

A Physics-Based Work-Energy Formulation for Real-Time Trajectory Guidance of a Lunar Lander

J. M. Munoz-Burgos,^{1, a)} P. J. McDonough,^{2, b)} and C. C. Proctor^{3, c)}

¹⁾Senior Navigation Engineer, Amentum Space Exploration Division, EV42, NASA MSFC: 4600-4321, Huntsville, AL 35812, USA.

²⁾Team Lead, S/C & Veh. Sys., Amentum Space Exploration Division, EV42, NASA MSFC: 4600-5204, Huntsville, AL 35812, USA.

³⁾Senior Aerospace Engineer, Amentum Space Exploration Division, EV42, NASA MSFC: 4600-4323, Huntsville, AL 35812, USA.

(Dated: January 22, 2026)

Many sophisticated mathematical approaches have been developed to calculate and optimized trajectory solutions for lunar landing systems, including but not limited to Hamilton's Principle of Variation, Pontryagin's maximum principle, and convex optimization techniques among others. Many of these approaches require expensive computational resources in order to achieve convergence to a solution, which may be problematic due to the well known limitations in computational power and memory capacity of space hardened computers when compared to modern computers. In an effort to reduce complexity and the computational load required to generate the real-time guidance commands that are necessary during manual piloting, a simple physics-based work-energy approach is formulated. This approach is based on the dissipation of mechanical energy of the vehicle in reference to the final energy state required to achieve a safe landing. The rocket engine(s) used during landing and other maneuvers dissipate mechanical energy both by doing work against the velocity vector of the vehicle and by jettisoning mass. Therefore, by analytically solving the energy dissipation problem during every step of the maneuver, a much simpler algebraic solution that naturally and quickly attains convergence is obtained. This concept is not limited to approach and landing, but it may also be applied to de-orbiting, braking-burn, and divert maneuvers. The principle also applies for other type of maneuvers such as ascent and orbital insertion, where instead of energy dissipation, the engine injects the necessary energy into the system to attain the desired final state.

I. INTRODUCTION

Trajectory calculation and optimization during lunar de-orbiting, approach, and landing is a relevant and ongoing area of research, particularly in this modern era when the United States and other countries are conducting efforts to return humans to the Moon. Due to mission constraints and weight limits, it is imperative to minimize propellant consumption while simultaneously achieving a safe lunar landing as close as possible to the desired landing site (LS). From the 1960's until recently, the primary numerical methods for propellant minimization during vertical descent to the lunar surface leverages Pontryagin's maximum principle and the calculus of variations, minimizing descent engine firing time as a proxy for minimization of propellant consumption.¹⁻⁴

One of simplest and most elegant algorithms is the Apollo powered descent guidance (APDG),⁵ which was developed over 50 years ago and employed by the Apollo Lunar Module (LM) to land on the Moon. In this algorithm, the acceleration vector is represented by a quadratic function of time; this has been extended to a

general fractional-polynomial form, in which a reference acceleration, velocity, and position vectors are calculated based on the final target conditions.⁶ This method has the advantage of being computationally efficient, which is required to calculate the real-time guidance commands necessary for manual control, but has the disadvantages that it may not always provide optimal solutions for propellant consumption, and that is highly sensitive to navigation errors.

With the advent of more powerful computational tools since the time of Apollo, more propellant-efficient approaches based on convex optimization techniques have been developed⁷. Two popular convex methods have arisen from the direct transcription of trajectory optimization for solving the general non-linear programming problem (NLP); sequential quadratic programming and successive convex programming.⁸⁻¹¹ Both methods attempt to solve the NLP by approximating it as convex and iterating for better solutions. This can be computationally expensive and mathematically complex, and neither method guarantees that the convex approximation will converge to the global solution of the NLP. With regard to spacecraft, the convex approximation is often comprised of a cost function (or objective function) with the goal saving fuel or time under the constraints of obeying dynamics and meeting mission requirements. These techniques benefit from, but do not require, a realistic initial guess of the solution, which is

^{a)}Electronic mail: jorge.m.munoz-burgos@nasa.gov.

^{b)}Electronic mail: peter.j.mcdonough@nasa.gov.

^{c)}Electronic mail: christopher.c.proctor@nasa.gov.

then iteratively improved upon until convergence, after which the solution is used to generate commands to the vehicle. In general, the problem formulation for convex optimization approaches is that of solving a system of non-linear equations whose solutions behave chaotically, such that any slight change in the initial conditions of the vehicle at any moment during the trajectory will change the optimal solution.¹² Because the solution is sensitive to small perturbations, the convex optimization must be repeated as the maneuver executes (similar to a receding horizon controller) to continually generate updated guidance commands. For real-time use, such as may be the case under manual control or when a change in landing site is required during flight, a large amount of on-board computing power is required. This cost hinders the applicability of real time optimization by limiting the number of constraints, states, or times of flight that an onboard optimization routine can handle when using sequential quadratic programming or successive convex programming. Although modern computers are becoming faster and more computationally efficient, space hardened computers are typically many years behind modern computers, thus limiting computational capacity and memory.

In this paper, a novel, simple, physics-based work-energy formulation for calculating optimal guidance commands is presented with the goals of providing optimal solutions at less computational cost than convex optimization approaches but with comparable performance. The physics-based formulation is based on the principle of energy conservation, where a lunar lander approaching the surface is guaranteed to achieve a soft landing by dissipating mechanical energy as long as the engine is able to provide the necessary thrust force required by the physics. Only two mechanisms can be used to dissipate energy: jettisoning mass and doing work against the velocity vector using the engine(s). By iteratively solving between the work done by the engine and the jettisoned mass, an exact solution of the constant engine thrust force vector at every instant of the trajectory can be obtained which guarantees the desired energy dissipation to reach the intended landing site conditions. Due to the nonlinear nature of the trajectory, small changes in the vehicle state render the most recently calculated constant thrust force vector invalid by the butterfly effect.¹² Because the vehicle's state is constantly evolving while pursuing the latest optimal work-energy solution, a new solution must be continuously recalculated as the vehicle state evolves to smoothly incorporate the latest optimal solution. However, in contrast with convex optimization methods, the work energy update is calculated very efficiently, making such calculations in real time feasible with much less required on-board computing power. This is because an entire optimal trajectory is not required during every update; instead, all that is required is calculating the present optimal energy dissipation state.

Note that the work-energy approach presented here should not be confused with work-energy Lyapunov-based guidance, where the energy of the vehicle is employed to construct a Lyapunov function that describes the error dynamics of the system. That technique is computationally expensive and solutions are usually not optimal.¹³ In contrast, the formulation in this work yields simple algebraic expressions without the use of linear algebra or numerical integration via a single optimization loop that always converges in a few iterations to the theoretical optimal state of energy dissipation between engine work and mass jettisoning.

The paper is organized as follows: Section II introduces the concept of constant optimal engine settings, defines the coordinate system, and presents the analytical derivation of the optimal solution of Newton's equations of motion. Section III presents the derivation of the generalized work-energy computational cycle, and its application to final approach and vertical descent maneuvers. Section IV includes the formulation and application of the work-energy solution to the braking-burn maneuver. A brief discussion of propellant consumption minimization from the crew-resource and risk management perspective is included in Sec. V. Section VI presents braking-burn and final approach Monte-Carlo (MC) trajectories calculations for a real moon-lander. However, due to the proprietary and sensitive nature of the data, only qualitative comparisons between the work-energy formulation and both fractional-polynomial and quadratic programming convex optimization are presented.^{6,9,10} Finally, the formulation of the work-energy solution to real-time divert-space calculation is introduced in Sec. VII.

Other important supplemental material is included in the appendixes. Appendix A contains the derivation of the optimal final attitude and yaw angles together with their respective angular rates during final rotational maneuvers for both vertical descent and trajectory centering during the end of the braking-burn maneuver. Appendix B includes the maneuvering timing calculations necessary to determine the theoretical energy boundaries during a divert maneuver before landing. The derivation of the angular rate frame transformation between local LS and vehicle coordinates is presented in Appx. C. Additionally, because the accuracy of all guidance methods is dependent on the quality of the navigation data, an engine-based lunar navigation calculation is suggested in Appx. D as a final approach and descent navigational aid to help improve final guidance commands and landing accuracy. Although this additional navigational aid has not been implemented in the work presented here, it is in the process of being implemented into the simulator, and represents important material for future work. For completeness and documentation purposes, this final appendix is included in the manuscript to make it available as a reference to the interested audience.

II. ANALYTICAL INTEGRATION OF NEWTON'S EQUATIONS ASSUMING CONSTANT OPTIMAL CONDITIONS

Figure 1 illustrates a sample LM with initial kinetic and potential energy in a parking orbit that initiates a descent trajectory towards the desired landing location. During the descent, it is necessary to achieve the required energy dissipation at the moment the LS is reached in order to avoid a crash.

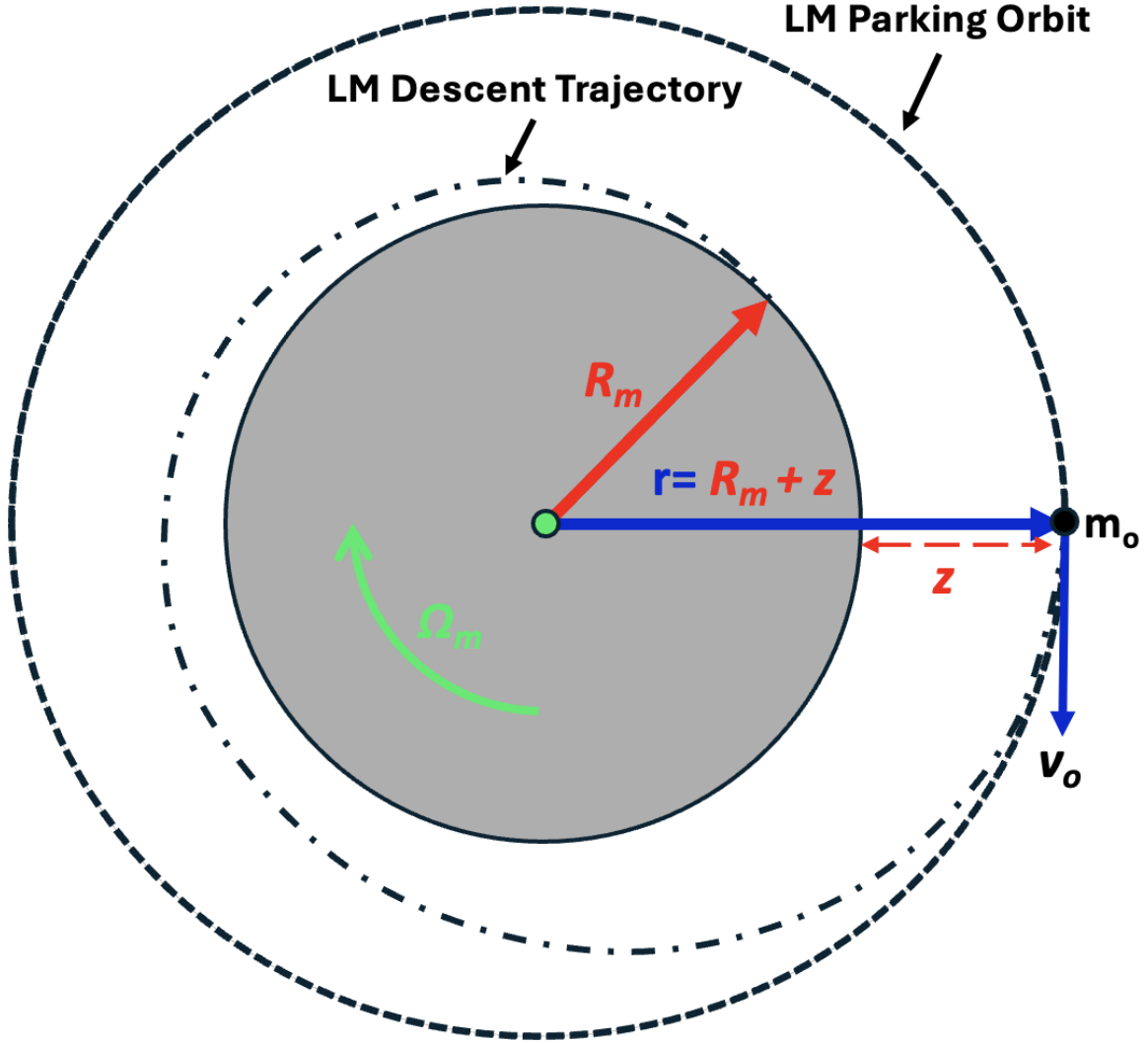


Figure 1: LM Descent Trajectory to desired LS. The vehicle has both kinetic and potential energy related to its initial mass m_o , initial velocity v_o , and altitude above the surface z . The goal is to dissipate the vehicle's total energy to the same energy state as that of the LS, which has kinetic energy related to its local lunar angular rotation Ω_m and gravitational potential energy, both functions of the local LS lunar radius R_m .

In general, the lunar gravitational potential energy may be expressed using the spherical-harmonics expansion form expressed as^{14,15}

$$U(z, \varphi, \lambda) = -m_o \cdot \frac{GM_m}{(R_m + z)} \cdot \sum_{n=0}^{\infty} \sum_{k=0}^n \left[\frac{R_m}{(R_m + z)} \right]^n \cdot P_n^k[\sin(\varphi)] \cdot [C_{n,k} \cdot \cos(k \cdot \lambda) + S_{n,k} \cdot \sin(k \cdot \lambda)] , \quad (1)$$

where G is Newton's universal gravitational constant, m_o is the mass of the lander at the present, M_m and R_m are the lunar mass and the average lunar radius, and z is the altitude above the lunar surface. The associated Legendre polynomials are represented by P_n^k , with Fourier spherical coefficients $C_{n,k}$ and $S_{n,k}$, and lunar latitude and longitude given by λ and φ .

By calculating the difference between the potential energy of any point along the trajectory and the LS and by employing algebra, it can be shown that the value of the gravitational acceleration g calculated at the average trajectory altitude $\bar{z}$ [see Eq. (19)] is given by

$$g(\bar{z}, \varphi, \lambda) = \frac{GM_m}{R_m^2(1+\frac{\bar{z}}{R_m})} \cdot \left\{ \left[1 + \frac{R_m}{\bar{z}} \right] \cdot \sum_{n=0}^{\infty} \sum_{k=0}^n P_n^k[\sin(\varphi_o)] \cdot [C_{n,k} \cdot \cos(k \cdot \lambda_o) + S_{n,k} \cdot \sin(k \cdot \lambda_o)] \right. \\ \left. - \sum_{n=0}^{\infty} \sum_{k=0}^n \left[\frac{R_m}{(R_m + \bar{z})} \right] \cdot P_n^k[\sin(\varphi)] \cdot [C_{n,k} \cdot \cos(k \cdot \lambda) + S_{n,k} \cdot \sin(k \cdot \lambda)] \right\}. \quad (2)$$

Figure 2 shows a summary of the physics-based Work-Energy (WE) formulation presented in this work by illustrating the two energy dissipation mechanisms necessary to achieve a safe landing during a vertical descent

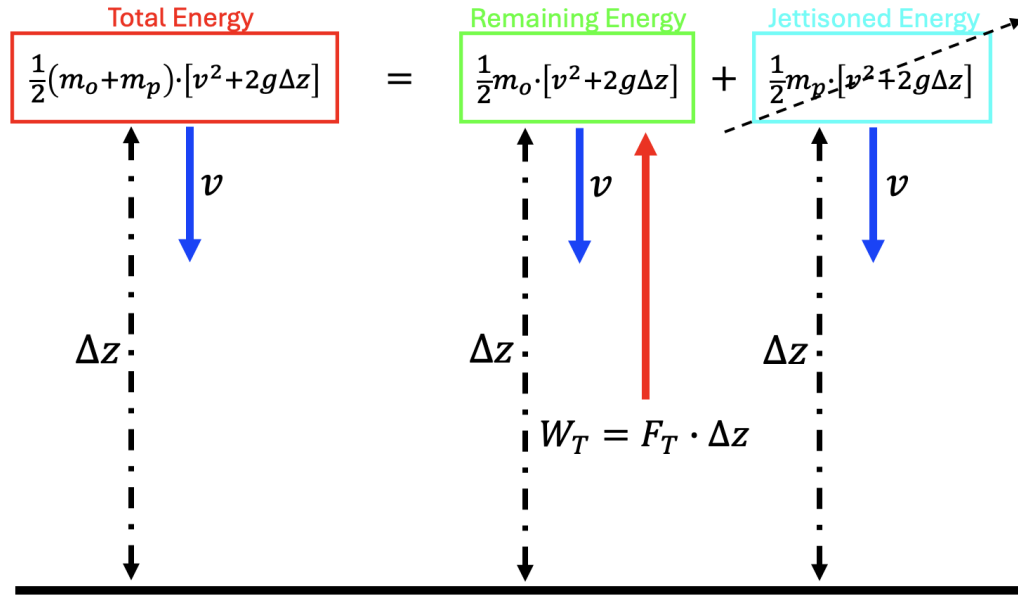


Figure 2: Illustration of the energy dissipation mechanisms during a vertical descent. The jettisoned energy is lost by the mass of burnt propellant m_p exhausted by the engine, while the rest of the energy is dissipated by the work done by the thrust force F_T of the engine along Δz . The WE formulation calculates the exact solution of the optimal constant value of the thrust force with constant jettisoned mass-rate, time, and thrust direction to achieve the energy dissipation required for a safe landing.

As illustrated in Fig. 2, the WE formulation assumes that there exist optimal constant engine settings [i.e., constant engine thrust force (throttle), mass flow, time, and thrust direction] along the vehicle displacement that guarantee the energy dissipation required to reach the LS. Throughout the paper, these calculated constant values are referred to as “constant settings”. The assumption of constant engine settings in the formulation yields analytical solutions that simplify the calculation of real-time guidance commands.

The local inertial right-hand (RH) coordinates with respect to the LS are defined at the moment the maneuver initiates as $\hat{\mathbf{x}}_{LS}$ -Cross-Range, $\hat{\mathbf{y}}_{LS}$ -Down-Range, and $\hat{\mathbf{z}}_{LS}$ -Altitude; these coordinates are kept constant throughout the whole maneuver. Figure 3 illustrates the definition of the inertial LS coordinates employed in the calculation.

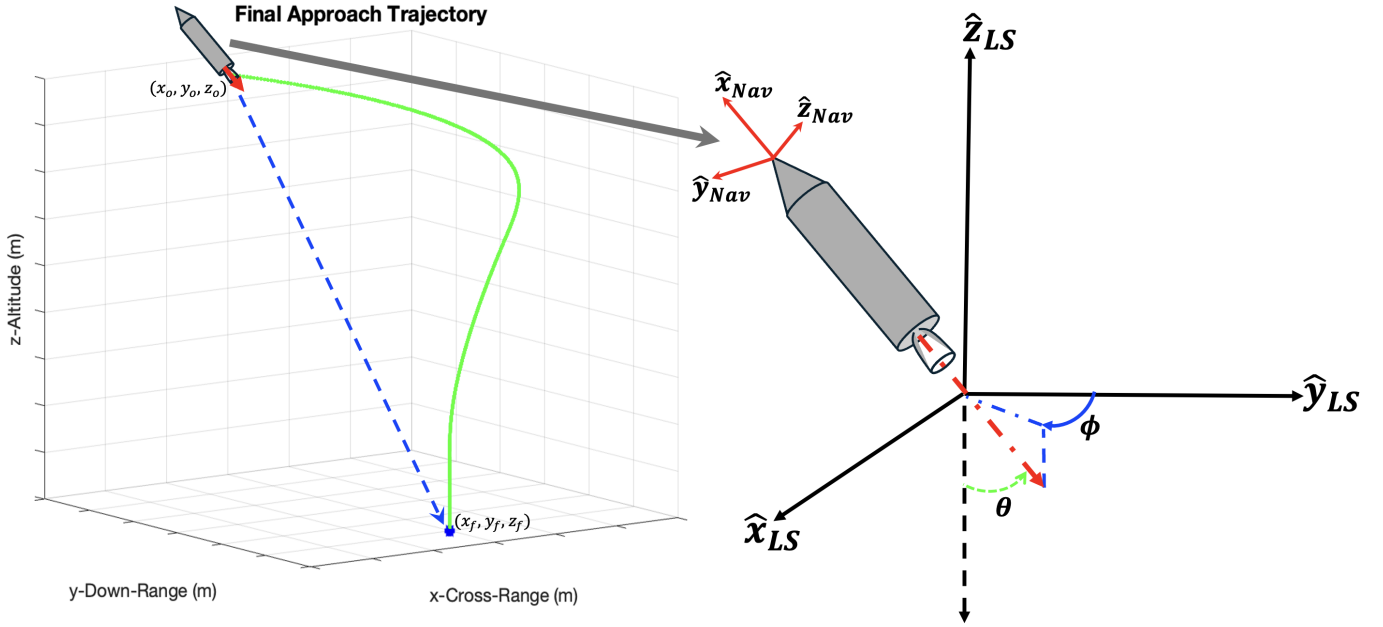


Figure 3: Definition of the local LS coordinates for the approach trajectory and the navigation coordinates of the vehicle. Then the approach maneuver begins, a local set of inertial RH coordinates with respect to the LS are defined and kept constant until landing. The $\hat{\mathbf{y}}_{LS}$ -Down-Range direction is defined as the horizontal vector from the starting horizontal position of the approach to the LS (the $\hat{\mathbf{y}}_{LS}$ -Down-Range direction may also be defined along the horizontal component of the vehicle's velocity vector), while the $\hat{\mathbf{z}}_{LS}$ -Altitude is defined as the vertical vector from the LS to the starting altitude, and the $\hat{\mathbf{x}}_{LS}$ -Cross-Range is defined by the vector product of the Down-Range and Altitude vectors ($\hat{\mathbf{x}}_{LS} = \hat{\mathbf{y}}_{LS} \times \hat{\mathbf{z}}_{LS}$). The attitude angle θ and yaw ϕ are defined with respect to the LS coordinates, while the navigation coordinates are defined with respect to the vehicle's body roll, pitch, and yaw. Appendix C contains the formulation of the angular-rates transformation from the LS into the navigation coordinates. Notice that at the beginning of the maneuver $x_o = x_f$, such that $\Delta x = 0$.

