

Aeroservoelastic Flight Dynamic Modeling of Multi-Rotor eVTOL Aircraft

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This paper presents an analytical formulation of aeroservoelastic-flight dynamic modeling of multi-rotor electric vertical takeoff and landing (eVTOL) aircraft in hover and transition flight. The method captures the analytical stability and control sensitivities of rotor unsteady aerodynamic loads about a trim solution due to the flexible airframe. The rotor aerodynamic forces include circulatory and non-circulatory components to account for unsteady aerodynamics. The circulatory forces are corrected for unsteady aerodynamics using the time-domain approximation of Theodorsen's function or alternatively the Leishmann's method. The integrated aeroservoelastic flight dynamic model includes the rigid-body aircraft flight dynamics, the structural dynamics of the flexible airframe, inflow dynamics, unsteady aerodynamics, blade pitch dynamics, and rotor dynamics. An analytical aeroservoelastic model of the NASA Lift+Cruise aircraft in hover is developed to illustrate the aeroelastic effect of the elastic airframe on the rigid-body modes of the aircraft.

I. Introduction

The flight dynamic motion of a rotorcraft is generally much more complex than that for a fixed-wing aircraft. The complex flight physics of a rotorcraft is difficult to predict precisely. The complex unsteady flow and unsteady wake interactions between rotors and airframe makes the flight dynamic modeling of electric vertical takeoff and landing (eVTOL) a particularly challenging task. Flight control system design for rotorcraft still largely depend on experimentally-derived flight dynamic modeling using experimentally derived system identification flight data. With the emerging Advanced Air Mobility (AAM), there exist a plethora of multi-rotor eVTOL aircraft designs with highly complex aerodynamic configurations. These complex configurations pose significant challenges in stability and control of AAM aircraft. Due to the need for computational efficiency, flight dynamic modeling of eVTOL aircraft still depends on relatively lower fidelity tools such as unsteady vortex lattice method which cannot capture complex unsteady flow fields with viscous dissipation generated by rotor wakes or transonic flow in the rotor tip region. Computational fluid dynamics (CFD) methods have been coupled to rotorcraft comprehensive analysis tools such as CAMRAD II¹ for use in analyzing rotorcraft dynamics, but the computational cost is quite high which limits the broad applicability of these high-fidelity tools in conceptual flight dynamic analysis.

In the current practice, general nonlinear flight dynamic models of eVTOL aircraft can be generated by rotorcraft comprehensive analysis tools such as CAMRAD II,¹⁴ CHARM,³ and FLIGHTLAB.⁴ These rotorcraft tools are widely used in the rotorcraft community. Linearized flight dynamic models are usually extracted from nonlinear models created by rotorcraft comprehensive analysis tools using system identification techniques or numerical differentiation. Due to nonlinearity, traditional system identification techniques using sinusoidal inputs results in signal distortion which could result in inaccurate identified models.

In accounting for the aeroelasticity of the airframe, the level of complexity in coupling aeroelasticity to flight dynamics of an eVTOL aircraft substantially increases due to the multi-rotor configurations which give rise to complex unsteady aerodynamic interactions between the individual rotors and between the rotors and

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the airframe. In this work, we present an analytical method for estimating stability and control derivatives for an aeroelastic multi-rotor eVTOL aircraft with an elastic airframe. The rotors are assumed to be rigid in this work. The unsteady aerodynamic effects are predominantly due to the rotor thrust. The method affords a computationally efficient process for modeling the flight dynamic motion of an aeroelastic eVTOL aircraft without resorting to high-fidelity CFD. A numerical tool has been developed to be used in conjunction with CHARM to automate the process of creating a linear coupled aeroelastic flight dynamic model in hover and transition flight. The tool is applied to the NASA Lift+Cruise eVTOL aircraft¹⁵ to evaluate the effects of the elastic airframe on the vehicle dynamic response.

II. Air Velocity and Acceleration Analysis

In this section, an air velocity and acceleration analysis is performed to develop the analytical stability and control derivatives of an eVTOL aircraft in hover and transition flights as well as the force and moment derivatives due to the elastic motion of the airframe. For the analysis purpose, the NASA Lift+Cruise VTOL aircraft is used.¹⁵ The development assumes a small perturbation about a nonlinear trim equilibrium solution which is obtained by a trim analysis provided by CHARM rotorcraft comprehensive analysis tool. Complex unsteady wake interactions between the rotors and fixed-wing airframe are neglected in the present analysis due to the lack of fidelity in the wake models employed by CHARM. The rotor-rotor unsteady aerodynamic interactions are assumed quasi-steady whereby the effects are captured in the lift curve slopes for the rotors as computed by CHARM.

A. Reference Frames

The various reference frames of the Lift+Cruise vehicle are shown in Figure 1. We assume a rigid blade system with no flapping motion in the present analysis, but this assumption will be relaxed in future work.

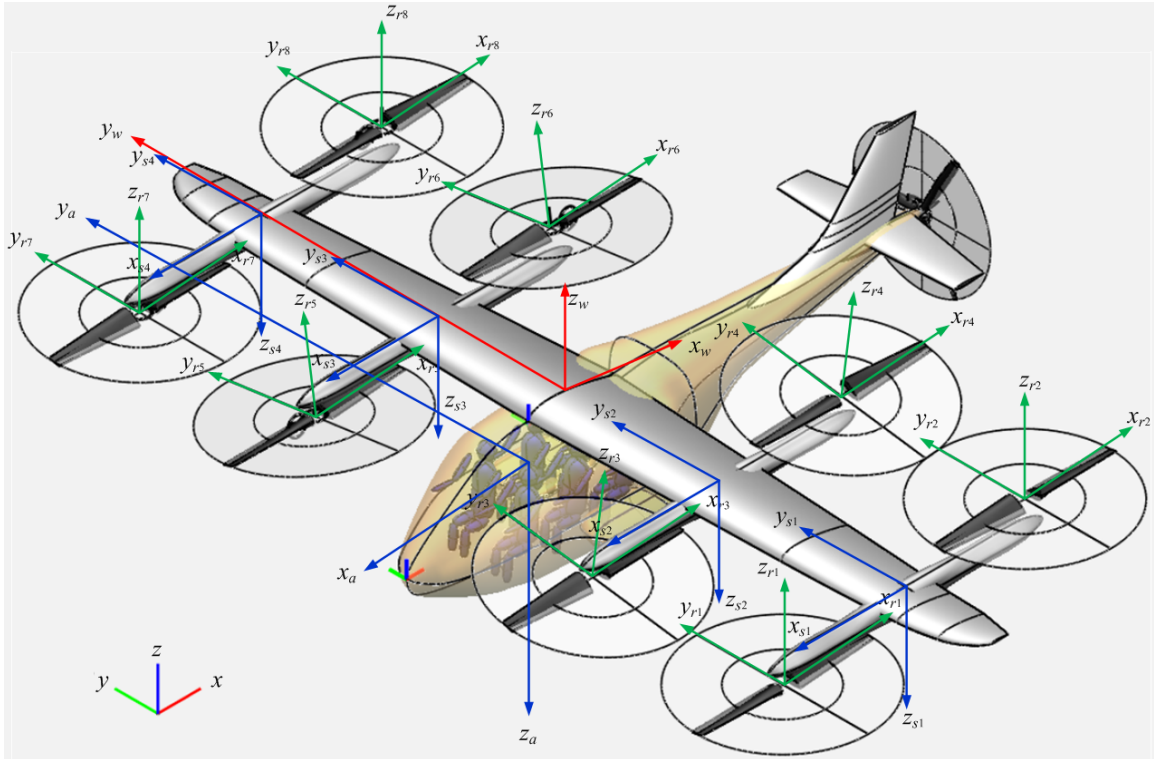


Figure 1. Lift+Cruise Aircraft Reference Frames

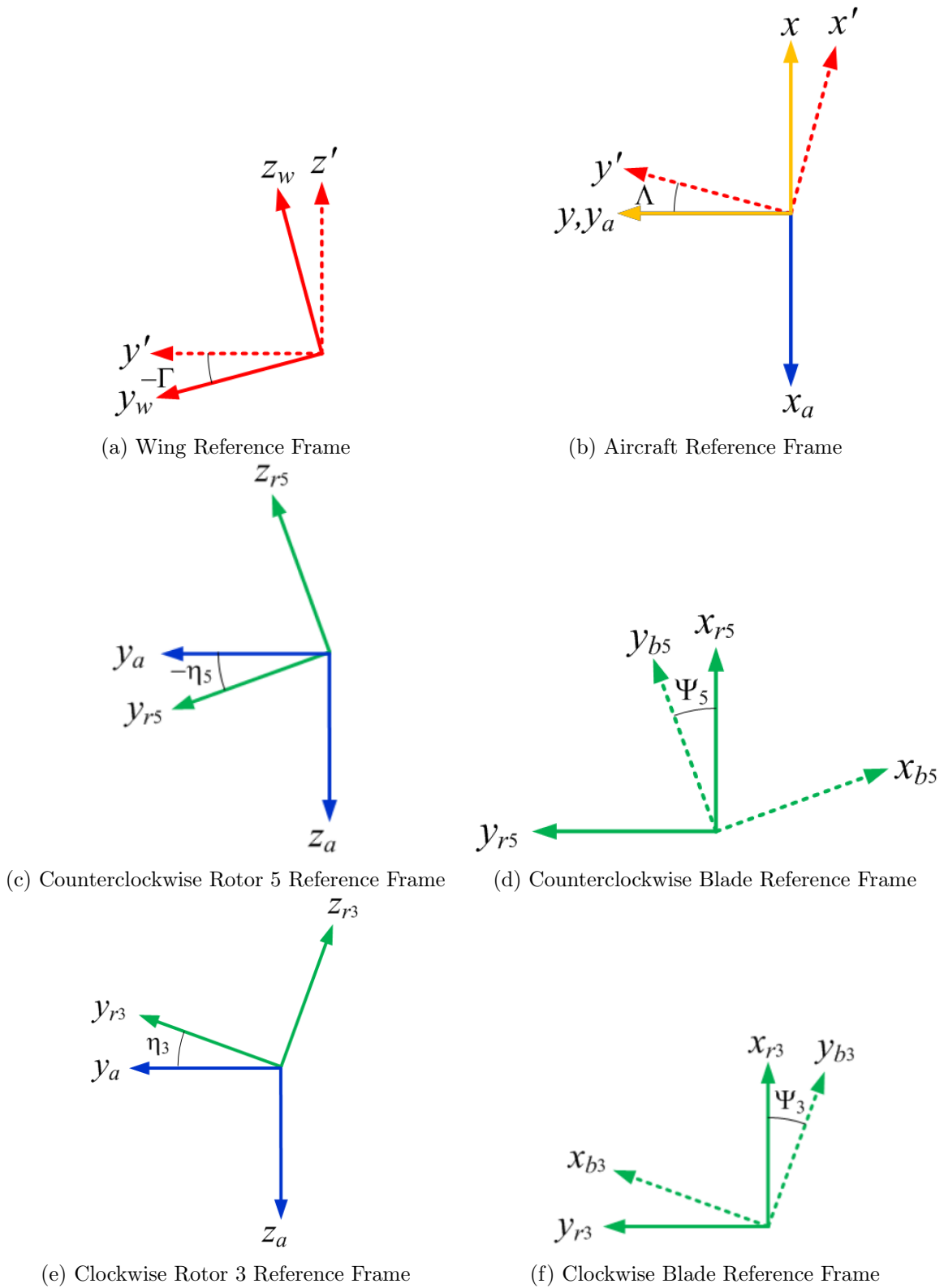


Figure 2. Wing, Aircraft, Rotor, and Blade Reference Frames

The wing has a negative dihedral angle $-\Gamma$ and a positive sweep angle Λ . The wing elastic reference frame is defined by (x_w, y_w, z_w) . The reference frame defined by (x', y', z') is obtained by a rotating (x_w, y_w, z_w)

about $x_w = x'$ by $-\Gamma$ where Γ is a negative value. This is shown in Figure 2(a). The negative value indicates that the dihedral angle is negative or equivalently a positive anhedral angle. Thus,

$$\begin{bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Gamma & -\sin \Gamma \\ 0 & \sin \Gamma & \cos \Gamma \end{bmatrix} \begin{bmatrix} \mathbf{i}_w \\ \mathbf{j}_w \\ \mathbf{k}_w \end{bmatrix} \quad (1)$$

where $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$ are unit vectors along the x' -axis, y' -axis, and z' -axis, respectively.

The z' -axis points in the upward vertical direction in the direction opposite to the aircraft z_a -axis. Referring to Figure 2(b), the (x', y', z') reference frame is related to the aircraft reference frame defined by (x_a, y_a, z_a) as

$$\begin{bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{bmatrix} = \begin{bmatrix} -\cos \Lambda & -\sin \Lambda & 0 \\ -\sin \Lambda & \cos \Lambda & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (2)$$

There are eight lift rotors on the Lift+Cruise aircraft, numbered 1 through 8. Rotors 1, 4, 5, and 8 are rotating in the counterclockwise (CCW) direction as viewed from the top of Figure 1. Rotors 2, 3, 6, and 7 are rotating in the clockwise (CW) direction. The rotors are canted by an angle η_i in the y_a - z_a plane where the positive angle is about the x_r -axis. Referring to Figures 2(c) and (e), the CCW and CW rotor reference frames defined by $(x_{r_i}, y_{r_i}, z_{r_i})$ are related to the aircraft reference frame (x_a, y_a, z_a) by

$$\begin{bmatrix} \mathbf{i}_{r_i} \\ \mathbf{j}_{r_i} \\ \mathbf{k}_{r_i} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos \eta_i & -\sin \eta_i \\ 0 & -\sin \eta_i & -\cos \eta_i \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (3)$$

Referring to Figures 2(d) and (f), the CCW blade reference frames is defined by $(x_{b_i}, y_{b_i}, z_{b_i})$, $i = 1, 4, 5, 8$, where the blade pitch axis is aligned with the y_{b_i} -axis. The CW blade reference frames is defined by $(x_{b_i}, y_{b_i}, z_{b_i})$, $i = 2, 3, 6, 7$, where the blade pitch axis is aligned with the y_{b_i} -axis. The CCW and CW rotations are defined by the angle Ψ_i where the positive angle is about the z_{b_i} -axis. The CCW and CW blade reference frames $(x_{b_i}, y_{b_i}, z_{b_i})$ are related to the rotor reference frames $(x_{r_i}, y_{r_i}, z_{r_i})$ by

$$\begin{bmatrix} \mathbf{i}_{b_i} \\ \mathbf{j}_{b_i} \\ \mathbf{k}_{b_i} \end{bmatrix} = \begin{bmatrix} \sin \Psi_i & \mp \cos \Psi_i & 0 \\ \cos \Psi_i & \pm \sin \Psi_i & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_{r_i} \\ \mathbf{j}_{r_i} \\ \mathbf{k}_{r_i} \end{bmatrix} \quad (4)$$

where the upper sign corresponds to the CCW blade reference frame.

The rotor support reference frames are defined by $(x_{s_i}, y_{s_i}, z_{s_i})$, $i = 1, 2, 3, 4$ in the same orientation as the aircraft reference frame (x_a, y_a, z_a) . Therefore,

$$\begin{bmatrix} \mathbf{i}_{s_i} \\ \mathbf{j}_{s_i} \\ \mathbf{k}_{s_i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (5)$$

The aircraft body station, buttline station, and waterline station are defined by the global reference frame (x, y, z) as shown in Figure 1. The global reference frame (x, y, z) is related to the aircraft reference frame (x_a, y_a, z_a) as

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{j}_a \\ \mathbf{k}_a \end{bmatrix} \quad (6)$$

The aircraft center of gravity (CG) is located at (x_{cg}, y_{cg}, z_{cg}) in the global reference frame (x, y, z) . The origin of the wing elastic reference frame is located at $(x_{w_0}, y_{w_0}, z_{w_0})$ relative to the CG location with respect to the global reference frame (x, y, z) . The origins of the rotor support reference frames are located at $(x_{s_{i0}}, y_{s_{i0}}, z_{s_{i0}})$ relative to the CG location with respect to the global reference frame (x, y, z) . The origins of the rotor and blade reference frames are located at $(x_{r_{i0}}, y_{r_{i0}}, z_{r_{i0}})$ relative to the CG location with respect to the global reference frame (x, y, z) .

B. Air Velocity and Acceleration

The wing and the rotor supports are assumed to deform elastically under combined bending, torsion, and axial loading. The lift forces are due to the air velocity and air acceleration. We consider the air velocities and accelerations on the rotors and the aircraft aerodynamic surfaces.

C. Air Velocity Analysis

The displacement and rotation of the wing structure under load are expressed as

$$\mathbf{u}_w = u_w \mathbf{i} + v_w \mathbf{j} + w_w \mathbf{k} \quad (7)$$

$$\boldsymbol{\phi}_w = \phi_w \mathbf{i} + \theta_w \mathbf{j} + \psi_w \mathbf{k} \quad (8)$$

where $u_w(y, t)$, $v_w(y, t)$, and $w_w(y, t)$ are the lateral bending displacement, extensional displacement, and vertical bending displacement, respectively; and ϕ_w , θ_w , and ψ_w are the vertical bending rotation, torsional rotation, and lateral bending rotation, respectively.

At the origin of the rotor support reference frames $(x_{s_i}, y_{s_i}, z_{s_i})$, the displacements and rotations of a rotor are denoted as

$$\mathbf{u}_w = \boldsymbol{\Phi}_w(y_{s_{i0}}) \mathbf{q}(t) \quad (9)$$

$$\boldsymbol{\phi}_w = \boldsymbol{\Psi}_w(y_{s_{i0}}) \mathbf{q}(t) \quad (10)$$

where $\boldsymbol{\Phi}_w(y_{s_{i0}})$ is the displacement mode shape matrix, $\boldsymbol{\Psi}_w(y_{s_{i0}})$ is the rotation mode shape matrix, and $\mathbf{q}(t) = \begin{bmatrix} q_1(t) & q_2(t) & \cdots & q_{n_{mode}}(t) \end{bmatrix}$ is the generalized displacement vector.

The displacement and rotation of a rotor support structure is expressed as

$$\mathbf{u}_{s_i} = u_{s_i} \mathbf{i} + v_{s_i} \mathbf{j} + w_{s_i} \mathbf{k} \quad (11)$$

$$\boldsymbol{\phi}_{s_i} = \phi_{s_i} \mathbf{i} + \theta_{s_i} \mathbf{j} + \psi_{s_i} \mathbf{k} \quad (12)$$

where $u_{s_i}(y, t)$, $v_{s_i}(y, t)$, and $w_{s_i}(y, t)$ are the extensional displacement, lateral bending displacement, and vertical bending displacement, respectively; and ϕ_{s_i} , θ_{s_i} , and ψ_{s_i} are the torsional rotation, vertical bending rotation, and lateral bending rotation, respectively.

At the origin of the rotor support reference frames $(x_{s_i}, y_{s_i}, z_{s_i})$, the displacements and rotations of a rotor are denoted as

$$\mathbf{u}_{s_i} = \boldsymbol{\Phi}_{s_i}(\Delta x_{rs_{i0}}) \mathbf{q}(t) \quad (13)$$

$$\boldsymbol{\phi}_{s_i} = \boldsymbol{\Psi}_{s_i}(\Delta x_{rs_{i0}}) \mathbf{q}(t) \quad (14)$$

where $\Delta x_{rs_{i0}} = x_{rs_{i0}} - x_{s_{i0}}$.

The wing bending rotation ϕ_w and the torsional rotation ϕ_s of the rotor support structure add to the rotor cant angle η_i . Let η_i^* be the total cant angle where

$$\eta_i^* = \eta_i + \phi_w + \phi_{s_i} \quad (15)$$

The global reference frame (x, y, z) is related to the blade reference frames $(x_{b_i}, y_{b_i}, z_{b_i})$ by

$$\begin{aligned} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos \eta_i^* & -\sin \eta_i^* \\ 0 & -\sin \eta_i^* & -\cos \eta_i^* \end{bmatrix} \begin{bmatrix} \sin \Psi_i & \cos \Psi_i & 0 \\ \mp \cos \Psi_i & \pm \sin \Psi_i & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_{b_i} \\ \mathbf{j}_{b_i} \\ \mathbf{k}_{b_i} \end{bmatrix} \\ &= \begin{bmatrix} \sin \Psi_i & \cos \Psi_i & 0 \\ \mp \cos \eta_i^* \cos \Psi_i & \pm \cos \eta_i^* \sin \Psi_i & \mp \sin \eta_i^* \\ \mp \sin \eta_i^* \cos \Psi_i & \pm \sin \eta_i^* \sin \Psi_i & \pm \cos \eta_i^* \end{bmatrix} \begin{bmatrix} \mathbf{i}_{b_i} \\ \mathbf{j}_{b_i} \\ \mathbf{k}_{b_i} \end{bmatrix} \end{aligned} \quad (16)$$

The air velocity on a rigid blade section is computed by a coordinate transformation from the global reference frame (x, y, z) to the blade reference frames $(x_{b_i}, y_{b_i}, z_{b_i})$ as

$$\mathbf{v}_{b_i} = \bar{\mathbf{v}}_{b_i} + \mathbf{v}_{r_i} + \tilde{\mathbf{v}}_{b_i} \quad (17)$$

where $\bar{\mathbf{v}}_{b_i}$ is the air velocity due to the aircraft motion, $\mathbf{v}_{r_i}$ is the relative air velocity due to the blade motion in the rotor reference frame, and $\tilde{\mathbf{v}}_{b_i}$ is the air velocity due to the airframe elastic motion. These air velocities are given by

$$\bar{\mathbf{v}}_{b_i} = \mathbf{v} + \boldsymbol{\omega} \times (\mathbf{r}_{r_i} + \Delta \mathbf{r}_{b_i}) \quad (18)$$

$$\mathbf{v}_{r_i} = \mathbf{v}_p + \boldsymbol{\omega}_{r_i} \times \Delta \mathbf{r}_{b_i} \quad (19)$$

$$\tilde{\mathbf{v}}_{b_i} = \mathbf{v}_e + \boldsymbol{\omega}_e \times (\mathbf{p}_{r_i} + \Delta \mathbf{r}_{b_i}) \quad (20)$$

The air velocities are further described as follows.

- $\mathbf{v}_a = u\mathbf{i}_a + v\mathbf{j}_a + w\mathbf{k}_a$ is the aircraft velocity.
- The air velocity of the incoming flow to the rotors is opposite to the aircraft velocity and also includes the air velocity due to the elastic motion of the wing and rotor supports. It is given by $\mathbf{v} = -u\mathbf{i}_a - v\mathbf{j}_a - w\mathbf{k}_a$ where

$$v^* = v - u(\psi_w + \psi_{s_i}) \quad (21)$$

$$w^* = w + u(\theta_w + \theta_{s_i}) \quad (22)$$

- $\mathbf{v}_p = -v_{p_i}\mathbf{k}_{r_i}$ is the inflow velocity normal to the rotor plane.
- $\mathbf{v}_e = -\dot{\mathbf{u}}_w - \dot{\mathbf{u}}_{s_i}$ is the air velocity due to the elastic motion.
- $\boldsymbol{\omega}_a = p\mathbf{i}_a + q\mathbf{j}_a + r\mathbf{k}_a$ is the aircraft angular velocity.
- $\boldsymbol{\omega} = -\boldsymbol{\omega}_a$ is the air angular velocity opposite to the aircraft angular velocity.
- $\boldsymbol{\omega}_{r_i} = -(\pm\dot{\theta}_{b_i}\mathbf{j}_{b_i} + \Omega_i\mathbf{k}_{b_i})$ is the air angular velocity due to rotor rotation where $\dot{\theta}_{b_i}\mathbf{j}_{b_i}$ is the blade pitch velocity about the blade pitch axis and $\Omega_i\mathbf{k}_{b_i}$ is the blade angular velocity.
- $\boldsymbol{\omega}_e = -(\dot{\phi}_w + \dot{\phi}_{s_i})$ is the air angular velocity due to the elastic motion.
- $\mathbf{r}_{r_i}$ is a position vector from the aircraft CG to the origin of a rotor reference frame given by

$$\mathbf{r}_{r_i} = \Delta x_{r_{i0}}\mathbf{i} + \Delta y_{r_{i0}}\mathbf{j} + \Delta z_{r_{i0}}\mathbf{k} \quad (23)$$

where $(\Delta x_{r_{i0}}, \Delta y_{r_{i0}}, \Delta z_{r_{i0}}) = (x_{r_{i0}} - x_{cg}, y_{r_{i0}} - y_{cg}, z_{r_{i0}} - z_{cg})$ is the coordinate of the origin of the rotor i -th reference frame relative to the aircraft CG.

