{\frac{2(s+3)}{s(5s-1)}}{1+\frac{2(s+3)}{s(5s-1)}}=\frac{2(s+3)}{s(5s-1)+2(s+3)}=\frac{2(s+3)}{5s^2+s+6}$$

Z=Number of poles in closed loop transfer function in right hand plane

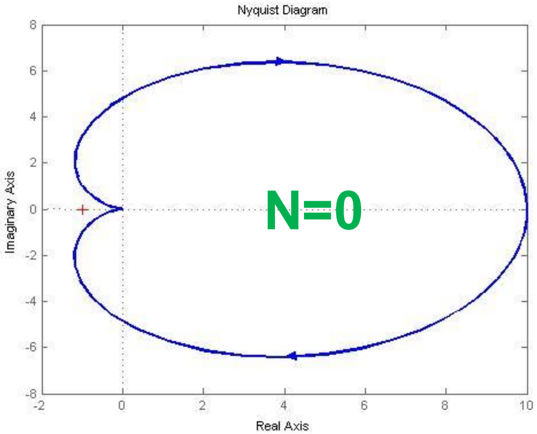
Nyquist Stability: A Visualization

- *Recall for a stable closed loop system, the characteristic equation 1+L(s)=0 must not have positive real roots:*



$$\frac{L(s)}{1+L(s)}$$

- *Substituting s=jω into the characteristic equation leads to the "critical point" L(jω)= -1 + j0, which can be shown to be critical to stability*
- *Hence the -1 + 0j "critical point" is then plotted on a Nyquist Plot and compared against the Nyquist path:*



- *If the Nyquist plot of L(jω) intersects the critical point the system is on the verge of instability because it is close to failing the necessary encirclement criterion.*

- *How close the curve comes from the critical point is called margin.*