Per the definition of LS-coordinates introduced in Fig. 3, the attitude angle θ is defined as the angle between the axial axis of the rocket $\hat{\mathbf{x}}_{Nav}$ and the $\hat{\mathbf{z}}_{LS}$ -axis, and the yaw ϕ is defined as the right-hand rotation angle on the $\hat{\mathbf{x}}_{LS}\hat{\mathbf{y}}_{LS}$ -plane along the negative $\hat{\mathbf{z}}_{LS}$ -axis.

The mass flow expelled from the rocket engine is defined as

$$\frac{dm}{dt} = \dot{m} = T_{hrtl} \cdot \dot{m}_{max}, \quad (3)$$

where $\dot{m} \leq 0$, and the engine throttle is defined as $0 \leq T_{hrtl} \leq 1$ (the physical lower limit may not be zero), and the magnitude of $\dot{m}_{max}$ represents the maximum mass flow possible at full engine throttle.

With the assumption that there exists a constant optimal mass flow $\dot{m}$, and an optimal thrust direction defined

by θ and ϕ , during a time interval t necessary to dissipate the required amount of energy, the vehicle mass as a function of time is given by

$$m(t) = m_o + \dot{m} \cdot t, \quad (4)$$

with initial vehicle mass m_o at $t = 0$. The magnitude of the rocket engine thrust force as derived by Konstantin E. Tsiolkovsky is¹⁶

$$F_T = -v_e \cdot \dot{m}, \quad (5)$$

with the effective exhaust velocity $v_e = g_o \cdot I_{sp}$, the standard gravity $g_o = 9.80665 \text{ m/s}^2$, and the engine's specific impulse I_{sp} . The maximum engine thrust force is given by

$$F_{T_{max}} = -v_e \cdot \dot{m}_{max}. \quad (6)$$

With respect to the inertial LS coordinate frame, the net force that the engine exerts on the vehicle (braking force) plus lunar gravity is given by

$$\begin{aligned} \mathbf{F}_T &= F_{T_x} \cdot \hat{\mathbf{x}}_{LS} + F_{T_y} \cdot \hat{\mathbf{y}}_{LS} + F_{T_z} \cdot \hat{\mathbf{z}}_{LS} \\ &= \dot{m} \cdot v_e \sin(\phi) \sin(\theta) \cdot \hat{\mathbf{x}}_{LS} \\ &\quad + \dot{m} \cdot v_e \cos(\phi) \sin(\theta) \cdot \hat{\mathbf{y}}_{LS} \\ &\quad - [m \cdot g + \dot{m} \cdot v_e \cos(\theta)] \cdot \hat{\mathbf{z}}_{LS} . \end{aligned} \quad (7)$$

The components of the acceleration vector are obtained by dividing Eq. (7) by the vehicle's mass as a function of time defined in Eq. (4)

$$a_x(t) = \frac{\dot{m}}{m} \cdot v_e \sin(\phi) \sin(\theta) , \quad (8a)$$

$$a_y(t) = \frac{\dot{m}}{m} \cdot v_e \cos(\phi) \sin(\theta) , \quad (8b)$$

$$a_z(t) = -g - \frac{\dot{m}}{m} \cdot v_e \cos(\theta) . \quad (8c)$$

Equations (8a) through (8c) are integrated with respect to time to obtain the velocity components

$$v_x(t) = v_{x_o} + v_e \sin(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (9a)$$

$$v_y(t) = v_{y_o} + v_e \cos(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (9b)$$

$$v_z(t) = v_{z_o} - g \cdot t - v_e \cos(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \quad (9c)$$

that in turn are also integrated to obtain the components of the position vector in the form

$$\begin{aligned} x(t) &= x_o + [v_{x_o} - v_e \sin(\phi) \sin(\theta)] \cdot t \\ &\quad + v_e \sin(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \end{aligned} \quad (10a)$$

$$\begin{aligned} y(t) &= y_o + [v_{y_o} - v_e \cos(\phi) \sin(\theta)] \cdot t \\ &\quad + v_e \cos(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] , \end{aligned} \quad (10b)$$

$$\begin{aligned} z(t) &= z_o + [v_{z_o} + v_e \cos(\theta)] \cdot t - \frac{1}{2} \cdot g \cdot t^2 \\ &\quad - v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot t \right] . \end{aligned} \quad (10c)$$

It can be shown that by combining Eqs. (9a) through (9c), with Eqs. (10a) through (10c), the time-intervals for each of the vector components to achieve their individual final states are given by

$$\tau_x = \frac{\left\{ \frac{m_o}{\dot{m}} \cdot [v_{x_o} - v_x(\tau_x)] - [x_o - x(\tau_x)] \right\}}{[v_x(\tau_x) - v_e \sin(\phi) \sin(\theta)]} , \quad (11a)$$

$$\tau_y = \frac{\left\{ \frac{m_o}{\dot{m}} \cdot [v_{y_o} - v_y(\tau_y)] - [y_o - y(\tau_y)] \right\}}{[v_y(\tau_y) - v_e \cos(\phi) \sin(\theta)]} , \quad (11b)$$

$$\begin{aligned} \tau_z &= - \left\{ \frac{1}{g} \cdot [v_z(\tau_z) + v_e \cos(\theta)] + \frac{m_o}{\dot{m}} \right\} \\ &\quad \pm \sqrt{\left\{ \frac{1}{g} \cdot [v_z(\tau_z) + v_e \cos(\theta)] + \frac{m_o}{\dot{m}} \right\}^2 - \frac{2}{g} \cdot \left\{ [z_o - z(\tau_z)] - \frac{m_o}{\dot{m}} \cdot [v_{z_o} - v_z(\tau_z)] \right\}} , \end{aligned} \quad (11c)$$

where $x(\tau_x)$, $y(\tau_y)$, $z(\tau_z)$, $v_x(\tau_x)$, $v_y(\tau_y)$, and $v_z(\tau_z)$ represent the final components of the position and velocity vectors of the vehicle at the end of the maneuver (e.g. landing), and τ_x , τ_y , and τ_z are the individual time intervals required to achieve each individual final state.

III. WORK-ENERGY THEORETICAL FORMULATION

The initial and final energy states of the lunar lander during the approach maneuver can be expressed as

$$E_o = \frac{1}{2} \cdot m_o \cdot [\mathbf{v}_o^2 + 2 \cdot g \cdot z_o] , \quad (12a)$$

$$E_f = \frac{1}{2} \cdot (m_o - m_p) \cdot [\mathbf{v}_f^2 + 2 \cdot g \cdot z_f] , \quad (12b)$$

where the final mass of the vehicle is reduced by the consumed propellant mass m_p , the horizontal components of

the velocity are set to zero upon landing, and the final vertical velocity is set to a small negative descent velocity v_{zf} at a given target altitude z_f . Therefore

$$\mathbf{v}_o^2 = v_{x_o}^2 + v_{y_o}^2 + v_{z_o}^2 , \quad (13a)$$

$$\mathbf{v}_f^2 = v_{zf}^2 . \quad (13b)$$

By assuming that the engine is operating with an optimal constant thrust and orientation along the trajectory (constant attitude, yaw, and throttle), the work done by the engine thrust along the displacement vector $\Delta \mathbf{r}$ is given by

$$W_T = \mathbf{F}_T \cdot \Delta \mathbf{r} \quad (14)$$

$$= F_{T_x} \cdot [x(\tau_x) - x_o] + F_{T_y} \cdot [y(\tau_y) - y_o] + F_{T_z} \cdot [z(\tau_z) - z_o] ,$$

where $y_f = y(\tau_y)$ and $z_f = z(\tau_z)$. And, by the definition of the cross-range direction illustrated in Fig. 3, the cross-range position $x_f = x(\tau_x) \approx x_o$ along the trajectory. Thus, the work-energy calculations are constrained to the horizontal and vertical directions as illustrated in Fig. 4.

The second energy dissipation mechanism is the jettisoning of mass in the form of burnt propellant with the assumption of a constant mass flow $\dot{m}$ during each of the time-intervals for the three different LS coordinate components in the form

$$E_{Jet} = \frac{1}{2} \cdot [m_{F_x} \cdot \bar{v}_x^2 + m_{F_y} \cdot \bar{v}_y^2 + m_{F_z} \cdot (\bar{v}_z^2 + 2 \cdot g \cdot \bar{z})] , \quad (15)$$

where the average squares of the velocity components and the average altitude ($\bar{v}_x^2$, $\bar{v}_y^2$, $\bar{v}_z^2$, and $\bar{z}$) must be constantly calculated along the entirety of the trajectory together with the jettisoned propellant mass during each of the i^{th} time intervals τ_i with constant $\dot{m}$ using

$$m_{P_i} = -\tau_i \cdot \dot{m} . \quad (16)$$

The average square of each of the i^{th} velocity components $\bar{v}_i^2$ and average altitude displacement $\bar{z}$ in Eq. (15) are calculated using the average integral functions

$$\bar{v}_i^2 = \frac{1}{\tau_i} \int_0^{\tau_i} dt \cdot v_i^2(t) , \quad (17a)$$

$$\bar{z} = \frac{1}{\tau_z} \int_0^{\tau_z} dt \cdot z(t) . \quad (17b)$$

By substituting in the velocities from Eqs. (9a) through (9c) into Eq. (17a), and by integrating along the i^{th} time interval τ_i , the average squares of the velocity components are obtained

$$\begin{aligned} \bar{v}_x^2 &= v_{x_o}^2 + 2 \cdot v_e^2 \sin^2(\phi) \sin^2(\theta) - 2 \cdot v_{x_o} v_e \sin(\phi) \sin(\theta) \\ &\quad + \frac{1}{\tau_x} \cdot v_e \sin(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right] \\ &\quad \times \left\{ 2 \cdot [v_{x_o} - v_e \sin(\phi) \sin(\theta)] + v_e \sin(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right] \right\} \\ &\quad \times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right] , \end{aligned} \quad (18a)$$

$$\begin{aligned} \bar{v}_y^2 &= v_{y_o}^2 + 2 \cdot v_e^2 \cos^2(\phi) \sin^2(\theta) - 2 \cdot v_{y_o} v_e \cos(\phi) \sin(\theta) \\ &\quad + \frac{1}{\tau_y} \cdot v_e \cos(\phi) \sin(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right] \\ &\quad \times \left\{ 2 \cdot [v_{y_o} - v_e \cos(\phi) \sin(\theta)] + v_e \cos(\phi) \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right] \right\} \\ &\quad \times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right] , \end{aligned} \quad (18b)$$

$$\begin{aligned} \bar{v}_z^2 &= v_{z_o}^2 + 2 \cdot v_e^2 \cos^2(\theta) + 2 \cdot v_{z_o} v_e \cos(\theta) \\ &\quad + g \tau_z \cdot \left\{ \frac{1}{3} \cdot g \tau_z - \left[v_{z_o} + \frac{1}{2} \cdot v_e \cos(\theta) \right] + \frac{1}{\tau_z} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \right\} \\ &\quad + \frac{1}{\tau_z} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right] \\ &\quad \times \left\{ g \cdot \left[\tau_z - \frac{m_o}{\dot{m}} \right] - 2 \cdot [v_{z_o} + v_e \cos(\theta)] + v_e \cos(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right] \right\} \\ &\quad \times \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right] . \end{aligned} \quad (18c)$$

Substituting the altitude provided in Eq. (10c) into Eq. (17b), and integrating along the time interval τ_z , the average altitude displacement is obtained in the form

$$\begin{aligned} \bar{z} &= z_o + \frac{1}{2} \cdot \tau_z \cdot \left[v_{z_o} + \frac{3}{2} \cdot v_e \cos(\theta) \right] - \frac{1}{6} \cdot g \cdot \tau_z^2 \\ &\quad + \frac{1}{2} \cdot v_e \cos(\theta) \cdot \frac{m_o}{\dot{m}} \cdot \left\{ 1 - \frac{1}{\tau_z} \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right]^2 \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right] \right\} . \end{aligned} \quad (19)$$

Therefore, the total energy dissipated by the engine work and mass jettisoning is expressed as

$$E_o - E_f = W_T + E_{jet} . \quad (20)$$

In general, the WE formulation works for solving the problem in three dimensions. However, in this work the problem is reduced to two dimensions. A 2D calculation is applicable due to the initial definition of the local LS coordinates in Fig. 3, where the net cross-range displacement is $\Delta x = 0$ along the trajectory, thus making the net work done along the cross-range very close to zero. Figure 4 illustrates the separation of the problem between horizontal and vertical stages.

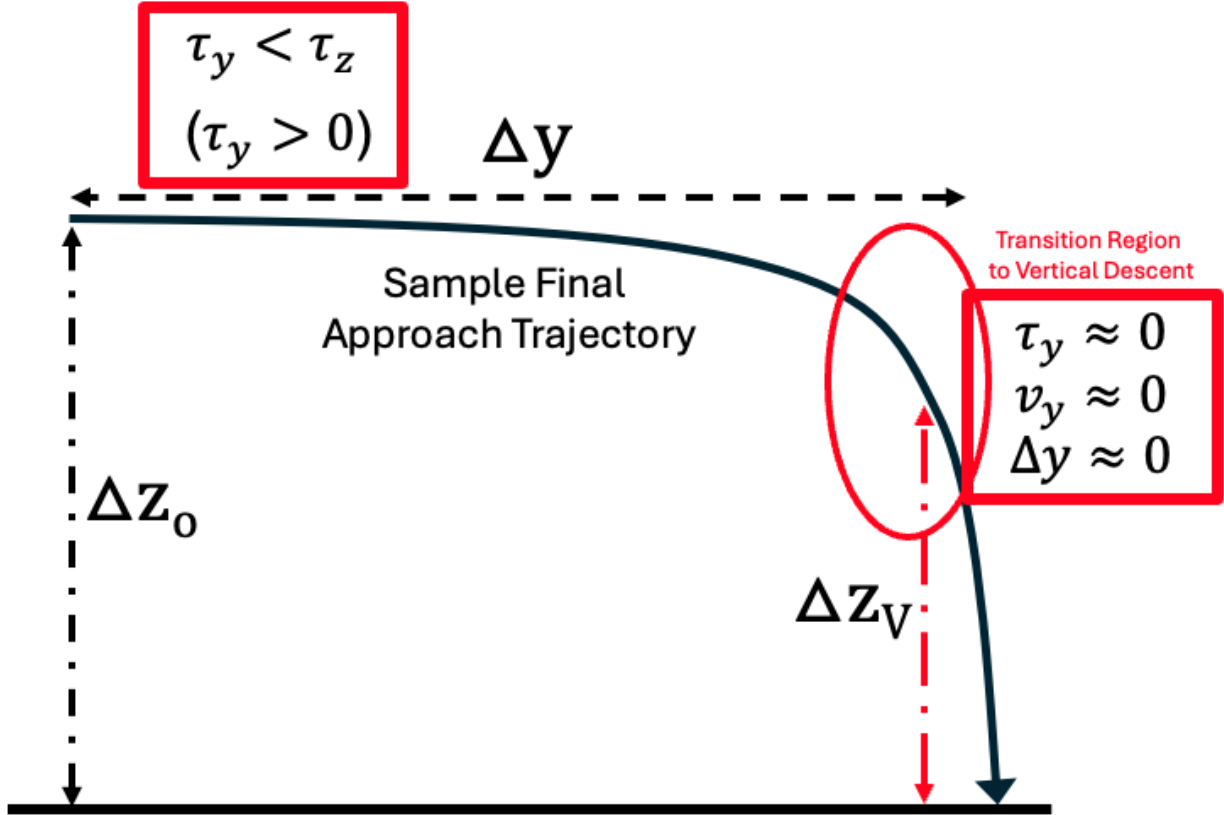


Figure 4: Separation of the approach trajectory maneuver into predominantly horizontal and vertical components. Typically during the approach $\tau_y < \tau_z$. Once $\tau_y \approx 0$, which occurs when $(y_f - y_o) \approx 0$, the WE formulation should not be applied to the horizontal direction because there is a risk of encountering a singularity in the calculation of the horizontal force [see Eq. (24a)]. Therefore, when reaching the transition region, the WE is only employed along the vertical descent calculation (1D), while setting $\tau_y = \tau_z$ and employing it in Eq. (A20) to calculate the optimal attitude angle that will force the final horizontal velocity to zero before landing. Similarly, to avoid singularities in calculating the cross-range (since $\Delta x = 0$), guidance commands are calculated analytically rather than using the WE formulation, thus constraining the WE calculation to 2D.

Therefore, the problem is simplified by separating Eq. (20) into both horizontal and vertical components in the form

$$E_{o_y} - E_{f_y}^0 = F_y \cdot \Delta y + E_{jet_y} , \quad (21a)$$

$$E_{o_z} - E_{f_z} = F_z \cdot \Delta z + E_{jet_z} , \quad (21b)$$

where the jettisoned energies are represented by

$$E_{jet_y} = \frac{1}{2} \cdot m_{p_y} \cdot \bar{v}_y^2, \quad (22a)$$

$$E_{jet_z} = \frac{1}{2} \cdot m_{p_z} \cdot [\bar{v}_z^2 + 2 \cdot g \cdot \bar{z}], \quad (22b)$$

and the mass-losses with respect to horizontal and vertical stages during the approach are given by

$$m_{p_y} = -\tau_y \cdot \dot{m}, \quad (23a)$$

$$m_{p_z} = -\tau_z \cdot \dot{m}. \quad (23b)$$

The horizontal and vertical force components during the approach maneuver are calculated using

$$F_y = \frac{[E_{Oy} - E_{jet_y}]}{[y_f - y_o]}, \quad (24a)$$

$$F_z = \frac{[E_{Oz} - E_{fz} - E_{jet_z}]}{|z_f - z_o|}, \quad (24b)$$

from which an optimal attitude approach angle is derived (see Fig. 3) in the form

$$\theta_{opt.} = \tan^{-1} \left[\frac{F_y}{F_z \cdot \cos(\phi)} \right]. \quad (25)$$

For the cross-range guidance commands, instead of employing the WE formulation, it can be shown that by combining Eqs. (9a) through (11c), and expanding the natural logarithm to its first order, two solutions for the optimal yaw angle $\phi_{opt.}$ are derived in the form

$$\begin{aligned} \phi_{opt.} &= \tan^{-1} \left\{ \frac{[x(\tau_x) - x_o] - v_{x_o} \cdot \tau_x}{[y(\tau_y) - y_o] - v_{y_o} \cdot \tau_y} \cdot \frac{[\tau_y - \frac{m_o}{\dot{m}} \cdot (1 + \frac{\dot{m}}{m_o} \cdot \tau_y) \cdot \ln(1 + \frac{\dot{m}}{m_o} \cdot \tau_y)]}{[\tau_x - \frac{m_o}{\dot{m}} \cdot (1 + \frac{\dot{m}}{m_o} \cdot \tau_x) \cdot \ln(1 + \frac{\dot{m}}{m_o} \cdot \tau_x)]} \right\} \\ &\approx \tan^{-1} \left\{ \frac{\tau_y^2 \cdot [x(\tau_x) - x_o] - v_{x_o} \cdot \tau_x}{\tau_x^2 \cdot [y(\tau_y) - y_o] - v_{y_o} \cdot \tau_y} \right\}, \end{aligned} \quad (26a)$$

$$\phi_{opt.} = \tan^{-1} \left\{ \frac{[(x_f - x_o) - \frac{m_o}{\dot{m}} \cdot (v_{x_f} - v_{x_o})]}{[(y_f - y_o) - \frac{m_o}{\dot{m}} \cdot (v_{y_f} - v_{y_o})]} \right\}, \quad (26b)$$

where $\tau_x \leq \tau_y$ to achieve full cross-range correction at or before horizontal energy dissipation is achieved, and τ_x is set in terms of a non-dimensional heuristic tuning cross-range time fraction $a_x \leq 1$ in the form $\tau_x = a_x \cdot \tau_y$. The magnitude of the optimal yaw angle is limited to a set maximum angle $\phi_{max.}$. The first solution is time-dependent and more applicable during the predominantly horizontal stage during braking-burn and approach, while the second may be applied during the final quasi-vertical descent stage.