- $\Delta \mathbf{r}_{b_i}$ is a position vector from the origin of a rotor reference frame to a point on a blade section given by

$$\Delta \mathbf{r}_{b_i} = (\xi - \xi_e)\mathbf{i}_{b_i} + r'\mathbf{j}_{b_i} \quad (24)$$

where ξ is the blade airfoil coordinate defined along the blade chord direction, pointing from the leading edge at $\xi = 0$ to the trailing edge at $\xi = c_b$ and r' is the blade radial station.

- $\mathbf{p}_{r_i}$ is a position vector from the origin of a rotor support to the origin of a rotor reference frame given by

$$\mathbf{p}_{r_i} = \Delta x_{rs_{i0}}\mathbf{i} + \Delta y_{rs_{i0}}\mathbf{j} + \Delta z_{rs_{i0}}\mathbf{k} \quad (25)$$

where $(\Delta x_{rs_{i0}}, \Delta y_{rs_{i0}}, \Delta z_{rs_{i0}}) = (x_{r_{i0}} - x_{s_{i0}}, y_{r_{i0}} - y_{s_{i0}}, z_{r_{i0}} - z_{s_{i0}})$ is the coordinate of the origin of the rotor i -th reference frame relative to the origin of the rotor support i -th reference frame.

The air velocity due to the aircraft motion is obtained in the blade reference frame $(x_{b_i}, y_{b_i}, z_{r_i})$ as

$$\begin{aligned} \bar{\mathbf{v}}_{b_i} &= (u - q\Delta z_{r_{i0}} - r\Delta y_{r_{i0}})\mathbf{i} + (-v^* - p\Delta z_{r_{i0}} + r\Delta x_{r_{i0}})\mathbf{j} + (w^* + p\Delta y_{r_{i0}} + q\Delta z_{r_{i0}})\mathbf{k} \\ &\quad + (p\mathbf{i} - q\mathbf{j} + r\mathbf{k}) \times [(\xi - \xi_e)\mathbf{i}_{b_i} + r'\mathbf{j}_{b_i}] \\ &= \bar{v}_{x_{b_i}}\mathbf{i}_{b_i} + \bar{v}_{y_{b_i}}\mathbf{j}_{b_i} + \bar{v}_{z_{b_i}}\mathbf{k}_{b_i} \end{aligned} \quad (26)$$

The relative air velocity due to the blade motion is

$$\begin{aligned}\mathbf{v}_{r_i} &= -v_{p_i} \mathbf{k}_{r_i} - \left(\pm \dot{\theta}_{b_i} \mathbf{j}_{b_i} + \Omega_i \mathbf{k}_{b_i} \right) \times \Delta \mathbf{r}_{b_i} \\ &= \Omega_i r'_{b_i} - \Omega_i (\xi - \xi_e) \mathbf{j}_{b_i} \mp \left[v_{p_i} - \dot{\theta}_{b_i} (\xi - \xi_e) \right] \mathbf{k}_{b_i}\end{aligned}\quad (27)$$

The air velocity due to the airframe elastic motion is computed as

$$\begin{aligned}\tilde{\mathbf{v}}_{b_i} &= -(\dot{u}_w + \dot{u}_{s_i}) \mathbf{i} - (\dot{v}_w + \dot{v}_{s_i}) \mathbf{j} - (\dot{w}_w + \dot{w}_{s_i}) \mathbf{k} \\ &\quad - \left[\left(\dot{\phi}_w + \dot{\phi}_{s_i} \right) \mathbf{i} + \left(\dot{\theta}_w + \dot{\theta}_{s_i} \right) \mathbf{j} + \left(\dot{\psi}_w + \dot{\psi}_{s_i} \right) \mathbf{k} \right] \times (\mathbf{p}_{r_i} + \Delta \mathbf{r}_{b_i}) \\ &= \tilde{v}_{x_{b_i}} \mathbf{i}_{b_i} + \tilde{v}_{y_{b_i}} \mathbf{j}_{b_i} + \tilde{v}_{z_{b_i}} \mathbf{k}_{b_i}\end{aligned}\quad (28)$$

The tangential, radial, and normal airspeeds on a rotor blade section are obtained

$$\begin{bmatrix} V_t(r', \xi, t) \\ V_r(r', \xi, t) \\ V_z(r', \xi, t) \end{bmatrix} = \begin{bmatrix} \bar{v}_{x_{b_i}} + \Omega_i r' + \tilde{v}_{x_{b_i}} \\ \bar{v}_{y_{b_i}} - \Omega_i (\xi - \xi_e) + \tilde{v}_{y_{b_i}} \\ \pm \bar{v}_{z_{b_i}} - v_{p_i} + \dot{\theta}_{b_i} (\xi - \xi_e) \pm \tilde{v}_{z_{b_i}} \end{bmatrix}\quad (29)$$

The inflow velocity is prescribed by

$$v_{p_i} = v_{0,p_i} + v_{s,p_i} \frac{r'}{R} \sin \Psi_i + v_{c,p_i} \frac{r'}{R} \cos \Psi_i \quad (30)$$

The Pitt-Peters three-state inflow model^{6,7} is used to describe the inflow velocity

$$\mathbf{M}_{p_i} \dot{\mathbf{v}}_{p_i} = -\mathbf{L}_{p_i} \mathbf{v}_{p_i} + \mathbf{C}_{p_i} \quad (31)$$

where $\mathbf{v}_{p_i} = \begin{bmatrix} v_{0,p_i} & v_{s,p_i} & v_{c,p_i} \end{bmatrix}^\top$, $\mathbf{C}_{p_i} = \begin{bmatrix} C_{T_{r_i}} & C_{l_{r_i}} & C_{m_{r_i}} \end{bmatrix}^\top$, $\mathbf{M}_{p_i} = \frac{1}{R\Omega_i^2} \text{diag} \left(\begin{bmatrix} \frac{8}{3\pi} & -\frac{16}{45\pi} \\ -\frac{16}{45\pi} \end{bmatrix} \right)$ is the apparent mass matrix, and $\mathbf{L}_{p_i} = \frac{R^2\Omega_i^2}{\bar{v}_{p_i}} \text{diag} \left(\begin{bmatrix} \frac{1}{2} & -1 & -1 \end{bmatrix} \right)$ for hover and $\mathbf{L}_{p_i} = \frac{R^2\Omega_i^2}{V} \begin{bmatrix} \frac{64}{75\pi} & 0 & -\frac{1}{2} \\ 0 & \frac{64}{45\pi} & 0 \\ \frac{2}{5} & 0 & 0 \end{bmatrix}$ is the reciprocal of the sensitivity matrix. The thrust and moment coefficients are defined as

$$C_{T_{r_i}} = \frac{T_{r_i}}{\rho_\infty \pi R^4 \Omega_i^2} \quad (32)$$

$$C_{l_{r_i}} = \frac{l_{r_i}}{\rho_\infty \pi R^5 \Omega_i^2} \quad (33)$$

$$C_{m_{r_i}} = \frac{m_{r_i}}{\rho_\infty \pi R^5 \Omega_i^2} \quad (34)$$

D. Air Acceleration Analysis

The air acceleration creates inertial forces, also called non-circulatory forces. The air acceleration is computed as

$$\mathbf{a}_{b_i} = \left(\frac{d\mathbf{v}_{b_i}}{dt} \right)_{b_i} + (\boldsymbol{\omega} + \boldsymbol{\omega}_{r_i} + \boldsymbol{\omega}_e) \times \mathbf{v}_{b_i} \quad (35)$$

where $\left(\frac{d}{dt} \right)_{b_i}$ denotes differentiation with respect to the blade reference frame.

The air acceleration is comprised of the air acceleration component $\bar{\mathbf{a}}_{b_i}$ due to the rigid-body aircraft motion, the air acceleration component $\mathbf{a}_{r_i}$ due to the rigid-body motion of the rotor, and the air acceleration component $\tilde{\mathbf{a}}_{b_i}$ due to the elastic motion of the airframe. These components are computed as

$$\bar{\mathbf{a}}_{b_i} = \left(\frac{d\bar{\mathbf{v}}_{b_i}}{dt} \right)_{b_i} + \boldsymbol{\omega} \times \bar{\mathbf{v}}_{b_i} \quad (36)$$

$$\mathbf{a}_{r_i} = \left(\frac{d\mathbf{v}_{r_i}}{dt} \right)_{b_i} + (\boldsymbol{\omega} + \boldsymbol{\omega}_{r_i}) \times \mathbf{v}_{r_i} + \boldsymbol{\omega}_{r_i} \times \bar{\mathbf{v}}_{b_i} \quad (37)$$

$$\tilde{\mathbf{a}}_{b_i} = \left(\frac{d\tilde{\mathbf{v}}_{b_i}}{dt} \right)_{b_i} + (\boldsymbol{\omega} + \boldsymbol{\omega}_{r_i}) \times \tilde{\mathbf{v}}_{b_i} + \boldsymbol{\omega}_e \times \mathbf{v}_{b_i} \quad (38)$$

Thus, the total air acceleration is expressed as

$$\mathbf{a}_{b_i} = \bar{\mathbf{a}}_{b_i} + \mathbf{a}_{r_i} + \tilde{\mathbf{a}}_{b_i} \quad (39)$$

We are interested in the vertical air acceleration component $a_{z_{b_i}} = \mathbf{a}_{b_i} \cdot \mathbf{k}_{r_i}$ in the rotor thrust axis along the $\mathbf{k}_{r_i}$ -direction which contributes to the non-circulatory force.

III. Unsteady Aerodynamic Forces

The unsteady aerodynamic forces comprise forces due to lift circulation, also called circulatory forces, and non-circulatory forces which are inertial reaction forces exerted by the air. These forces are treated in the following sections.

A. Lifting Line Theory

The lift circulation of a rotor blade section in forward flight is a function of the radius r' and azimuthal angle Ψ given by

$$\frac{d\Gamma(r', \Psi)}{d\Psi} = C(k) \frac{1}{2} V(r', \Psi) c_b(r') c_l(r', \Psi) = C(k) \frac{1}{2} (r'\Omega + u \sin \Psi) c_b c_l \quad (40)$$

where u is the axial aircraft velocity component, $C(k)$ is the Theodorsen's function⁸ and $k = \frac{\Omega \bar{c}_b}{2\Omega \bar{r}} = \frac{2\bar{c}_b}{3R}$ is the reduced frequency evaluated using the mean chord $\bar{c}_b$ at the mean radius $\bar{r} = \frac{3R}{4}$ and R is the rotor radius.

The lift coefficient is expressed as

$$c_l = c_{l_\alpha}^* \left[\alpha(r') - \alpha_0^*(r') + \frac{w_i(r', \Psi)}{r'\Omega + V_x \sin \Psi} \right] \quad (41)$$

where α is the angle of attack along the blade radial station, $c_{l_\alpha}^* = \frac{2\pi}{\sqrt{1-M_\infty^2}}$, α_0^* is the angle of attack at zero lift of the blade airfoil, and w_i is the inflow velocity.

Thus,

$$\frac{d\Gamma(r', \Psi)}{d\Psi} = C(k) \frac{1}{2} (r'\Omega + u \sin \Psi) c_b c_{l_\alpha}^* (\alpha - \alpha_0^*) + \frac{1}{2} C(k) c_b c_{l_\alpha}^* w_i \quad (42)$$

The following inflow model is used:

$$w_i = w_0(r') + w_s(r') \sin \Psi + w_c(r') \cos \Psi \quad (43)$$

Since $C(k) = F(k) + iG(k)$ is a complex expression and $\Psi = \Omega t$, we can express the lift circulation in the time domain as

$$\frac{d\Gamma(r', \Psi)}{d\Psi} = \frac{1}{2} [F(k) (r'\Omega + u \sin \Psi) + G(k) u \cos \Psi] c_b c_{l_\alpha}^* (\alpha - \alpha_0^*) + \frac{1}{2} c_b c_{l_\alpha}^* \left[F(k) w_i + \frac{G(k)}{\Omega} \dot{w}_i \right] \quad (44)$$

Integrating the lift circulation over one revolution yields the circumferentially averaged lift circulation

$$\begin{aligned} \Gamma(r') &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\Gamma}{d\Psi} d\Psi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} [F(k) (r'\Omega + u \sin \Psi) + G(k) u \cos \Psi] c_b c_{l_\alpha}^* (\alpha - \alpha_0^*) d\Psi \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} c_b c_{l_\alpha}^* \left[F(k) w_i + \frac{G(k)}{\Omega} \dot{w}_i \right] d\Psi \\ &= \frac{1}{2} F(k) [r'\Omega c_b c_{l_\alpha}^* (\alpha - \alpha_0^*) + w_0] \end{aligned} \quad (45)$$

The lift circulation is expressed as a Fourier sine series as

$$\Gamma(r') = \sum_{n=1}^N A_n \sin n\phi \quad (46)$$

where $r' = \frac{R+R_i}{2} - \frac{R-R_i}{2} \cos \phi$ and R_i is the rotor hub radius. Thus, $r' \in [R_i, R]$ is mapped to $\phi \in [0, \pi]$.

The induced vertical airspeed is obtained from the lifting line theory as

$$w_0 = -\frac{1}{4\pi} \int_{R_i}^R \frac{\frac{d\Gamma}{d\eta} d\eta}{r' - \eta} \quad (47)$$

where $\eta = \frac{R+R_i}{2} - \frac{R-R_i}{2} \cos \varphi$.

Hence,

$$w_0 = -\frac{1}{\pi(R-R_i)} \int_0^\pi \frac{\sum_{n=1}^N A_n n \cos n\varphi d\varphi}{\cos \varphi - \cos \phi} = -\frac{1}{2(R-R_i)} \sum_{n=1}^N A_n \frac{n \sin n\phi}{\sin \phi} \quad (48)$$

Therefore, we have

$$\Gamma = \sum_{n=1}^N A_n \sin n\phi = \frac{1}{2} F(k) r' \Omega c_b c_{l_\alpha}^* (\alpha - \alpha_0^*) - \frac{F(k) c_b c_{l_\alpha}^*}{4(R-R_i)} \sum_{n=1}^N A_n \frac{n \sin n\phi}{\sin \phi} \quad (49)$$

Integrating Eq. (49) upon multiplying it by $\sin m\phi$ yields

$$\sum_{n=1}^N \int_0^\pi A_n \left[\sin n\phi + \frac{F(k) c_b c_{l_\alpha}^*}{4(R-R_i)} \frac{n \sin n\phi}{\sin \phi} \right] \sin m\phi d\phi = \int_0^\pi \frac{1}{2} F(k) r' \Omega c_b c_{l_\alpha}^* (\alpha - \alpha_0^*) \sin m\phi d\phi \quad (50)$$

This results in the matrix equation

$$\sum_{n=1}^N \left(\frac{\pi}{2} \delta_{mn} + a_{mn} \right) A_n = b_m \quad (51)$$

where

$$a_{mn} = \int_0^\pi \frac{F(k) c_b c_{l_\alpha}^*}{4(R-R_i)} \frac{n \sin n\phi \sin m\phi}{\sin \phi} d\phi \quad (52)$$

$$b_m = \int_0^\pi \frac{1}{2} F(k) r' \Omega c_b c_{l_\alpha}^* (\alpha - \alpha_0^*) \sin m\phi d\phi \quad (53)$$

Taking the partial derivative of Γ with respect to α and solving for c_{l_α} result in

$$c_{l_\alpha} = \frac{2}{r' \Omega c_b} \sum_{n=1}^N \frac{\partial A_n}{\partial \alpha} \sin n\phi \quad (54)$$

B. Circulatory Lift Analysis

The air velocity $\mathbf{v}(r', \frac{3c_b}{4}, t)$ at the three quarter-chord location is obtained by letting $\xi - \xi_e = \frac{3c_b}{4} - \xi_e = e_{cb}$ where e_{cb} is the distance between the three-quarter-chord and the pitch center, positive when the three quarter-chord location is aft of the pitch center.

The blade angle of attack is defined as

$$\alpha_{b_i} \left(r', \frac{3c_b}{4}, t \right) = \tan^{-1} \frac{V_z \left(r', \frac{3c_b}{4}, t \right)}{V_t \left(r', \frac{3c_b}{4}, t \right)} \quad (55)$$

The circulatory lift is given by

$$\frac{dl_{c,b_i}}{d\Psi_i} = \frac{1}{2} \rho_\infty V^2(r', \xi_e, t) c_b c_l \quad (56)$$

The lift coefficient is given by

$$c_l = c_{l_0} + c_{l_\alpha} \alpha \quad (57)$$

where α is the angle of attack on a blade section given by

$$\alpha = \theta_{b_i}(r', t) + \alpha_{b_i} \left(r', \frac{3c_b}{4}, t \right) \quad (58)$$

The blade pitch angle is given by

$$\theta_{b_i}(r', t) = \bar{\theta}_{b_i}(r') + \tilde{\theta}_{b_i}(r', \Psi, t) \quad (59)$$

where $\tilde{\theta}_{b_i}(r', t)$ is the blade pitch control input which can be expressed as

$$\tilde{\theta}_{b_i} = \sum_{n=0}^N \theta_{s_{i,n}} \sin n\Omega t + \theta_{c_{i,n}} \cos n\Omega t \quad (60)$$

The circulatory drag is modeled as

$$\frac{dd_{c,b_i}}{d\Psi_i} = \frac{1}{2} \rho_\infty V^2(r', \xi_e, t) c_b c_d \quad (61)$$

where the drag coefficient c_d is given by

$$c_d = c_{d0} + c_{d_\alpha} \alpha + c_{d_{\alpha^2}} \alpha^2 \quad (62)$$

The circulatory force on a blade section in the blade reference frame is expressed as

$$\mathbf{f}_{c,b_i} = \frac{df_{x_{c,b_i}}}{d\Psi_i} \mathbf{i}_{b_i} \pm \frac{df_{z_{c,b_i}}}{d\Psi_i} \mathbf{k}_{b_i} \quad (63)$$

The circulatory force on a blade section in the aircraft reference frame is

$$\mathbf{f}_{c,b_i} = \frac{df_{x_{c,a_i}}}{d\Psi_i} \mathbf{i}_a + \frac{df_{y_{c,a_i}}}{d\Psi_i} \mathbf{j}_a + \frac{df_{z_{c,a_i}}}{d\Psi_i} \mathbf{k}_a \quad (64)$$

The blade reference frame is related to the rotor reference frame by Eq. (4). The circulatory force on a blade section in the rotor reference frame is

$$\mathbf{f}_{c,b_i} = \frac{df_{x_{c,r_i}}}{d\Psi_i} \mathbf{i}_{r_i} + \frac{df_{y_{c,r_i}}}{d\Psi_i} \mathbf{j}_{r_i} + \frac{df_{z_{c,r_i}}}{d\Psi_i} \mathbf{k}_{r_i} \quad (65)$$

The circulatory rotor thrust force in the rotor reference frame is

$$\frac{d^2 T_{c,r_i}}{dr' d\Psi_i} = \frac{df_{z_{c,r_i}}}{d\Psi_i} = \frac{df_{z_{c,b_i}}}{d\Psi_i} = \frac{dl_{c,b_i}}{d\Psi_i} \cos \alpha_{b_i} + \frac{dd_{c,b_i}}{d\Psi_i} \sin \alpha_{b_i} \quad (66)$$

The position vector of the quarter-chord point on a blade from the origin of the rotor reference frame is expressed as

$$\Delta \mathbf{r}_{b_i} \left(\frac{c_b}{4} \right) = r' \mathbf{j}_{b_i} - e_b \mathbf{i}_{b_i} = r' (\cos \Psi_i \mathbf{i}_{r_i} \pm \sin \Psi_i \mathbf{j}_{r_i}) - e_b (\sin \Psi_i \mathbf{i}_{r_i} \mp \cos \Psi_i \mathbf{j}_{r_i}) \quad (67)$$

where $e_b = \xi_e - \frac{c_b}{4}$ is the offset between the blade elastic center and the aerodynamic center at the quarter-chord location.

The circulatory rotor moment about the origin of the i -th rotor reference frame is

$$\mathbf{M}_{c,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \Delta \mathbf{r}_{b_i} \left(\frac{c_b}{4} \right) \times \mathbf{f}_{c,b_i} d\Psi_i dr' = -l_{c,r_i} \mathbf{i}_{r_i} + m_{c,r_i} \mathbf{j}_{r_i} - Q_{c,r_i} \mathbf{k}_{r_i} \quad (68)$$

where

$$l_{c,r_i} = \mp \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} (r' \sin \Psi_i + e_b \cos \Psi_i) \frac{df_{z_{c,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (69)$$

$$m_{c,r_i} = -\frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} (r' \cos \Psi_i - e_b \sin \Psi_i) \frac{df_{z_{c,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (70)$$

$$Q_{c,r_i} = \pm \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} r' \frac{df_{x_{c,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (71)$$

with $N_b = 2$.

Note that the negative sign in front of the rolling moment l_{c,r_i} and yawing moment or rotor torque Q_{c,r_i} reflects the sign convention in flight dynamics where the positive rolling moment points forward and positive yawing moment points down.

The circulatory rotor force acting on the aircraft due to the i -th rotor is obtained as

$$\mathbf{F}_{c,a_i} = X_{c,a_i} \mathbf{i}_a + Y_{c,a_i} \mathbf{j}_a + Z_{c,a_i} \mathbf{k}_a \quad (72)$$

where

$$X_{c,a_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{x_{c,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (73)$$

$$Y_{c,a_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{y_{c,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (74)$$

$$Z_{c,a_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{z_{c,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (75)$$

The circulatory rotor moment about the aircraft CG due to the i -th rotor is obtained as

$$\mathbf{M}_{c,a_i} = \mathbf{M}_{c,r_i} + \mathbf{r}_{r_i} \times \mathbf{F}_{c,r_i} = l_{c,a_i} \mathbf{i}_a + m_{c,a_i} \mathbf{j}_a + n_{c,a_i} \mathbf{k}_a \quad (76)$$

where

$$l_{c,a_i} = l_{c,r_i} + Y_{c,a_i} \Delta z_{r_{i0}} + Z_{c,a_i} \Delta y_{r_{i0}} \quad (77)$$

$$m_{c,a_i} = m_{c,r_i} \cos \eta_i + Q_{c,r_i} \sin \eta_i - X_{c,a_i} \Delta z_{r_{i0}} + Z_{c,a_i} \Delta x_{r_{i0}} \quad (78)$$

$$n_{c,a_i} = -m_{c,r_i} \sin \eta_i + Q_{c,r_i} \cos \eta_i - X_{c,a_i} \Delta y_{r_{i0}} - Y_{c,a_i} \Delta x_{r_{i0}} \quad (79)$$

C. Non-Circulatory Lift Analysis

The air vertical acceleration can be separated into two components

$$a_{z_{b_i}}(r', \xi, t) = a_{z_{0,b_i}}(r', t) + a_\xi(r', t)(\xi - \xi_e) \quad (80)$$

where $a_{z_0}(r', t)$ is a translational vertical acceleration component and $a_\xi(r', t)$ is a rotational angular or pitch acceleration component.