The value of the engine throttle for the approach maneuver is obtained using

$$T_{hrtl} = \frac{F_z}{F_{T_{max.}} \cdot \cos(\theta_{opt.})}, \quad (27)$$

which is employed in Eq. (3) to calculate the value of the engine mass flow inside the WE calculation cycle as illustrated in Fig. 5

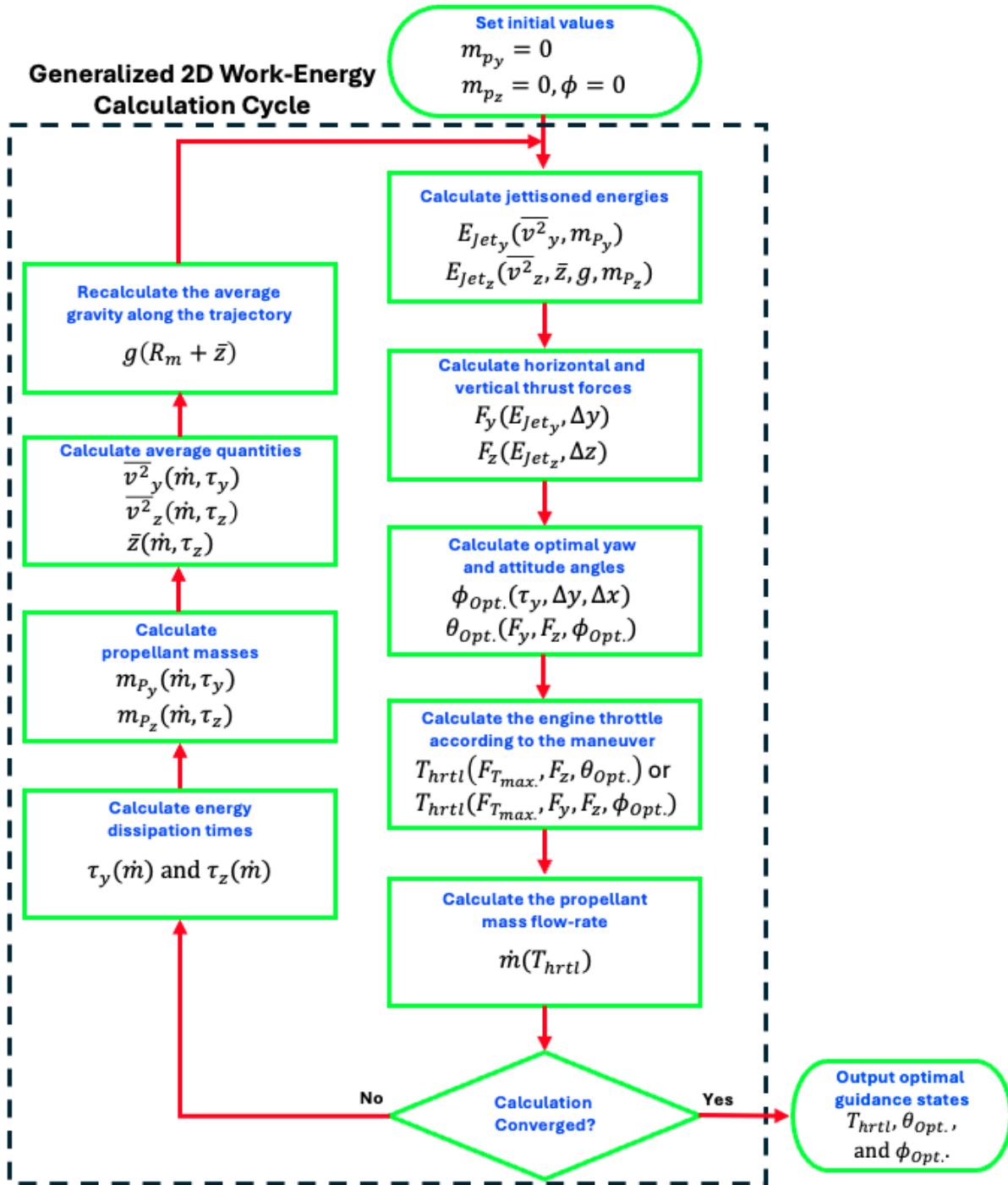


Figure 5: Generalized WE calculation cycle. Both horizontal and vertical forces from Eqs. (24a) and (24b) are calculated recursively by initially setting the propellant masses and yaw angle to zero during the first cycle, that initially sets $E_{Jet_y} = 0$ and $E_{Jet_z} = 0$. The optimal attitude and yaw angles are calculated using Eq. (25) and Eq. (26a), the engine throttle is calculated according to the maneuver (using either Eq. (27) or Eq. (36)), with the mass flow calculated using Eq. (3). These values are employed to calculate both horizontal and vertical time intervals [Eqs. (11b) and (11c)], which are needed to calculate consumed propellant masses [Eqs. (23a) and (23b)], average squares of the horizontal and vertical velocities [Eqs. (18b) and (18c)], average altitude $\bar{z}$ [Eq. (19)], and acceleration due to gravity at $\bar{z}$ [Eq. (2)]. The jettisoned energies are calculated with Eqs. (22a) and (22b), and the whole calculation cycle is repeated until the desired precision in the optimal values is achieved.

The summary of the WE calculation cycle presented in Fig. 5 solves for the necessary energy dissipation conditions by employing the vehicle's states as initial conditions at the moment of the calculation as illustrated in the example given in Fig. 2. This produces an exact solution of the constant optimal guidance commands to achieve the necessary energy dissipation and fulfill the desired landing conditions. Since it is very unlikely that the vehicle is already configured at the optimal WE state at the moment of the calculation, the WE cycle is constantly repeated, and the optimal states change continuously as the vehicle pursues the most current optimal states. Within the WE cycle, convergence is defined as obtaining the desired precision of the optimal guidance commands. Per the energy conservation principle, the calculation will always converge, although the calculated throttle may be outside the physical capabilities of the engine. For example, if the WE yields a lower throttle value than the lowest physical limit of the engine, the engine will not be ignited and the WE calculation will continually yield an increasing throttle value as the vehicle approaches the LS until the physical lowest limit of the engine is reached. If the WE yields a higher value than the maximum physical throttle limit, the horizontal distance to the LS must be extended, which would result in an overshooting of the intended LS. In the work presented here, it has been observed that 5 to 8 iterations are sufficient to achieve convergence in the WE calculation, and a default maximum value of 10 iterations is employed.

Although the WE cycle yields optimal conditions for energy dissipation, the desired trajectory may be shaped by defining different energy states or waypoints along the path to satisfy certain constraints (e.g., attitude, velocity, altitude, glide-slope). Another simple way to reshape the trajectory is through adding small corrections to the optimal attitude angle $\theta_{opt.}$ in order to modify both down-range and vertical time intervals (τ_y and τ_z). For example, it is typical that $\tau_y < \tau_z$ during the final approach trajectory, which means that the LS will first be reached horizontally, and then a vertical descent will complete the landing maneuver. Sometimes, mission constraints may require a shorter vertical descent time, or require the LS is reached at the same time horizontally and vertically. Alternately, during the braking-burn maneuver, $\tau_y > \tau_z$, and it may be desirable to reach both altitude and distance from the LS at the same time in order to transition to the approach maneuver at the mission required altitude and down-range distance. Therefore, in order to approach convergence when $\tau_y \approx \tau_z$, an empirical form for the optimal attitude angle corrections is formulated as

$$\delta_{\theta_1} = c_1 \cdot \frac{|\tau_z - \tau_y|}{\tau_z}, \quad (28a)$$

$$\delta_{\theta_2} = c_2 \cdot \frac{|\tau_z - \tau_y|}{\tau_y}, \quad (28b)$$

where the constant correction angles c_1 and c_2 (in degrees

or radians) are determined heuristically, and the corrections are applied to the optimal attitude angle in the form

$$\theta_{opt.} = \begin{cases} \tau_y > \tau_z, & \theta_{opt.} - \delta_{\theta_1} \\ \tau_y < \tau_z, & \theta_{opt.} + \delta_{\theta_2} \end{cases} \quad (29)$$

The logic supporting these small attitude corrections is that with a slight decrease in the attitude angle, the vertical component of thrust increases, which increases the descent time while simultaneously reducing the down-range time. With a slight increase of the attitude angle, the opposite results are obtained.

After calculating the WE optimal attitude and yaw angles, the vehicle's physical characteristics (e.g., mass, moment of inertia, propellant slosh, center of mass) require that optimal angular rates are calculated in order to smoothly steer the vehicle to the optimal orientation while setting the throttle to the optimal value (referred in this paper as pursuing the optimal state). Therefore, the commanded angular velocities in the LS frame are calculated using

$$\omega_{\theta} = k_{\theta} \cdot \frac{(\theta_{opt.} - \theta_o)}{\delta\tau}, \quad (30a)$$

$$\omega_{\phi} = k_{\phi} \cdot \frac{(\phi_{opt.} - \phi_o)}{\delta\tau}, \quad (30b)$$

where the vehicle's current attitude and yaw angles with respect to the LS (see Fig. 3) are represented by θ_o and ϕ_o , and the constants k_{θ} and k_{ϕ} are determined heuristically according to the physical characteristics of the vehicle. The time interval $\delta\tau$ is defined according to the intended maneuver as shown in Figs. 6 and 7.

The commanded attitude and yaw angles for the vehicle during the guidance time interval Δt are given by

$$\theta_{CMD} = \theta_o + \omega_{\theta} \cdot \Delta t, \quad (31a)$$

$$\phi_{CMD} = \phi_o + \omega_{\phi} \cdot \Delta t, \quad (31b)$$

where Δt is the reciprocal of the guidance calculation frequency ($\Delta t = 0.1 \text{ sec.}$ in this work).

Finally, the commanded engine throttle is calculated using

$$T_{hrtl}^{CMD} = \frac{F_z}{F_{T_{max.}} \cdot \cos(\theta_o + \frac{1}{2} \cdot \omega_{\theta} \cdot \Delta t)}, \quad (32)$$

where during the approach maneuver, the commanded throttle is always calculated in reference to the vertical force F_z in order to prioritize full vertical energy dissipation at every point of the maneuver to avoid a crash.

The commanded angular rates with respect to the vehicle's navigation reference (roll, pitch, and yaw as illustrated in Fig. 3) are calculated using [for the derivation see Appx. C, Eq. (C9)]

$$\omega_{LS}^{Nav.} = -\omega_\phi \cdot \cos(\theta) \cdot \hat{x}_{Nav.} + \omega_\theta \cdot \hat{y}_{Nav.} - \omega_\phi \cdot \sin(\theta) \cdot \hat{z}_{Nav.} . \quad (33)$$

Figure 6 summarizes the optimal WE guidance commands calculated during the first stage of the approach maneuver ($\tau_y < \tau_z$) before the final vertical descent as represented in Fig. 4.

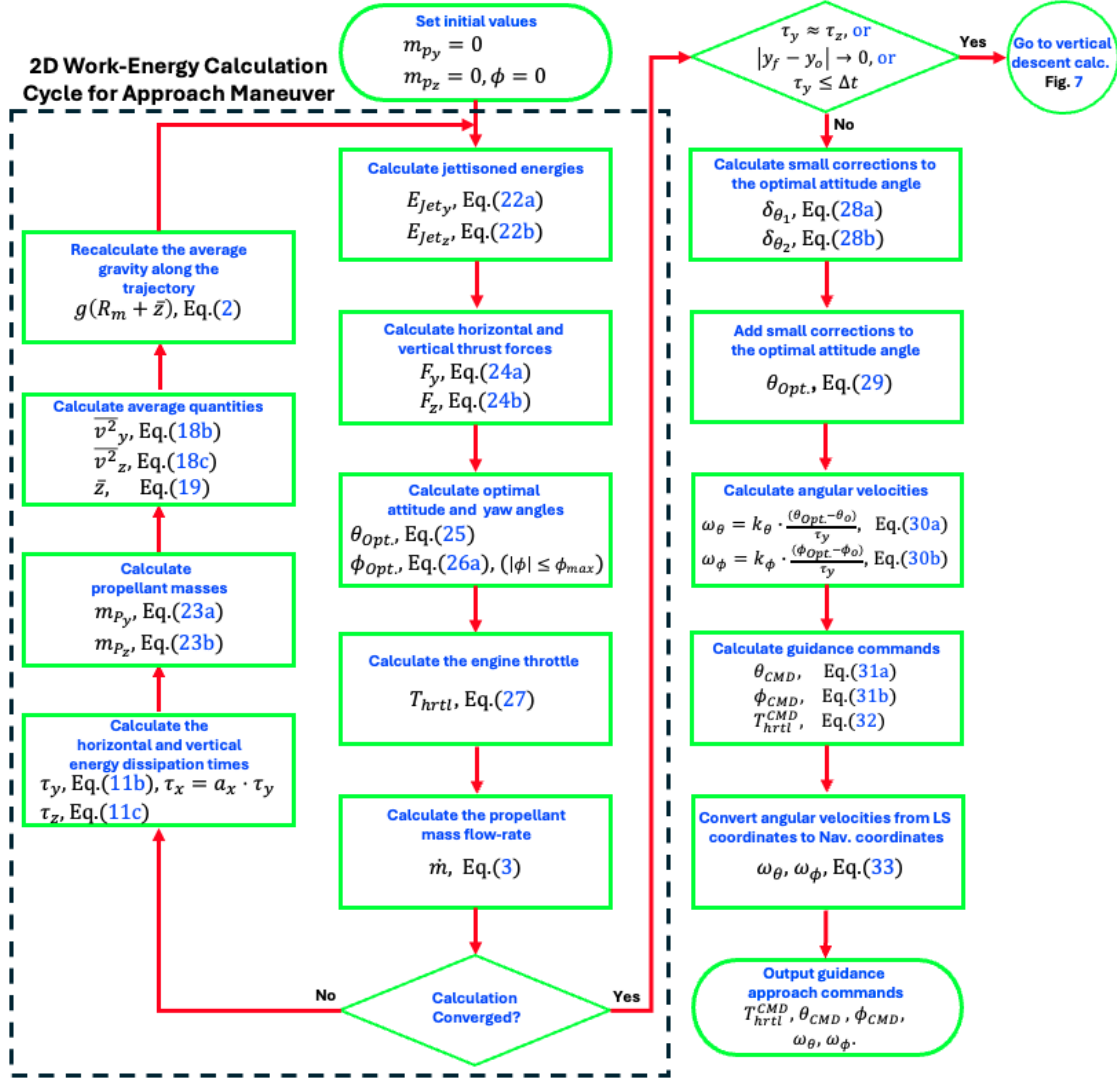


Figure 6: Summary of the WE guidance calculation employed during the final approach maneuver when $\tau_y < \tau_z$ (or $\tau_y > 0$). In this work, the cross-range time fraction inside the WE cycle is set to $a_x = 0.76$ in order to reach a full cross-range correction when 76% of the horizontal time dissipation is achieved, before reaching vertical descent. For the simulated vehicle, the yaw angle is limited to $|\phi| \leq \phi_{max} = 15^\circ$ in order to avoid overcorrecting the cross-range position and velocity during both the first stage and vertical descent of the approach. The constant correction angles employed in Eqs. (28a) and (28b) to adjust time convergence ($\tau_z \approx \tau_y$) are obtained heuristically according to the vehicle's characteristics and desired approach profile, with values $c_1 = 100.0^\circ$ and $c_2 = 2.525^\circ$. The non-dimensional constants employed in Eqs. (30a) and (30b) to calculate both LS-referenced attitude and yaw angular velocities are $k_\theta = 6.70$ and $k_\phi = 5.68$, which are heuristically determined according to the vehicle's physical characteristics, and the time interval in Eqs. (30a) and (30b) is set to the horizontal time value $\delta\tau = \tau_y$. Both magnitudes of the attitude and yaw angular velocities are limited to the vehicle's maximum angular velocity of $5^\circ/sec$. In this work, the differential time defined by the reciprocal of the guidance calculation frequency is $\Delta t = 0.1 \text{ sec}$.¹⁷

The WE cycle illustrated in Fig. 6 is performed as long as $\tau_y < \tau_z$ and $\tau_y > \Delta t$. Once $\tau_y \approx 0$ ($\tau_y \leq \Delta t$), the vehicle is approaching the LS along the down-range direction ($y_f - y_o \approx 0$). The WE formulation should no longer be applied to horizontal guidance due to the risk of singularity in Eq. (24a). As illustrated in Fig. 23, when the vehicle is closely located above the LS, the priority for both down-range and cross-range guidance is to ensure zero horizontal velocities at landing to avoid toppling the vehicle upon touchdown. Therefore, the final attitude angular velocity and optimal attitude and yaw angles are calculated using Eqs. (A12), (A20), and (A21) as described in Fig. 7, and the down-range displacement during the final rotation maneuver may be obtained from Eq. (A19) (all derived in Appx. A)

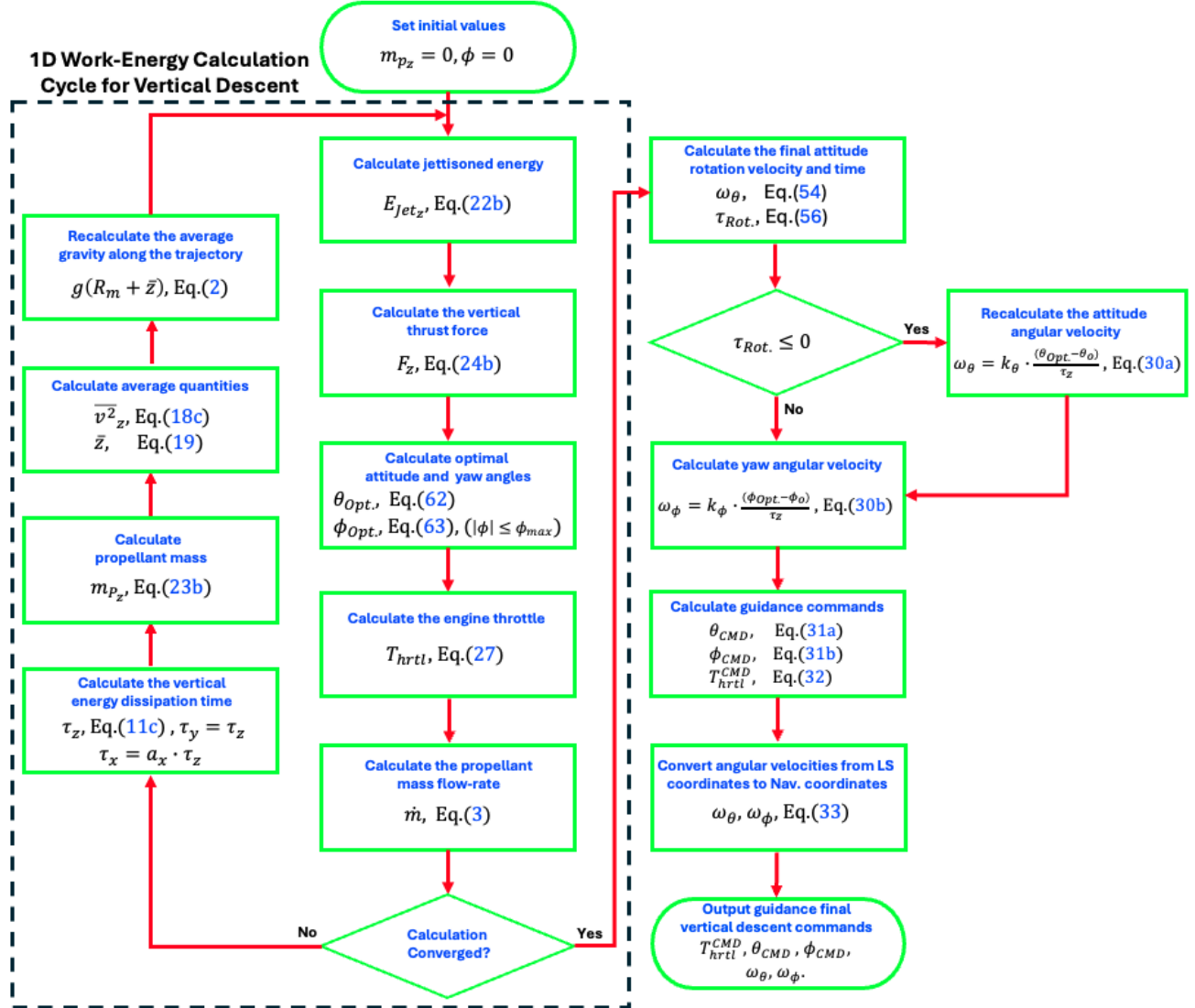


Figure 7: The WE cycle only calculates the optimal vertical descent state, while optimal attitude and yaw angles are calculated using Eqs. (A20) and (A21) to guarantee close-to-zero horizontal velocity at touchdown. For this maneuver, $\tau_y = \tau_z$, and the cross-range time fraction is set to $a_x = 0.38$ to reach zero cross-range velocity during 38% vertical dissipation time. The optimal attitude angular velocity is calculated using Eq. (A12), and when valid ($\tau_{Rot.} > 0$), $\theta_{Opt.} = 0$ (see Fig. 23). For the case when $\tau_{Rot.} \leq 0$, the attitude angular velocity is calculated using Eq. (30a), with $\delta\tau = \tau_z$ used to calculate the optimal yaw angular velocity from Eq. (30b), which employs the same non-dimensional constant values for k_θ and k_ϕ shown in Fig. 6.

IV. BRAKING-BURN WORK-ENERGY CALCULATION

The WE formulation is also employed in calculating guidance commands for orbital-and-sub-orbital-type maneuvers where, due to the Moon's rotation, the displacements are calculated by estimating the location of the LS at the moment of landing. Lunar rotational corrections may also be applied to the final approach maneuver described in the previous section (Figs. 6 and 7). Figure 8 illustrates the application of the WE formulation to the calculation of a braking-burn sub-orbital maneuver, and Fig. 9 summarizes the WE guidance calculation.

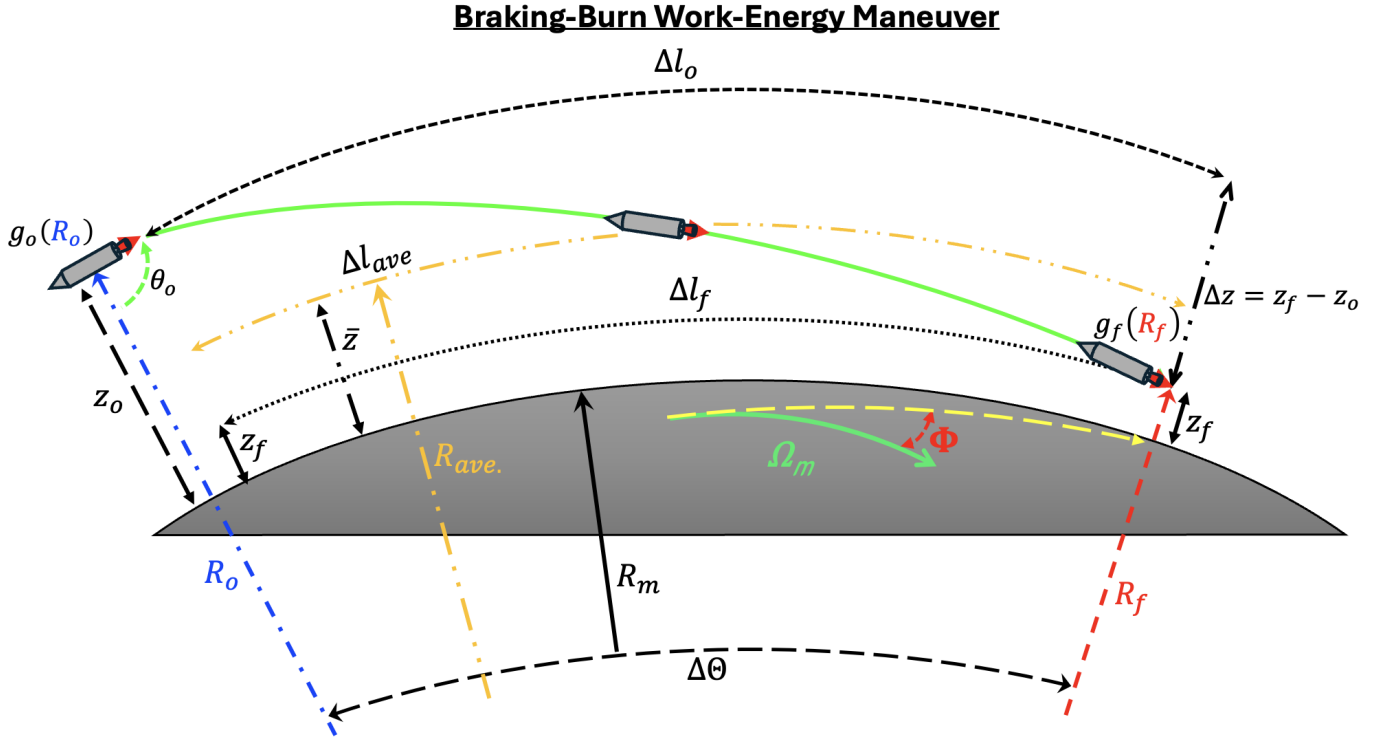


Figure 8: The initial and final gravitational accelerations $g_o(R_o)$ and $g_f(R_f)$ are a function of the radial distances R_o and R_f with respect to the lunar center of mass, and R_m is the average radius of the Moon along the ground path. Gravity may also be a function of initial and final latitudes and longitudes [see Eq. (2)]. The higher and lower down-range paths are a function of the total angular displacement $\Delta\Theta$: $\Delta l_o = R_o \cdot \Delta\Theta$ and $\Delta l_f = R_f \cdot \Delta\Theta$. The total vertical displacement is represented by Δz , while the angle Φ represents the angle between the ground-track of the trajectory and the lunar angular rotation path Ω_m . $R_{ave.}$ is the average radius along the trajectory, which is needed to calculate the net down-range displacement $\Delta l_{ave.} = R_{ave.} \cdot \Delta\Theta$, that is employed in the WE formulation.