The non-circulatory lift contribution due to the air acceleration at the mid-chord location according to Theodorsen's theory^{8,9} is computed as

$$\frac{dl_{nc,b_i}}{d\Psi_i} = \frac{\pi}{4} \rho_\infty c_b^2 a_{z_{b_i}} \left(r', \frac{c_b}{2}, t \right) \quad (81)$$

The non-circulatory pitching moment contribution about the blade pitch axis is computed as

$$\frac{dm_{nc,b_i}}{d\Psi_i} = \frac{dm_{ac,b_i}}{d\Psi_i} - e_{mb} \frac{dl_{nc,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \quad (82)$$

where $e_{mb} = \frac{c_b}{2} - \xi_e$ is the offset between the blade pitch axis and the mid-chord location and $\frac{dm_{ac,b_i}}{d\Psi_i}$ is the pitching moment about the aerodynamic center of a blade section given by

$$\frac{dm_{ac,b_i}}{d\Psi_i} = c_{mac} \frac{1}{2} \rho_\infty V^2(r', \xi_e, t) c_b^2 \quad (83)$$

The pitching moment about the aerodynamic center does not depend on lift, but is a function of the effective blade mean camber angle which is expressed as

$$\frac{dz}{d\xi} = -\tan^{-1} \frac{V_z}{V_t} \approx -\alpha_{b_i}^*(r', t) - \alpha_\xi(r', t)(\xi - \xi_e) \quad (84)$$

where

$$\alpha_\xi(r', t) = \frac{\bar{V}_t}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_z}{\partial(\xi - \xi_e)} - \frac{\bar{V}_z}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_t}{\partial(\xi - \xi_e)} \quad (85)$$

Using the transformation $\xi = \frac{c_b}{2}(1 - \cos \phi)$, the pitching moment coefficient about the quarter-chord aerodynamic center is evaluated as¹⁰

$$\begin{aligned} c_{m_{ac}} &= \frac{1}{2} \int_0^\pi \frac{dz}{dx} (\cos 2\theta - \cos \theta) d\theta \\ &= -\frac{1}{2} \int_0^\pi \left[\alpha_{b_i}^* + \alpha_\xi \frac{c_b}{2} (\cos \phi_e - \cos \phi) \right] (\cos 2\phi - \cos \phi) d\phi = -\frac{\pi}{8} c_b \alpha_\xi \end{aligned} \quad (86)$$

We assume the non-circulatory drag is zero. Then, the non-circulatory force on a blade section in the blade reference frame is expressed as

$$\mathbf{f}_{nc, b_i} = \frac{df_{x_{nc, b_i}}}{d\Psi_i} \mathbf{i}_{b_i} \pm \frac{df_{z_{nc, b_i}}}{d\Psi_i} \mathbf{k}_{b_i} \quad (87)$$

where

$$\frac{df_{x_{nc, b_i}}}{d\Psi_i} = -\frac{dl_{nc, b_i}}{d\Psi_i} \sin \alpha_{b_i} \quad (88)$$

$$\frac{df_{z_{nc, b_i}}}{d\Psi_i} = \frac{dl_{nc, b_i}}{d\Psi_i} \cos \alpha_{b_i} \quad (89)$$

The non-circulatory force on a blade section in the aircraft reference frame is

$$\mathbf{f}_{nc, b_i} = \frac{df_{x_{nc, a_i}}}{d\Psi_i} \mathbf{i}_a + \frac{df_{y_{nc, a_i}}}{d\Psi_i} \mathbf{j}_a + \frac{df_{z_{nc, a_i}}}{d\Psi_i} \mathbf{k}_a \quad (90)$$

where

$$\frac{df_{x_{nc, a_i}}}{d\Psi_i} = -\frac{df_{x_{nc, b_i}}}{d\Psi_i} \sin \Psi_i \quad (91)$$

$$\frac{df_{y_{nc, a_i}}}{d\Psi_i} = \mp \frac{df_{x_{nc, b_i}}}{d\Psi_i} \cos \eta_i \cos \Psi_i - \frac{df_{z_{nc, b_i}}}{d\Psi_i} \sin \eta_i \quad (92)$$

$$\frac{df_{z_{nc, a_i}}}{d\Psi_i} = \pm \frac{df_{x_{nc, b_i}}}{d\Psi_i} \sin \eta_i \cos \Psi_i - \frac{df_{z_{nc, b_i}}}{d\Psi_i} \cos \eta_i \quad (93)$$

The non-circulatory force on a blade section in the rotor reference frame is

$$\mathbf{f}_{nc, b_i} = \frac{df_{x_{nc, r_i}}}{d\Psi_i} \mathbf{i}_{r_i} + \frac{df_{y_{nc, r_i}}}{d\Psi_i} \mathbf{j}_{r_i} + \frac{df_{z_{nc, r_i}}}{d\Psi_i} \mathbf{k}_{r_i} \quad (94)$$

where

$$\frac{df_{x_{nc, r_i}}}{d\Psi_i} = \frac{df_{x_{nc, b_i}}}{d\Psi_i} \sin \Psi_i \quad (95)$$

$$\frac{df_{y_{nc, r_i}}}{d\Psi_i} = \mp \frac{df_{x_{nc, b_i}}}{d\Psi_i} \cos \Psi_i \quad (96)$$

$$\frac{df_{z_{nc, r_i}}}{d\Psi_i} = \frac{df_{z_{nc, b_i}}}{d\Psi_i} \quad (97)$$

The non-circulatory rotor thrust force in the rotor reference frame is

$$\frac{d^2 T_{nc, r_i}}{dr' d\Psi_i} = \frac{df_{z_{nc, r_i}}}{d\Psi_i} = \frac{df_{z_{nc, b_i}}}{d\Psi_i} = \frac{dl_{nc, b_i}}{d\Psi_i} \cos \alpha_{b_i} \quad (98)$$

The position vector of the mid-chord point on a blade from the origin of the rotor reference frame is expressed as

$$\Delta \mathbf{r}_{b_i} \left(\frac{c}{2} \right) = r' \mathbf{j}_{b_i} + e_{mb} \mathbf{i}_{b_i} = r' (\cos \Psi_i \mathbf{i}_{r_i} \pm \sin \Psi_i \mathbf{j}_{r_i}) + e_{mb} (\sin \Psi_i \mathbf{i}_{r_i} \mp \cos \Psi_i \mathbf{j}_{r_i}) \quad (99)$$

The non-circulatory rotor moment about the origin of the i -th rotor reference frame is

$$\begin{aligned}\mathbf{M}_{nc,r_i} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[\Delta \mathbf{r}_{b_i} \left(\frac{c_b}{2} \right) \times \mathbf{f}_{nc,b_i} + \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \mathbf{j}_{b_i} \right] d\Psi_i dr' \\ &= -l_{nc,r_i} \mathbf{i}_{r_i} + m_{nc,r_i} \mathbf{j}_{r_i} - Q_{nc,r_i} \mathbf{k}_{r_i}\end{aligned}\quad (100)$$

where

$$\begin{aligned}l_{nc,r_i} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[\mp (r' \sin \Psi_i - e_{mb} \cos \Psi_i) \frac{df_{z_{nc,b_i}}}{d\Psi_i} \right. \\ &\quad \left. - \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \cos \Psi_i \right] d\Psi_i dr'\end{aligned}\quad (101)$$

$$\begin{aligned}m_{nc,r_i} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[- (r' \cos \Psi_i + e_{mb} \sin \Psi_i) \frac{df_{z_{nc,b_i}}}{d\Psi_i} \right. \\ &\quad \left. \pm \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \sin \Psi_i \right] d\Psi_i dr'\end{aligned}\quad (102)$$

$$Q_{nc,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \pm r' \frac{df_{x_{nc,b_i}}}{d\Psi_i} d\Psi_i dr' \quad (103)$$

The non-circulatory force acting on the aircraft due to the i -th rotor is obtained as

$$\mathbf{F}_{nc,a_i} = X_{nc,a_i} \mathbf{i}_a + Y_{nc,a_i} \mathbf{j}_a + Z_{nc,a_i} \mathbf{k}_a \quad (104)$$

where

$$X_{nc,a_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{x_{nc,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (105)$$

$$Y_{nc,a_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{y_{nc,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (106)$$

$$Z_{nc,a_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{df_{z_{nc,a_i}}}{d\Psi_i} d\Psi_i dr' \quad (107)$$

The non-circulatory moment about the aircraft CG due to the i -th rotor is obtained as

$$\mathbf{M}_{nc,a_i} = \mathbf{M}_{nc,r_i} + \mathbf{r}_{r_i} \times \mathbf{F}_{nc,r_i} = l_{nc,a_i} \mathbf{i}_a + m_{nc,a_i} \mathbf{j}_a + n_{nc,a_i} \mathbf{k}_a \quad (108)$$

where

$$l_{nc,a_i} = l_{nc,r_i} + Y_{nc,a_i} \Delta z_{r_{i0}} + Z_{nc,a_i} \Delta y_{r_{i0}} \quad (109)$$

$$m_{nc,a_i} = m_{nc,r_i} \cos \eta_i + Q_{nc,r_i} \sin \eta_i - X_{nc,a_i} \Delta z_{r_{i0}} + Z_{nc,a_i} \Delta x_{r_{i0}} \quad (110)$$

$$n_{nc,a_i} = -m_{nc,r_i} \sin \eta_i + Q_{nc,r_i} \cos \eta_i - X_{nc,a_i} \Delta y_{r_{i0}} - Y_{nc,a_i} \Delta x_{r_{i0}} \quad (111)$$

IV. Partial Derivatives of Forces and Moments

The partial derivatives of forces and moments are to be computed as linear contributions to the forces and moments by the state variables about a trim point. These partial derivatives are referred to as stability and control derivatives. By summing the partial derivative contributions, the total forces and moments are obtained. The dynamic stability of the aircraft motion is influenced by the stability derivatives.

A. Partial Derivatives of Circulatory Forces and Moments

We write the lift force as

$$\frac{dl_{c,b_i}}{d\Psi_i} = \frac{dl_{c,b_i}}{d\Psi_i} \left(V \left(\mathbf{x}_a, \mathbf{v}_{p_i}, \mathbf{q}, \dot{\mathbf{q}}, \dot{\theta}_{b_i}, \Omega_i \right), c_l \left(M(V), \alpha \left(\tilde{\theta}_{b_i}, \alpha_{b_i}(V) \right) \right) \right) \quad (112)$$

where $\mathbf{x}_a(t) = \begin{bmatrix} u(t) & v(t) & w(t) & p(t) & q(t) & r(t) \end{bmatrix}^\top$ is the aircraft state vector.

Let $x(t)$ be a dummy variable representing a state or control variable or their time derivatives. The partial derivative of $\frac{dl_{c,b_i}}{d\Psi_i}$ with respect to $x(t)$ is evaluated as

$$\frac{\partial}{\partial x} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) = \frac{\partial}{\partial V} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) \frac{\partial V}{\partial x} + \frac{\partial}{\partial c_l} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) \left(\frac{\partial c_l}{\partial M} \frac{\partial M}{\partial x} + \frac{\partial c_l}{\partial \alpha} \frac{\partial \alpha}{\partial x} \right) \quad (113)$$

The partial derivatives of $\frac{dl_{c,b_i}}{d\Psi_i}$ are

$$\frac{\partial}{\partial V} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) = \bar{c}_l \frac{1}{2} \rho_\infty 2\bar{V} c_b \quad (114)$$

$$\frac{\partial}{\partial c_l} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) = \frac{1}{2} \rho_\infty \bar{V}^2 c_b \quad (115)$$

where

$$\bar{V} = \sqrt{\bar{V}_t^2 + \bar{V}_r^2 + \bar{V}_z^2} \quad (116)$$

The partial derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial M}{\partial x}$ are

$$\frac{\partial V}{\partial x} = \frac{\bar{V}_t}{\bar{V}} \frac{\partial V_t}{\partial x} + \frac{\bar{V}_r}{\bar{V}} \frac{\partial V_r}{\partial x} + \frac{\bar{V}_z}{\bar{V}} \frac{\partial V_z}{\partial x} \quad (117)$$

$$\frac{\partial M}{\partial x} = \frac{1}{a(h)} \frac{\partial V}{\partial x} = \frac{1}{a(h)} \frac{\partial V}{\partial x} \quad (118)$$

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha_{b_i}}{\partial x} \quad (119)$$

$$\frac{\partial \alpha_{b_i}}{\partial x} = \frac{\bar{V}_t}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_z}{\partial x} - \frac{\bar{V}_z}{\bar{V}_t^2 + \bar{V}_z^2} \frac{\partial V_t}{\partial x} \quad (120)$$

The lift coefficient is expressed as¹⁰

$$c_l = c_{l_\alpha} (\alpha - \alpha_0) = c_{l_\alpha}^* \left[\alpha - \alpha_0 - \frac{C_{L\kappa}(\phi)}{\pi \mathcal{R}} \right] \quad (121)$$

where

$$\kappa(\phi) = 1 + \sum_{n=2}^N \frac{n A_n \sin n\phi}{A_1 \sin \phi} \quad (122)$$

Taking the partial derivative of c_l with respect to α yields

$$c_{l_\alpha} = c_{l_\alpha}^* \left[1 - \frac{C_{L\kappa}(\phi)}{\pi \mathcal{R}} \right] \quad (123)$$

The lift curve slope C_{L_α} is given by

$$C_{L_\alpha} = \frac{c_{l_\alpha}^*}{1 + \frac{c_{l_\alpha}^* \bar{\kappa}}{\pi \mathcal{R}}} \quad (124)$$

where $\bar{\kappa}$ is the mean value of κ .

Hence,

$$c_{l_\alpha} = c_{l_\alpha}^* \left[1 - \frac{\frac{c_{l_\alpha}^* \kappa(\phi)}{\pi \mathcal{R}}}{1 + \frac{c_{l_\alpha}^* \bar{\kappa}}{\pi \mathcal{R}}} \right] = c_{l_\alpha}^* \frac{1 + \frac{c_{l_\alpha}^* [\bar{\kappa} - \kappa(\phi)]}{\pi \mathcal{R}}}{1 + \frac{c_{l_\alpha}^* \bar{\kappa}}{\pi \mathcal{R}}} \quad (125)$$

Making an assumption that $\kappa(\phi) \approx 1$. Then,

$$c_{l_\alpha} = \frac{c_{l_\alpha}^*}{1 + \frac{c_{l_\alpha}^*}{\pi \mathcal{R}}} = \frac{2\pi}{\sqrt{1 - M^2} + \frac{2}{\mathcal{R}}} \quad (126)$$

Hence, we write

$$c_l = \frac{c_l^*}{\sqrt{1-M^2} + \frac{2}{\mathcal{R}}} \quad (127)$$

where c_l^* is the ideal incompressible lift coefficient and $\mathcal{R}$ is the aspect ratio.

The partial derivative $\frac{\partial c_l}{\partial M}$ is evaluated as

$$\frac{\partial c_l}{\partial M} = \frac{c_l^* M}{\sqrt{1-M^2} \left(\sqrt{1-M^2} + \frac{2}{\mathcal{R}} \right)^2} = \frac{c_l M}{1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \quad (128)$$

The partial derivative of $\frac{dl_{c,b_i}}{d\Psi_i}$ with respect to $x(t)$ becomes

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) &= \bar{c}_l \frac{1}{2} \rho_\infty 2\bar{V} c_b \frac{\partial V}{\partial x} + \frac{\bar{c}_l M^2}{1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \frac{1}{2} \rho_\infty \bar{V} c_b \frac{\partial V}{\partial x} + \bar{c}_{l_\alpha} \frac{1}{2} \rho_\infty \bar{V}^2 c_b \frac{\partial \alpha_{b_i}}{\partial x} \\ &= \bar{c}_l \frac{1}{2} \rho_\infty \bar{V}^2 c_b \left\{ \left[2 + \frac{M^2}{1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \right] \frac{1}{\bar{V}} \frac{\partial V}{\partial x} + \frac{\bar{c}_{l_\alpha}}{\bar{c}_l} \frac{\partial \alpha_{b_i}}{\partial x} \right\} \\ &= \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{\partial f_{b_i}}{\partial x} \end{aligned} \quad (129)$$

where

$$\frac{d\bar{l}_{c,b_i}}{d\Psi_i} = \bar{c}_l \frac{1}{2} \rho_\infty \bar{V}^2 c_b \quad (130)$$

$$\frac{\partial f_{b_i}}{\partial x} = \frac{k_{c_l}}{\bar{V}} \frac{\partial V}{\partial x} + \frac{\bar{c}_{l_\alpha}}{\bar{c}_l} \frac{\partial \alpha_{b_i}}{\partial x} \quad (131)$$

$$k_{c_l} = 2 + \frac{M^2}{1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \quad (132)$$

$$\bar{c}_{l_\alpha} = c_{l_\alpha} + 2c_{l_\alpha 2} \bar{\alpha} \quad (133)$$

The drag coefficient is expressed approximately as

$$c_d \approx c_{d_0} + \frac{c_l^2}{\pi \mathcal{R}} \quad (134)$$

The partial derivative of c_d with respect to M is evaluated as

$$\begin{aligned} \frac{\partial c_d}{\partial M} &= \frac{2c_l}{\pi \mathcal{R}} \frac{\partial c_l}{\partial M} = \frac{2c_l^2 M}{\pi \mathcal{R} \left(1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}} \right)} \\ &= \frac{2(c_d - c_{d_0}) M}{1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \end{aligned} \quad (135)$$

The partial derivative of $\frac{dd_{c,b_i}}{d\Psi_i}$ with respect to $x(t)$ is evaluated as

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{dd_{c,b_i}}{d\Psi_i} \right) &= \bar{c}_d \frac{1}{2} \rho_\infty 2\bar{V} c_b \frac{\partial V}{\partial x} + \frac{2(\bar{c}_d - c_{d_0}) M^2}{1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \frac{1}{2} \rho_\infty \bar{V} c_b \frac{\partial V}{\partial x} + \bar{c}_{d_\alpha} \frac{1}{2} \rho_\infty \bar{V}^2 c_b \frac{\partial \alpha_{b_i}}{\partial x} \\ &= \bar{c}_d \frac{1}{2} \rho_\infty \bar{V}^2 c_b \left\{ \left[2 + \frac{2 \left(1 - \frac{c_{d_0}}{\bar{c}_d} \right) M^2}{1-M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \right] \frac{1}{\bar{V}} \frac{\partial V}{\partial x} + \frac{\bar{c}_{d_\alpha}}{\bar{c}_d} \frac{\partial \alpha_{b_i}}{\partial x} \right\} \\ &= \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \frac{\partial g_{b_i}}{\partial x} \end{aligned} \quad (136)$$

where

$$\frac{d\bar{d}_{c,b_i}}{d\Psi_i} = \bar{c}_d \frac{1}{2} \rho_\infty \bar{V}^2 c_b \quad (137)$$

$$\frac{\partial g_{b_i}}{\partial x} = \frac{k_{c_d}}{V} \frac{\partial V}{\partial x} + \frac{\bar{c}_{d_\alpha}}{\bar{c}_d} \frac{\partial \alpha_{b_i}}{\partial x} \quad (138)$$

$$k_{c_d} = 2 + \frac{2 \left(1 - \frac{c_{d_0}}{\bar{c}_d}\right) M^2}{1 - M^2 + \frac{2\sqrt{1-M^2}}{\mathcal{R}}} \quad (139)$$

$$\bar{c}_{d_\alpha} = c_{d_\alpha} + 2c_{d_\alpha^2} \bar{\alpha} \quad (140)$$

We express $\frac{\partial f_{b_i}}{\partial x}$, $\frac{\partial g_{b_i}}{\partial x}$, and $\frac{\partial \alpha_{b_i}}{\partial x}$ as

$$\frac{\partial f_{b_i}}{\partial x} = \frac{\partial f_{0,b_i}}{\partial x} + \frac{\partial f_{s,b_i}}{\partial x} \sin \Psi_i + \frac{\partial f_{c,b_i}}{\partial x} \cos \Psi_i \quad (141)$$

$$\frac{\partial g_{b_i}}{\partial x} = \frac{\partial g_{0,b_i}}{\partial x} + \frac{\partial g_{s,b_i}}{\partial x} \sin \Psi_i + \frac{\partial g_{c,b_i}}{\partial x} \cos \Psi_i \quad (142)$$

$$\frac{\partial \alpha_{b_i}}{\partial x} = \frac{\partial \alpha_{0,b_i}}{\partial x} + \frac{\partial \alpha_{s,b_i}}{\partial x} \sin \Psi_i + \frac{\partial \alpha_{c,b_i}}{\partial x} \cos \Psi_i \quad (143)$$

The partial derivatives of the circulatory axial and normal forces in the blade reference frame are

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{df_{x_{c,b_i}}}{d\Psi_i} \right) &= - \frac{\partial}{\partial x} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) \sin \bar{\alpha}_{b_i} + \frac{\partial}{\partial x} \left(\frac{dd_{c,b_i}}{d\Psi_i} \right) \cos \bar{\alpha}_{b_i} \\ &\quad - \left(\frac{d\bar{l}_{c,b_i}}{d\Psi_i} \cos \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \sin \bar{\alpha}_{b_i} \right) \frac{\partial \alpha_{b_i}}{\partial x} \\ &= - \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{\partial f_{b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \frac{\partial g_{b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \frac{d\bar{f}_{x_{c,b_i}}}{d\Psi_i} \frac{\partial \alpha_{b_i}}{\partial x} \end{aligned} \quad (144)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{df_{z_{c,b_i}}}{d\Psi_i} \right) &= \frac{\partial}{\partial x} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) \cos \bar{\alpha}_{b_i} + \frac{\partial}{\partial x} \left(\frac{dd_{c,b_i}}{d\Psi_i} \right) \sin \bar{\alpha}_{b_i} \\ &\quad - \left(\frac{d\bar{l}_{c,b_i}}{d\Psi_i} \sin \bar{\alpha}_{b_i} - \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \frac{\partial g_{b_i}}{\partial x} \cos \bar{\alpha}_{b_i} \right) \\ &= \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{\partial f_{b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \frac{\partial g_{b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \frac{d\bar{f}_{x_{c,b_i}}}{d\Psi_i} \frac{\partial \alpha_{b_i}}{\partial x} \end{aligned} \quad (145)$$

where

$$\frac{d\bar{f}_{x_{c,b_i}}}{d\Psi_i} = - \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \sin \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \cos \bar{\alpha}_{b_i} \quad (146)$$

$$\frac{d\bar{f}_{z_{c,b_i}}}{d\Psi_i} = \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \cos \bar{\alpha}_{b_i} + \frac{d\bar{d}_{c,b_i}}{d\Psi_i} \sin \bar{\alpha}_{b_i} \quad (147)$$

The partial derivatives of the integrated circulatory thrust force and rotor moments which affect the inflow velocity are computed as

$$\begin{aligned} \frac{\partial T_{c,r_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{df_{z_{c,b_i}}}{d\Psi_i} \right) d\Psi_i dr' \\ &= \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_{c,b_i}} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr' \end{aligned} \quad (148)$$

$$\begin{aligned} \frac{\partial l_{c,r_i}}{\partial x} &= \mp \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \int_0^{2\pi} \frac{\partial}{\partial x} \left[(r' \sin \Psi_i + e_b \cos \Psi_i) \frac{df_{z_{c,b_i}}}{d\Psi_i} \right] d\Psi_i dr \\ &= \mp \frac{1}{2} \int_{R_i}^R r' \left(\bar{l}_{c,b_i} \frac{\partial f_{s,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{s,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_{c,b_i}} \frac{\partial \alpha_{s,b_i}}{\partial x} \right) dr' \\ &\quad \mp \frac{1}{2} \int_{R_i}^R e_b \left(\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_{c,b_i}} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr' \end{aligned} \quad (149)$$

$$\begin{aligned}
\frac{\partial m_{c,r_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left[- (r' \cos \Psi_i - e_b \sin \Psi_i) \frac{df_{z_{c,b_i}}}{d\Psi_i} \pm \frac{dm_{c,b_i}}{d\Psi_i} \sin \Psi_i \right] d\Psi_i dr' \\
&= -\frac{1}{2} \int_{R_i}^R r' \left(\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_{c,b_i}} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr' \\
&\quad + \frac{1}{2} \int_{R_i}^R e_b \left(\bar{l}_{c,b_i} \frac{\partial f_{s,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{s,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_{c,b_i}} \frac{\partial \alpha_{s,b_i}}{\partial x} \right) dr'
\end{aligned} \tag{150}$$

$$\begin{aligned}
\frac{\partial Q_{c,r_i}}{\partial x} &= \pm \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(r' \frac{df_{x_{c,b_i}}}{d\Psi_i} \right) d\Psi_i dr' \\
&= \pm \int_{R_i}^R r' \left(-\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z_{c,b_i}} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr'
\end{aligned} \tag{151}$$

where

$$\bar{l}_{c,b_i} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{l}_{c,b_i}}{\partial \Psi_i} d\Psi_i \tag{152}$$

$$\bar{d}_{c,b_i} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{d}_{c,b_i}}{\partial \Psi_i} d\Psi_i \tag{153}$$

$$\bar{f}_{x_{c,b_i}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{f}_{x_{c,b_i}}}{\partial \Psi_i} d\Psi_i \tag{154}$$

$$\bar{f}_{z_{c,b_i}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \bar{f}_{z_{c,b_i}}}{\partial \Psi_i} d\Psi_i \tag{155}$$