By taking into account the curvature of the trajectory and the variation of the gravitational acceleration with respect to altitude during the initial WE calculation cycle (before knowing the average altitude $\bar{z}$ and τ_y), the average gravitational acceleration along the trajectory and down-range displacement are approximated using

$$g_{ave.} \approx \frac{1}{2} \cdot [g_o(R_o) + g_f(R_f)] , \quad (34a)$$

$$R_{ave.} \approx \frac{1}{2} \cdot [R_o + R_f] , \quad (34b)$$

$$\Delta l_{ave.} \approx \frac{1}{2} \cdot [R_o + R_f] \cdot \Delta\Theta . \quad (34c)$$

During subsequent WE cycles, the average altitude $\bar{z}$ along the trajectory is calculated using Eq. (19), and the

average radius with respect to the lunar center of mass is given by $R_{ave.} = R_m + \bar{z}$.

The gravitational acceleration with both cross-range and down-range displacements during the time interval τ_y (plus the estimated final approach maneuver time $\tau_{apch.}$), including lunar rotation displacements as a function of local latitude λ , are calculated using

$$g_{ave.} = g(R_{ave.}) , \quad (35a)$$

$$\Delta x = (\tau_y + \tau_{apch.}) \cdot \Omega_m \cdot R_{ave.} \cdot \cos(\lambda) \sin(\Phi) , \quad (35b)$$

$$\Delta y = R_{ave.} \cdot \left[\Delta\Theta + (\tau_y + \tau_{apch.}) \cdot \Omega_m \cdot \cos(\lambda) \cos(\Phi) \right] . \quad (35c)$$

These quantities along with the vertical displacement Δz are employed in the WE calculation cycles. The horizontal and vertical force components are calculated using Eqs. (24a) and (24b), the optimal attitude and yaw angles are calculated using Eqs. (25) and (26a), and the engine optimal WE throttle is obtained from

$$T_{hrtl} = \frac{F_{T_{max.}}}{\sqrt{F_z^2 + [F_y / \cos(\phi_{opt.})]^2}}, \quad (36)$$

which is employed in Eq. (3) to calculate the value of the engine mass flow.

The commanded angular velocities are calculated using Eqs. (37a) and (37b), in which a set maximum value ω_{max} and non-dimensional heuristic parameters k_θ and k_ϕ are employed. When the cross-range displacement [Eq. (A38)] is within the limits of the LS radius (see Fig. 24), the yaw angular velocity is calculated using Eq. (A33).

$$\omega_\theta = k_\theta \cdot \omega_{max} \cdot \frac{(\theta_{opt.} - \theta_o)}{|\theta_{opt.} - \theta_o|}, \quad (37a)$$

$$\omega_\phi = k_\phi \cdot \omega_{max} \cdot \frac{(\phi_{opt.} - \phi_o)}{|\phi_{opt.} - \phi_o|}. \quad (37b)$$

Due to the changing orientation of the vector normal to the surface with respect to the vehicle's motion along the

curved surface, a curvature angular velocity correction during the guidance time interval Δt is calculated using

$$\omega_c = -\frac{v_y}{(R_m + z + \frac{1}{2} \cdot v_z \cdot \Delta t)}, \quad (38)$$

which is added into the commanded attitude angular velocity ($\omega_\theta \rightarrow \omega_\theta + \omega_c$).

The commanded throttle for the vehicle is obtained with

$$T_{hrtl}^{CMD} = \frac{F_y}{F_{T_{max.}} \cdot \sin(\theta_o + \frac{1}{2} \cdot \omega_\theta \cdot \Delta t) \cdot \cos(\phi_o + \frac{1}{2} \cdot \omega_\phi \cdot \Delta t)}, \quad (39)$$

in which the throttle is always calculated in reference to the horizontal force F_y in order to prioritize horizontal energy dissipation. Horizontal energy dissipation is prioritized because in general, during orbital and sub-orbital maneuvers, the horizontal kinetic energy is much higher than the vertical potential energy and $\tau_y \gg \tau_z$. During braking-burn, the time intervals are forced to converge $\tau_y \approx \tau_z$ at the end of the maneuver in order to satisfy final altitude, distance, and velocity constraints by employing attitude correction angles calculated from Eqs. (28a) and (28b).

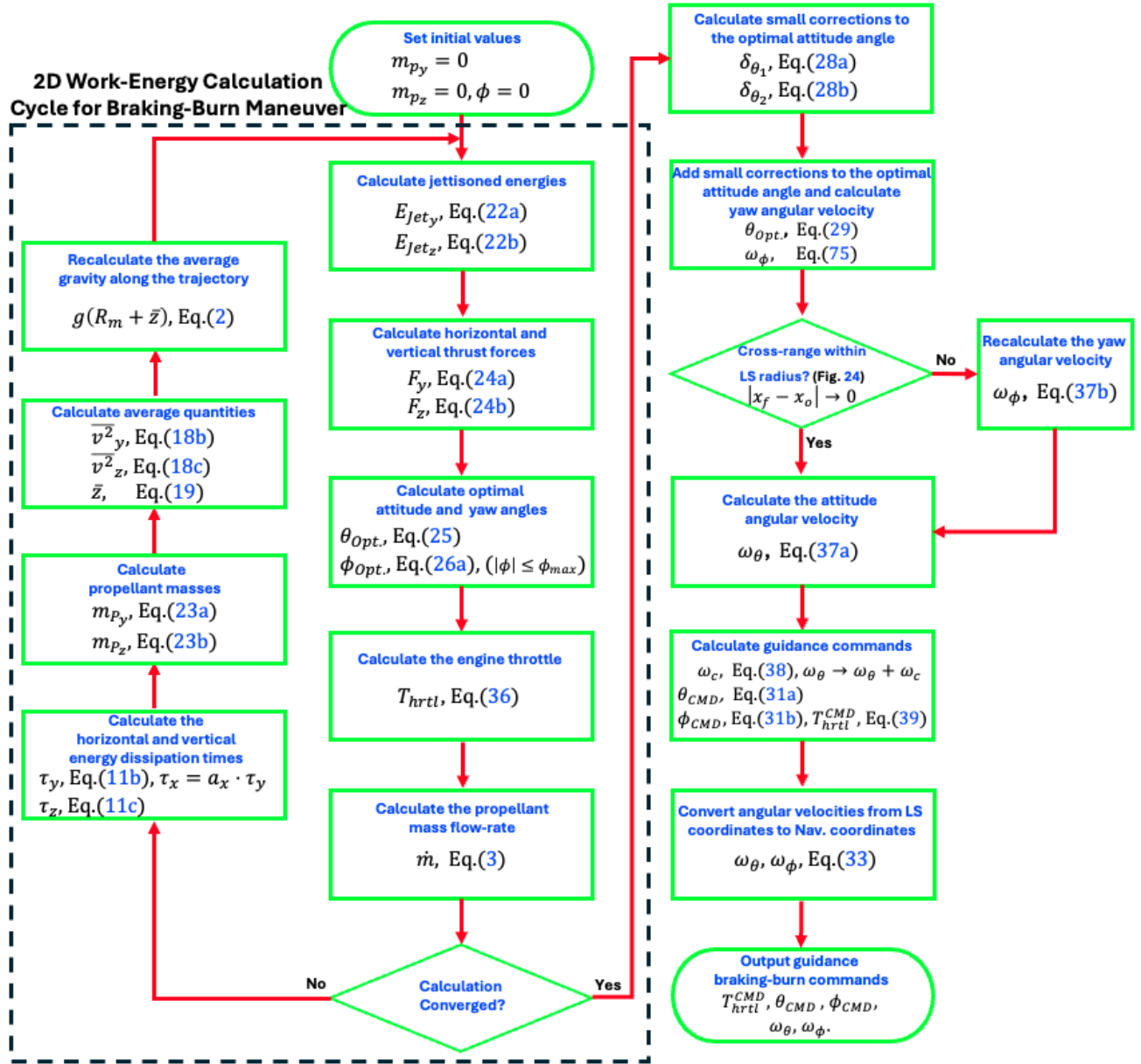


Figure 9: Summary of the WE calculation employed during the braking-burn suborbital maneuver. The cross-range time fraction inside the WE cycle is set to $a_x = 0.10$, thus intending to reach a full cross-range correction when 10% of the horizontal time dissipation is achieved, and the yaw angle is limited to $|\phi| \leq \phi_{max} = 4.5^\circ$ in order to avoid overcorrection of the cross-range position and velocity. Because $\tau_y \gg \tau_z$ during the braking-burn maneuver, the constant correction angles employed in Eqs. (28a) and (28b) to force time convergence ($\tau_y \approx \tau_z$) before the end of the braking-burn are set to $c_1 = 13.5^\circ$ and $c_2 = 1000.0^\circ$. The non-dimensional constants used in Eqs. (37a) and (37b) to calculate both attitude and yaw angular velocities are set to $k_\theta = 0.475$ and $k_\phi = 0.475$, with a maximum angular velocity value set to $\omega_{max} = 5^\circ/sec$. All these values are obtained heuristically according to the vehicle's physical characteristics and desired approach profile. The differential guidance time is the same as the one employed during the final approach trajectory, with a value of $\Delta t = 0.1 \text{ sec}$.

V. PROPELLANT CONSUMPTION MINIMIZATION AND ENGINE EFFICIENCY

As shown by previous work, Pontryagin's maximization principle is employed to minimize fuel consumption by reducing the amount of time the engine is fired.^{1,2,4} This same time minimization principle is commonly employed in convex-optimization calculations to minimize propellant consumption.^{8,9} Since the WE formulation directly calculates both the necessary engine throttle (or engine thrust force) and the engine firing duration to achieve the required energy dissipation, the time minimization principle is easily implemented into the calculation. This is accomplished by defining a minimum throttle threshold (or range of thresholds) to ignite the engine. The lower the throttle threshold for ignition, the longer the burn-time needed, and the more propellant consumed during the maneuver. Conversely, the higher the throttle threshold for ignition, the shorter the burn-time, and the less propellant consumed. Therefore, increasing the throttle threshold is equivalent to delaying the ignition of the engine, and increases propellant efficiency at the cost of valuable time. The highest propellant efficiency is attained at 100% minimum throttle threshold, leaving no time left for corrections. Figure 10 illustrates the propellant consumption minimization scheme from the WE perspective.

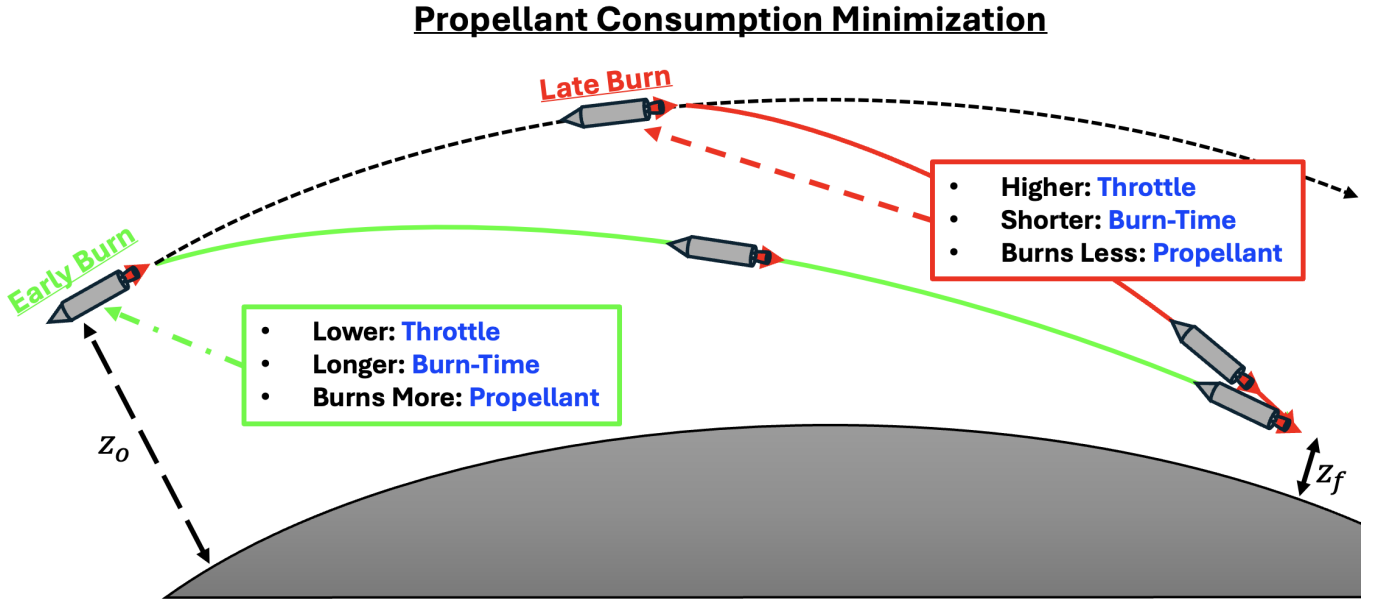


Figure 10: Illustration of the propellant minimization scheme from the WE formulation perspective. In this case, propellant minimization is no longer determined by a time minimization assumption (where burn-time is minimized in order to minimize propellant consumption).^{8,9} Instead, the optimization is determined from a crew-resource/risk management perspective. Both time and usable propellant are finite resources which are important to conserve to achieve mission objectives. In the case of an engine failure or misfire, for example, having extra time for the crew to troubleshoot may be as important as saving propellant. By solely considering the burn-time minimization principle, the most propellant-efficient throttle threshold would be 100%; this approach minimizes both propellant consumption and time but reduces control-authority, and does not allow extra time for further corrective maneuvering in case of a mishap which increases the risk of mission failure. The WE formulation has the advantage of real-time calculation of the necessary throttle and engine firing duration needed to accomplish the maneuver with the use of a few simple analytical expressions that help to quantify both fuel consumption and time throughout the whole maneuver.

Another important consideration to minimize propellant consumption, is the actual efficiency and performance of the engine. A typical assumption is that the thrust force is a linear function with respect to the mass flow with constant slope, defined as effective exhaust velocity v_e [Eq. (5)].¹⁶ However, engine performance is better characterized as a combination of affine and/or polynomial functions as illustrated in Fig. 11

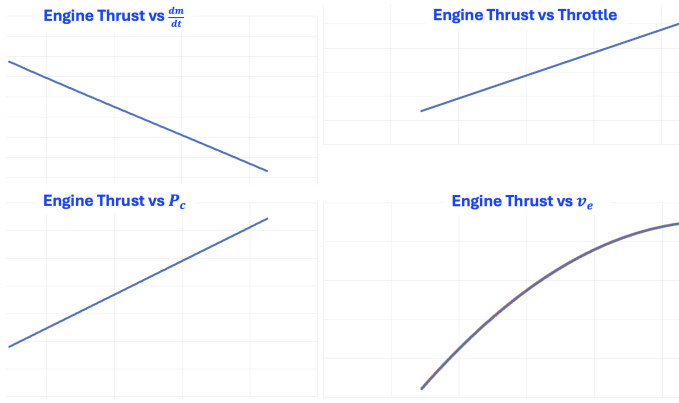


Figure 11: Sample of the characteristic curves of a rocket engine. The figure shows the relationships between engine thrust to negative mass flow, engine throttle, chamber pressure, and effective exhaust velocity v_e . Most quantities are parametrized as affine functions, while the effective exhaust velocity is parametrized as a polynomial. Notice that for this specific sample engine, the effective exhaust velocity attains its highest value at maximum throttle, representing a maximum thrust force in relation to mass flow. Therefore, the maximum engine performance and propellant efficiency is attained at maximum throttle, a feature which may not apply to all engines.

Real engine characterization is easily implemented into the WE formulation, since WE calculates the total engine thrust required to accomplish the desired energy dissipation. The thrust force is used in conjunction with the engine parametrization curves to obtain engine throttle, effective exhaust velocity, and mass flow, which in turn are incorporated back into the WE calculation cycle.

VI. MONTE-CARLO BRAKING-BURN AND APPROACH LUNAR DESCENT TRAJECTORIES

The WE formulation described in this paper has been implemented into the NASA Marshall Space Flight Center Generalized Aerospace Simulation in SIMULINK Framework (GLASS).¹⁷ GLASS is a 6DoF lunar-lander simulator that encompasses all maneuvers from lunar de-orbiting to landing and includes navigation and sensor errors in its computations. To protect sensitive proprietary data, the comparisons presented here are purely qualitative rather than quantitative; however, it is the opinion of the authors that the general trends shown are applicable to other real-world scenarios. In this work, lunar trajectories that include both braking-burn and final approach maneuvers are calculated and compared to results from other methods also implemented in the GLASS simulation, namely a Fractional-Polynomial (FP) Apollo braking-burn powered descent guidance algorithm,^{5,6} and a 6-DoF Quadratic-Programming (QP) convex optimization guidance algorithm.^{9,10}

To generate a convex optimization benchmark case, the QP algorithm is employed during flight to generate minimum time trajectories (as a proxy for minimum propellant).⁹ The algorithm is called at fixed time intervals (2 Hz due to its comp. intensity) during the descent to capture off nominal conditions and changes in mass properties. The underlying optimization problem is stated as:

$$\begin{aligned} & \min_{\mathbf{x}(t), \mathbf{u}(t)} t \\ & \text{subject to: } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) , \\ & \mathbf{x}_o = \mathbf{x}(0) , \\ & \mathbf{x} \in \mathbf{X} , \\ & \mathbf{u} \in \mathbf{U} , \\ & \mathbf{x}_{t_f} = \mathbf{x}(t_f) , \end{aligned} \quad (40)$$

where the state, $\mathbf{x}$ and input, $\mathbf{u}$, belong to the closed sets $\mathcal{X}$ and $\mathcal{U}$ respectively. To solve the underlying QP underlying optimal control problem during every call, a direct transcription is employed by making the following assumptions (for both QP and FP guidance):

1. The LS is considered an inertial frame;
2. The vehicle can be represented by rigid body equations of motion;
3. The dynamics are linearized and represented as a discrete time system;
4. The moment of inertia of the vehicle is unchanging over a single solution;
5. The system constraints can be represented as linear functions of state and control.

In contrast with both the FP and QP algorithms, the WE approach does not assume that the LS is in an inertial frame (as illustrated in Figs. 1 and 8). Therefore, the WE guidance calculation is based on the projected position of the LS at the moment of landing. Both the FP and WE are called every tenth of a second (10 Hz) to provide real-time guidance commands. While WE and QP employ the same minimum throttle threshold value and similar final target conditions during the braking-burn maneuver, while the FP employs a very different parametrization.¹⁷

To compare the FP, QP, and WE methods, a Monte-Carlo (MC) calculation that consists of 101 seeds (1 nominal case plus 100 seeds) is conducted. All methods are qualitatively compared and evaluated with respect to predetermined boundaries and limits, such as a specified radius near the LS and horizontal and vertical velocity limits at touchdown. Landing within the determined velocity limits is critical since they represent a margin of safety for the lander in order to ensure the vehicle will not topple or crash during landing.

The first comparison includes a perfect navigation and vehicle mass case, where navigation and mass-determination errors are not included. This comparison is intended to assess solely the performance of the guidance algorithms without being “contaminated” by navigation errors. The second comparison is performed using baseline navigation and mass that includes errors in both navigation and vehicle mass-determination. This allows for evaluation of the robustness and resilience of the methods when varying navigation performance is present. Figures 12, 13, and 14 show the MC results for the landing performance of the modeled lunar lander employed in this work, while Figs. 15 and 16 show the baseline Nav./Mass Braking-Burn and Approach trajectories.