The partial derivatives of the aircraft integrated circulatory forces are evaluated as

$$\begin{aligned}
\frac{\partial X_{c,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(-\frac{df_{x_{c,b_i}}}{d\Psi_i} \right) \sin \Psi_i d\Psi_i dr' \\
&= -\frac{1}{2} \int_{R_i}^R \left(-\bar{l}_{c,b_i} \frac{\partial f_{s,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{s,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z_{c,b_i}} \frac{\partial \alpha_{s,b_i}}{\partial x} \right) dr'
\end{aligned} \tag{156}$$

$$\begin{aligned}
\frac{\partial Y_{c,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\mp \frac{df_{x_{c,b_i}}}{d\Psi_i} \cos \eta_i \cos \Psi_i - \frac{df_{z_{c,b_i}}}{d\Psi_i} \sin \eta_i \right) d\Psi_i dr' \\
&= \mp \frac{1}{2} \cos \bar{\eta}_i \int_{R_i}^R \left(-\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z_{c,b_i}} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr' \\
&\quad - \sin \bar{\eta}_i \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_{c,b_i}} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr'
\end{aligned} \tag{157}$$

$$\begin{aligned}
\frac{\partial Z_{c,a_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\pm \frac{df_{x_{c,b_i}}}{d\Psi_i} \sin \eta_i \cos \Psi_i - \frac{df_{z_{c,b_i}}}{d\Psi_i} \cos \eta_i \right) d\Psi_i dr' \\
&= \pm \frac{1}{2} \sin \bar{\eta}_i \int_{R_i}^R \left(-\bar{l}_{c,b_i} \frac{\partial f_{c,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{c,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} - \bar{f}_{z_{c,b_i}} \frac{\partial \alpha_{c,b_i}}{\partial x} \right) dr' \\
&\quad - \cos \bar{\eta}_i \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{\partial f_{0,b_i}}{\partial x} \cos \bar{\alpha}_{b_i} + \bar{d}_{c,b_i} \frac{\partial g_{0,b_i}}{\partial x} \sin \bar{\alpha}_{b_i} + \bar{f}_{x_{c,b_i}} \frac{\partial \alpha_{0,b_i}}{\partial x} \right) dr'
\end{aligned} \tag{158}$$

The partial derivatives of the aircraft integrated circulatory moments are evaluated as

$$\frac{\partial l_{c,a_i}}{\partial x} = \frac{\partial l_{c,r_i}}{\partial x} + \frac{\partial Y_{c,a_i}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{c,a_i}}{\partial x} \Delta y_{r_{i0}} \tag{159}$$

$$\frac{\partial m_{c,a_i}}{\partial x} = \frac{\partial m_{c,r_i}}{\partial x} \cos \bar{\eta}_i + \frac{\partial Q_{c,r_i}}{\partial x} \sin \bar{\eta}_i - \frac{\partial X_{c,r_i}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{c,r_i}}{\partial x} \Delta x_{r_{i0}} \tag{160}$$

$$\frac{\partial n_{c,a_i}}{\partial x} = -\frac{\partial m_{c,r_i}}{\partial x} \sin \bar{\eta}_i + \frac{\partial Q_{c,r_i}}{\partial x} \cos \bar{\eta}_i - \frac{\partial X_{c,r_i}}{\partial x} \Delta y_{r_{i0}} - \frac{\partial Y_{c,r_i}}{\partial x} \Delta x_{r_{i0}} \tag{161}$$

B. Partial Derivatives of Non-Circulatory Forces and Moments

We write the non-circulatory lift force and the quarter-chord pitching moment as

$$\frac{dl_{nc,b_i}}{d\Psi_i} = \frac{\pi}{4} \rho_\infty c_b^2 a_{z_{b_i}} \left(\mathbf{x}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{v}}_{p_i}, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \dot{\theta}_{b_i}, \ddot{\theta}_{b_i} \right) \quad (162)$$

$$\frac{dm_{ac,b_i}}{d\Psi_i} = c_{mac} \frac{1}{2} \rho_\infty V^2 \left(\mathbf{x}_a, \mathbf{v}_{p_i}, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \dot{\theta}_{b_i}, \Omega_i \right) c_b^2 \quad (163)$$

The partial derivatives of $\frac{dl_{c,b_i}}{d\Psi_i}$ and $\frac{dm_{ac,b_i}}{d\Psi_i}$ are evaluated as

$$\frac{\partial}{\partial x} \left(\frac{dl_{nc,b_i}}{d\Psi_i} \right) = \frac{\pi}{4} \rho_\infty c_b^2 \frac{\partial a_{z_{b_i}}}{\partial x} \quad (164)$$

$$\frac{\partial}{\partial x} \left(\frac{dm_{ac,b_i}}{d\Psi_i} \right) = \frac{\partial c_{mac}}{\partial x} \frac{1}{2} \rho_\infty \bar{V}^2 c_b^2 = \frac{d\bar{l}_{c,b_i}}{d\Psi_i} \frac{c_b}{\bar{c}_l} \frac{\partial c_{mac}}{\partial x} \quad (165)$$

where $x(t)$ is a dummy variable.

The partial derivatives of the non-circulatory axial and normal forces in the blade reference frame are

$$\frac{\partial}{\partial x} \left(\frac{df_{x_{nc,b_i}}}{d\Psi_i} \right) = - \frac{\partial}{\partial x} \left(\frac{dl_{nc,b_i}}{d\Psi_i} \right) \sin \bar{\alpha}_{b_i} = - \frac{\pi}{4} \rho_\infty c_b^2 \sin \bar{\alpha}_{b_i} \frac{\partial a_{z_{b_i}}}{\partial x} \quad (166)$$

$$\frac{\partial}{\partial x} \left(\frac{df_{z_{nc,b_i}}}{d\Psi_i} \right) = \frac{\partial}{\partial x} \left(\frac{dl_{c,b_i}}{d\Psi_i} \right) \cos \bar{\alpha}_{b_i} = \frac{\pi}{4} \rho_\infty c_b^2 \cos \bar{\alpha}_{b_i} \frac{\partial a_{z_{b_i}}}{\partial x} \quad (167)$$

We express $\frac{\partial a_{z_{b_i}}}{\partial x}$, $\frac{\partial a_\xi}{\partial x}$, and $\frac{\partial c_{mac}}{\partial x}$ as

$$\frac{\partial a_{z_{b_i}}}{\partial x} = \frac{\partial a_{0,z_{b_i}}}{\partial x} + \frac{\partial a_{s,z_{b_i}}}{\partial x} \sin \Psi_i + \frac{\partial a_{c,z_{b_i}}}{\partial x} \cos \Psi_i \quad (168)$$

$$\frac{\partial a_\xi}{\partial x} = \frac{\partial a_{0,\xi}}{\partial x} + \frac{\partial a_{s,\xi}}{\partial x} \sin \Psi_i + \frac{\partial a_{c,\xi}}{\partial x} \cos \Psi_i \quad (169)$$

$$\frac{\partial c_{mac}}{\partial x} = \frac{\partial c_{0,mac}}{\partial x} + \frac{\partial c_{s,mac}}{\partial x} \sin \Psi_i + \frac{\partial c_{c,mac}}{\partial x} \cos \Psi_i \quad (170)$$

The partial derivative of the integrated non-circulatory thrust force and rotor moments are computed as

$$\frac{\partial T_{nc,r_i}}{\partial x} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{df_{z_{nc,b_i}}}{d\Psi_i} \right) d\Psi_i dr' = \frac{\pi}{4} \rho_\infty \int_{R_i}^R c_b^2 \cos \bar{\alpha}_{b_i} \frac{\partial a_{z_{0,b_i}}}{\partial x} dr' \quad (171)$$

$$\begin{aligned} \frac{\partial l_{nc,r_i}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[\mp (r' \sin \Psi_i - e_{mb} \cos \Psi_i) \frac{df_{z_{nc,b_i}}}{d\Psi_i} - \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \cos \Psi_i \right] d\Psi_i dr' \\ &= \mp \frac{\pi}{8} \rho_\infty \int_{R_i}^R \left(r' \frac{\partial a_{s,z_{b_i}}}{\partial x} - e_{mb} \frac{\partial a_{c,z_{b_i}}}{\partial x} \right) c_b^2 \cos \bar{\alpha}_{b_i} dr' \\ &\quad - \frac{1}{2} \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{c_b}{\bar{c}_l} \frac{\partial c_{c,mac}}{\partial x} - \frac{\pi}{128} \rho_\infty c_b^4 \frac{\partial a_{c,\xi}}{\partial x} \right) dr' \end{aligned} \quad (172)$$

$$\begin{aligned} m_{nc,r_i} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left[- (r' \cos \Psi_i + e_{mb} \sin \Psi_i) \frac{df_{z_{nc,b_i}}}{d\Psi_i} \pm \left(\frac{dm_{ac,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) \sin \Psi_i \right] d\Psi_i dr' \\ &= - \frac{\pi}{8} \rho_\infty \int_{R_i}^R \left(r' \frac{\partial a_{c,z_{b_i}}}{\partial x} + e_{mb} \frac{\partial a_{s,z_{b_i}}}{\partial x} \right) c_b^2 \cos \bar{\alpha}_{b_i} dr' \\ &\quad \pm \frac{1}{2} \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{c_b}{\bar{c}_l} \frac{\partial c_{s,mac}}{\partial x} - \frac{\pi}{128} \rho_\infty c_b^4 \frac{\partial a_{s,\xi}}{\partial x} \right) dr' \end{aligned} \quad (173)$$

$$Q_{nc,r_i} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \pm r' \frac{df_{x_{nc,b_i}}}{d\Psi_i} d\Psi_i dr' = \mp \frac{\pi}{4} \rho_\infty \int_{R_i}^R r' \frac{\partial a_{0,z_{b_i}}}{\partial x} c_b^2 \sin \bar{\alpha}_{b_i} dr' \quad (174)$$

The partial derivatives of the aircraft integrated non-circulatory forces are evaluated as

$$\frac{\partial X_{nc,ai}}{\partial x} = \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(-\frac{df_{x_{nc,b_i}}}{d\Psi_i} \right) \sin \Psi_i d\Psi_i dr' = \frac{\pi}{8} \rho_\infty \int_{R_i}^R c_b^2 \frac{\partial a_{s,z_{b_i}}}{\partial x} \sin \bar{\alpha}_{b_i} dr' \quad (175)$$

$$\begin{aligned} \frac{\partial Y_{nc,ai}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\mp \frac{df_{x_{nc,b_i}}}{d\Psi_i} \cos \eta_i \cos \Psi_i - \frac{df_{z_{nc,b_i}}}{d\Psi_i} \sin \eta_i \right) d\Psi_i dr' \\ &= \frac{\pi}{8} \rho_\infty \int_{R_i}^R c_b^2 \left[\pm \frac{\partial a_{c,z_{b_i}}}{\partial x} \sin \bar{\alpha}_{b_i} \cos \bar{\eta}_i - 2 \frac{\partial a_{0,z_{b_i}}}{\partial x} \cos \bar{\alpha}_{b_i} \sin \bar{\eta}_i \right] dr' \end{aligned} \quad (176)$$

$$\begin{aligned} \frac{\partial Z_{nc,ai}}{\partial x} &= \frac{1}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\pm \frac{df_{x_{nc,b_i}}}{d\Psi_i} \sin \eta_i \cos \Psi_i - \frac{df_{z_{nc,b_i}}}{d\Psi_i} \cos \eta_i \right) d\Psi_i dr' \\ &= \frac{\pi}{8} \rho_\infty \int_{R_i}^R c_b^2 \left[\mp \frac{\partial a_{c,z_{b_i}}}{\partial x} \sin \bar{\alpha}_{b_i} \sin \bar{\eta}_i - 2 \frac{\partial a_{0,z_{b_i}}}{\partial x} \cos \bar{\alpha}_{b_i} \cos \bar{\eta}_i \right] dr' \end{aligned} \quad (177)$$

The partial derivatives of the aircraft integrated non-circulatory moments are evaluated as

$$\frac{\partial l_{nc,ai}}{\partial x} = \frac{\partial l_{nc,ri}}{\partial x} + \frac{\partial Y_{nc,ai}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{nc,ai}}{\partial x} \Delta y_{r_{i0}} \quad (178)$$

$$\frac{\partial m_{nc,ai}}{\partial x} = \frac{\partial m_{nc,ri}}{\partial x} \cos \bar{\eta}_i + \frac{\partial Q_{nc,ri}}{\partial x} \sin \bar{\eta}_i - \frac{\partial X_{nc,ri}}{\partial x} \Delta z_{r_{i0}} + \frac{\partial Z_{nc,ri}}{\partial x} \Delta x_{r_{i0}} \quad (179)$$

$$\frac{\partial n_{nc,ai}}{\partial x} = -\frac{\partial m_{nc,ri}}{\partial x} \sin \bar{\eta}_i + \frac{\partial Q_{nc,ri}}{\partial x} \cos \bar{\eta}_i - \frac{\partial X_{nc,ri}}{\partial x} \Delta y_{r_{i0}} - \frac{\partial Y_{nc,ri}}{\partial x} \Delta x_{r_{i0}} \quad (180)$$

V. Aircraft and Rotor Forces and Moments

Let $\mathbf{v}_p(t) = \begin{bmatrix} \mathbf{v}_{p_1}^\top(t) & \mathbf{v}_{p_2}^\top(t) & \cdots & \mathbf{v}_{p_{n_{rotor}}}^\top(t) \end{bmatrix}^\top$, $\mathbf{x}_q(t) = \begin{bmatrix} \mathbf{q}^\top(t) & \dot{\mathbf{q}}^\top(t) \end{bmatrix}^\top$, $\mathbf{u}_\theta(t) = \begin{bmatrix} \tilde{\boldsymbol{\theta}}_b(t) & \dot{\boldsymbol{\theta}}_b(t) \end{bmatrix}^\top$ where $\tilde{\boldsymbol{\theta}}_b(t) = \begin{bmatrix} \tilde{\theta}_{b_1}(t) & \tilde{\theta}_{b_2}(t) & \cdots & \tilde{\theta}_{b_{n_{rotor}}}(t) \end{bmatrix}^\top$, and $\boldsymbol{\Omega}(t) = \begin{bmatrix} \Omega_1(t) & \Omega_2(t) & \cdots & \Omega_{n_{rotor}}(t) \end{bmatrix}^\top$. We express the total axial, side, and vertical circulatory forces and moments acting on the aircraft at the aircraft CG as

$$\mathbf{F}_{c,a} = \sum_{i=1}^{n_{rotor}} \left(\frac{\partial \mathbf{F}_{c,ai}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{c,ai}}{\partial \mathbf{v}_p} \mathbf{v}_p + \frac{\partial \mathbf{F}_{c,ai}}{\partial \mathbf{x}_q} \mathbf{x}_q + \frac{\partial \mathbf{F}_{c,ai}}{\partial \mathbf{u}_\theta} \mathbf{u}_\theta + \frac{\partial \mathbf{F}_{c,ai}}{\partial \boldsymbol{\Omega}} \boldsymbol{\Omega} \right) \quad (181)$$

$$\mathbf{M}_{c,a} = \sum_{i=1}^{n_{rotor}} \left(\frac{\partial \mathbf{M}_{c,ai}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{c,ai}}{\partial \mathbf{v}_p} \mathbf{v}_p + \frac{\partial \mathbf{M}_{c,ai}}{\partial \mathbf{x}_q} \mathbf{x}_q + \frac{\partial \mathbf{M}_{c,ai}}{\partial \mathbf{u}_\theta} \mathbf{u}_\theta + \frac{\partial \mathbf{M}_{c,ai}}{\partial \boldsymbol{\Omega}} \boldsymbol{\Omega} \right) \quad (182)$$

where $\mathbf{F}_{c,a} = \begin{bmatrix} X_{c,a} & Y_{c,a} & Z_{c,a} \end{bmatrix}^\top$ and $\mathbf{M}_{c,a} = \begin{bmatrix} l_{c,a} & m_{c,a} & n_{c,a} \end{bmatrix}^\top$.

The total axial, side, and vertical non-circulatory forces and moments acting on the aircraft at the aircraft CG are expressed as

$$\mathbf{F}_{nc,a} = \sum_{i=1}^{n_{rotor}} \left(\frac{\partial \mathbf{F}_{nc,ai}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{nc,ai}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{F}_{nc,ai}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \frac{\partial \mathbf{F}_{nc,ai}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \frac{\partial \mathbf{F}_{nc,ai}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta \right) \quad (183)$$

$$\mathbf{M}_{nc,a} = \sum_{i=1}^{n_{rotor}} \left(\frac{\partial \mathbf{M}_{nc,ai}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{nc,ai}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{M}_{nc,ai}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \frac{\partial \mathbf{M}_{nc,ai}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \frac{\partial \mathbf{M}_{nc,ai}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta \right) \quad (184)$$

where $\mathbf{F}_{nc,a} = \begin{bmatrix} X_{nc,a} & Y_{nc,a} & Z_{nc,a} \end{bmatrix}^\top$, $\mathbf{M}_{nc,a} = \begin{bmatrix} l_{nc,a} & m_{nc,a} & n_{nc,a} \end{bmatrix}^\top$.

The total force and moment vectors are

$$\mathbf{F}_a = \mathbf{F}_{c,a} + \mathbf{F}_{nc,a} \quad (185)$$

$$\mathbf{M}_a = \mathbf{M}_{c,a} + \mathbf{M}_{nc,a} \quad (186)$$

The rotor circulatory thrust and moments acting at the origin of the i -th rotor reference frame are expressed as

$$T_{c,r_i} = \frac{\partial T_{c,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial T_{c,r_i}}{\partial \mathbf{v}_{p_i}} \mathbf{v}_{p_i} + \frac{\partial T_{c,r_i}}{\partial \mathbf{x}_q} \mathbf{x}_q + \frac{\partial T_{c,r_i}}{\partial \mathbf{u}_{\theta_i}} \mathbf{u}_{\theta_i} + \frac{\partial T_{c,r_i}}{\partial \Omega_i} \Omega_i \quad (187)$$

$$\mathbf{M}_{c,r_i} = \frac{\partial \mathbf{M}_{c,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{c,r_i}}{\partial \mathbf{v}_{p_i}} \mathbf{v}_{p_i} + \frac{\partial \mathbf{M}_{c,r_i}}{\partial \mathbf{x}} \mathbf{x}_q + \frac{\partial \mathbf{M}_{c,r_i}}{\partial \mathbf{u}_{\theta_i}} \mathbf{u}_{\theta_i} + \frac{\partial \mathbf{M}_{c,r_i}}{\partial \Omega_i} \Omega_i \quad (188)$$

where $\mathbf{M}_{c,r_i} = \begin{bmatrix} l_{c,r_i} & m_{c,r_i} & Q_{c,r_i} \end{bmatrix}^\top$.

The rotor non-circulatory thrust and moments acting at the origin of the i -th rotor reference frame are expressed as

$$T_{nc,r_i} = \frac{\partial T_{nc,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial T_{nc,r_i}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial T_{nc,r_i}}{\partial \dot{\mathbf{v}}_{p_i}} \dot{\mathbf{v}}_{p_i} + \frac{\partial T_{nc,r_i}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \frac{\partial T_{nc,r_i}}{\partial \dot{\mathbf{u}}_{\theta_i}} \dot{\mathbf{u}}_{\theta_i} \quad (189)$$

$$\mathbf{M}_{nc,r_i} = \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\mathbf{v}}_{p_i}} \dot{\mathbf{v}}_{p_i} + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\mathbf{u}}_{\theta_i}} \dot{\mathbf{u}}_{\theta_i} \quad (190)$$

where $\mathbf{M}_{nc,r_i} = \begin{bmatrix} l_{nc,r_i} & m_{nc,r_i} & Q_{nc,r_i} \end{bmatrix}^\top$.

The partial derivatives with respect to the generalized displacement and velocity vectors $\dot{\mathbf{q}}(t)$ and $\ddot{\mathbf{q}}(t)$ are evaluated as

$$\frac{\partial T_{nc,r_i}}{\partial \dot{\mathbf{q}}} = \frac{\partial T_{nc,r_i}}{\partial \dot{\phi}_w} \Psi_w(y_{s_{i0}}) + \frac{\partial T_{nc,r_i}}{\partial \dot{\phi}_{s_i}} \Psi_{s_i}(\Delta x_{rs_{i0}}) \quad (191)$$

$$\frac{\partial T_{nc,r_i}}{\partial \ddot{\mathbf{q}}} = \frac{\partial T_{nc,r_i}}{\partial \ddot{\mathbf{u}}_w} \Phi_w(y_{s_{i0}}) + \frac{\partial T_{nc,r_i}}{\partial \ddot{\phi}_w} \Psi_w(y_{s_{i0}}) + \frac{\partial T_{nc,r_i}}{\partial \ddot{\mathbf{u}}_{s_i}} \Phi_{s_i}(\Delta x_{rs_{i0}}) + \frac{\partial T_{nc,r_i}}{\partial \ddot{\phi}_{s_i}} \Psi_{s_i}(\Delta x_{rs_{i0}})$$

$$\frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\mathbf{q}}} = \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\phi}_w} \Psi_w(y_{s_{i0}}) + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \dot{\phi}_{s_i}} \Psi_{s_i}(\Delta x_{rs_{i0}}) \quad (192)$$

$$\frac{\partial \mathbf{M}_{nc,r_i}}{\partial \ddot{\mathbf{q}}} = \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \ddot{\mathbf{u}}_w} \Phi_w(y_{s_{i0}}) + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \ddot{\phi}_w} \Psi_w(y_{s_{i0}}) + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \ddot{\mathbf{u}}_{s_i}} \Phi_{s_i}(\Delta x_{rs_{i0}}) + \frac{\partial \mathbf{M}_{nc,r_i}}{\partial \ddot{\phi}_{s_i}} \Psi_{s_i}(\Delta x_{rs_{i0}})$$

The coefficient vector $\mathbf{C}_{p_i} = \begin{bmatrix} C_{T_{r_i}} & C_{l_{r_i}} & C_{m_{r_i}} \end{bmatrix}^\top$ for the Pitt-Peters three-state inflow model is comprised of the circulatory and non-circulatory components expressed as

$$\mathbf{C}_{p_i} = \mathbf{C}_{c,p_i} + \mathbf{C}_{nc,p_i} \quad (193)$$

where $\mathbf{C}_{c,p_i}$ are obtained from T_{c,r_i} and $\mathbf{M}_{c,r_i}$ and $\mathbf{C}_{nc,p_i}$ are obtained from T_{nc,r_i} and $\mathbf{M}_{nc,r_i}$

VI. Generalized Unsteady Aerodynamic Forces

A NASTRAN finite-element model (FEM) of the all-electric Lift+Cruise aircraft is available.¹⁵ Figure 3 shows the NASTRAN FEM of the Lift+Cruise aircraft. The NASTRAN FEM is created in the global reference frame (x, y, z) . Therefore, all the forces acting on the structures are to be expressed in this reference frame. The mass properties computed by the NASTRAN FEM are shown in Table 1.