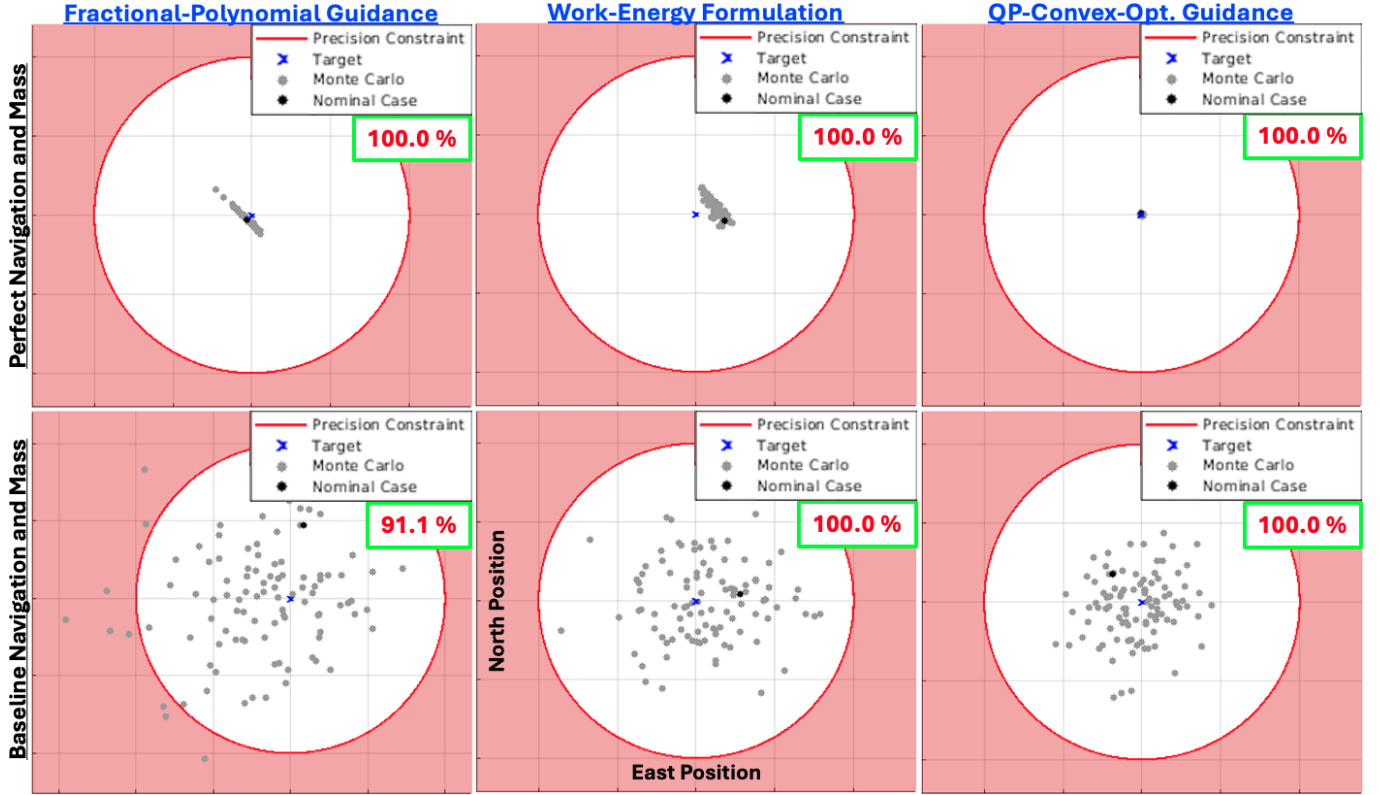


Figure 12: Position performance with respect to LS boundaries. All three methods achieve 100% when perfect Nav./Mass is present, while the FP method degrades to 91.1% when baseline Nav./Mass is present compared to both WE and QP, which both continue to achieve 100%.

It is observed that the FP method is highly sensitive to the accuracy of the navigation data when compared to WE and QP methods. As shown in Figs. 12 through 14, when the baseline navigation and mass are employed, the FP method misses a significant portion of the final landing conditions. While both WE and QP methods constantly calculate guidance commands with respect to the best estimate of the vehicle’s position, velocity, and mass, the FP method is tuned by coefficient vectors in $\mathbf{R}^3$ with respect to an ideal trajectory,^{5,6} which causes significant errors in the guidance solution when the true state of the trajectory deviates from the ideal. The navigation solution could be further improved by the implementation of the engine-based lunar navigational aid formulated in Appx. D, and which is the subject of future work.

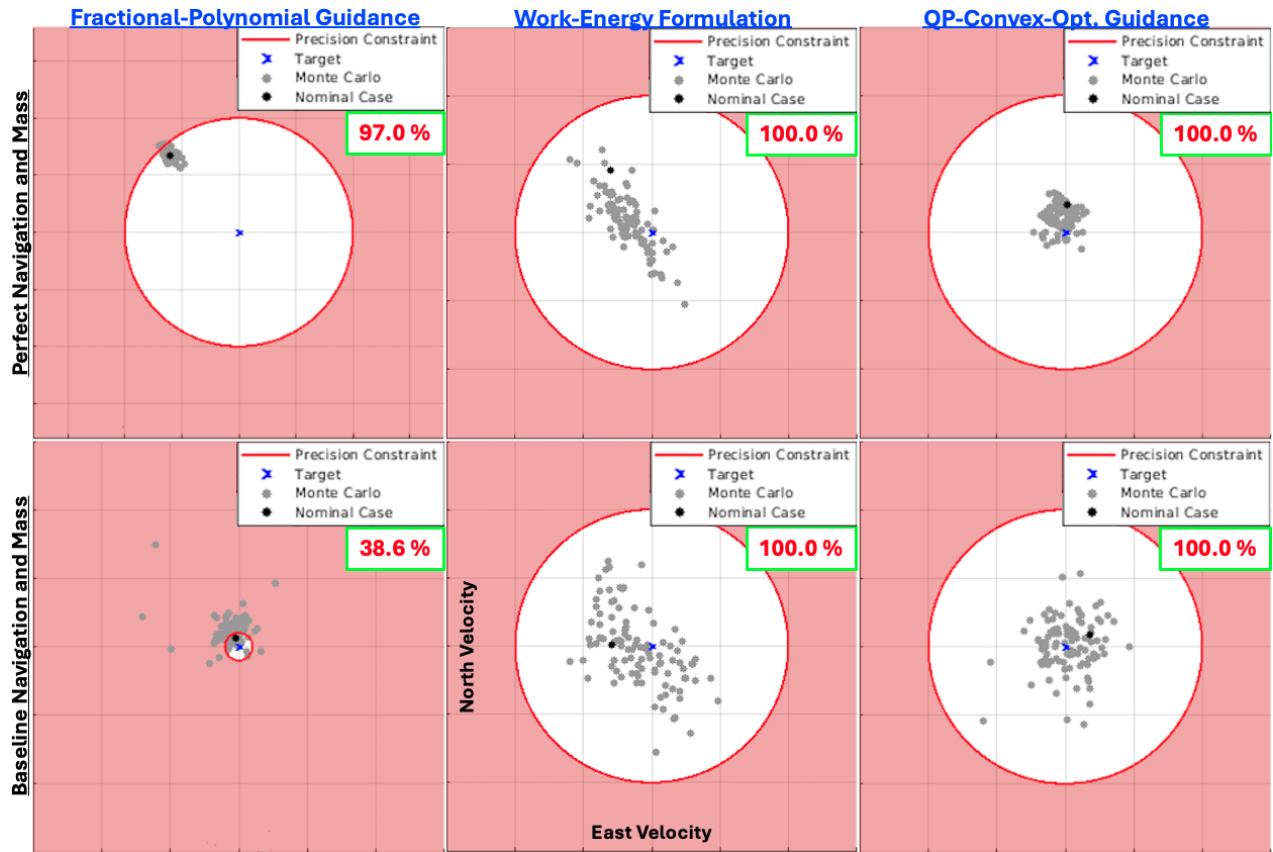


Figure 13: Horizontal velocity performance with respect to LS limits. Both WE and QP methods achieve 100% when perfect Nav./Mass is present, while FP only achieves 97%. When baseline Nav./Mass is present, the FP method further degrades to 38.6%, while both WE and QP still achieve 100%.

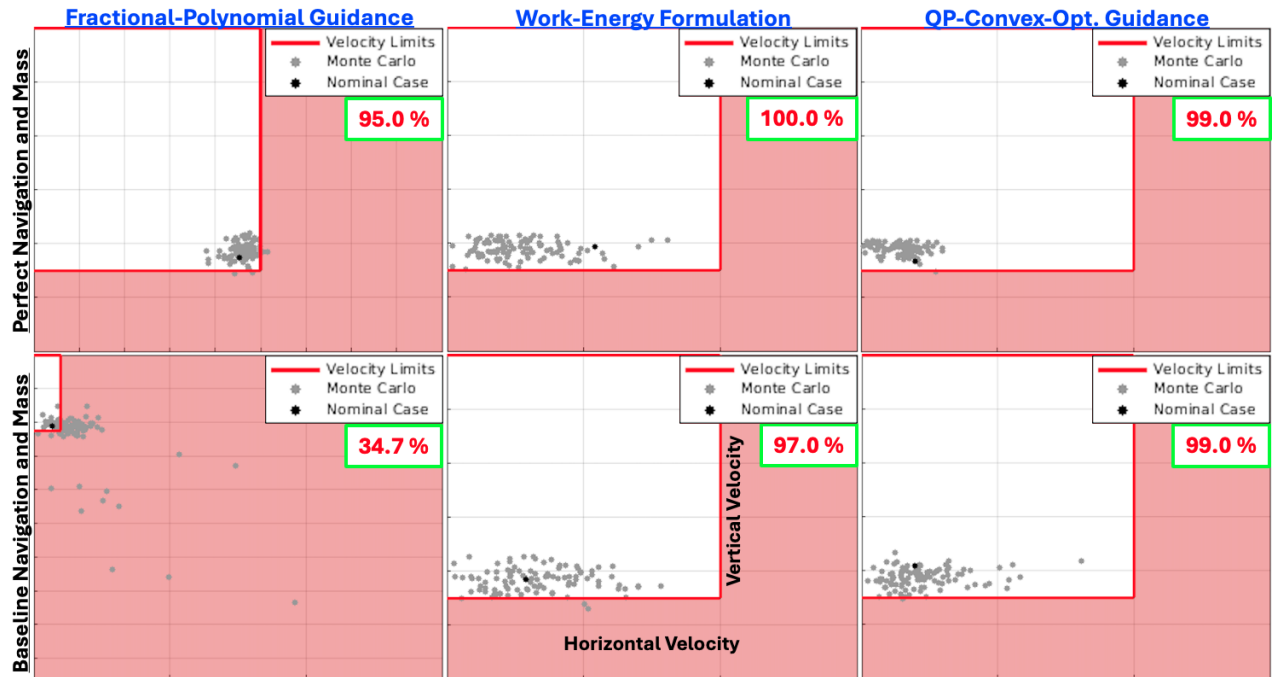


Figure 14: Horizontal vs vertical velocity performance with respect to LS limits. FP and QP methods achieve below 100% (95% and 99%) respectively when perfect Nav./Mass is present, while WE achieves 100%. When baseline Nav./Mass is present, the FP method further degrades to 34.7%, WE degrades to 97%, and QP does not change.

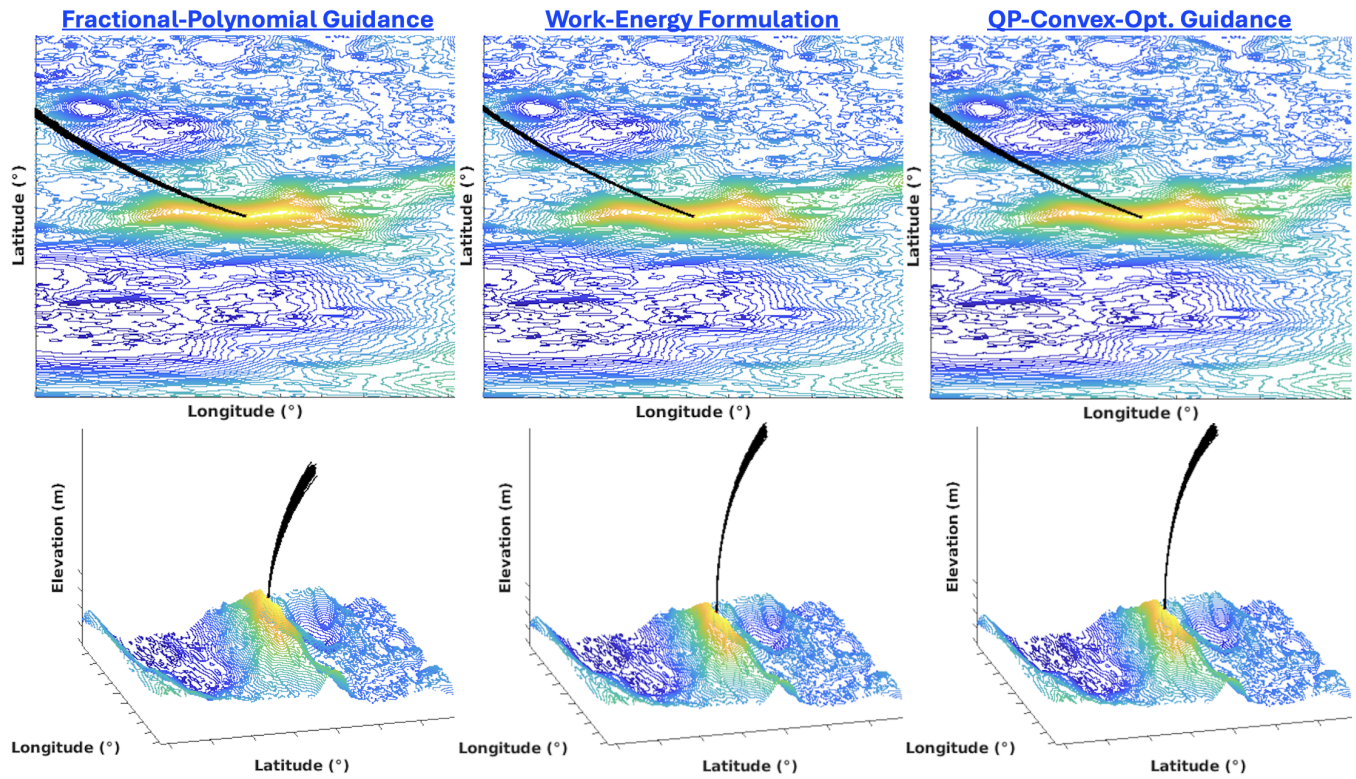


Figure 15: Braking-burn trajectories calculated using FP, WE, and QP methods. The FP trajectory has a lower target altitude in comparison to WE and QP with very similar final targets. Both WE and QP yield very similar trajectory profiles that satisfy glide-slope mission constraints.

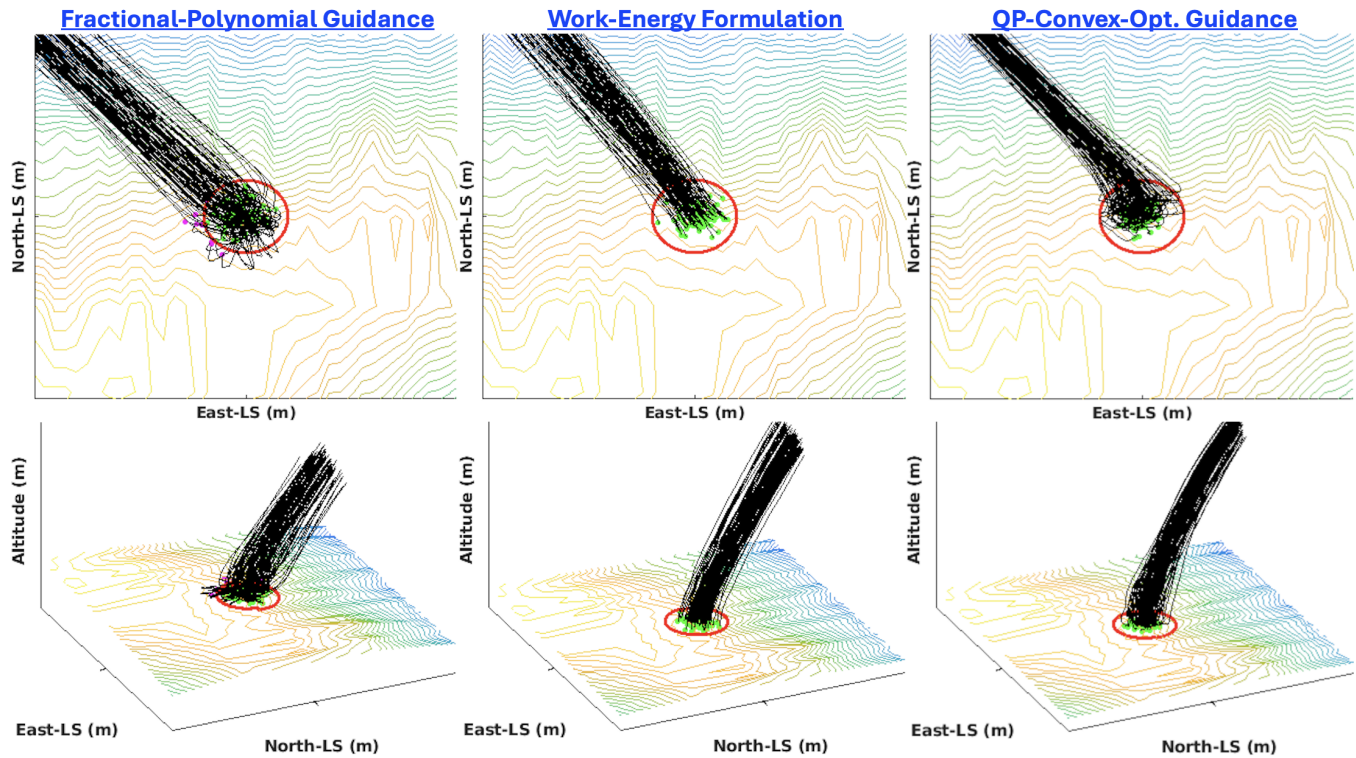


Figure 16: Approach trajectories calculated using FP, WE, and QP methods. The FP trajectory begins at a lower altitude in comparison to both WE and QP, which have very similar initial conditions, performance, and trajectory shapes.

The three methods are compared with respect to propellant consumption. The total spent propellant mass after the approach maneuver is compared to the initial propellant mass before braking-burn. In order to accurately compare the three methods, only the cases that achieved both LS position boundaries and velocity limits are included.

The FP calculation commands a lower throttle during the braking-burn, and a full throttle during most of the duration of the approach maneuver. This high maximum throttle setting yields the lowest net propellant consumption compared WE and QP, while greatly reducing control-authority, thus inhibiting the possibility of further corrections and yielding higher percentage of landing failures. As such, it is observed that FP yields the lowest performance with respect to the LS boundary limits. Figure 17 shows the net propellant consumption for all three methods.

During the braking-burn maneuver, both WE and QP employ the same minimum throttle threshold to ignite the engine. However, it is observed that WE yields a smooth monotonically increasing throttle command during braking-burn, followed by a smooth monotonically decreasing throttle command during the approach maneuver compared to QP. This smooth monotonically increasing and decreasing throttle and angular commands produced by the WE formulation have the added benefit that astronauts may easily follow them during manual control, although the plotted solution of the throttle has been omitted from this work because is considered to be sensitive proprietary data.

As discussed in Sec. V, overall propellant consumption may be reduced by increasing the throttle ignition threshold during the braking-burn maneuver; Fig. 18 shows the reduction of both consumed propellant and maneuvering time as a function of minimum throttle threshold for the WE during braking-burn.

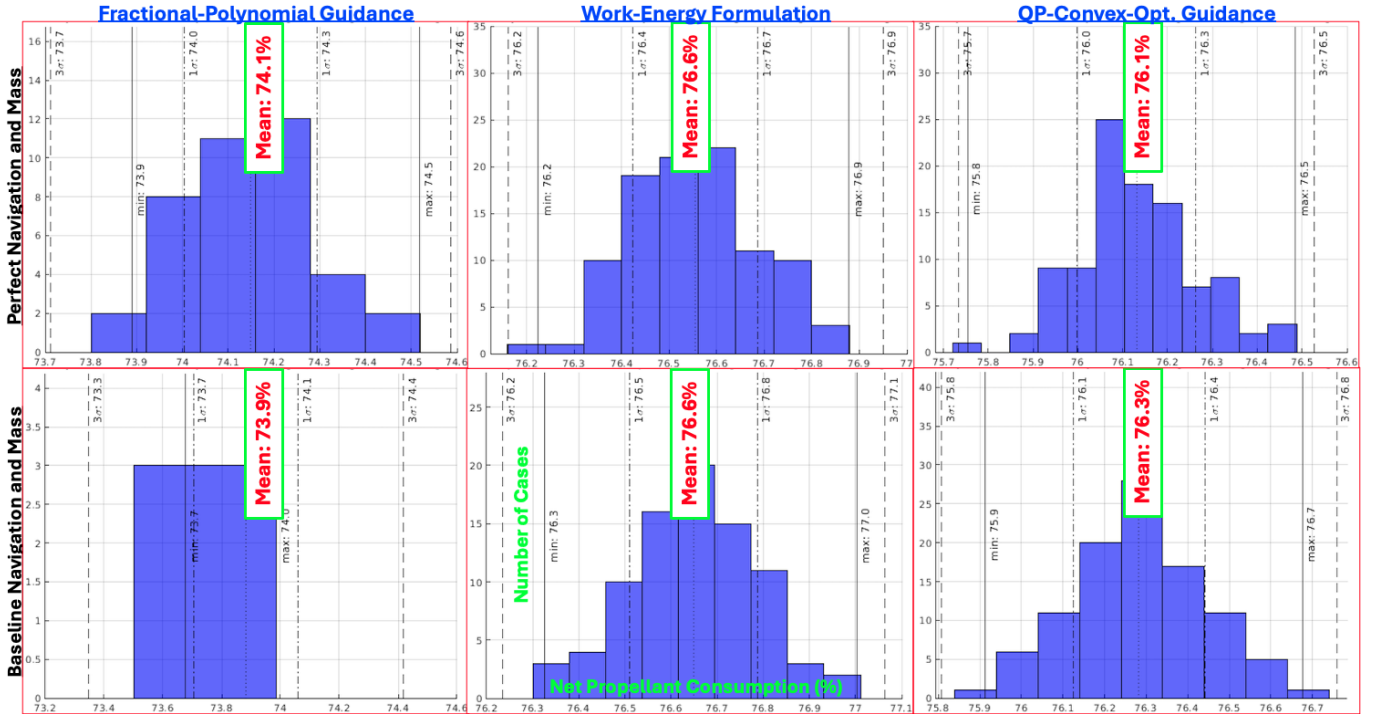


Figure 17: Net propellant consumption during braking-burn and approach maneuvers for both perfect and baseline Nav./Mass cases. Due to the maximum throttle commanded by the FP calculation during most of the approach maneuver, the FP yields both the lowest propellant consumption and the lowest performance with respect to the LS position and velocity constraints. Both WE and QP yield very similar propellant consumption with differences within 0.3%.