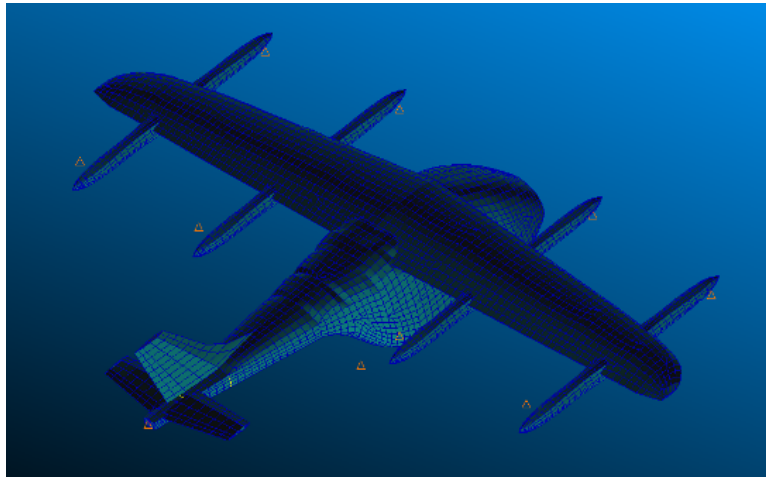


Figure 3. NASTRAN Finite-Element Model of All-Electric Lift+Cruise Aircraft

Weight (lb)	8,394.7
x_{cg} (ft)	13.323325
y_{cg} (ft)	0.00466
z_{cg} (ft)	6.188667
I_{xx} (ft-lb-sec ²)	507,714
I_{yy} (ft-lb-sec ²)	426,360
I_{zz} (ft-lb-sec ²)	865,301
I_{xy} (ft-lb-sec ²)	93
I_{yz} (ft-lb-sec ²)	-63
I_{xz} (ft-lb-sec ²)	-47999

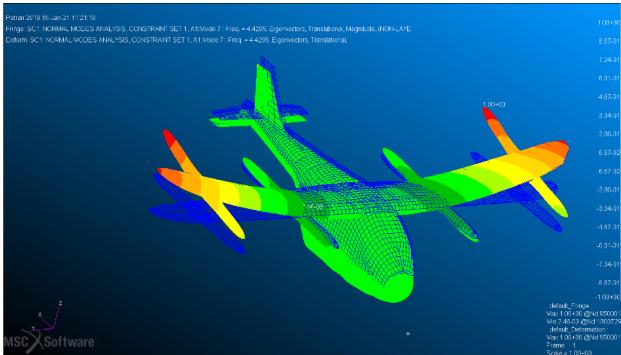
Table 1. NASTRAN Mass Properties of All-Electric Lift+Cruise Aircraft

A. Frequencies and Mode Shapes

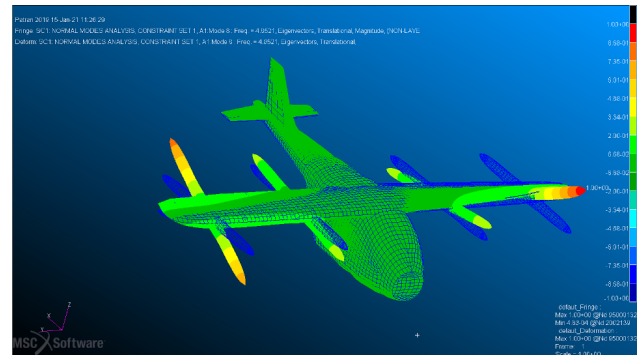
MODE NO.	EIGENVALUE	FREQUENCY RAD/S	FREQUENCY HZ	GENERALIZED MASS	GENERALIZED STIFFNESS
1	0.0	0.0	0.0	1.6795E+02	0.0
2	0.0	0.0	0.0	4.0037E+01	0.0
3	0.0	0.0	0.0	1.9112E+02	0.0
4	0.0	0.0	0.0	6.3491E+01	0.0
5	0.0	0.0	0.0	6.0104E+01	0.0
6	0.0	0.0	0.0	4.0233E+01	0.0
7	7.7460E+02	2.7832E+01	4.4295E+00	1.7886E+01	1.3855E+04
8	9.2943E+02	3.0487E+01	4.8521E+00	1.6978E+01	1.5780E+04
9	1.0684E+03	3.2687E+01	5.2023E+00	1.3628E+01	1.4561E+04
10	2.4543E+03	4.9541E+01	7.8846E+00	1.5860E+01	3.8925E+04
11	2.5168E+03	5.0168E+01	7.9845E+00	1.4342E+01	3.6096E+04
12	2.6978E+03	5.1941E+01	8.2666E+00	2.8445E+01	7.6740E+04
13	2.7814E+03	5.2739E+01	8.3937E+00	1.9590E+01	5.4489E+04
14	3.2202E+03	5.6747E+01	9.0316E+00	2.5244E+01	8.1293E+04
15	3.6364E+03	6.0303E+01	9.5975E+00	1.4101E+01	5.1277E+04
16	3.7013E+03	6.0838E+01	9.6827E+00	1.6076E+01	5.9504E+04
17	4.4168E+03	6.6459E+01	1.0577E+01	1.2656E+01	5.5901E+04
18	4.6238E+03	6.7998E+01	1.0822E+01	1.6832E+01	7.7825E+04
19	4.6501E+03	6.8191E+01	1.0853E+01	3.1833E+01	1.4803E+05
20	5.0233E+03	7.0875E+01	1.1280E+01	1.6266E+01	8.1709E+04

Table 2. Frequency, Generalized Mass, and Generalized Stiffness

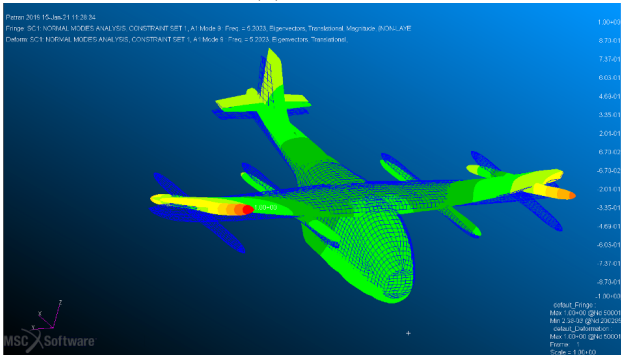
The frequencies and generalized mass and stiffness values of the NASTRAN model of the all electric Lift+Cruise aircraft are shown in Table 2. The first six modes are the rigid-body modes of the aircraft without aerodynamic forces. The mode shapes of the first 14 structural dynamic modes are shown in Figure 4.



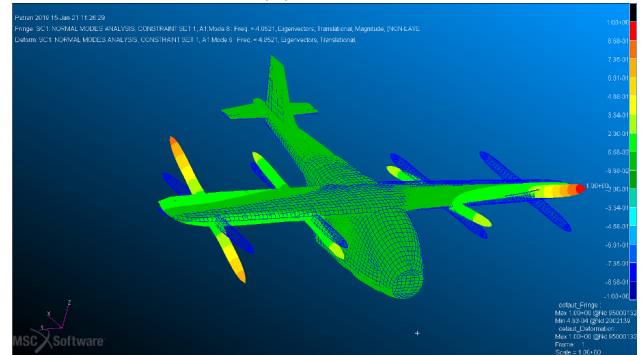
(a) Mode 1



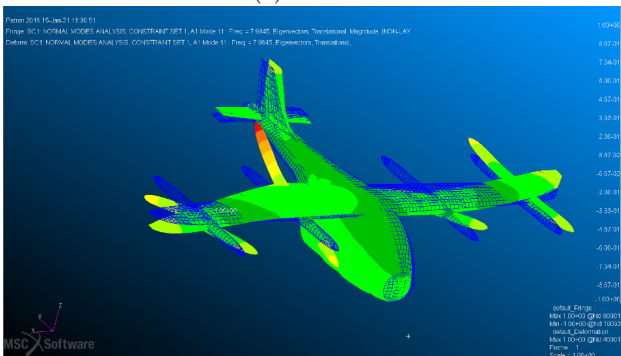
(b) Mode 2



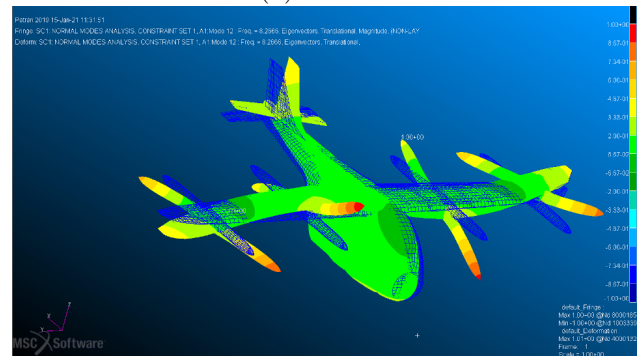
(c) Mode 3



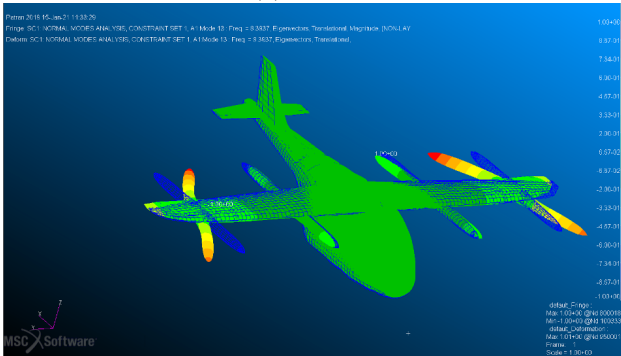
(d) Mode 4



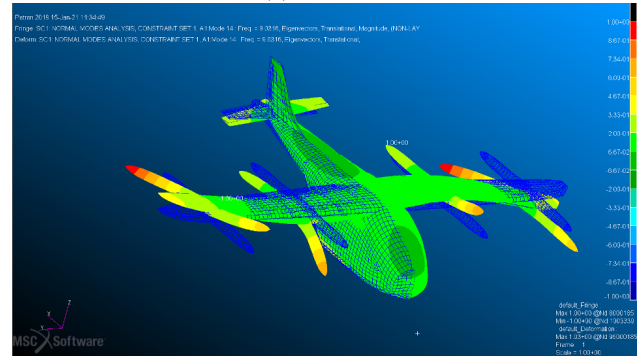
(e) Mode 5



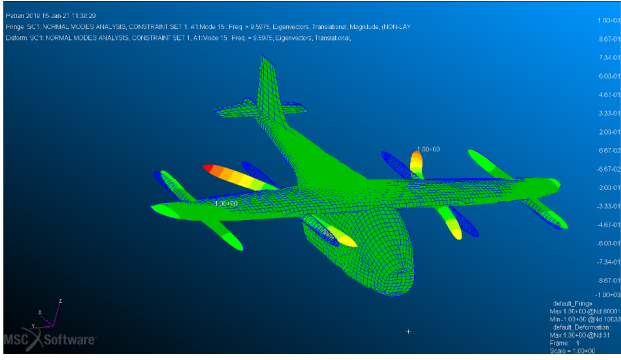
(f) Mode 6



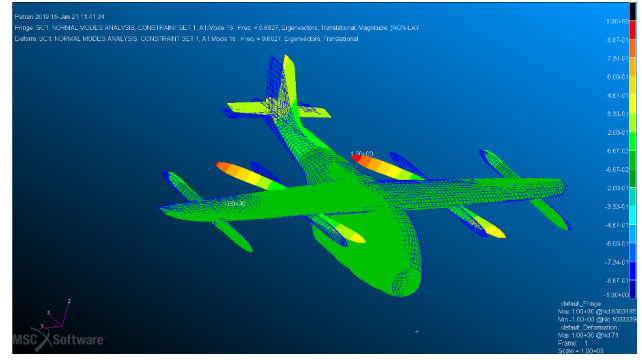
(g) Mode 7



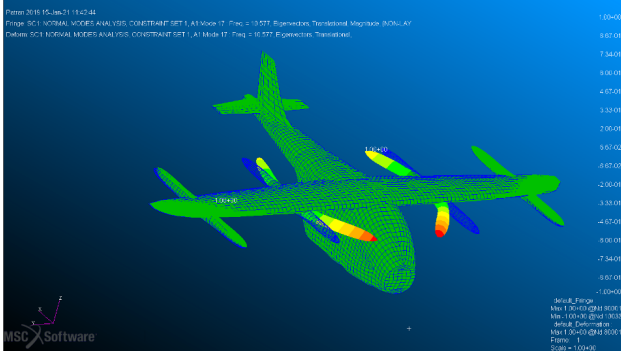
(h) Mode 8



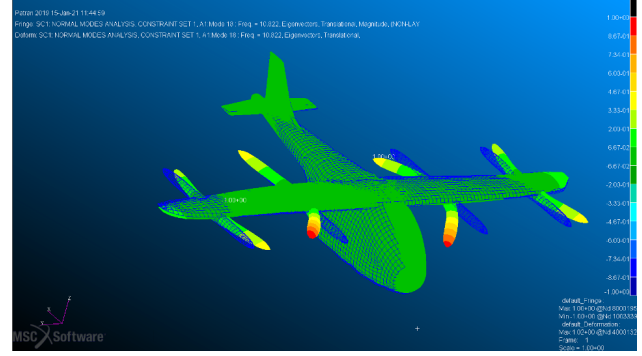
(i) Mode 9



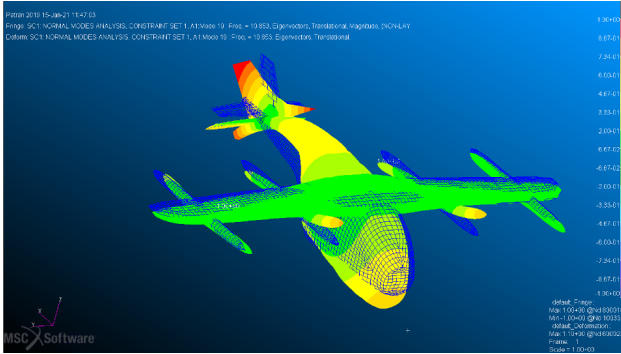
(j) Mode 10



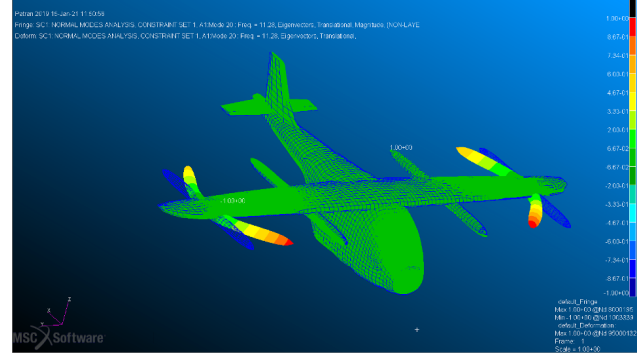
(k) Mode 11



(l) Mode 12



(m) Mode 13



(n) Mode 14

Figure 4. Mode Shapes of All-Electric Lift+Cruise Aircraft

B. Generalized Aerodynamic Force

The rotor forces are transmitted to the wing structure at four locations corresponding to

- The intersection of the elastic axis of the wing structure and the elastic axes of rotor support structures 1 and 2, denoted as $j = 1$,
- The intersection of the elastic axis of the wing structure and the elastic axes of rotor support structures 3 and 4, denoted as $j = 2$,
- The intersection of the elastic axis of the wing structure and the elastic axes of rotor support structures 5 and 6, denoted as $j = 3$, and
- The intersection of the elastic axis of the wing structure and the elastic axes of rotor support structures 7 and 8, denoted as $j = 4$.

The forces acting on the rotor support structures create moments acting on the wing structure. The moments produced by the rotors are the sum of the circulatory components and the non-circulatory components expressed as

$$\mathbf{M}_{w_j} = C(k) \mathbf{M}_{c,w_j} + \mathbf{M}_{nc,w_j} \quad (194)$$

where

$$\begin{aligned} \mathbf{M}_{c,w_j} &= \sum_{i=2j-1}^{2j} (\mathbf{M}_{c,r_i} + \mathbf{p}_{r_i} \times \mathbf{F}_{c,r_i}) \\ &= \sum_{i=2j-1}^{2j} \left[-(l_{c,r_i} + Y_{c,a_i} \Delta z_{rs_{i0}} + Z_{c,a_i} \Delta y_{rs_{i0}}) \mathbf{i} \right. \\ &\quad + (m_{c,r_i} \cos \eta_i + Q_{c,r_i} \sin \eta_i - X_{c,a_i} \Delta z_{rs_{i0}} + Z_{c,a_i} \Delta x_{rs_{i0}}) \mathbf{j} \\ &\quad \left. + (m_{c,r_i} \sin \eta_i - Q_{c,r_i} \cos \eta_i + X_{c,a_i} \Delta y_{rs_{i0}} + Y_{c,a_i} \Delta x_{rs_{i0}}) \mathbf{k} \right] \end{aligned} \quad (195)$$

$$\begin{aligned} \mathbf{M}_{nc,w_j} &= \sum_{i=2j-1}^{2j} (\mathbf{M}_{nc,r_i} + \mathbf{p}_{r_i} \times \mathbf{F}_{nc,r_i}) \\ &= \sum_{i=2j-1}^{2j} \left[-(l_{nc,r_i} + Y_{nc,a_i} \Delta z_{rs_{i0}} + Z_{nc,a_i} \Delta y_{rs_{i0}}) \mathbf{i} \right. \\ &\quad + (m_{nc,r_i} \cos \eta_i + Q_{nc,r_i} \sin \eta_i - X_{nc,a_i} \Delta z_{rs_{i0}} + Z_{nc,a_i} \Delta x_{rs_{i0}}) \mathbf{j} \\ &\quad \left. + (m_{nc,r_i} \sin \eta_i - Q_{nc,r_i} \cos \eta_i + X_{nc,a_i} \Delta y_{rs_{i0}} + Y_{nc,a_i} \Delta x_{rs_{i0}}) \mathbf{k} \right] \end{aligned} \quad (196)$$

The function $C(k)$ is the Theodorsen's function to account for unsteady aerodynamic effects at high frequencies, $k = \frac{\omega \bar{c}_b}{2V_t} = \frac{\omega \bar{c}_b}{2\bar{r}\Omega_i}$ with the average blade chord $\bar{c}_b$ taken at the average blade radial station $\bar{r} = \frac{3}{4}R$. The unsteady aerodynamic correction by the Theodorsen's theory in aeroelasticity is important because it properly accounts for phase delay and amplitude attenuation of the aerodynamic forces and moments as functions of the frequency. For low-frequency rigid-body flight dynamics, this correction is usually ignored since $C(k) = 1$ as $k \rightarrow 0$. Other unsteady aerodynamic theories such as the Leishman's model could be used in lieu of the Theodorsen's theory.

The moments acting on the wing structure are evaluated as

$$\mathbf{M}_{w_j} = C(k) \mathbf{M}_{c,w_j} + \mathbf{M}_{nc,w_j} \quad (197)$$

where

$$\begin{aligned} \mathbf{M}_{c,w_j} &= \sum_{i=2j-1}^{2j} (\mathbf{M}_{c,r_i} + \mathbf{p}_{r_i} \times \mathbf{F}_{c,r_i}) \\ &= \sum_{i=2j-1}^{2j} \left[-(l_{c,r_i} + Y_{c,a_i} \Delta z_{rs_{i0}} + Z_{c,a_i} \Delta y_{rs_{i0}}) \mathbf{i} \right. \\ &\quad + (m_{c,r_i} \cos \eta_i + Q_{c,r_i} \sin \eta_i - X_{c,a_i} \Delta z_{rs_{i0}} + Z_{c,a_i} \Delta x_{rs_{i0}}) \mathbf{j} \\ &\quad \left. + (m_{c,r_i} \sin \eta_i - Q_{c,r_i} \cos \eta_i + X_{c,a_i} \Delta y_{rs_{i0}} + Y_{c,a_i} \Delta x_{rs_{i0}}) \mathbf{k} \right] \end{aligned} \quad (198)$$

$$\begin{aligned} \mathbf{M}_{nc,w_j} &= \sum_{i=2j-1}^{2j} (\mathbf{M}_{nc,r_i} + \mathbf{p}_{r_i} \times \mathbf{F}_{nc,r_i}) \\ &= \sum_{i=2j-1}^{2j} \left[-(l_{nc,r_i} + Y_{nc,a_i} \Delta z_{rs_{i0}} + Z_{nc,a_i} \Delta y_{rs_{i0}}) \mathbf{i} \right. \\ &\quad + (m_{nc,r_i} \cos \eta_i + Q_{nc,r_i} \sin \eta_i - X_{nc,a_i} \Delta z_{rs_{i0}} + Z_{nc,a_i} \Delta x_{rs_{i0}}) \mathbf{j} \\ &\quad \left. + (m_{nc,r_i} \sin \eta_i - Q_{nc,r_i} \cos \eta_i + X_{nc,a_i} \Delta y_{rs_{i0}} + Y_{nc,a_i} \Delta x_{rs_{i0}}) \mathbf{k} \right] \end{aligned} \quad (199)$$

The rotor forces and moments are transmitted to the rotor support structures at the intersections of the rotor thrust lines and the elastic axes of the rotor support structures. The position vector from the intersection to the origin of the i -rotor reference frame is given by

$$\mathbf{r}_{s_i} = \Delta y_{rs_{i0}} \mathbf{j} + \Delta z_{rs_{i0}} \mathbf{k} \quad (200)$$

The forces acting on the i -th rotor support produced by the rotors are

$$\mathbf{F}_{s_i} = C(k) \mathbf{F}_{c,s_i} + \mathbf{F}_{nc,s_i} \quad (201)$$

where

$$\mathbf{F}_{c,s_i} = \mathbf{F}_{c,r_i} \quad (202)$$

$$\mathbf{F}_{nc,s_i} = \mathbf{F}_{nc,r_i} \quad (203)$$

The moments acting on the rotor support structures are evaluated as

$$\mathbf{M}_{s_i} = C(k) \mathbf{M}_{c,s_i} + \mathbf{M}_{nc,s_i} \quad (204)$$

where

$$\begin{aligned} \mathbf{M}_{c,s_i} &= \mathbf{M}_{c,r_i} + \mathbf{r}_{s_i} \times \mathbf{F}_{c,r_i} \\ &= \sum_{i=2j-1}^{2j} \left[-(l_{c,r_i} + Y_{c,a_i} \Delta z_{rs_{i0}} + Z_{c,a_i} \Delta y_{rs_{i0}}) \mathbf{i} + (m_{c,r_i} \cos \eta_i + Q_{c,r_i} \sin \eta_i - X_{c,a_i} \Delta z_{rs_{i0}}) \mathbf{j} \right. \\ &\quad \left. + (m_{c,r_i} \sin \eta_i - Q_{c,r_i} \cos \eta_i + X_{c,a_i} \Delta y_{rs_{i0}}) \mathbf{k} \right] \end{aligned} \quad (205)$$

$$\begin{aligned} \mathbf{M}_{nc,s_i} &= \mathbf{M}_{nc,r_i} + \mathbf{r}_{s_i} \times \mathbf{F}_{nc,r_i} \\ &= \sum_{i=2j-1}^{2j} \left[-(l_{nc,r_i} + Y_{nc,a_i} \Delta z_{rs_{i0}} + Z_{nc,a_i} \Delta y_{rs_{i0}}) \mathbf{i} + (m_{nc,r_i} \cos \eta_i + Q_{nc,r_i} \sin \eta_i - X_{nc,a_i} \Delta z_{rs_{i0}}) \mathbf{j} \right. \\ &\quad \left. + (m_{nc,r_i} \sin \eta_i - Q_{nc,r_i} \cos \eta_i + X_{nc,a_i} \Delta y_{rs_{i0}}) \mathbf{k} \right] \end{aligned} \quad (206)$$

The generalized force is defined as

$$\mathbf{F}_{gen} = \sum_{j=1}^4 \mathbf{F}_{gen,w_j} + \sum_{i=1}^{n_{rotor}} \mathbf{F}_{gen,s_i} \quad (207)$$

where $\mathbf{F}_{gen_{w_j}}(t)$ is the generalized force contribution by the elastic motion of the wing at the location y_j and $\mathbf{F}_{s_i,gen}(t)$ is the generalized force contribution by the elastic motion of the i -th rotor support structure.

If a structure experiences a concentrated force $\mathbf{F}_{w_j}(t)$ and a concentrated moment $\mathbf{M}_{w_j}(t)$ at the location y_j where the elastic deformation is described by the structural displacement $\mathbf{u}_w(y_j, t)$ and rotation $\phi_w(y_j, t)$, then the generalized force is effectively the work done on the structure by the force and moment per unit generalized displacement $\mathbf{q}(t)$. This work done is expressed as

$$\mathbf{q} \cdot \mathbf{F}_{gen_{w_j}} = \mathbf{u}_w(y_j, t) \cdot \mathbf{F}_{w_j} + \phi_w(y_j, t) \cdot \mathbf{M}_{w_j} \quad (208)$$

The displacement and rotation can be expressed in terms of the generalized displacement $\mathbf{q}(t)$ according to Eqs. (9) and (10). The force $\mathbf{F}_w(t)$ and moment $\mathbf{M}_w(t)$ are generally functions of the elastic motion described by $\dot{\mathbf{u}}_w(y_j, t)$, $\ddot{\mathbf{u}}_w(y_j, t)$, $\dot{\phi}_w(y_j, t)$, $\ddot{\phi}_w(y_j, t)$, and $\ddot{\phi}_w(y_j, t)$ as well as functions of the aircraft motion described by $\mathbf{x}_a(t)$, $\dot{\mathbf{x}}_a(t)$, $\mathbf{v}_p(t)$, $\dot{\mathbf{v}}_p(t)$, $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, and $\ddot{\mathbf{u}}(t)$.

The generalized force of the wing structure due to the moments is decomposed into a circulatory component and non-circulatory component as

$$\mathbf{F}_{gen,w_j} = C(k) \mathbf{F}_{c,gen,w_j} + \mathbf{F}_{nc,gen,w_j} \quad (209)$$

The circulatory generalized force for the wing structure is evaluated as

$$\mathbf{F}_{c,gen,w_j} = \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{v}_p} \mathbf{v}_p + \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{x}_q} \mathbf{x}_q + \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{u}_\theta} \mathbf{u}_\theta + \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \Omega} \Omega \quad (210)$$

where $\frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{x}_q} = \begin{bmatrix} \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{q}} & \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \dot{\mathbf{q}}} \end{bmatrix}$.

The partial derivatives are evaluated as

$$\frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{x}_a} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{c,w_j}}{\partial \mathbf{x}_a} \quad (211)$$

$$\frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{v}_p} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{c,w_j}}{\partial \mathbf{v}_p} \quad (212)$$

$$\frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{q}} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{c,w_j}}{\partial \phi_w} \Psi_w(y_j) \quad (213)$$

$$\frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \dot{\mathbf{q}}} = \Psi_w^\top(y_j) \left[\frac{\partial \mathbf{M}_{c,w_j}}{\partial \dot{\mathbf{u}}_w} \Phi_w(y_j) + \frac{\partial \mathbf{M}_{c,w_j}}{\partial \dot{\phi}_w} \Psi_w(y_j) \right] \quad (214)$$

$$\frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \mathbf{u}_\theta} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{c,w_j}}{\partial \mathbf{u}_\theta} \quad (215)$$

$$\frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \Omega} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{c,w_j}}{\partial \Omega} \quad (215)$$

The non-circulatory generalized force for the wing structure is evaluated as

$$\mathbf{F}_{nc,gen,w_j} = \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta \quad (216)$$

where $\frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{x}}_q} = \begin{bmatrix} \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{q}}} & \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \ddot{\mathbf{q}}} \end{bmatrix}$.

The partial derivatives are evaluated as

$$\frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \mathbf{x}_a} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{nc,w_j}}{\partial \mathbf{x}_a} \quad (217)$$

$$\frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{x}}_a} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{nc,w_j}}{\partial \dot{\mathbf{x}}_a} \quad (218)$$

$$\frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{v}}_p} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{nc,w_j}}{\partial \dot{\mathbf{v}}_p} \quad (219)$$

$$\frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{q}}} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{nc,w_j}}{\partial \dot{\phi}_w} \Psi_w(y_j) \quad (220)$$

$$\frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \ddot{\mathbf{q}}} = \Psi_w^\top(y_j) \left[\frac{\partial \mathbf{M}_{nc,w_j}}{\partial \ddot{\mathbf{u}}_w} \Phi_w(y_j) + \frac{\partial \mathbf{M}_{nc,w_j}}{\partial \ddot{\phi}_w} \Psi_w(y_j) \right] \quad (221)$$

$$\frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial \dot{\mathbf{u}}_\theta} = \Psi_w^\top(y_j) \frac{\partial \mathbf{M}_{nc,w_j}}{\partial \dot{\mathbf{u}}_\theta} \quad (221)$$

The generalized forces of the rotor support structures are decomposed into circulatory components and non-circulatory components as

$$\mathbf{F}_{gen,s_i} = C(k) \mathbf{F}_{c,gen,s_i} + \mathbf{F}_{nc,gen,s_i} \quad (222)$$

The circulatory generalized forces for the rotor support structure are evaluated as

$$\mathbf{F}_{c,gen,s_i} = \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{v}_p} \mathbf{v}_p + \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{x}_q} \mathbf{x}_q + \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{u}_\theta} \mathbf{u}_\theta + \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \Omega} \Omega \quad (223)$$

where $\frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{x}_q} = \begin{bmatrix} \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{q}} & \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \dot{\mathbf{q}}} \end{bmatrix}$.

The partial derivatives are evaluated as

$$\frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{x}_a} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{c,s_i}}{\partial \mathbf{x}_a} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{c,s_i}}{\partial \mathbf{x}_a} \quad (224)$$

$$\frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{v}_p} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{c,s_i}}{\partial \mathbf{v}_p} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{c,s_i}}{\partial \mathbf{v}_p} \quad (225)$$

$$\frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{q}} = \left[\Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{c,s_i}}{\partial \phi_{s_i}} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{c,s_i}}{\partial \phi_{s_i}} \right] \Psi_{s_i} (\Delta x_{rs_{i0}}) \quad (226)$$

$$\begin{aligned} \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial \dot{\mathbf{q}}} &= \left[\Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{c,s_i}}{\partial \dot{\mathbf{u}}_{s_i}} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{c,s_i}}{\partial \dot{\mathbf{u}}_{s_i}} \right] \Phi_{s_i} (\Delta x_{rs_{i0}}) \\ &\quad + \left[\Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{c,s_i}}{\partial \dot{\phi}_{s_i}} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{c,s_i}}{\partial \dot{\phi}_{s_i}} \right] \Psi_{s_i} (\Delta x_{rs_{i0}}) \end{aligned} \quad (227)$$

$$\frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \mathbf{u}_\theta} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{c,s_i}}{\partial \mathbf{u}_\theta} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{c,s_i}}{\partial \mathbf{u}_\theta} \quad (228)$$

$$\frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \Omega} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{c,s_i}}{\partial \Omega} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{c,s_i}}{\partial \Omega} \quad (229)$$

The non-circulatory generalized forces for the rotor support structures are evaluated as

$$\mathbf{F}_{nc,gen,s_i} = \frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta \quad (230)$$

where $\frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{x}}_q} = \left[\frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{q}}} \quad \frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{q}}} \right]$.

The partial derivatives are evaluated as

$$\frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \mathbf{x}_a} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{nc,s_i}}{\partial \mathbf{x}_a} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{nc,s_i}}{\partial \mathbf{x}_a} \quad (231)$$

$$\frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{x}}_a} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{nc,s_i}}{\partial \dot{\mathbf{x}}_a} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{nc,s_i}}{\partial \dot{\mathbf{x}}_a} \quad (232)$$

$$\frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{v}}_p} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{nc,s_i}}{\partial \dot{\mathbf{v}}_p} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{nc,s_i}}{\partial \dot{\mathbf{v}}_p} \quad (233)$$

$$\frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{q}}} = \left[\Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{nc,s_i}}{\partial \dot{\phi}_{s_i}} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{nc,s_i}}{\partial \dot{\phi}_{s_i}} \right] \Psi_{s_i} (\Delta x_{rs_{i0}}) \quad (234)$$

$$\begin{aligned} \frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \ddot{\mathbf{q}}} &= \left[\Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{nc,s_i}}{\partial \ddot{\mathbf{u}}_{s_i}} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{nc,s_i}}{\partial \ddot{\mathbf{u}}_{s_i}} \right] \Phi_{s_i} (\Delta x_{rs_{i0}}) \\ &\quad + \left[\Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{nc,s_i}}{\partial \ddot{\phi}_{s_i}} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{nc,s_i}}{\partial \ddot{\phi}_{s_i}} \right] \Psi_{s_i} (\Delta x_{rs_{i0}}) \end{aligned} \quad (235)$$

$$\frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial \dot{\mathbf{u}}_\theta} = \Phi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{F}_{nc,s_i}}{\partial \dot{\mathbf{u}}_\theta} + \Psi_{s_i}^\top (\Delta x_{rs_{i0}}) \frac{\partial \mathbf{M}_{nc,s_i}}{\partial \dot{\mathbf{u}}_\theta} \quad (236)$$

VII. Unsteady Aerodynamic Correction

We consider two unsteady aerodynamic models: Theodorsen's theory for incompressible flow with the R. T. Jones' approximation and the Leishman's model for subsonic flow.

A. Theodorsen's Theory

The unsteady aerodynamic effect of a rotor in a dynamic motion is much more complex than that of a fixed wing. The dynamic inflow models are generally used in flight dynamic modeling of rotors to capture the unsteady aerodynamics of the induced flow. Since the flight dynamic motion of a rigid-body rotorcraft involves very low frequencies, the dynamic inflow models are sufficient. For motions that involve high frequency dynamics such as structural modes when those modes are coupled to the rigid-body modes, the unsteady aerodynamic behaviors created by the high frequency motion which involved pitching and plunging of the entire rotor are less well understood. Theodorsen's unsteady aerodynamic theory for an oscillating

airfoil in free air is well-known,⁸ but there exist gaps in the theoretical knowledge of unsteady aerodynamics of a rotating airfoil in a pitching and plunging motion. The unsteady aerodynamic response of an aircraft can be estimated by conducting forced oscillation numerical experiments using CFD¹¹ from which a flight dynamic model can be constructed using system identification methods,¹² but such studies are computationally very expensive especially for a multi-rotor eVTOL aircraft configuration. In this study, we propose to use the Theodorsen's theory and Leishmann's method to capture the effect of unsteady flow on an airfoil in a rotating periodic motion subject to high-frequency dynamic motion in the vertical plane. While the present study pertains to flight dynamic model development, the use of the Theodorsen's theory creates a general framework for future consideration of high frequency dynamic motion such as coupled flight dynamic motion with aeroelasticity for eVTOL aircraft. The simplicity of the Theodorsen's theory facilitates the analytical flight dynamic model development. The Leishman's method¹³ is also considered.

The circulatory lift according to Theodorsen⁸ is given by

$$\frac{dl_{c,b_i}}{d\Psi} = [\bar{c}_l + C(k) c_{l_\alpha} \alpha] \frac{1}{2} \rho V^2 c_b \quad (237)$$

The aerodynamic lag expressed as a transport delay is

$$t_d = \frac{1}{\omega} \tan^{-1} \left[-\frac{G(k)}{F(k)} \right] \quad (238)$$

R. T. Jones proposed the following approximation of the Theodorsen's function:¹⁴

$$C(k) \approx \hat{C}(\bar{s}) = \frac{0.5\bar{s}^2 + a_1\bar{s} + a_2}{\bar{s}^2 + a_3\bar{s} + a_2} \quad (239)$$

where $\bar{s} = \frac{s\bar{c}_b}{2V_t}$ is a dimensionless Laplace transform variable, and $a_1 = 0.2808$, $a_2 = 0.01365$, and $a_3 = 0.3455$.

The largest expected reduced frequency of interest is based on the highest active structural mode. For the all electric Lift+Cruise aircraft, the first 14 structural modes occur in a frequency bandwidth of 71 rad/sec. This corresponds to a reduced frequency of $k = 0.1$ for the mean chord of $\bar{c}_b = 1.2126$ ft at the average blade station of $\bar{r} = 3.75$ which gives $C(k) = 0.83 - i0.17$. To increase accuracy or $k \in [0, 0.1]$, we obtain the following approximation of the Theodorsen's function by a frequency-domain least square regression:

$$\hat{C}(\bar{s}) = \frac{0.5\bar{s}^2 + 0.2226\bar{s} + 0.0074006}{\bar{s}^2 + 0.2603\bar{s} + 0.007400} \quad (240)$$

$C(k)$ can be expressed in the partial fraction form as

$$C(\bar{s}) = r_0 + \frac{r_1}{\bar{s} - p_1} + \frac{r_2}{\bar{s} - p_2} \quad (241)$$

where r_i , $i = 1, 2, 3$ are the residues and p_i , $i = 1, 2$ are the poles of the unsteady aerodynamic transfer function given in Table 3.

	$\hat{C}(\bar{s})$
r_0	0.5
r_1	0.08851
r_2	0.003620
p_1	-0.2278
p_2	-0.03249

Table 3. Residues and Poles of Unsteady Aerodynamic Transfer Function

Figure 5 shows the comparison of the exact Theodorsen's function, the R. T. Jones' approximation, and the least-square approximation.

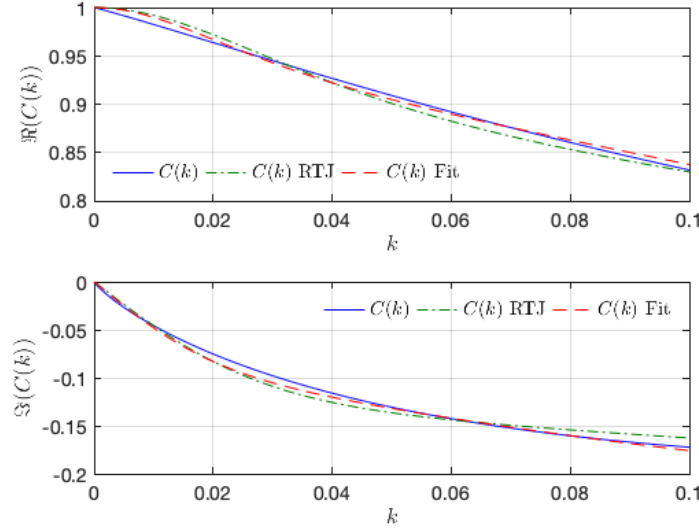


Figure 5. Approximation of Theodorsen's Functions $C(k)$

The unsteady lift is expressed as

$$c_l = c_{l_\alpha} C(k) \alpha = c_{l_\alpha} \left(r_0 + \frac{r_1}{\bar{s} - p_1} + \frac{r_2}{\bar{s} - p_2} \right) \alpha \quad (242)$$

Let

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{2\bar{V}_t}{\bar{c}_b} \left(\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \alpha \right) \quad (243)$$

Then, the unsteady lift coefficient is obtained as

$$c_l = c_{l_\alpha} (r_0 \alpha + r_1 x_1 + r_2 x_2) \quad (244)$$

We generalize the unsteady aerodynamic analysis for the circulatory force acting on the aircraft which can be expressed as

$$\mathbf{F}_{c,a_i} = \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{x}_a} C(k) \mathbf{x}_a + \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{v}_p} C(k) \mathbf{v}_p + \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{x}_q} C(k) \mathbf{x}_q + \frac{\partial \mathbf{F}_{c,a_i}}{\partial \mathbf{u}_\theta} C(k) \mathbf{u}_\theta + \frac{\partial \mathbf{F}_{c,a_i}}{\partial \boldsymbol{\Omega}} C(k) \boldsymbol{\Omega} \quad (245)$$

where

$$C(k) \mathbf{x}_a = r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a \quad (246)$$

$$C(k) \mathbf{v}_p = r_0 \mathbf{v}_p + \mathbf{r}_p \mathbf{y}_p \quad (247)$$

$$C(k) \mathbf{x}_q = r_0 \mathbf{x}_q + \mathbf{r}_q \mathbf{y}_q \quad (248)$$

$$C(k) \mathbf{u}_\theta = r_0 \mathbf{u}_\theta + \mathbf{r}_\theta \mathbf{y}_\theta \quad (249)$$

$$C(k) \boldsymbol{\Omega} = r_0 \boldsymbol{\Omega} + \mathbf{r}_\Omega \mathbf{y}_\Omega \quad (250)$$

The quantities $\mathbf{y}_a$, $\mathbf{y}_p$, $\mathbf{y}_q$, $\mathbf{y}_\theta$, and $\mathbf{y}_\Omega$ are the unsteady aerodynamic state vectors which are computed from

$$\dot{\mathbf{y}}_a = \mathbf{A} \mathbf{y}_a + \mathbf{B} \mathbf{x}_a \quad (251)$$

$$\dot{\mathbf{y}}_p = \mathbf{A} \mathbf{y}_p + \mathbf{B} \mathbf{v}_p \quad (252)$$

$$\dot{\mathbf{y}}_q = \mathbf{A} \mathbf{y}_q + \mathbf{B} \mathbf{x}_q \quad (253)$$

$$\dot{\mathbf{y}}_\theta = \mathbf{A} \mathbf{y}_\theta + \mathbf{B} \mathbf{u}_\theta \quad (254)$$

$$\dot{\mathbf{y}}_\Omega = \mathbf{A} \mathbf{y}_\Omega + \mathbf{B} \boldsymbol{\Omega} \quad (255)$$

where $\mathbf{r} = \begin{bmatrix} r_1 \mathbf{I} & r_2 \mathbf{I} \end{bmatrix}$, $\mathbf{A} = \frac{2\bar{V}_t}{c_b} \text{diag} \left(\begin{bmatrix} p_1 \mathbf{I} & p_2 \mathbf{I} \end{bmatrix} \right)$, and $\mathbf{B} = \frac{2\bar{V}_t}{c_b} \begin{bmatrix} \mathbf{I} & \mathbf{I} \end{bmatrix}^\top$.

The circulatory moment $\mathbf{M}_{c,a_i}$ and the circulatory generalized forces $\mathbf{F}_{c,gen,w_j}$ and $\mathbf{F}_{c,gen,s_i}$ are treated in the same manner.

B. Leishman's Method

Leishman applies the indicial functions to obtain the circulatory lift and non-circulatory lift.¹³ The unsteady lift coefficient is given by

$$c_l = \left[\frac{2\pi}{\beta} \phi_\alpha^C(\tau, M) + \frac{4}{M} \phi_\alpha^I(\tau, M) \right] \alpha \quad (256)$$

where $M = \frac{\bar{V}_t}{a(h)}$ is the Mach number at the average radial station $\bar{r}$, and $\beta = \sqrt{1 - M^2}$ is a compressibility correction term. The compressible lift curve slope c_{l_α} accounts for the compressibility effect automatically in many aerodynamic solvers. Therefore, we can write

$$c_l = \left[c_{l_\alpha} \phi_\alpha^C(\tau, M) + \frac{4}{M} \phi_\alpha^I(\tau, M) \right] \alpha \quad (257)$$

The indicial functions $\phi_\alpha^C(\tau, M)$ and $\phi_\alpha^I(\tau, M)$ are the circulatory and non-circulatory responses to the unit step angle of attack, respectively, given by

$$\phi_\alpha^C(\tau, M) = 1 - A_1 e^{-b_1 \beta^2 \tau} - A_2 e^{-b_2 \beta^2 \tau} \quad (258)$$

$$\phi_\alpha^I(\tau, M) = e^{-\frac{c_b \tau}{2\bar{V}_t T_\alpha}} \quad (259)$$

where $\tau = \frac{2\bar{V}_t t}{c_b}$ is a non-dimensional time or distance traveled by the airfoil in semi-chords, $A_1 = 0.3$, $A_2 = 0.7$, $b_1 = 0.14$, $b_2 = 0.53$, and

$$T_\alpha = K_\alpha T_I \quad (260)$$

$$K_\alpha = \frac{1}{1 - M + \pi \beta M^2 (A_1 b_1 + A_2 b_2)} \quad (261)$$

$$T_I = \frac{c_b}{a(h)} \quad (262)$$

The Laplace transforms of the indicial functions are obtained as

$$\Phi_\alpha^C(\bar{s}, M) = \frac{A_1 b_1 \beta^2}{\bar{s} + b_1 \beta^2} + \frac{A_2 b_2 \beta^2}{\bar{s} + b_2 \beta^2} \quad (263)$$

$$\Phi_\alpha^I(\bar{s}, M) = \frac{\bar{s}}{\bar{s} + \frac{c_b}{2\bar{V}_t T_\alpha}} \quad (264)$$

The altitude at the design condition is $h = 4000$ ft which corresponds to $M = 0.3747$. This gives

$$\Phi_\alpha^C(\bar{s}) = \frac{r_1}{\bar{s} - p_1} + \frac{r_2}{\bar{s} - p_2} \quad (265)$$

$$\Phi_\alpha^I(\bar{s}) = \frac{\bar{s}}{\bar{s} - p_3} \quad (266)$$

where $r_1 = 0.03610$, $r_2 = 0.3189$, $p_1 = -0.1203$, $p_2 = -0.4556$, $p_3 = -1.0597$.

The circulatory lift given by Leishman's method is obtained as

$$c_{l_c} = c_{l_\alpha} (r_1 x_1 + r_2 x_2) \quad (267)$$

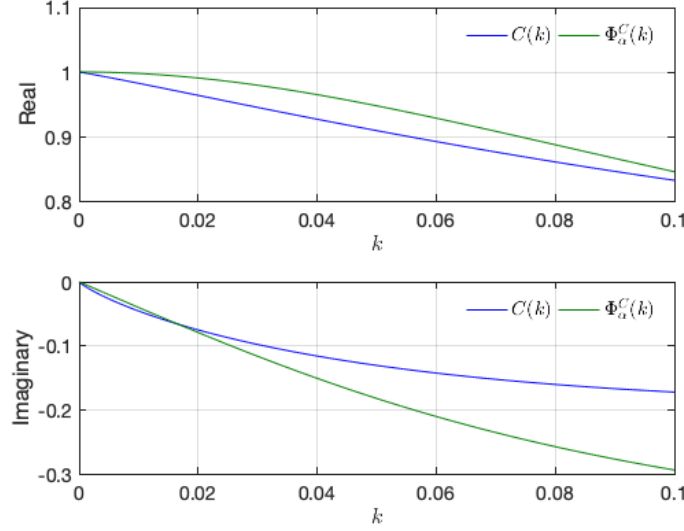


Figure 6. Theodorsen's Function and Leishman's Indicial Function

Figure 6 shows the comparison between the Theodorsen's function $C(k)$ for incompressible flow and Leishman's indicial function $\Phi_\alpha^C(k)$ for subsonic flow. The magnitude of $\Phi_\alpha^C(k)$ is larger than that of $C(k)$ for $k \in [0, 0.1]$. The phase delay of $\Phi_\alpha^C(k)$ is greater than that of $C(k)$. The Leishman's indicial function $\Phi_\alpha^C(\bar{s})$ has the asymptotic limit of $\lim_{k \rightarrow \infty} \Phi_\alpha^C(\bar{s}) \rightarrow 0$.

The non-circulatory lift coefficient is given by the Leishman's method as

$$c_{l_{nc}} = \frac{4}{M} \Phi_\alpha^I(\bar{s}, M) \alpha = \frac{4}{M} \frac{\bar{s} \alpha}{\bar{s} - p_3} = \frac{4}{M} \frac{1}{\bar{s} - p_3} \frac{\dot{\alpha} \bar{c}_b}{2\bar{V}_t} \quad (268)$$

where $p_3 = -\frac{\bar{c}_b}{2\bar{V}_t T_\alpha} = -\frac{1}{2MK_\alpha}$.

The steady-state solution of the non-circulatory lift is obtained as

$$c_{l_{nc}} = -\frac{4\bar{c}_b}{2M\bar{V}_t p_3} \dot{\alpha} = \frac{4K_\alpha \bar{c}_b}{Ma(h)} \dot{\alpha} \approx \frac{4K_\alpha \bar{c}_b}{\bar{V}} \dot{\alpha} \quad (269)$$

The angle of attack on a blade section is approximately equal to

$$\alpha = \frac{w}{V} \quad (270)$$

Thus, the non-circulatory lift coefficient is given by

$$c_{l_{nc}} = \frac{4K_\alpha \bar{c}_b \dot{w}}{\bar{V}^2} \quad (271)$$

The non-circulatory lift coefficient given by the Theodorsen's theory is

$$c_{l_{nc}} = \frac{\frac{\pi}{4} \rho_\infty \bar{c}_b^2 \dot{w}}{\frac{1}{2} \rho_\infty \bar{V}^2 \bar{c}_b} = \frac{\pi}{2} \frac{\bar{c}_b \dot{w}}{\bar{V}^2} \quad (272)$$

Thus, the Leishman's method predicts a larger non-circulatory lift than the Theodorsen's theory. A correction factor can be defined to adjust the non-circulatory lift to match that predicted by the Leishman's method as

$$K_{l_{nc}} = \frac{8K_\alpha}{\pi} \quad (273)$$

The non-circulatory lift is modified by the Leishman's method as

$$\frac{dl_{nc, b_i}}{d\Psi_i} = -K_{l_{nc}} p_3 \frac{\Phi_\alpha^I(\bar{s})}{\bar{s}} \frac{\pi}{4} \rho_\infty \bar{c}_b^2 a_{z_{b_i}} \left(r', \frac{c_b}{2}, t \right) \quad (274)$$

We express the non-circulatory lift as

$$\frac{dl_{nc,b_i}}{d\Psi_i} = \frac{r_3}{\bar{s} - p_3} \frac{\pi}{4} \rho_\infty c_b^2 a_{z_{b_i}} \left(r', \frac{c_b}{2}, t \right) \quad (275)$$

where $r_3 = -K_{l_{nc}} p_3 = \frac{4}{\pi M}$. For the Lift+Cruise aircraft, $r_3 = 3.3979$.