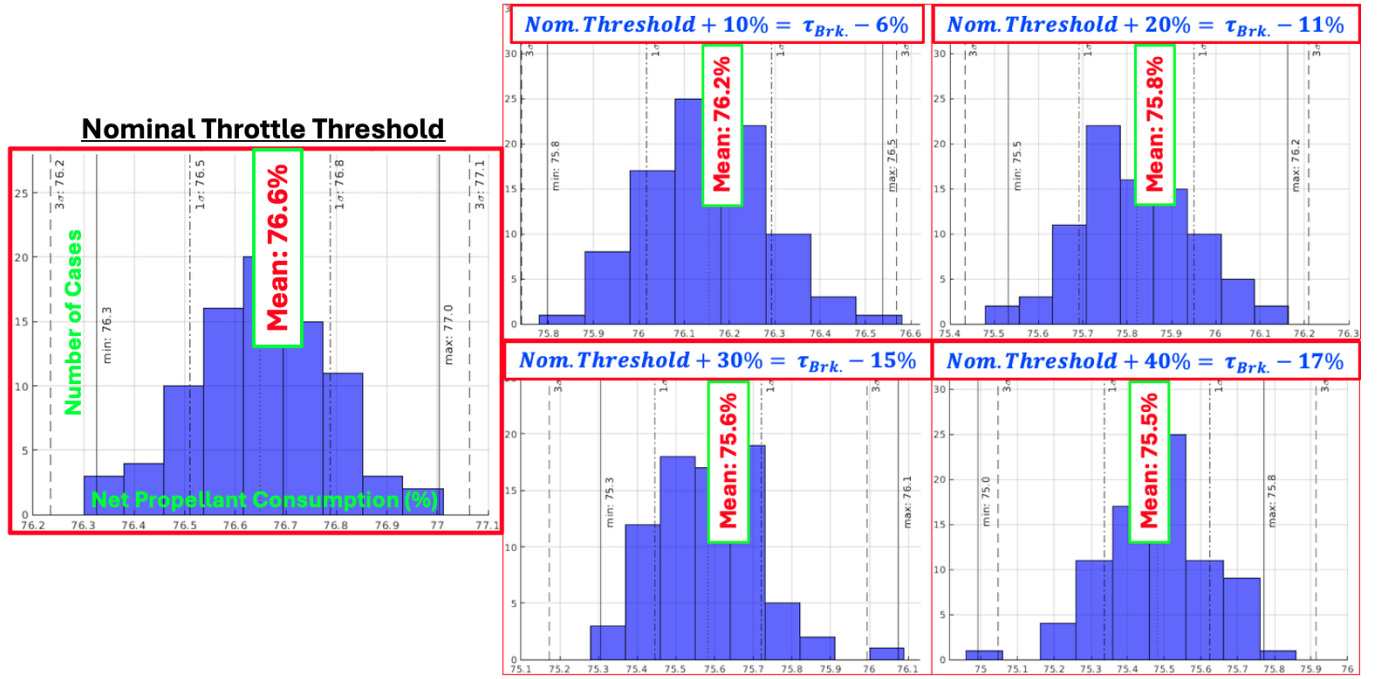


Figure 18: WE net propellant consumption as a function of minimum ignition throttle threshold. The threshold is varied in increments of 10%, with the highest threshold kept below 100%. The consumption includes both braking-burn and approach maneuvers, but the throttle threshold is only increased during braking-burn. The time reduction percentage is calculated in reference to the nominal braking-burn maneuvering time. Note that by increasing the threshold by 10%, propellant consumption is reduced by 0.4% and the maneuver time is reduced by 6%. In this example, maximum propellant reduction is obtained by increasing the threshold by 40%, thus reducing propellant consumption by 1.1% and reducing maneuvering time by 17%.

An observed source of inefficiency during the WE braking-burn maneuver, is the oscillatory nature of the optimal yaw commands as calculated in Eq. (26a) when $(x_f - x_o) \approx 0$. As shown in Eq. (26a), when $(x_f - x_o) \approx 0$ and $|v_{x_o}| > 0$, the yaw forces v_{x_o} to zero while drifting away from the cross-range centerline. Once $v_{x_o} \approx 0$, the yaw forces the vehicle to the centerline, which is repeated cyclically, leading to wasted propellant. This effect is observed during braking-burn once the vehicle reaches the centerline along the cross-range, and may explain the 0.3% propellant consumption difference between WE and QP. Implementing a low-pass filter for the yaw guidance commands once the centerline is reached during braking-burn may increase fuel efficiency, and an increase in the minimum throttle threshold during braking-burn may serve to increase efficiency even further.

Figure 19 compares the computational time (CPU time) required for WE, QP, and FP methods to calculate the first 100 guidance commands at the start of the nominal approach maneuver. For this work, the GLASS simulation is run in an AMD Ryzen Threadripper 3970X Model 49 cluster with 32-Core x86_64 processors (2 threads per core), with a maximum clock of 4549.1211 MHz (minimum 2200.0000 MHz), 256GB DDR4 Ram at 3200MHz with 43 bits of physical memory (48 bits of virtual memory). The cluster is run using Linux Ubuntu 22.04.3 LTS (Jammy Jellyfish) operative system.

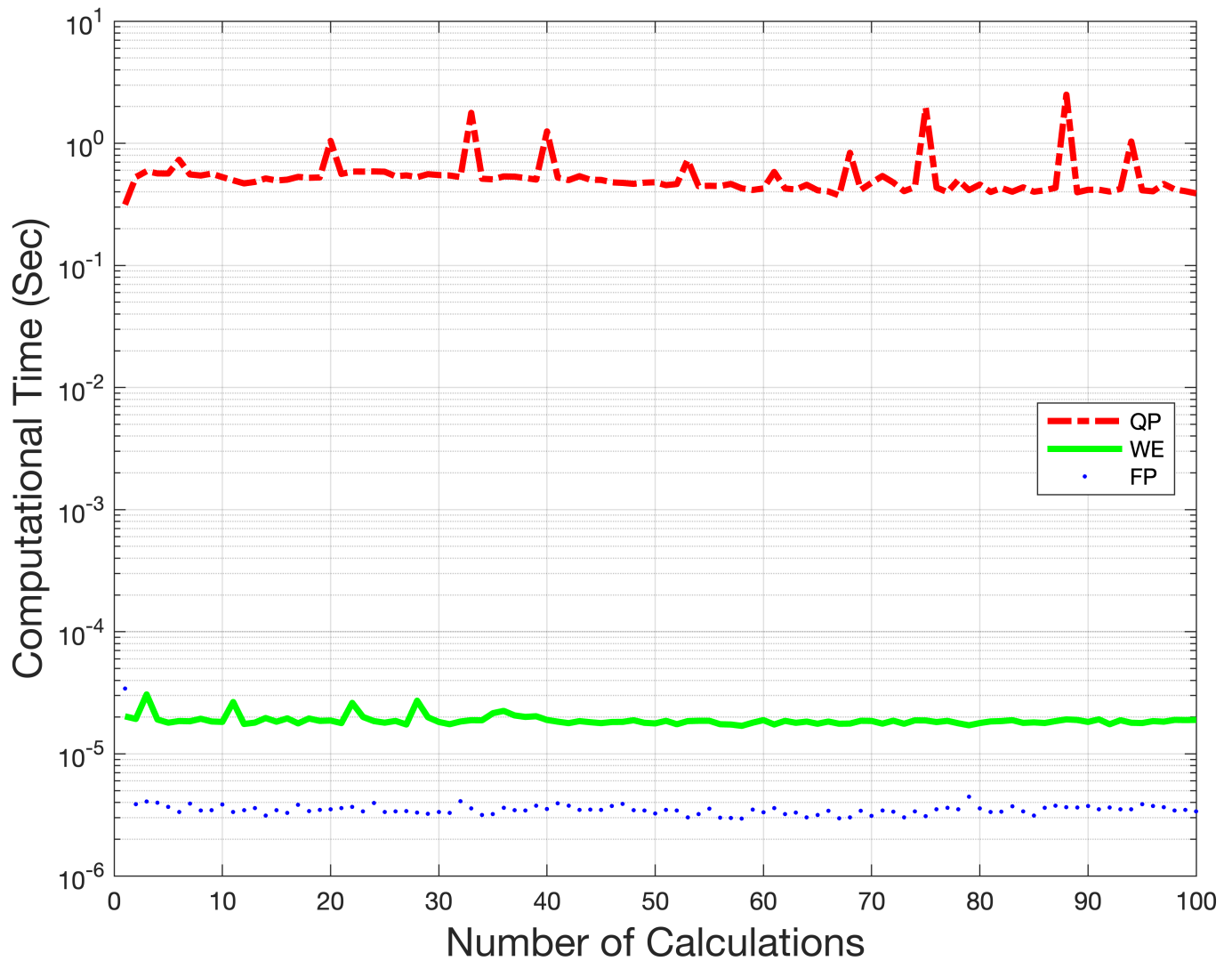


Figure 19: Comparison between the computational times required to calculate the first 100 guidance commands during the start of the nominal approach. The WE performs each of the calculations on the order of tens of a microseconds compared to the QP, which takes in between tenths of a second to over a second to complete each calculation. Due to its simplicity, the FP takes in the order of tenths of microseconds to complete each calculation. The 4-orders of magnitude difference between WE and QP indicates WE is significantly more able to calculate the real-time/high-frequency guidance commands necessary for an astronaut to follow during manual control, while still maintaining comparable performance and efficiency with QP.

VII. WORK-ENERGY-BASED REAL-TIME DIVERT-SPACE DETERMINATION

The WE formulation is also employed in real-time calculation of the divert-space around the LS and can be considered an alternate answer to the EDL Challenge Problem stated during the NESC Trajectory Optimization Workshop.¹⁸ This is efficiently accomplished by calculating an energy boundary contour map around the LS that satisfies the energy conservation principle of the vehicle at every instant of flight. A divert landing is a maneuver that is performed when astronauts may consider the current LS to be unfeasible or unsafe and an alternate LS must quickly be determined.

Figure 20 illustrates a general WE divert maneuver.

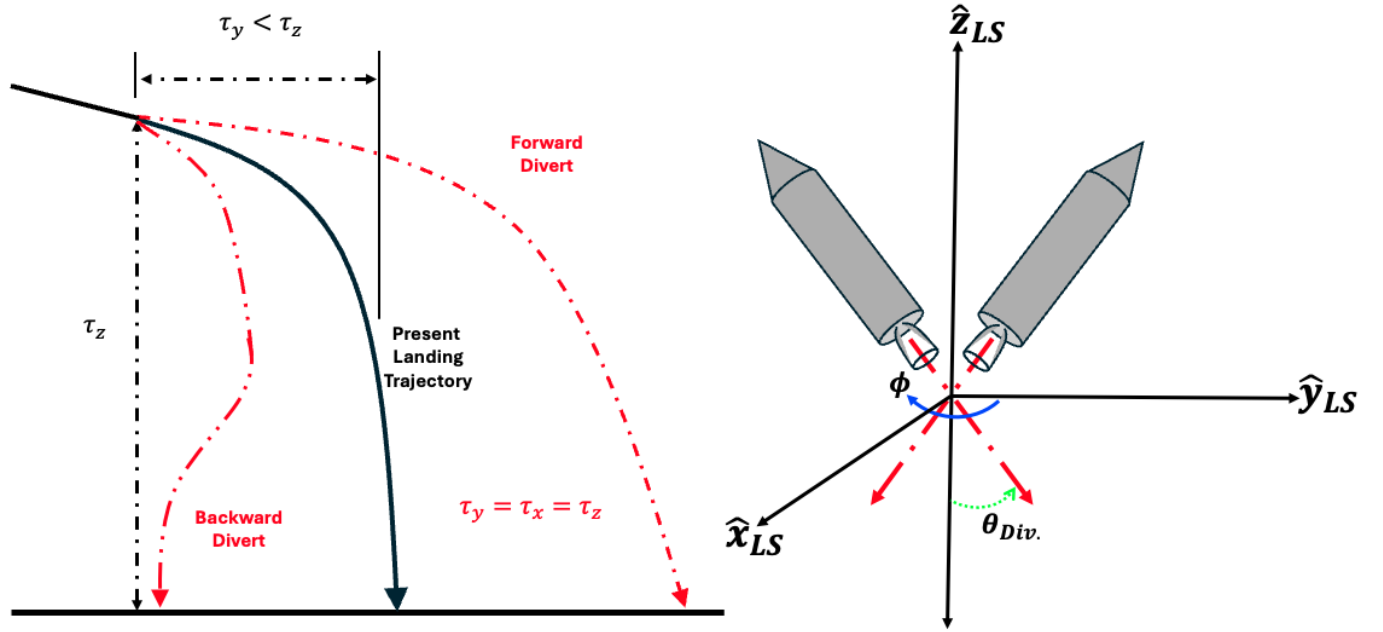


Figure 20: WE divert-space determination. The total maneuvering time along the down-range and cross-range directions must be equal to the total flight (“airborne”) time τ_z in order to achieve zero horizontal velocity at touchdown. Both total “airborne” time τ_z and optimal attitude angle $\theta_{Div.}$ are calculated while managing precise timing on the yaw control to guarantee effective energy dissipation and a safe landing.

The divert maneuver is initiated by setting the throttle to the selected maximum optimal value that satisfies both full energy dissipation and propellant constraints:

$$\frac{dm}{dt} = \dot{m} = \dot{m}_{max}. \quad (41)$$

As shown in Fig. 20, a higher throttle value translates into a higher thrust-vector, and thus a higher optimal divert angle $\theta_{Div.}$. The higher divert angle leads to a higher horizontal thrust vector, and therefore higher horizontal control authority is available to expand the divert-space, which also translates into higher propellant consumption.

The optimal divert attitude angle may be calculated simply using

$$\theta_{Div.} = \cos^{-1} \left[\frac{F_z}{F_{T_{max}}} \right], \quad (42)$$

where $F_{T_{max}}$ is the maximum optimal thrust force that corresponds to the selected maximum value of the mass flow $\dot{m}_{max}$.

Figure 21 shows the WE approach cycle used to determine the optimal divert attitude $\theta_{Div.}$, and the total flight time τ_z available to calculate the boundary of the divert maneuver.

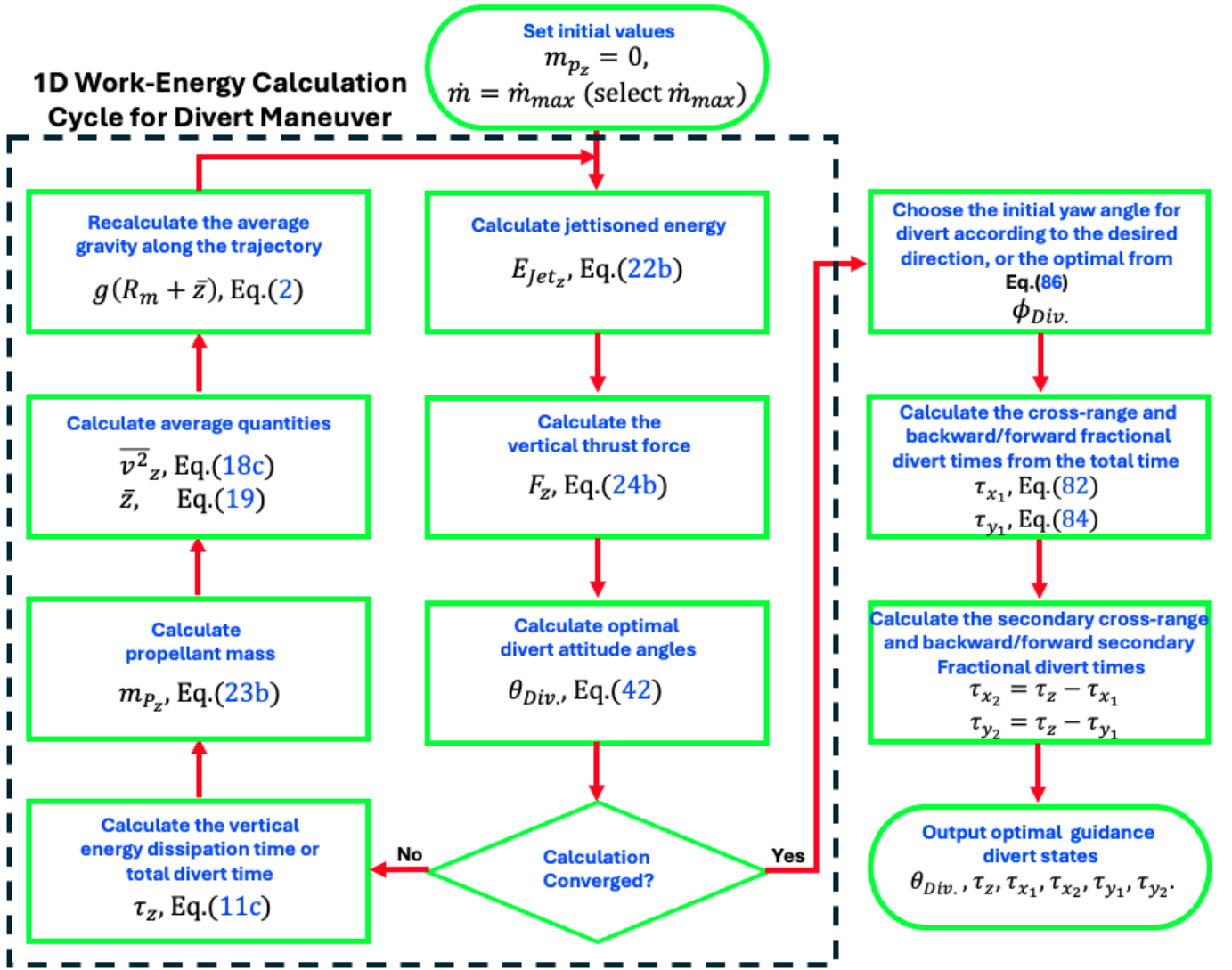


Figure 21: Divert WE calculation of the optimal attitude angle $\theta_{Div.}$, and the total available maneuvering time τ_z for the selected $\dot{m}_{max.}$. The calculation begins by selecting the maximum mass flow during divert. This provides a total flight time τ_z , which is used to calculate the required propellant during the maneuver. If propellant consumption falls above constraints, a lower $\dot{m}_{max.}$ value is chosen, which also reduces the divert range. A divert yaw angle with respect to the horizontal velocity vector $\phi_{Div.}$ is chosen according to the desired divert direction. The yaw orientation must be accurately timed with respect to the calculated τ_{x1} and τ_{y1} [Eqs. (B2) and (B4), derived in Appx. B], before reversing yaw from direction to dissipate both cross-range and down-range velocities before landing. The optimal pair of yaw angles from Eq. (B6) may be chosen to both minimize vehicle's maneuvering and maximizes divert range.

The divert-space boundary is symmetric with respect to the horizontal velocity vector at the moment the maneuver is initiated. This simplifies the calculation since initial yaw divert angles in between $0 \leq \phi_{Div.} \leq \pi$ are chosen to generate half of the the maximum divert-space contour, which can be mirrored onto $\pi < \phi_{Div.} < 2\pi$ and then mapped to the local LS coordinates.

The advantage of the WE approach for divert-space determination is that it becomes possible to calculate the entire divert space in real-time. This method is computationally efficient, and assists astronauts when switching to manual control and guiding the vehicle to a safe landing located within the boundary. Note that the divert-space contour or “energy contour map” represents the ideal maximum range possible to reach with respect to the instant energy state of the vehicle, which is not reachable in practice due to the real maneuverability of the vehicle and time delays. However, by minimizing vehicle maneuvering to reach a specific divert area, the effective range can be kept as close as possible to the ideal range. During approach, the engine thrust vector is pointing against the velocity vector of the vehicle to dissipate energy. During divert, reaching the closest set of angles in relation to the initial

velocity vector (right of left of the velocity vector) serves to minimize maneuvering. The optimal divert pair of yaw angles can be calculated using Eq. (B6), which is derived in Appx. B.

Figure 22 shows calculated divert-space boundaries (“divert energy maps”) using the WE formulation during equally delayed time intervals along a sample nominal approach trajectory.

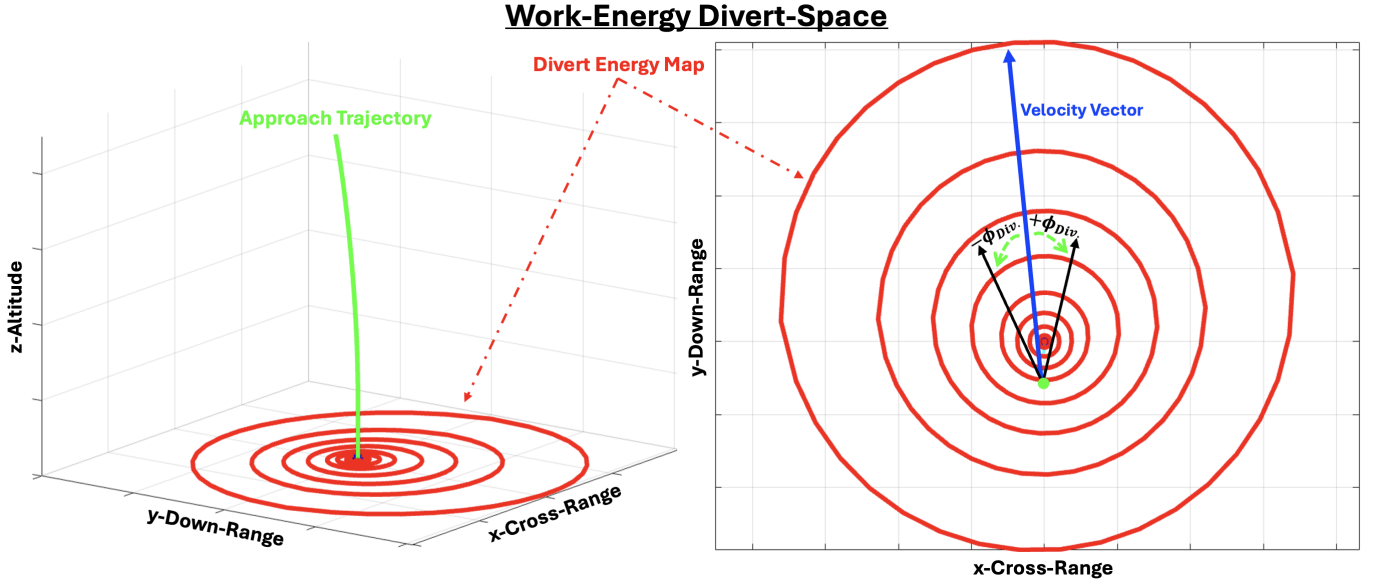


Figure 22: Divert-space boundaries calculated using the WE formulation. The figure on the left shows the 3D nominal approach trajectory (green-line), while the ovals around the LS represent the divert boundaries initiated at equally delayed time intervals along the trajectory. The right plot shows a top view of the divert boundaries that are symmetric with respect to each of their initial horizontal velocity vectors when divers are initiated. The blue arrow represents the direction of the horizontal velocity vector at the start of the approach trajectory. As expected, the divert boundaries shrink around the LS the later the divert maneuver is initiated, and become more circular as the horizontal velocity component is reduced. Note also that the divert boundary is much larger along the horizontal velocity direction, and shorter in the backward direction.