The solution of the non-circulatory lift is given by

$$\frac{dl_{nc,b_i}}{d\Psi_i} = r_3 x_3 \quad (276)$$

where

$$\dot{x}_3 = \frac{2\bar{V}_t}{\bar{c}_b} \left[p_3 x_3 + \frac{\pi}{4} \rho_\infty c_b^2 a_{z_{b_i}} \left(r', \frac{c_b}{2}, t \right) \right] \quad (277)$$

The circulatory forces using the Leishman's method is modified by the same equation (245) for the Theodorsen's theory with $r_0 = 0$.

The Leishman's method modifies the non-circulatory force acting on the aircraft as

$$\mathbf{F}_{nc,a_i} = \frac{\partial \mathbf{F}_{nc,a_i}}{\partial \mathbf{x}_a} \mathbf{x}_a + r_3 \left(\frac{\partial \mathbf{F}_{nc,a_i}}{\partial \dot{\mathbf{x}}_a} \mathbf{z}_a + \frac{\partial \mathbf{F}_{nc,a_i}}{\partial \dot{\mathbf{v}}_p} \mathbf{z}_p + \frac{\partial \mathbf{F}_{nc,a_i}}{\partial \dot{\mathbf{x}}_q} \mathbf{z}_q + \frac{\partial \mathbf{F}_{nc,a_i}}{\partial \dot{\mathbf{u}}_\theta} \mathbf{z}_\theta \right) \quad (278)$$

where $\mathbf{z}_a$, $\mathbf{z}_p$, $\mathbf{z}_q$, and $\mathbf{z}_\theta$ are the unsteady aerodynamic vectors which are computed from

$$\dot{\mathbf{z}}_a = \frac{2\bar{V}_t}{\bar{c}_b} p_3 \mathbf{z}_a + \dot{\mathbf{x}}_a \quad (279)$$

$$\dot{\mathbf{z}}_p = \frac{2\bar{V}_t}{\bar{c}_b} p_3 \mathbf{z}_p + \dot{\mathbf{v}}_p \quad (280)$$

$$\dot{\mathbf{z}}_q = \frac{2\bar{V}_t}{\bar{c}_b} p_3 \mathbf{z}_q + \dot{\mathbf{x}}_q \quad (281)$$

$$\dot{\mathbf{z}}_\theta = \frac{2\bar{V}_t}{\bar{c}_b} p_3 \mathbf{z}_\theta + \dot{\mathbf{u}}_\theta \quad (282)$$

The rest of the non-circulatory forces are treated in the same manner.

C. Unsteady Aerodynamic Corrections to Rotor Forces and Moments

The inflow dynamics depend on the rotor thrust, rolling moment, and pitching moment which are entirely functions of the lift force acting on the rotor blades. We express the circulatory rotor coefficient vector

$\mathbf{C}_{c,p} = \left[\mathbf{C}_{c,p_1}^\top \quad \mathbf{C}_{c,p_2}^\top \quad \cdots \quad \mathbf{C}_{c,p_{n_{rotor}}}^\top \right]^\top$ for inflow dynamics as

$$C(k) \mathbf{C}_{c,p} = \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} \mathbf{v}_p + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_q} \mathbf{x}_q + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} \mathbf{u}_\theta + \frac{\partial \mathbf{C}_{c,p}}{\partial \boldsymbol{\Omega}} \boldsymbol{\Omega} \quad (283)$$

The unsteady aerodynamic correction to the circulatory rotor coefficients is expressed as

$$\begin{aligned} C(k) \mathbf{C}_{c,p} = & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_q} (r_0 \mathbf{x}_q + \mathbf{r}_a \mathbf{y}_q) \\ & + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \frac{\partial \mathbf{C}_{c,p}}{\partial \boldsymbol{\Omega}} (r_0 \boldsymbol{\Omega} + \mathbf{r}_a \mathbf{y}_\Omega) \end{aligned} \quad (284)$$

where $r_0 = 0$ if the Leishman's method is applied.

The non-circulatory rotor coefficient vector $\mathbf{C}_{nc,p} = \left[\mathbf{C}_{nc,p_1}^\top \quad \mathbf{C}_{nc,p_2}^\top \quad \cdots \quad \mathbf{C}_{nc,p_{n_{rotor}}}^\top \right]^\top$ is expressed as

$$\mathbf{C}_{nc,p} = \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta \quad (285)$$

The unsteady aerodynamic correction by the Leishman's method to the non-circulatory rotor coefficients is expressed as

$$\mathbf{C}_{nc,p} = \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} \mathbf{x}_a + r_3 \left(\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_a} \mathbf{z}_a + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} \mathbf{z}_p + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_q} \mathbf{z}_q + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} \mathbf{z}_\theta \right) \quad (286)$$

VIII. Aeroservoelastic Model of Elastic Aircraft

An aeroservoelastic (ASE) model of an elastic aircraft comprises of the following essential elements:

- A flight dynamic model of the rigid-body aircraft which includes the aerodynamic force contributions by the rigid-aircraft motion and the deformation of the aircraft structure,
- A rotor inflow dynamic model which includes the aerodynamic force contributions by the rigid-aircraft motion and the deformation of the aircraft structure,
- A structural dynamic model of the aircraft structure which includes the aerodynamic force contributions by the rigid-aircraft motion and the deformation of the aircraft structure,
- A closed-loop servo-actuator dynamic model of the control effectors which includes a closed-loop PID controller that tracks an actuator command, and
- A unsteady aerodynamic model which models the phase delay and amplitude attenuation by a state-space realization of a transfer function approximation of the frequency-domain Theodorsen's function or alternatively the Leishman's method.

Each of these elements are described as follows.

A. Flight Dynamic Model of Rigid Aircraft

The nonlinear flight dynamic equation of a rigid-body aircraft is described by

$$m_a \mathbf{I}_3 \dot{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{v} = C(k) \mathbf{F}_{c,a} + \mathbf{F}_{nc,a} + \mathbf{F}_b \quad (287)$$

$$\mathbf{I}_a \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_a \boldsymbol{\omega} = C(k) \mathbf{M}_{c,a} + \mathbf{M}_{nc,a} + \mathbf{M}_b \quad (288)$$

where the circulatory and non-circulatory force and moment vectors have already been developed.

The body and propulsive force and moment vectors $\mathbf{F}_b$ and $\mathbf{M}_b$ are given by

$$\mathbf{F}_b = \begin{bmatrix} T - m_a g \sin \theta \\ m_a g \cos \theta \sin \phi \\ m_a g \cos \theta \cos \phi \end{bmatrix} \quad (289)$$

$$\mathbf{M}_b = \begin{bmatrix} l_T \\ m_T \\ n_T \end{bmatrix} \quad (290)$$

where T denotes forward flight thrust force, l_T , m_T , and n_T are thrust-induced moments, θ is the aircraft pitch angle, and ϕ is the aircraft bank angle.

The nonlinear kinematic equation is given by

$$\dot{\boldsymbol{\sigma}} = \mathbf{H} \boldsymbol{\omega} \quad (291)$$

where $\boldsymbol{\sigma} = \begin{bmatrix} \phi & \theta \end{bmatrix}^\top$ and

$$\mathbf{H} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \end{bmatrix} \quad (292)$$

We are interested in the linearized equations of motion about the equilibrium condition given by $\bar{\boldsymbol{\sigma}} = \mathbf{0}$ and $\boldsymbol{\omega} = \mathbf{0}$. Upon linearization, we obtain

$$m_a \mathbf{I}_3 \dot{\mathbf{v}} = \frac{\partial \mathbf{F}}{\partial \boldsymbol{\omega}} \boldsymbol{\omega} + C(k) \mathbf{F}_{c,a} + \mathbf{F}_{nc,a} + \frac{\partial \mathbf{F}_b}{\partial \boldsymbol{\sigma}} \boldsymbol{\sigma} \quad (293)$$

$$\bar{\mathbf{I}}_a \dot{\boldsymbol{\omega}} = C(k) \mathbf{M}_{c,a} + \mathbf{M}_{nc,a} \quad (294)$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} = \bar{\mathbf{H}} \mathbf{x}_a \quad (295)$$

where $\boldsymbol{\sigma}(t) = \begin{bmatrix} \phi(t) & \theta(t) & \psi(t) \end{bmatrix}^\top$, $\psi(t)$ is the heading angle, $\bar{\mathbf{I}}_a = \text{diag}(\mathbf{I}_a)$, $\frac{\partial \mathbf{F}}{\partial \boldsymbol{\omega}} = \begin{bmatrix} 0 & -\bar{w} \\ \bar{w} & 0 \\ -\bar{v} & \bar{u} \end{bmatrix}$

$$\begin{bmatrix} \bar{v} \\ -\bar{u} \\ 0 \end{bmatrix}, \frac{\partial \mathbf{F}_b}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} 0 & -m_a g & 0 \\ m_a g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \bar{\mathbf{H}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Let $\mathbf{S}_a = \begin{bmatrix} \mathbf{F}_a^\top & \mathbf{M}_a^\top \end{bmatrix}^\top$ and $\mathbf{M}_v = \text{diag} \left(\begin{bmatrix} m_a \mathbf{I}_3 & \bar{\mathbf{I}}_a \end{bmatrix} \right)$. Moving all the terms containing the time derivatives in the non-circulatory forces and moments to the left hand side according to the simplified approach according to the Theodorsen's theory, we have

$$\begin{aligned} \left(\mathbf{M}_v - \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_a} \right) \dot{\mathbf{x}}_a - \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p - \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q - \frac{\partial \mathbf{S}_{na}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta &= \frac{\partial \mathbf{S}}{\partial \mathbf{x}_a} \mathbf{x}_a + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) \\ &+ \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_q} (r_0 \mathbf{x}_q + \mathbf{r}_a \mathbf{y}_q) + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) \\ &+ \frac{\partial \mathbf{S}_{c,a}}{\partial \boldsymbol{\Omega}} (r_0 \boldsymbol{\Omega} + \mathbf{r}_a \mathbf{y}_\Omega) + \frac{\partial \mathbf{S}_{nc,a}}{\partial \mathbf{x}_a} \mathbf{x}_a \mathbf{x}_a + \frac{\partial \mathbf{S}_b}{\partial \boldsymbol{\sigma}} \boldsymbol{\sigma} \end{aligned} \quad (296)$$

where $\frac{\partial \mathbf{S}}{\partial \mathbf{x}_a} = \begin{bmatrix} \mathbf{0} & \frac{\partial \mathbf{F}}{\partial \boldsymbol{\omega}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\frac{\partial \mathbf{S}_b}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \frac{\partial \mathbf{F}_b}{\partial \boldsymbol{\sigma}} \\ \mathbf{0} \end{bmatrix}$.

If the simplified approach according to the Leishman's method is applied, then

$$\begin{aligned} \mathbf{M}_v \dot{\mathbf{x}}_a &= \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_a} \mathbf{r}_a \mathbf{y}_a + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{v}_p} \mathbf{r}_a \mathbf{y}_p + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_q} \mathbf{r}_a \mathbf{y}_q + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{u}_\theta} \mathbf{r}_a \mathbf{y}_\theta + \frac{\partial \mathbf{S}_{c,a}}{\partial \boldsymbol{\Omega}} \mathbf{r}_a \mathbf{y}_\Omega + \frac{\partial \mathbf{S}_{nc,a}}{\partial \mathbf{x}_a} \mathbf{x}_a \\ &+ r_3 \left(\frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_a} \mathbf{z}_a + \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{v}}_p} \mathbf{z}_p + \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_q} \mathbf{z}_q + \frac{\partial \mathbf{S}_{na}}{\partial \dot{\mathbf{u}}_\theta} \mathbf{z}_\theta \right) + \frac{\partial \mathbf{F}_b}{\partial \boldsymbol{\sigma}} \boldsymbol{\sigma} \end{aligned} \quad (297)$$

B. Inflow Dynamic Model

The linearized Pitt-Peters three-state inflow dynamic model is described by

$$\mathbf{M}_p \dot{\mathbf{v}}_p = -\mathbf{L}_p^{-1} \mathbf{v}_p + C(k) \mathbf{C}_{c,p} + \mathbf{C}_{nc,p} \quad (298)$$

The unsteady aerodynamic correction by the simplified approach according to the Theodorsen's theory yields

$$\begin{aligned} -\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \left(\mathbf{M}_p - \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} \right) \dot{\mathbf{v}}_p - \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q - \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta &= -\mathbf{L}_p^{-1} \mathbf{v}_p \\ &+ \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_q} (r_0 \mathbf{x}_q + \mathbf{r}_a \mathbf{y}_q) \\ &+ \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \frac{\partial \mathbf{C}_{c,p}}{\partial \boldsymbol{\Omega}} (r_0 \boldsymbol{\Omega} + \mathbf{r}_a \mathbf{y}_\Omega) + \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} \mathbf{x}_a \end{aligned} \quad (299)$$

The unsteady aerodynamic correction by the simplified approach according to the Leishman's method yields

$$\begin{aligned} \mathbf{M}_p \dot{\mathbf{v}}_p &= -\mathbf{L}_p^{-1} \mathbf{v}_p + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} \mathbf{r}_a \mathbf{y}_a + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} \mathbf{r}_a \mathbf{y}_p + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_q} \mathbf{r}_a \mathbf{y}_q + \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} \mathbf{r}_a \mathbf{y}_\theta + \frac{\partial \mathbf{C}_{c,p}}{\partial \boldsymbol{\Omega}} \mathbf{r}_a \mathbf{y}_\Omega \\ &+ \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} \mathbf{x}_a + r_3 \left(\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_a} \mathbf{z}_a + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} \mathbf{z}_p + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_q} \mathbf{z}_q + \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} \mathbf{z}_\theta \right) \end{aligned} \quad (300)$$

C. Structural Dynamic Model

The structural dynamic model of the airframe structure is described by

$$\mathbf{M}_{gen} \ddot{\mathbf{q}} + \mathbf{C}_{gen} \dot{\mathbf{q}} + \mathbf{K}_{gen} \mathbf{q} = \mathbf{F}_{gen} \quad (301)$$

where $\mathbf{M}_{gen}$ and $\mathbf{K}_{gen}$ are the diagonal generalized mass and stiffness matrices obtained from the NASTRAN model of the Lift+Cruise aircraft, $\mathbf{C}_{gen} = \zeta \mathbf{K}_{gen}$ is the structural damping matrix with an assumed viscous damping ratio of $\zeta = 0.02$, and $\mathbf{F}_{gen}$ is the generalized aerodynamic force.

The structural dynamics can be expressed as

$$\mathbf{M}_s \dot{\mathbf{x}}_q = \mathbf{S}_s \mathbf{x}_q + \mathbf{B}_s C(k) \mathbf{F}_{c,gen} + \mathbf{B}_s \mathbf{F}_{nc,gen} \quad (302)$$

where $\mathbf{M}_s = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{gen} \end{bmatrix}$, $\mathbf{S}_s = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_{gen} & -\mathbf{C}_{gen} \end{bmatrix}$, and $\mathbf{B}_s = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$.

The unsteady aerodynamic correction by the simplified approach according to the Theodorsen's theory yields

$$\begin{aligned} -\mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a - \mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p + \left(\mathbf{M}_s - \mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{x}}_q} \right) \dot{\mathbf{x}}_q - \mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta = \mathbf{S}_s \mathbf{x}_q \\ + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_q} (r_0 \mathbf{x}_q + \mathbf{r}_a \mathbf{y}_q) \\ + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \Omega} (r_0 \Omega + \mathbf{r}_a \mathbf{y}_\Omega) + \mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{x}_a} \mathbf{x}_a \end{aligned} \quad (303)$$

where

$$\frac{\partial \mathbf{F}_{c,gen}}{\partial x} = \sum_{j=1}^4 \frac{\partial \mathbf{F}_{c,gen,w_j}}{\partial x} + \sum_{i=1}^8 \frac{\partial \mathbf{F}_{c,gen,s_i}}{\partial x} \quad (304)$$

$$\frac{\partial \mathbf{F}_{nc,gen}}{\partial x} = \sum_{j=1}^4 \frac{\partial \mathbf{F}_{nc,gen,w_j}}{\partial x} + \sum_{i=1}^8 \frac{\partial \mathbf{F}_{nc,gen,s_i}}{\partial x} \quad (305)$$

The unsteady aerodynamic correction by the simplified approach according to the Leishman's method yields

$$\begin{aligned} \mathbf{M}_s \dot{\mathbf{x}}_q = \mathbf{S}_s \mathbf{x}_q + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_a} \mathbf{r}_a \mathbf{y}_a + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{v}_p} \mathbf{r}_a \mathbf{y}_p + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_q} \mathbf{r}_a \mathbf{y}_q + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{u}_\theta} \mathbf{r}_a \mathbf{y}_\theta \\ + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \Omega} \mathbf{r}_a \mathbf{y}_\Omega + \mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{x}_a} \mathbf{x}_a + \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{x}}_a} \mathbf{z}_a + \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{v}}_p} \mathbf{z}_p \\ + \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{x}}_q} \mathbf{z}_q + \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{u}}_\theta} \mathbf{z}_\theta \end{aligned} \quad (306)$$

D. Actuator Dynamic Models

The control effectors include the aerodynamic control surfaces $\delta_e(t)$, $\delta_a(t)$, and $\delta_r(t)$, the blade pitch angle θ_{b_i} , and the rotor speed Ω_i . For hover, the blade pitch and rotor speed are used as control effectors. The actuator dynamic model of the control surfaces has been developed in the forward-flight aeroservoelastic model.

1. Blade Pitch Actuator Dynamic Model

The blade pitch angle is driven by a DC drive motor which is modeled as

$$\mathbf{J}_b \ddot{\boldsymbol{\theta}}_b + \mathbf{G}_b \dot{\boldsymbol{\theta}}_b = \mathbf{m}_b + \mathbf{T}_b \quad (307)$$

where $\mathbf{J}_b$ is the motor inertia matrix, $\mathbf{G}_b$ is a matrix representing the motor friction windage loss, $\mathbf{m}_b$ is the blade pitching moment vector about the pitch axis, and $\mathbf{T}_b$ is the motor torque vector. The blade pitch motor inertia is estimated to be 8.5835 slug-ft² and the motor friction windage loss is estimated to be 21.5729 slug-ft²/sec.

The blade pitch actuator dynamic model is also expressed as

$$\mathbf{M}_b \dot{\mathbf{u}}_\theta = \mathbf{S}_b \mathbf{u}_\theta + \mathbf{B}_b C(k) \mathbf{m}_{c,b} + \mathbf{B}_b \mathbf{m}_{nc,b} + \mathbf{B}_b \mathbf{T}_b \quad (308)$$

where $\mathbf{M}_b = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_b \end{bmatrix}$, $\mathbf{S}_b = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & -\mathbf{G}_b \end{bmatrix}$, and $\mathbf{B}_b = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$.

The circulatory and non-circulatory blade pitching moments are computed as

$$m_{c,b_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} e_b \frac{df_{z_c,b_i}}{d\Psi_i} d\Psi_i dr' \quad (309)$$

$$m_{nc,b_i} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \left(\frac{dm_{ac,b_i}}{d\Psi_i} - e_{mb} \frac{dl_{nc,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) d\Psi_i dr' \quad (310)$$

The partial derivatives of the circulatory and non-circulatory pitching moments are obtained as

$$\frac{\partial m_{c,b_i}}{\partial x} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{dm_{c,b_i}}{d\Psi_i} \right) d\Psi_i dr' = N_b \int_{R_i}^R \bar{l}_{c,b_i} e_b \frac{\partial f_{0,b_i}}{\partial x} dr' \quad (311)$$

$$\frac{\partial m_{nc,b_i}}{\partial x} = \frac{N_b}{2\pi} \int_{R_i}^R \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{dm_{ac,b_i}}{d\Psi_i} - e_{mb} \frac{dl_{nc,b_i}}{d\Psi_i} - \frac{\pi}{128} \rho_\infty c_b^4 a_\xi \right) d\Psi_i dr' \quad (312)$$

$$= N_b \int_{R_i}^R \left(\bar{l}_{c,b_i} \frac{c_b}{c_l} \frac{\partial c_{0,m_{ac}}}{\partial x} dr' - \frac{\pi}{4} \rho_\infty c_b^2 e_{mb} \frac{\partial a_{0,z_{b_i}}}{\partial x} - \frac{\pi}{128} \rho_\infty c_b^4 \frac{\partial a_{0,\xi}}{\partial x} \right) dr' \quad (313)$$

The blade pitch angle is to be controlled by a PI controller. The control motor torque is given by

$$\mathbf{T}_b = \mathbf{u}_b - \mathbf{K}_{b,p} \mathbf{u}_\theta - \mathbf{K}_{b,i} \mathbf{e}_b \quad (314)$$

where $\mathbf{u}_b(t)$ is the motor torque control, $\mathbf{e}_b(t)$ is the the blade pitch angle integral error and the controller gain matrices $\mathbf{K}_{b,p}$ and $\mathbf{K}_{b,i}$ are computed by the linear quadratic regulator (LQR) optimal control.

The error dynamics are given by

$$\dot{\mathbf{e}}_b = \mathbf{C}_b \mathbf{u}_\theta - \theta_{b,c} \quad (315)$$

where $\mathbf{C}_b = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$.

The equation of blade pitch dynamics is expressed using the simplified approach according to the Theodorsen's theory as

$$\begin{aligned} & -\mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a - \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p - \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q + \left(\mathbf{M}_b - \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{u}}_\theta} \right) \dot{\mathbf{u}}_\theta = \mathbf{S}_b \mathbf{u}_\theta \\ & + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_q} (r_0 \mathbf{x}_q + \mathbf{r}_a \mathbf{y}_q) + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) \\ & + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \Omega} (r_0 \Omega + \mathbf{r}_a \mathbf{y}_\Omega) + \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \mathbf{x}_a} \mathbf{x}_a + \mathbf{B}_b \mathbf{u}_b - \mathbf{B}_b \mathbf{K}_{b,p} \mathbf{u}_\theta - \mathbf{B}_b \mathbf{K}_{b,i} \mathbf{e}_b \end{aligned} \quad (316)$$

The equation of blade pitch dynamics is expressed using the simplified approach according to the Leishman's method as

$$\begin{aligned} \mathbf{M}_b \dot{\mathbf{u}}_\theta &= \mathbf{S}_b \mathbf{u}_\theta + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_a} \mathbf{r}_a \mathbf{y}_a + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{v}_p} \mathbf{r}_a \mathbf{y}_p + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_q} \mathbf{r}_a \mathbf{y}_q + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{u}_\theta} \mathbf{r}_a \mathbf{y}_\theta \\ & + \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \Omega} \mathbf{r}_a \mathbf{y}_\Omega + \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \mathbf{x}_a} \mathbf{x}_a + \mathbf{B}_b r_3 \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_a} \mathbf{z}_a + \mathbf{B}_b r_3 \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{v}}_p} \mathbf{z}_p \\ & + \mathbf{B}_b r_3 \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_q} \mathbf{z}_q + \mathbf{B}_b r_3 \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{u}}_\theta} \mathbf{z}_\theta + \mathbf{B}_b \mathbf{u}_b - \mathbf{B}_b \mathbf{K}_{b,p} \mathbf{u}_\theta - \mathbf{B}_b \mathbf{K}_{b,i} \mathbf{e}_b \end{aligned} \quad (317)$$

2. Rotor Actuator Dynamic Model

The rotor speed is driven by a DC drive motor which is modeled as

$$\mathbf{J}_r \dot{\Omega} + \mathbf{B}_r \Omega = \mathbf{m}_r + \mathbf{T}_r \quad (318)$$

where $\mathbf{J}_r$ is the motor inertia matrix, $\mathbf{B}_r$ is the motor friction matrix, $\mathbf{m}_r = \begin{bmatrix} \pm Q_1 & \pm Q_2 & \cdots \\ \pm Q_{n_{rotor}} \end{bmatrix}$ is the rotor torque vector, and $\mathbf{T}_r$ is the motor torque vector. The blade pitch motor inertia

is estimated to be 10.1239 slug-ft² and the motor friction windage loss is estimated to be 668.1763 slug-ft²/sec.