VIII. CONCLUSION

The WE formulation applies the energy conservation principle to derive very simple analytical algebraic expressions, which can be used to calculate the instant guidance commands necessary at any point during the trajectory to reach the desired final state. In this formulation, all integral equations are solved analytically and do not require any forward time-propagation, linearization, or computationally intensive linear algebra to be employed in order to calculate possible optimal reference trajectories, as is required by more complex methods such as convex optimization.

Due to the implementation of simple algebraic expressions, a very significant reduction in the computational load is attained. This allows real-time and high-frequency guidance commands to be computed, thus aiding astronauts with manual control when performing different types of maneuvers such as braking-burn,

landing, ascent, etc., as well as providing them with real-time boundary limits of the available divert-space to assist in finding alternate landing-sites during the critical final approach.

The physics-based WE formulation is intuitive with respect to vehicle dynamics and seeks to satisfy the instant work-energy conditions that guarantee a safe landing, thus achieving similar propellant consumption within 0.3% when compared with convex optimization methods. The efficiency of the WE method may also be increased further by operating at a higher frequency, which is possible due to the reduced computational complexity. The WE guidance calculation has also proved to be very robust with respect to navigation errors. Accurate navigation data is necessary to determine the total distances, altitude, and velocities required by all guidance algorithms to provide correct guidance commands (see Figs. 12, 13, and 14). Further corrections to navigation may be possible by employing

the engine-based lunar navigation as a complement to inertial-based navigation, derived in Appx. D and described by Eqs. (D1a) through (D4d).

This approach is tunable with respect to vehicle control-response, therefore the angular velocities needed to orient the vehicle into its optimal attitude and yaw may be employed as tuning parameters. Future work includes to find a more methodical way to determine turning parameters with respect to vehicle characteristics. Possible numerical singularities when reaching the final state points are easily avoided by employing simple analytical expressions. Different trajectory shapes and constraints are easily implemented using the WE formulation by dividing the trajectory into different energy waypoints and adding small corrections to the optimal attitude angle. This may be employed to include glide-slope, and/or different velocity limits and constraints.

In summary, employing the WE formulation allows a more intuitive physical understanding of the trajectory and the consequences of changes due to perturbations and manual control, eliminates the need for constant recalculation and optimization of reference trajectories, allows real-time high-frequency calculation of guidance commands at any point along the flight, and allows real-time high-frequency calculation of the boundaries of the divert-space for alternate landing.

A similar WE formulation for calculating real-time guidance commands in atmospheric conditions (such as Earth or Mars) is currently being formulated.

ACKNOWLEDGMENTS

The authors would like to acknowledge the team of Amentum Space Exploration Division (ASED), particularly Michael M. Sanders from ASED for his review of this manuscript, as well as the NASA Human Landing System team, and the entire EV42 group at MSFC.

The work presented here was funded under NASA's ESSCA Contract Number: 80MSFC18C0011.

DATA AVAILABILITY STATEMENT

The data used in this article and supplemental materials are available from the corresponding author upon reasonable request.

AUTHOR DECLARATIONS

Conflict of Interest:

The authors have no conflicts to disclose.

Appendix A: Formulation of Final Rotation Maneuvers

The transition region between the horizontal to the final vertical decent maneuver as shown in Fig. 4, is critical when trying to achieve complete horizontal energy dissipation. The goal of this final rotation maneuver is to successfully achieve vertical landing (attitude $\theta \approx 0$) with minimum horizontal velocity. From the WE perspective, this is achieved with a constant vertical force F_z calculated using Eq. (24b) as illustrated in Fig. 23.

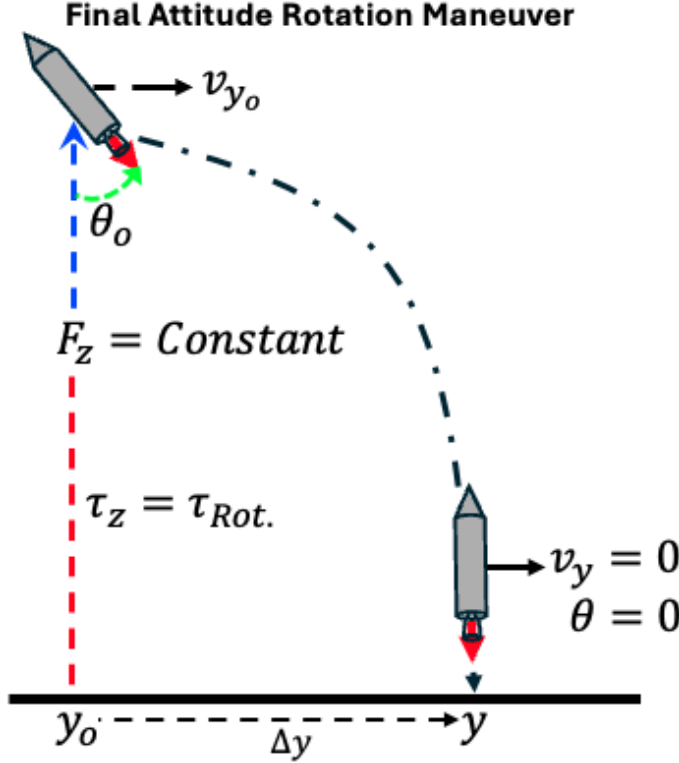


Figure 23: Final attitude rotation maneuver before landing. The maneuver starts with initial attitude θ_o and initial down-range velocity v_{y_o} . A constant vertical force F_z is necessary to dissipate the total vertical kinetic and potential energy, thus achieving a safe landing with final down-range velocity $v_y = 0$, with total displacement Δy , and final attitude angle $\theta = 0$.

In order to formulate the solution with constant rotation, the down-range acceleration from Eq. (8b) is written as

$$\frac{dv_y}{dt} = \frac{\dot{m}}{m} v_e \cos(\phi) \sin(\theta) , \quad (\text{A1})$$

where by assuming a constant vertical force F_z , Eq. (5) is written in the form

$$F_T = \frac{F_z}{\cos(\theta)} = -v_e \dot{m} , \quad (\text{A2})$$

and the mass flow is expressed as a function of the attitude angle θ as

$$\dot{m} = -\frac{F_z}{v_e \cos(\theta)} . \quad (\text{A3})$$

By substituting Eq. (A3) into Eq. (A1), and by assuming a constant angular attitude rate ω_θ

$$\frac{dv_y}{dt} = \frac{\partial v_y}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} = \omega_\theta \cdot \frac{dv_y}{d\theta} = -F_z \cdot \cos(\phi) \cdot \frac{\tan(\theta)}{m(\theta)} . \quad (\text{A4})$$

The mass as a function of the attitude angle $m(\theta)$ is obtained from Eq. (4). By setting the time $t = \frac{(\theta - \theta_o)}{\omega_\theta}$, and by employing Eq. (A3), the mass is expressed as

$$m(\theta) = m_o \cdot \left[1 - \frac{F_z}{\omega_\theta m_o v_e} \cdot \frac{(\theta - \theta_o)}{\cos(\theta)} \right] . \quad (\text{A5})$$

The difference between down-range velocities is obtained by solving and integrating Eq. (A4) with respect to the attitude angle in the form

$$v_y(\theta) - v_y(\theta_o) = -\frac{F_z}{\omega_\theta} \cdot \cos(\phi) \cdot \int_{\theta_o}^{\theta} d\theta' \cdot \frac{\tan(\theta')}{m(\theta')} . \quad (\text{A6})$$

By expanding to the first order, the reciprocal of the mass yields

$$\frac{1}{m(\theta')} \approx \frac{1}{m_o} \cdot \left[1 + \frac{F_z}{\omega_\theta m_o v_e} \cdot \frac{(\theta' - \theta_o)}{\cos(\theta')} \right] , \quad (\text{A7})$$

and the integral Eq. (A6) is expressed as a simple trigonometric integral

$$\begin{aligned} v_y(\theta) - v_y(\theta_o) = & -\frac{F_z}{\omega_\theta m_o} \cdot \cos(\phi) \\ & \times \int_{\theta_o}^{\theta} d\theta' \cdot \tan(\theta') \cdot \left[1 + \frac{F_z}{\omega_\theta m_o v_e} \cdot \frac{(\theta' - \theta_o)}{\cos(\theta')} \right] , \end{aligned} \quad (\text{A8})$$

with solution given by

$$v_y(\theta) - v_y(\theta_o) = \frac{F_z}{\omega_\theta m_o} \cdot \cos(\phi) \cdot \left\{ \ln \left[\frac{\cos(\theta)}{\cos(\theta_o)} \right] - \frac{F_z}{\omega_\theta m_o v_e} \cdot \left[\frac{(\theta - \theta_o)}{\cos(\theta)} - \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \cdot \frac{\cos(\theta)}{1 - \sin(\theta)} \right] \right] \right\}. \quad (\text{A9})$$

By setting $\theta = 0$ and $v_y(\theta) \rightarrow v_{y_o}$ [See Fig. 23], with $v_{y_o} = v_y(\theta_o)$, Eq. (A9) is simplified to

$$v_{y_o} = \frac{F_z}{\omega_\theta m_o} \cdot \cos(\phi) \cdot \left\{ \ln[\cos(\theta_o)] - \frac{F_z}{\omega_\theta m_o v_e} \cdot \left[\theta_o + \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \right] \right\}. \quad (\text{A10})$$

Equation (A10) is rewritten in the quadratic form with respect to ω_θ as

$$\omega_\theta^2 - \frac{F_z}{m_o v_{y_o}} \cdot \cos(\phi) \cdot \ln[\cos(\theta_o)] \cdot \omega_\theta + \frac{F_z^2}{m_o^2 v_{y_o} v_e} \cdot \cos(\phi) \cdot \left\{ \theta_o + \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \right\} = 0, \quad (\text{A11})$$

and by completing the square, the solution of the optimal constant attitude angular rate is given by

$$\omega_\theta = \frac{F_z}{2 \cdot m_o v_{y_o}} \cdot \left\{ \cos(\phi) \cdot \ln[\cos(\theta_o)] \pm \sqrt{\cos^2(\phi) \cdot \ln^2[\cos(\theta_o)] - 4 F_z^2 \cdot \cos(\phi) \cdot \left[\frac{v_{y_o}}{v_e} \right] \cdot \left\{ \theta_o + \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \right\}} \right\}. \quad (\text{A12})$$

A further approximation for the angular rate may be obtained by noticing that the vertical force $F_z \sim m_o \cdot g$, where g is the local value of the lunar gravity acceleration. Therefore $\frac{F_z}{m_o v_e} \sim \frac{g}{v_e} \ll 1$, and the second term on the RHS of Eq. (A10) may be ignored. The approximated angular rate is thus given by

$$\omega_\theta \approx \frac{F_z}{m_o v_{y_o}} \cdot \cos(\phi) \cdot \ln[\cos(\theta_o)]. \quad (\text{A13})$$

The estimated time for the final rotation maneuver is simply calculated using

$$\tau_{Rot.} = -\frac{\theta_o}{\omega_\theta}. \quad (\text{A14})$$

The down-range total displacement as a function of the angle is formulated from Eq. (A9) in the form

$$\frac{dy}{dt} = \frac{\partial y}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} = \omega_\theta \cdot \frac{dy}{d\theta} = v_y(\theta), \quad (\text{A15})$$

that is integrated and expressed as

$$y(\theta) - y(\theta_o) = \frac{(\theta - \theta_o)}{\omega_\theta} \cdot v_{y_o} + \frac{F_z^2}{\omega_\theta^3 m_o^2 v_e} \cdot \cos(\phi) \cdot (\theta - \theta_o) \cdot \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \cdot \frac{\cos(\theta)}{1 - \sin(\theta)} \right] + \frac{F_z}{\omega_\theta^2 m_o} \cdot \cos(\phi) \cdot \int_{\theta_o}^{\theta} d\theta' \cdot \ln \left[\frac{\cos(\theta')}{\cos(\theta_o)} \right]. \quad (\text{A16})$$

The logarithmic integral with trigonometric terms can be solved using a Taylor expansion with respect to θ_o using

$$\begin{aligned} f(\theta') &= \ln \left[\frac{\cos(\theta')}{\cos(\theta_o)} \right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(\theta_o)}{n!} \cdot (\theta' - \theta_o)^n \\ &\approx -\frac{\sin(\theta_o)}{\cos(\theta_o)} \cdot (\theta' - \theta_o) - \frac{1}{2} \cdot \frac{1}{\cos^2(\theta_o)} \cdot (\theta' - \theta_o)^2 - \frac{1}{3} \cdot \frac{\sin(\theta_o)}{\cos^2(\theta_o)} \cdot (\theta' - \theta_o)^3 \\ &\quad - \frac{1}{12} \cdot \frac{[1 + \sin^2(\theta_o)]}{\cos^3(\theta_o)} \cdot (\theta' - \theta_o)^4 - \frac{1}{60} \cdot \frac{\sin(\theta_o) \cdot [5 + \sin^2(\theta_o)]}{\cos^4(\theta_o)} \cdot (\theta' - \theta_o)^5 \\ &\quad - \frac{1}{360} \cdot \frac{[5 + 18 \cdot \sin^2(\theta_o) + \sin^4(\theta_o)]}{\cos^5(\theta_o)} \cdot (\theta' - \theta_o)^6, \end{aligned} \quad (\text{A17})$$

where in this work, the expansion is expressed as a 6th order polynomial. By substituting the truncated expansion from Eq. (A17) into Eq. (A16), and by integrating with respect to θ' , the following solution is obtained

$$\begin{aligned}
 y(\theta) - y(\theta_o) \approx & \frac{(\theta - \theta_o)}{\omega_\theta} \cdot v_{y_o} + \frac{F_z^2}{\omega_\theta^3 m_o^2 v_e} \cdot \cos(\phi) \cdot (\theta - \theta_o) \cdot \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \cdot \frac{\cos(\theta)}{1 + \sin(\theta)} \right] \\
 & - \frac{F_z}{2 \cdot \omega_\theta^2 m_o} \cdot \cos(\phi) \cdot \left\{ \frac{\sin(\theta_o)}{\cos(\theta_o)} \cdot (\theta - \theta_o)^2 + \frac{1}{3} \cdot \frac{1}{\cos^2(\theta_o)} \cdot (\theta - \theta_o)^3 \right. \\
 & + \frac{1}{6} \cdot \frac{\sin(\theta_o)}{\cos^2(\theta_o)} \cdot (\theta - \theta_o)^4 + \frac{1}{30} \cdot \frac{[1 + \sin^2(\theta_o)]}{\cos^3(\theta_o)} \cdot (\theta - \theta_o)^5 \\
 & + \frac{1}{180} \cdot \frac{\sin(\theta_o) \cdot [5 + \sin^2(\theta_o)]}{\cos^4(\theta_o)} \cdot (\theta - \theta_o)^6 \\
 & \left. + \frac{1}{1260} \cdot \frac{[5 + 18 \cdot \sin^2(\theta_o) + \sin^4(\theta_o)]}{\cos^5(\theta_o)} \cdot (\theta - \theta_o)^7 \right\}. \quad (A18)
 \end{aligned}$$

The total displacement during the final rotation maneuver is obtained when the attitude angle reaches zero ($\theta = 0$), thus giving

$$\begin{aligned}
 y(0) - y(\theta_o) \approx & -\frac{\theta_o}{\omega_\theta} \cdot v_{y_o} - \frac{F_z^2}{\omega_\theta^3 m_o^2 v_e} \cdot \cos(\phi) \cdot \theta_o \cdot \ln \left[\frac{\cos(\theta_o)}{1 + \sin(\theta_o)} \right] \\
 & - \frac{F_z}{2 \cdot \omega_\theta^2 m_o} \cdot \cos(\phi) \cdot \left\{ \frac{\sin(\theta_o)}{\cos(\theta_o)} \cdot \theta_o^2 - \frac{1}{3} \cdot \frac{1}{\cos^2(\theta_o)} \cdot \theta_o^3 + \frac{1}{6} \cdot \frac{\sin(\theta_o)}{\cos^2(\theta_o)} \cdot \theta_o^4 \right. \\
 & - \frac{1}{30} \cdot \frac{[1 + \sin^2(\theta_o)]}{\cos^3(\theta_o)} \cdot \theta_o^5 \\
 & + \frac{1}{180} \cdot \frac{\sin(\theta_o) \cdot [5 + \sin^2(\theta_o)]}{\cos^4(\theta_o)} \cdot \theta_o^6 \\
 & \left. - \frac{1}{1260} \cdot \frac{[5 + 18 \cdot \sin^2(\theta_o) + \sin^4(\theta_o)]}{\cos^5(\theta_o)} \cdot \theta_o^7 \right\}. \quad (A19)
 \end{aligned}$$

Just before landing, the WE formulation may approach a singularity when the denominator in Eq. (24a) approaches zero $[(y_f - y_o) \rightarrow 0]$ along the down-range. For the vertical component, this problem may be avoided by just keeping the force F_z at its constant previous value. In these type of conditions, it is no longer suitable to calculate the optimal approach angle using Eq. (25). Therefore, in order to drive the horizontal velocity to its final desired value, the attitude angle is directly solved from Eq. (9b) inside the same calculation cycle illustrated in Fig. 7 using

$$\theta_{opt.} = \sin^{-1} \left\{ \frac{[v_y(\tau_y) - v_{y_o}]}{v_e \cos(\phi) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_y \right]} \right\}, \quad (A20)$$

with $\tau_y = \tau_z$, and the optimal yaw angle is solved from Eq. (9a) in the form

$$\phi_{opt.} = \sin^{-1} \left\{ \frac{[v_x(\tau_x) - v_{x_o}]}{v_e \sin(\theta) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_x \right]} \right\}, \quad (A21)$$

with $\tau_x = a_x \cdot \tau_z$, where the positive constant $a_x \leq 1$, and its value is adjusted heuristically as a tuning parameter. In order not to induce oscillations into the small horizontal component of the velocity, the optimal attitude angle obtained from Eq. (A20) is truncated to a maximum angle $\theta_{Max.} = \gamma \cdot |v_{y_o}| \text{ deg.}$, with the positive constant γ obtained heuristically ($\gamma = 1.35^\circ \text{ sec/m}$ in this work).

In a similar analysis, a final yaw rotation maneuver performed during braking-burn in order to ensure that the cross-range velocity reaches zero at the center-line position of the down-range motion is formulated. During braking-burn, the energy-dissipating force is concentrated against the down-range motion of the vehicle. The constant down-range force F_y is calculated using Eq. (24a), and is illustrated in Fig. 24.

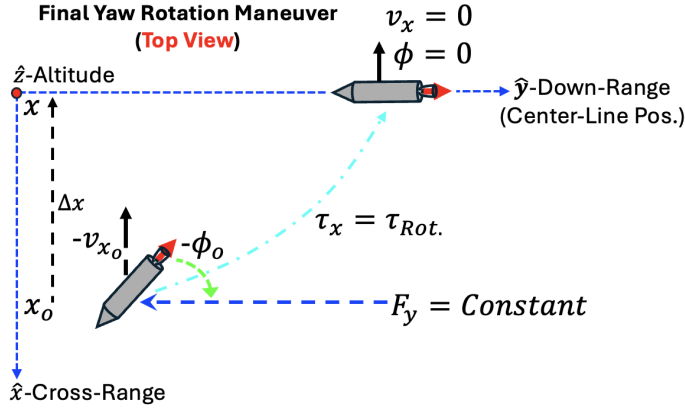


Figure 24: Final yaw rotation maneuver during braking-burn. The maneuver starts with initial yaw ϕ_o and initial cross-range velocity $-v_{x_o}$ (approximating the center-line position). A constant down-range force F_y is necessary to dissipate most of the total vertical kinetic energy in order brake the vehicle, and achieve the required velocity for the final approach maneuver. The desired final cross-range velocity is $v_x = 0$ when the yaw angle reaches $\phi = 0$, and a total displacement Δx during the rotation in order to end as close as possible to the center-line.