The rotor speed is to be controlled by a PI controller. The control motor torque is given by

$$\mathbf{T}_r = \mathbf{u}_r - \mathbf{K}_{r,p}\boldsymbol{\Omega} - \mathbf{K}_{r,i}\mathbf{e}_r \quad (319)$$

where $\mathbf{u}_r(t)$ is the motor torque control $\mathbf{e}_r(t)$ is the rotor speed integral error and the controller gain matrices $\mathbf{K}_{r,p}$ and $\mathbf{K}_{r,i}$ are computed by the LQR optimal control.

The error dynamics are given by

$$\dot{\mathbf{e}}_r = \boldsymbol{\Omega} - \boldsymbol{\Omega}_c \quad (320)$$

The equation of the rotor dynamics is expressed using the simplified approach according to the Theodorsen's theory as

$$\begin{aligned} -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a - \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{v}}_p} \dot{\mathbf{v}}_p - \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_q} \dot{\mathbf{x}}_q - \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{u}}_\theta} \dot{\mathbf{u}}_\theta + \mathbf{J}_r \dot{\boldsymbol{\Omega}} = -\mathbf{B}_r \boldsymbol{\Omega} \\ + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_a} (r_0 \mathbf{x}_a + \mathbf{r}_a \mathbf{y}_a) + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{v}_p} (r_0 \mathbf{v}_p + \mathbf{r}_a \mathbf{y}_p) + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_q} (r_0 \mathbf{x}_q + \mathbf{r}_a \mathbf{y}_q) \\ + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{u}_\theta} (r_0 \mathbf{u}_\theta + \mathbf{r}_a \mathbf{y}_\theta) + \frac{\partial \mathbf{m}_{c,r}}{\partial \boldsymbol{\Omega}} (r_0 \boldsymbol{\Omega} + \mathbf{r}_a \mathbf{y}_\Omega) + \frac{\partial \mathbf{m}_{nc,r}}{\partial \mathbf{x}_a} \mathbf{x}_a + \mathbf{u}_r - \mathbf{K}_{r,p}\boldsymbol{\Omega} - \mathbf{K}_{r,i}\mathbf{e}_r \end{aligned} \quad (321)$$

The equation of rotor dynamics is expressed using the simplified approach according to the Leishman's method as

$$\begin{aligned} \mathbf{J}_r \dot{\boldsymbol{\Omega}} = -\mathbf{B}_r \boldsymbol{\Omega} + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_a} \mathbf{r}_a \mathbf{y}_a + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{v}_p} \mathbf{r}_a \mathbf{y}_p + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_q} \mathbf{r}_a \mathbf{y}_q + \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{u}_\theta} \mathbf{r}_a \mathbf{y}_\theta + \frac{\partial \mathbf{m}_{c,r}}{\partial \boldsymbol{\Omega}} \mathbf{r}_a \mathbf{y}_\Omega + \frac{\partial \mathbf{m}_{nc,r}}{\partial \mathbf{x}_a} \mathbf{x}_a \\ + r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_a} \mathbf{z}_a + r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{v}}_p} \mathbf{z}_p + r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_q} \mathbf{z}_q + r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{u}}_\theta} \mathbf{z}_\theta + \mathbf{u}_r - \mathbf{K}_{r,p}\boldsymbol{\Omega} - \mathbf{K}_{r,i}\mathbf{e}_r \end{aligned} \quad (322)$$

E. Aeroservoelastic Model

We form the aeroservoelastic model by coupling together the flight dynamic model, inflow dynamic model, structural dynamic model, actuator dynamic model, and unsteady aerodynamic model. The integrated aeroservoelastic model is expressed compactly in a physical state-space form as

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{S}\mathbf{x} + \mathbf{T}\mathbf{u} \quad (323)$$

In the physical state-space form, $\mathbf{M}$ is the mass matrix, $\mathbf{S}$ is the force-moment stability derivative matrix, and $\mathbf{T}$ is the force-moment control derivative matrix.

By inverting the mass matrix $\mathbf{M}$, we obtain the standard state-space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \mathbf{v} \quad (324)$$

where $\mathbf{v}(t) = \begin{bmatrix} \mathbf{v}_c^\top(t) & \mathbf{0}^\top & \mathbf{0}^\top \end{bmatrix}^\top$ and $\mathbf{v}_c(t) = \begin{bmatrix} \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \tilde{\boldsymbol{\theta}}_{b,c}^\top(t) & \boldsymbol{\Omega}_c^\top(t) \end{bmatrix}^\top$ is a vector of the blade pitch and rotor speed command. For aircraft state tracking control, we set $\mathbf{v}_c = \mathbf{0}$ to maintain trim blade pitch and rotor speed. We then obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (325)$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ may not have any physical meaning since the mass matrix includes the non-circulatory unsteady aerodynamic contributions due to the Theodorsen's theory.

The state-space equation can also be expressed in a partitioned form using the simplified approach according to the Theodorsen's theory as

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{re} & \mathbf{0} \\ \mathbf{M}_{er} & \mathbf{M}_{ee} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_r \\ \dot{\mathbf{x}}_q \\ \dot{\mathbf{x}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{rr} & \mathbf{S}_{re} & \mathbf{S}_{rc} \\ \mathbf{S}_{er} & \mathbf{S}_{ee} & \mathbf{S}_{ec} \\ \mathbf{S}_{cr} & \mathbf{S}_{ce} & \mathbf{S}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_q \\ \mathbf{x}_c \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{rb} & \mathbf{T}_{rr} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_r \end{bmatrix} - \begin{bmatrix} \mathbf{v}_c \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (326)$$

where $\mathbf{x}_r(t) = \begin{bmatrix} \boldsymbol{\sigma}^\top(t) & \mathbf{x}_a^\top(t) & \mathbf{v}_p^\top(t) & \mathbf{u}_\theta^\top(t) & \boldsymbol{\Omega}^\top(t) & \mathbf{e}_b^\top(t) & \mathbf{e}_r^\top(t) \end{bmatrix}^\top$ and $\mathbf{x}_c(t) = \begin{bmatrix} \mathbf{y}_a^\top(t) & \mathbf{y}_p^\top(t) & \mathbf{y}_q^\top(t) & \mathbf{y}_\theta^\top(t) & \mathbf{y}_\Omega^\top(t) \end{bmatrix}^\top$

The mass matrices are given by

$$\mathbf{M}_{rr} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v - \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_a} & -\frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{v}}_p} & -\frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_a} & \mathbf{M}_p - \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} & -\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_a} & -\mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{v}}_p} & \mathbf{M}_b - \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_a} & -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{v}}_p} & -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{J}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (327)$$

$$\mathbf{M}_{re} = \begin{bmatrix} \mathbf{0} \\ -\frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_q} \\ -\frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_q} \\ -\mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_q} \\ -\frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_q} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (328)$$

$$\mathbf{M}_{er} = \begin{bmatrix} \mathbf{0} & -\mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{x}}_a} & -\mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{v}}_p} & -\mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{u}}_\theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (329)$$

$$\mathbf{M}_{ee} = \mathbf{M}_s - \mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \dot{\mathbf{x}}_q} \quad (330)$$

The stability derivative matrices are given by

$$\mathbf{S}_{rr} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{H}} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{F}_b}{\partial \boldsymbol{\sigma}} & \frac{\partial \mathbf{S}}{\partial \mathbf{x}_a} + \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{S}_{nc,a}}{\partial \mathbf{x}_a} & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{v}_p} r_0 & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{u}_\theta} r_0 \\ \mathbf{0} & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} r_0 - \mathbf{L}_p & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} r_0 \\ \mathbf{0} & \mathbf{B}_b \left(\frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{m}_{nc,b}}{\partial \mathbf{x}_a} \right) & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{v}_p} r_0 & \mathbf{B}_b \left(\frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{u}_\theta} r_0 + \mathbf{S}_b - \mathbf{K}_{b,p} \right) \\ \mathbf{0} & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{m}_{nc,r}}{\partial \mathbf{x}_a} & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{v}_p} r_0 & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{u}_\theta} r_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_b \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (331)$$

$$\mathbf{S}_{re} = \begin{bmatrix} \mathbf{0} \\ \frac{\partial \mathbf{S}_{c,a}}{\partial \boldsymbol{\Omega}} r_0 \\ \frac{\partial \mathbf{C}_{c,p}}{\partial \boldsymbol{\Omega}} r_0 \\ \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \boldsymbol{\Omega}} r_0 \\ \frac{\partial \mathbf{m}_{c,r}}{\partial \boldsymbol{\Omega}} r_0 - \mathbf{B}_r - \mathbf{K}_{r,p} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_b \mathbf{K}_{b,i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{K}_{r,i} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (332)$$

$$\mathbf{S}_{rc} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_a} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{v}_p} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{x}_q} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \frac{\partial \mathbf{S}_{c,a}}{\partial \Omega} \mathbf{r}_a \\ \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_a} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{v}_p} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{x}_q} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \frac{\partial \mathbf{C}_{c,p}}{\partial \Omega} \mathbf{r}_a \\ \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_a} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{v}_p} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{x}_q} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \mathbf{B}_b \frac{\partial \mathbf{m}_{c,b}}{\partial \Omega} \mathbf{r}_a \\ \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_a} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{v}_p} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{x}_q} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \frac{\partial \mathbf{m}_{c,r}}{\partial \Omega} \mathbf{r}_a \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (333)$$

$$\mathbf{S}_{er} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_s \left(\frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_a} r_0 + \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{x}_a} \right) & \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{v}_p} r_0 & \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{u}_\theta} r_0 & \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \Omega} r_0 & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (334)$$

$$\mathbf{S}_{ee} = \mathbf{S}_s + \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_q} r_0 \quad (335)$$

$$\mathbf{S}_{ec} = \begin{bmatrix} \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_a} \mathbf{r}_a & \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{v}_p} \mathbf{r}_a & \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{x}_q} \mathbf{r}_a & \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \mathbf{u}_\theta} \mathbf{r}_a & \mathbf{B}_s \frac{\partial \mathbf{F}_{c,gen}}{\partial \Omega} \mathbf{r}_a \end{bmatrix} \quad (336)$$

$$\mathbf{S}_{cr} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_\theta & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_\Omega & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (337)$$

$$\mathbf{S}_{ce} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{B}_q \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (338)$$

$$\mathbf{S}_{cc} = \begin{bmatrix} \mathbf{A}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_\theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_\Omega \end{bmatrix} \quad (339)$$

The control derivative matrices are given by

$$\mathbf{T}_{rb} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{B}_b \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (340)$$

$$\mathbf{T}_{rr} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (341)$$

The state-space equation can also be expressed in a partitioned form using the simplified approach

according to the Leishman's method as

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{ee} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{M}_{nr} & \mathbf{M}_{ne} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_r \\ \dot{\mathbf{x}}_q \\ \dot{\mathbf{x}}_c \\ \dot{\mathbf{x}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{rr} & \mathbf{0} & \mathbf{S}_{rc} & \mathbf{S}_{rn} \\ \mathbf{S}_{er} & \mathbf{S}_{ee} & \mathbf{S}_{ec} & \mathbf{S}_{en} \\ \mathbf{S}_{cr} & \mathbf{S}_{ce} & \mathbf{S}_{cc} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_q \\ \mathbf{x}_c \\ \mathbf{x}_n \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{rb} & \mathbf{T}_{rr} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_r \end{bmatrix} \quad (342)$$

where $\mathbf{x}_n(t) = \begin{bmatrix} \mathbf{z}_a^\top(t) & \mathbf{z}_p^\top(t) & \mathbf{z}_q^\top(t) & \mathbf{z}_\theta^\top(t) \end{bmatrix}^\top$.

The mass matrices are given by

$$\mathbf{M}_{rr} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (343)$$

$$\mathbf{M}_{ee} = \mathbf{M}_s \quad (344)$$

$$\mathbf{M}_{nr} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (345)$$

$$\mathbf{M}_{ne} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} \quad (346)$$

The stability derivative matrices are given by

$$\mathbf{S}_{rr} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{H}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \mathbf{F}_b}{\partial \boldsymbol{\sigma}} & \frac{\partial \mathbf{S}_{nc,a}}{\partial \mathbf{x}_a} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} & -\mathbf{L}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \mathbf{x}_a} & \mathbf{0} & \mathbf{S}_b - \mathbf{B}_b \mathbf{K}_{b,p} & \mathbf{0} & -\mathbf{B}_b \mathbf{K}_{b,i} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{m}_{nc,r}}{\partial \mathbf{x}_a} & \mathbf{0} & \mathbf{0} & -\mathbf{B}_r - \mathbf{K}_{r,p} & \mathbf{0} & -\mathbf{K}_{r,i} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (347)$$

$$\mathbf{S}_{rn} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ r_3 \frac{\partial \mathbf{S}_{nc,a}}{\partial \mathbf{x}_a} & r_3 \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{v}}_p} & r_3 \frac{\partial \mathbf{S}_{nc,a}}{\partial \dot{\mathbf{x}}_q} & r_3 \frac{\partial \mathbf{S}_{na}}{\partial \dot{\mathbf{u}}_\theta} \\ r_3 \frac{\partial \mathbf{C}_{nc,p}}{\partial \mathbf{x}_a} & r_3 \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{v}}_p} & r_3 \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{x}}_q} & r_3 \frac{\partial \mathbf{C}_{nc,p}}{\partial \dot{\mathbf{u}}_\theta} \\ \mathbf{B}_b r_3 \frac{\partial \mathbf{m}_{nc,b}}{\partial \mathbf{x}_a} & \mathbf{B}_b r_3 \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{v}}_p} & \mathbf{B}_b r_3 \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{x}}_q} & \mathbf{B}_b \frac{\partial \mathbf{m}_{nc,b}}{\partial \dot{\mathbf{u}}_\theta} \\ r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \mathbf{x}_a} & r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{v}}_p} & r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{x}}_q} & r_3 \frac{\partial \mathbf{m}_{nc,r}}{\partial \dot{\mathbf{u}}_\theta} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (348)$$

$$\mathbf{S}_{er} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_s \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{x}_a} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (349)$$

$$\mathbf{S}_{ee} = \mathbf{S}_s \quad (350)$$

$$\mathbf{S}_{en} = \begin{bmatrix} \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{x}_a} & \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{v}_p} & \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{x}_q} & \mathbf{B}_s r_3 \frac{\partial \mathbf{F}_{nc,gen}}{\partial \mathbf{u}_\theta} \end{bmatrix} \quad (351)$$

$$\mathbf{S}_{nn} = \frac{2\bar{V}_t}{\bar{c}_b} p_3 \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (352)$$

Note that the $\mathbf{S}_{rc}$, $\mathbf{S}_{ec}$, $\mathbf{S}_{cr}$, $\mathbf{S}_{ce}$, $\mathbf{S}_{cc}$, $\mathbf{T}_{rb}$, and $\mathbf{T}_{rr}$ matrices are the same for both approaches.

IX. Simulation Results

The Lift+Cruise eVTOL aircraft is a six-passenger aircraft¹⁵ as shown in Figure 1. It employs 8 vertical lift rotors for eVTOL and an aft push fan. All the 8 vertical lift rotors operate at a constant rotor speed of 110 rad/sec. In forward flight, the LPC vehicle operates like a conventional fixed-wing aircraft with the 8 vertical lift rotors stopped and locked in place. Two versions of the Lift+Cruise vehicle are conceptualized: a turboelectric version and an all-electric version.¹⁵ The all-electric Lift+Cruise vehicle has a design gross weight of 8,210 lbs and cruises at an average 100 knots. This model is used in the current study.

An aerodynamic trim analysis for hover flight is performed in CHARM,³ a rotorcraft aeromechanics comprehensive analysis tool. The trim solution is used to compute the analytical derivatives of the forces and moments created by the rotors. In addition, the non-circulatory forces and moments generated by the aircraft lifting surfaces are also computed.

Figure 7 shows the rotor thrust radial distributions of the eight rotors. The front rotors 1, 3, 5, 7 are trimmed at higher thrust values than the aft rotors 2, 4, 6, 8.

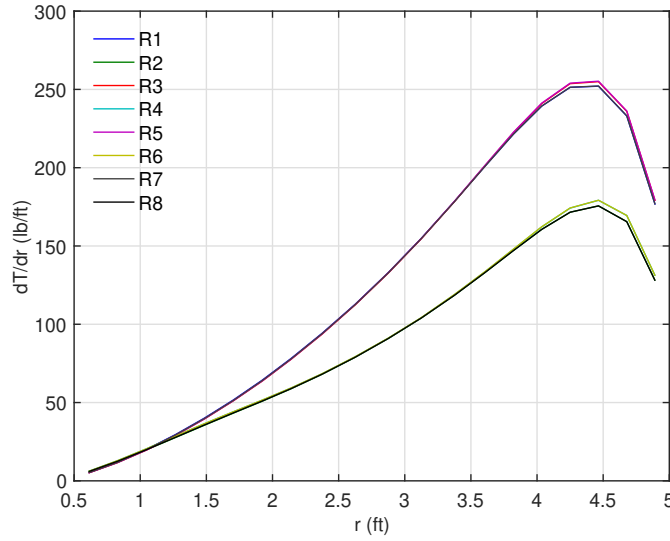


Figure 7. Rotor Thrust Distributions

The trim solution is also used to compute the static aeroelastic deformation of the wing. Table 4 shows the vertical bending and torsional rotation at the attachment points of the 8 vertical lift rotors to the wing. The static aeroelastic deflections indicate that the wing structure is quite stiff.

	Vertical Bending (ft)	Torsional Rotation (deg)
Rotors 1 and 2	1.19	-0.068
Rotors 3 and 4	0.021	-0.050
Rotors 5 and 6	0.0011	-0.046
Rotors 7 and 8	1.21	-0.065

Table 4. Static Aeroelastic Deformation of Lift+Cruise Aircraft Wing in Hover

The aeroservoelastic model of the Lift+Cruise aircraft in hover at 4000 ft has been developed. A critical damping ratio of 1% is assumed for all the structural modes. The unsteady aerodynamic correction is based on the Theodorsen's theory. There are altogether 264 states which include 8 rigid-body aircraft states (heading angle not included), 24 inflow states (3 per rotor), 16 blade pitch actuator states (2 per rotor), 8 integral error states for the blade pitch integral control, 8 rotor actuator states (1 per rotor), 8 integral error states for the rotor speed integral control, 28 structural dynamic states corresponding to the generalized displacements and velocities for the first 14 structural modes, and 164 unsteady aerodynamic states (2 per state except for roll angle, heading angle, and integral error states).

Figure 8 is the plot of the pole locations of the aeroservoelastic model of the Lift+Cruise aircraft in hover. The effect of the elastic airframe on the rigid-body modes of the aircraft is evident in Figure 9 which shows a comparison of the pole locations for the elastic aircraft to those of the rigid aircraft. All the structural modes are stable. The wind-off critical damping ratio $\zeta = 0.01$ is used for all the structural modes. The aerodynamic damping ratios of the structural modes are all greater than $\zeta = 0.01$. The poles associated with the inflow states and the unsteady aerodynamic states have large negative real values ranging from -20 to -165 which imply that the dynamic inflow and unsteady aerodynamic effects are mostly decoupled from the aircraft response. Table 5 shows the eigenvalues of the rigid-body modes with the elastic airframe compared to those with the rigid airframe. The elastic airframe causes an increase in the damping of the lateral-directional modes but at the same time a slight decrease in the damping of the longitudinal modes. There are two unstable modes: the phugoid mode and the spiral mode. These two modes can be stabilized by feedback control.¹⁵

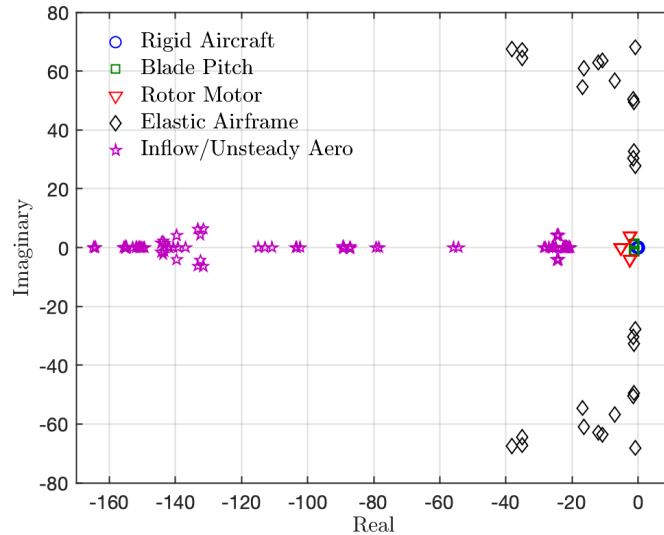


Figure 8. Pole Locations of Aeroservoelastic Model of Lift+Cruise Aircraft in Hover

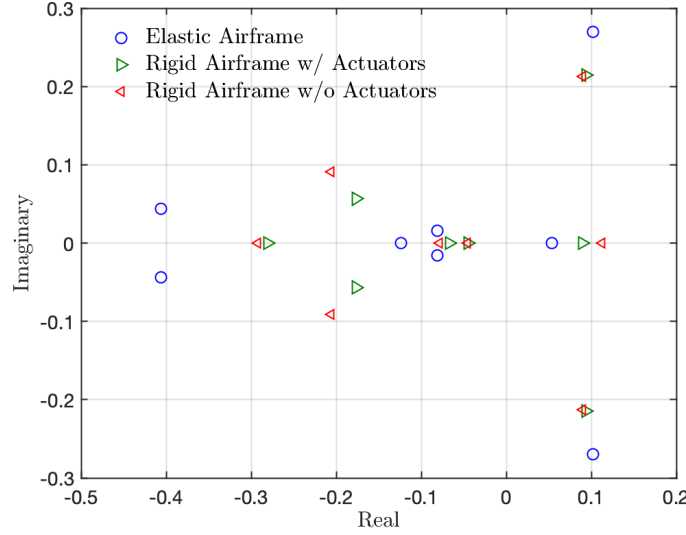


Figure 9. Pole Locations of Rigid-Body Modes of Lift+Cruise Aircraft in Hover

Rigid-Body Mode	Elastic Airframe	Rigid Airframe w/ Actuators	Rigid Airframe w/o Actuators
Phugoid	$0.1018 \pm 0.2694i$	$0.09378 \pm 0.2145i$	$0.09026 \pm 0.2131i$
Short-Period	$-0.4061 \pm 0.04407i$	$-0.06708, -0.2811$	$-0.07839, -0.2922$
Roll	-0.1240	-0.04526	-0.04541
Spiral	0.05390	0.08922	0.1128
Dutch-Roll	$-0.08076 \pm 0.01597i$	$-0.1769 \pm 0.05657i$	$-0.2064 \pm 0.09143i$

Table 5. Eigenvalues of Rigid-Body Modes of Lift+Cruise Aircraft in Hover with Elastic and Rigid Airframes

X. Conclusions

This paper presents an analytical aeroservoelastic modeling method for multi-rotor eVTOL aircraft. The method presents a general procedure for computing the circulatory and non-circulatory forces and moments acting on the aircraft and the elastic airframe. The partial derivatives of the forces and moments are evaluated to linearize the model. The unsteady aerodynamic effect is accounted for by Theodorsen's theory and the Leishmann's method. The aeroservoelastic model includes a three-state dynamic inflow model and actuator models of the blade pitch drive motors and rotor drive motors. An aeroservoelastic model of the NASA Lift+Cruise aircraft has been developed. The model illustrates the effect of the elastic airframe on the rigid-body modes of the aircraft.

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