As done previously, the cross-range acceleration from Eq. (8a) is written in the form

$$\frac{dv_x}{dt} = \frac{\dot{m}}{m} \cdot v_e \sin(\phi) \sin(\theta) . \quad (\text{A22})$$

By assuming a constant down-range force F_y (as it is the case during braking-burn), Eq. (5) rewritten as

$$F_T = \frac{F_y}{\cos(\phi) \sin(\theta)} = -v_e \cdot \dot{m} , \quad (\text{A23})$$

with solution given by

$$v_x(\phi) - v_x(\phi_o) = \frac{F_y}{\omega_\phi m_o} \cdot \left\{ \ln \left[\frac{\cos(\phi)}{\cos(\phi_o)} \right] - \frac{F_y}{\omega_\phi m_o v_e \cdot \sin(\theta)} \cdot \left[\frac{(\phi - \phi_o)}{\cos(\phi)} - \ln \left[\frac{\cos(\phi_o)}{1 + \sin(\phi_o)} \cdot \frac{\cos(\phi)}{1 - \sin(\phi)} \right] \right] \right\} . \quad (\text{A30})$$

and the mass flow is expressed as a function of the yaw angle ϕ as

$$\dot{m} = -\frac{F_y}{v_e \cdot \cos(\phi) \sin(\theta)} . \quad (\text{A24})$$

By substituting Eq. (A24) into Eq. (A22), and by assuming a constant angular yaw rate ω_ϕ

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial \phi} \cdot \frac{\partial \phi}{\partial t} = \omega_\phi \cdot \frac{dv_x}{d\phi} = -F_y \cdot \frac{\tan(\phi)}{m(\phi)} . \quad (\text{A25})$$

The mass as a function of the yaw angle $m(\phi)$ is obtained from Eq. (4). By setting the time $t = \frac{(\phi - \phi_o)}{\omega_\phi}$, and by employing Eq. (A24), the mass is expressed as

$$m(\theta) = m_o \cdot \left[1 - \frac{F_y}{\omega_\phi m_o v_e \cdot \sin(\theta)} \cdot \frac{(\phi - \phi_o)}{\cos(\phi)} \right] . \quad (\text{A26})$$

The difference between down-range velocities is obtained by solving and integrating Eq. (A25) with respect to the attitude angle in the form

$$v_x(\phi) - v_x(\phi_o) = -\frac{F_y}{\omega_\phi} \cdot \int_{\phi_o}^{\phi} d\phi' \cdot \frac{\tan(\phi')}{m(\phi')} . \quad (\text{A27})$$

By expanding to the first order, the reciprocal of the mass yields

$$\frac{1}{m(\phi')} \approx \frac{1}{m_o} \cdot \left[1 + \frac{F_y}{\omega_\phi m_o v_e \cdot \sin(\theta)} \cdot \frac{(\phi' - \phi_o)}{\cos(\phi')} \right] , \quad (\text{A28})$$

and the integral Eq. (A27) is expressed as a simple trigonometric integral

$$v_x(\phi) - v_x(\phi_o) = -\frac{F_y}{\omega_\phi m_o} \times \int_{\phi_o}^{\phi} d\phi' \cdot \tan(\phi') \cdot \left[1 + \frac{F_y}{\omega_\phi m_o v_e \cdot \sin(\theta)} \cdot \frac{(\phi' - \phi_o)}{\cos(\phi')} \right] , \quad (\text{A29})$$

By setting $\phi = 0$ and $v_x(0) \rightarrow v_{x_o}$ [See Fig. 24], with $v_{x_o} = v_x(\phi_o)$, Eq. (A30) is simplified to

$$v_{x_o} = \frac{F_y}{\omega_\phi m_o} \cdot \left\{ \ln[\cos(\phi_o)] - \frac{F_y}{\omega_\phi m_o v_e \sin(\theta)} \cdot \left[\phi_o + \ln \left[\frac{\cos(\phi_o)}{1 + \sin(\phi_o)} \right] \right] \right\} . \quad (\text{A31})$$

Equation (A31) is rewritten in the quadratic form with respect to ω_ϕ as

$$\omega_\phi^2 - \frac{F_y}{m_o v_{x_o}} \cdot \ln[\cos(\phi_o)] \cdot \omega_\phi + \frac{F_y^2}{m_o^2 v_{x_o} v_e \sin(\theta)} \cdot \left\{ \phi_o + \ln \left[\frac{\cos(\phi_o)}{1 + \sin(\phi_o)} \right] \right\} = 0 , \quad (\text{A32})$$

and by completing the square, the solution of the optimal constant yaw angular rate is given by

$$\omega_\phi = \frac{F_y}{2 m_o v_{x_o}} \cdot \left\{ \ln[\cos(\phi_o)] \pm \sqrt{\ln^2[\cos(\phi_o)] - 4 F_y^2 \cdot \left[\frac{v_{x_o}}{v_e \sin(\theta)} \right] \cdot \left\{ \phi_o + \ln \left[\frac{\cos(\phi_o)}{1 + \sin(\phi_o)} \right] \right\}} \right\} . \quad (\text{A33})$$

A further approximation for the angular rate may be obtained for the case that the down-range force $F_y \ll m_o v_e$ in Eq. (A31) thus giving

$$\omega_\phi \approx \frac{F_y}{m_o v_{x_o}} \cdot \ln[\cos(\phi_o)] . \quad (\text{A34})$$

The down-range total displacement as a function of the angle is formulated using Eq. (A30) in the form

$$\frac{dx}{dt} = \frac{\partial x}{\partial \phi} \cdot \frac{\partial \phi}{\partial t} = \omega_\phi \cdot \frac{dx}{d\phi} = v_x(\phi) , \quad (\text{A35})$$

that is integrated and expressed as

$$\begin{aligned} x(\phi) - x(\phi_o) &= \frac{(\phi - \phi_o)}{\omega_\phi} \cdot v_{x_o} + \frac{F_y^2}{\omega_\phi^3 m_o^2 v_e \sin(\theta)} \cdot (\phi - \phi_o) \cdot \ln \left[\frac{\cos(\phi_o)}{1 + \sin(\phi_o)} \cdot \frac{\cos(\phi)}{1 + \sin(\phi)} \right] \\ &\quad + \frac{F_y}{\omega_\phi^2 m_o} \cdot \int_{\phi_o}^{\phi} d\phi' \cdot \ln \left[\frac{\cos(\phi')}{\cos(\phi_o)} \right] . \end{aligned} \quad (\text{A36})$$

As previously, the logarithmic integral with trigonometric terms is solved using the same Taylor expansion given in Eq. (A17), but with respect to the angle ϕ_o . Thus the following solution is obtained

$$\begin{aligned} x(\phi) - x(\phi_o) &\approx \frac{(\phi - \phi_o)}{\omega_\phi} \cdot v_{x_o} + \frac{F_y^2}{\omega_\phi^3 m_o^2 v_e \sin(\theta)} \cdot (\phi - \phi_o) \cdot \ln \left[\frac{\cos(\phi_o)}{1 + \sin(\phi_o)} \cdot \frac{\cos(\phi)}{1 + \sin(\phi)} \right] \\ &\quad - \frac{F_y}{2 \omega_\phi^2 m_o} \cdot \left\{ \frac{\sin(\phi_o)}{\cos(\phi_o)} \cdot (\phi - \phi_o)^2 + \frac{1}{3} \cdot \frac{1}{\cos^2(\phi_o)} \cdot (\phi - \phi_o)^3 \right. \\ &\quad + \frac{1}{6} \cdot \frac{\sin(\phi_o)}{\cos^2(\phi_o)} \cdot (\phi - \phi_o)^4 + \frac{1}{30} \cdot \frac{[1 + \sin^2(\phi_o)]}{\cos^3(\phi_o)} \cdot (\phi - \phi_o)^5 \\ &\quad + \frac{1}{180} \cdot \frac{\sin(\phi_o) \cdot [5 + \sin^2(\phi_o)]}{\cos^4(\phi_o)} \cdot (\phi - \phi_o)^6 \\ &\quad \left. + \frac{1}{1260} \cdot \frac{[5 + 18 \sin^2(\phi_o) + \sin^4(\phi_o)]}{\cos^5(\phi_o)} \cdot (\phi - \phi_o)^7 \right\} . \end{aligned} \quad (\text{A37})$$

The total displacement during the final rotation maneuver is obtained when the yaw angle reaches zero ($\phi = 0$), thus giving

$$\begin{aligned}
 x(0) - x(\phi_o) \approx & -\frac{\phi_o}{\omega_\phi} \cdot v_{x_o} - \frac{F_y^2}{\omega_\phi^3 m_o^2 v_e \sin(\theta)} \cdot \phi_o \cdot \ln \left[\frac{\cos(\phi_o)}{1 + \sin(\phi_o)} \right] \\
 & - \frac{F_y}{2 \cdot \omega_\phi^2 m_o} \cdot \left\{ \frac{\sin(\phi_o)}{\cos(\phi_o)} \cdot \phi_o^2 - \frac{1}{3} \cdot \frac{1}{\cos^2(\phi_o)} \cdot \phi_o^3 + \frac{1}{6} \cdot \frac{\sin(\phi_o)}{\cos^2(\phi_o)} \cdot \phi_o^4 \right. \\
 & - \frac{1}{30} \cdot \frac{[1 + \sin^2(\phi_o)]}{\cos^3(\phi_o)} \cdot \phi_o^5 \\
 & + \frac{1}{180} \cdot \frac{\sin(\phi_o) \cdot [5 + \sin^2(\phi_o)]}{\cos^4(\phi_o)} \cdot \phi_o^6 \\
 & \left. - \frac{1}{1260} \cdot \frac{[5 + 18 \cdot \sin^2(\phi_o) + \sin^4(\phi_o)]}{\cos^5(\phi_o)} \cdot \phi_o^7 \right\}. \quad (A38)
 \end{aligned}$$

Appendix B: Divert-Space Time Management Calculation

The boundaries of the divert-space are limited to the total vertical descent (“airborne”) time τ_z calculated using Eq. (11c), and the optimal attitude divert angle $\theta_{Div.}$ obtained from Eq. (42). When the divert maneuver is initiated, the divert-space is symmetric with respect to the latest horizontal velocity vector. Therefore, for divert calculations purposes, the cross-range position is defined as to the right/left of the current velocity vector (not with respect to the LS) defined with a yaw angle ϕ' , initial cross-range velocity $v_{x_o} = 0$, and initial down-range velocity v_{y_o} . Once the divert-space is calculated with respect to the velocity vector, it is then transformed into the LS coordinates.

In order to maneuver the lander to the maximum boundaries, the yaw angle is changed according to the desired divert direction, and the change of the yaw angle orientation must be timed precisely to guarantee a zero horizontal velocity at the time of landing. Therefore, Eq. (9a) is broken into two maneuvers with different yaw angles (pointing right/left), and times for the cross-range in the form

$$v_{x_1} = \cancel{v_{x_o}}^0 + v_e \sin(\phi'_1) \sin(\theta_{Div.}) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_{x_1} \right], \quad (B1a)$$

$$\cancel{v_{x_2}}^0 = v_{x_1} + v_e \sin(\phi'_2) \sin(\theta_{Div.}) \cdot \ln \left[1 + \frac{\dot{m}}{m_1} \cdot \tau_{x_2} \right], \quad (B1b)$$

where $\phi'_1 = \phi'$ and $\phi'_2 = -\phi'$ (or $\phi'_2 = \pm \phi' + \pi$), and $m_1 = m_o + \dot{m} \cdot \tau_{x_1}$.

By noticing that $\tau_z = \tau_{x_1} + \tau_{x_2}$ (with $\tau_{x_2} = \tau_z - \tau_{x_1}$), Eq. (B1a) is plugged into Eq. (B1b) and solved for τ_{x_1} giving

$$\tau_{x_1} = -\frac{m_o}{\dot{m}} \cdot \left[1 - \sqrt{1 + \frac{\dot{m}}{m_o} \cdot \tau_z} \right]. \quad (B2)$$

In a similar manner, Eq. (9b) is broken into two maneuvers with two different yaw angles (forward/backward)

for the down-range (forward/backward) component in the form

$$v_{y_1} = v_{y_o} + v_e \cos(\phi'_1) \sin(\theta_{Div.}) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_{y_1} \right], \quad (B3a)$$

$$\cancel{v_{y_2}}^0 = v_{y_1} + v_e \cos(\phi'_2) \sin(\theta_{Div.}) \cdot \ln \left[1 + \frac{\dot{m}}{m_1} \cdot \tau_{y_2} \right], \quad (B3b)$$

with $\phi'_1 = \phi'$ and $\phi'_2 = -\phi'$ (or $\phi'_2 = \pm \phi' + \pi$), $m_1 = m_o + \dot{m} \cdot \tau_{y_1}$, and $\tau_z = \tau_{y_1} + \tau_{y_2}$, Eq. (B3a) is plugged into Eq. (B3b) to solved for τ_{y_1} (with $\tau_{y_2} = \tau_z - \tau_{y_1}$)

$$\tau_{y_1} = -\frac{m_o}{\dot{m}} \cdot \left[1 - \sqrt{1 + \frac{m_o}{\dot{m}} \cdot \tau_z} \cdot e^{\frac{v_{y_o}}{2 v_e \cos(\phi') \sin(\theta_{Div.})}} \right]. \quad (B4)$$

The optimal divert yaw angle is the angle that naturally makes the down-range time equals to the airborne time (vertical time of flight). By setting $\tau_y = \tau_z$, and the final down-range velocity to zero [$y(\tau_y) = 0$] in Eq. (11b), the following expression is obtained

$$-\tau_z \cdot v_e \cos(\phi) \sin(\theta_{Div.}) = \frac{m_o}{\dot{m}} \cdot v_{y_o} + [y(\tau_z) - y_o]. \quad (B5)$$

By solving Eq. (B5) for the yaw angle yields

$$\phi_{Div.} = \pm \cos^{-1} \left[\frac{-v_{y_o}}{v_e \sin(\theta_{Div.}) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \tau_z \right]} \right], \quad (B6)$$

where this angle is calculated with respect to the local velocity vector when performing the divert maneuver. Therefore, the spacecraft may be stirred either to the right or to the left, preferable towards the same direction where the engine is already pointing in order to reduce the maneuver workload and to maximize the divert range.

Appendix C: LS-to-Body Angular-Rates Transformation

The attitude and yaw angles (θ and ϕ) defined in Fig. 3, greatly simplify the mathematics employed to calculate angular-rate commands. However, these angular-rates are given with respect to the local LS coordinate frame, while the lander needs to perform these rotations with respect to its body-reference frame (or navigation). In this derivation, the angular-rates in the navigation frame are separated into two simple rotations as illustrated in Fig. 25.

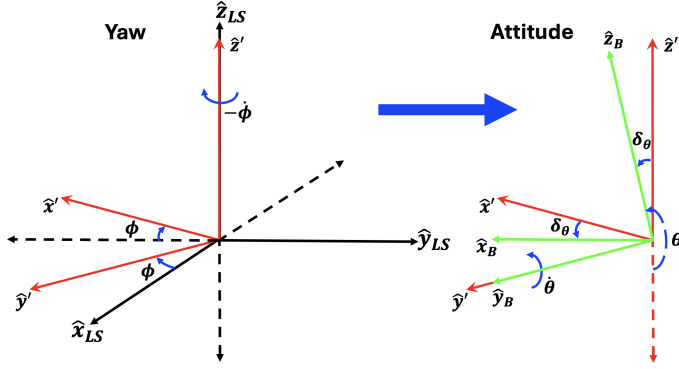


Figure 25: Yaw and attitude angular rotation-rate transformation from LS-coordinates into the body frame. First rotation (Yaw) is performed from the LS frame into the intermediate prime frame, while the second rotation (Attitude) is performed from the intermediate prime frame into the body (navigation) frame of the lander.

As shown on the left of Fig. 25, the Yaw rotation consists on a left-hand (negative) angular rotation of magnitude ϕ with respect to the $\hat{z}_{LS}$ axis, and that coincides with the intermediate prime reference $\hat{z}'$ expressed in the form

$$\omega_{LS}^{r'} = -\dot{\phi} \cdot \hat{z}_{LS} = -\dot{\phi} \cdot \hat{z}', \quad (C1)$$

with the frame-rotation from the LS frame into the intermediate prime frame represented by the matrix operation

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} -\sin(\phi) & -\cos(\phi) & 0 \\ \cos(\phi) & -\sin(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_{LS} \\ \hat{y}_{LS} \\ \hat{z}_{LS} \end{bmatrix}. \quad (C2)$$

The attitude rotation shown on the right of Fig. 25, consists on a right-hand angular rotation of magnitude δ_θ along the intermediate prime $\hat{y}'$ axis that coincides with the body $\hat{y}_B$ axis, and that is expressed in the form

$$\omega_{r'}^B = \dot{\theta} \cdot \hat{y}' = \dot{\theta} \cdot \hat{y}_B, \quad (C3)$$

with the frame-rotation from the intermediate prime frame into the body frame represented by

$$\begin{bmatrix} \hat{x}_B \\ \hat{y}_B \\ \hat{z}_B \end{bmatrix} = \begin{bmatrix} \cos(\delta_\theta) & 0 & -\sin(\delta_\theta) \\ 0 & 1 & 0 \\ \sin(\delta_\theta) & 0 & \cos(\delta_\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}. \quad (C4)$$

Therefore, the total angular-rate transformation between the LS and body is given by

$$\omega_{LS}^B = \omega_{LS}^{r'} + \omega_{r'}^B = -\dot{\phi} \cdot \hat{z}' + \dot{\theta} \cdot \hat{y}_B, \quad (C5)$$

where the prime intermediate frame is expressed in terms of the body frame by inverting the matrix from Eq. (C4)

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} \cos(\delta_\theta) & 0 & \sin(\delta_\theta) \\ 0 & 1 & 0 \\ -\sin(\delta_\theta) & 0 & \cos(\delta_\theta) \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_B \\ \hat{y}_B \\ \hat{z}_B \end{bmatrix}, \quad (C6)$$

and by expressing the $\hat{z}'$ axis in terms of the body frame yields

$$\hat{z}' = -\sin(\delta_\theta) \cdot \hat{x}_B + 0 \cdot \hat{y}_B + \cos(\delta_\theta) \cdot \hat{z}_B. \quad (C7)$$

By substituting the solution from Eq. (C7) into Eq. (C5), the angular-rates transformation is expressed in the form

$$\omega_{LS}^B = \dot{\phi} \sin(\delta_\theta) \cdot \hat{x}_B + \dot{\theta} \cdot \hat{y}_B - \dot{\phi} \cos(\delta_\theta) \cdot \hat{z}_B. \quad (C8)$$

Notice from Fig. 25, that $\delta_\theta = \theta - \frac{\pi}{2}$. Therefore, the final form of the frame transformation of the angular-rates from LS into body is given by

$$\omega_{LS}^B = -\dot{\phi} \cos(\theta) \cdot \hat{x}_B + \dot{\theta} \cdot \hat{y}_B - \dot{\phi} \sin(\theta) \cdot \hat{z}_B. \quad (C9)$$

Appendix D: Formulation of an Engine-Based Lunar Navigation

During the final descent phase, accurate guidance calculations rely on the fusion of both Inertial Navigation System (INS) and Radar Altimeter through Kalman filtering to accurately estimate the altitude and vertical velocity needed to achieve a safe landing. In case of radar blackout or failure when performing this critical maneuver, the INS integration becomes less reliable with respect to time due to the accumulation of process noise during the forward state propagation process.

In an effort to reduce error estimation and improve guidance, an engine-based altimeter to help correct the INS during final descent is suggested. While the INS is based on the integration and double integration of a drifting accelerometer data with highly accurate gyroscopic data to estimate velocity and position vectors, the engine-based altimeter is based on the same gyroscopic data combined with the expected acceleration from the well characterized operating engines instead of relying on accelerometers.

For an interval Δt with constant net engine mass flow $\dot{m}$, the vertical states are calculated using Eqs. (9c) and (10c) in the form

$$v_z = v_{z_o} - g \cdot \Delta t - v_e \cos(\theta_i) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] , \quad (\text{D1a})$$

$$z = z_o + \left[v_{z_o} + v_e \cos(\theta_i) \right] \cdot \Delta t - \frac{1}{2} \cdot g \cdot \Delta t^2 - v_e \cos(\theta_i) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] , \quad (\text{D1b})$$

where v_{z_o} , z_o , θ_o , m_o , and ω_θ are the latest vertical velocity, altitude, attitude, estimated mass, effective exhaust velocity, and attitude-rate states of the lander with standard deviations: $\sigma_{v_{z_o}}$, σ_{z_o} , σ_{θ_o} , σ_{m_o} , σ_{v_e} , σ_{ω_θ} , and σ_{ω_ϕ} .

The average attitude and yaw angles during the integration interval Δt are estimated using $\theta_i = \theta_o + \frac{1}{2} \cdot \omega_\theta \cdot \Delta t$ and $\phi_i = \phi_o + \frac{1}{2} \cdot \omega_\phi \cdot \Delta t$, and the variances for the estimated states are calculated using

$$\begin{aligned} \sigma_{v_z}^2 = & \sigma_{v_{z_o}}^2 + \sigma_g^2 \cdot \Delta t^2 + \sigma_{v_e}^2 \cdot \cos^2(\theta_i) \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] + \frac{v_e^2 \cos^2(\theta_i) \cdot \Delta t^2}{[m_o + \dot{m} \cdot \Delta t]^2} \cdot \left[\sigma_{\dot{m}}^2 + \frac{\dot{m}^2}{m_o^2} \cdot \sigma_{m_o}^2 \right] \\ & + v_e^2 \sin^2(\theta_i) \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] , \end{aligned} \quad (\text{D2a})$$

$$\begin{aligned} \sigma_z^2 = & \sigma_{z_o}^2 + \sigma_{v_{z_o}}^2 \cdot \Delta t^2 + \frac{1}{4} \cdot \sigma_g^2 \cdot \Delta t^4 + \sigma_{v_e}^2 \cdot \cos^2(\theta_i) \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 \\ & + v_e^2 \sin^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 \\ & + v_e^2 \cos^2(\theta_i) \cdot \left[\frac{\sigma_{\dot{m}}^2}{\dot{m}^2} + \frac{\sigma_{m_o}^2}{m_o^2} \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 , \end{aligned} \quad (\text{D2b})$$

with the standard deviations of the modeled gravity and mass flow represented by σ_g and $\sigma_{\dot{m}}$.

In general, the same engine-based navigation formulation may be extended to estimate both cross-range and down-range horizontal positions and velocities, which is accomplished by expressing Eqs. (9a) and (9b) together with Eqs. (10a) and (10b) in the form

$$v_x = v_{x_o} + v_e \sin(\phi_i) \sin(\theta_i) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] , \quad (\text{D3a})$$

$$v_y = v_{y_o} + v_e \cos(\phi_i) \sin(\theta_i) \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] , \quad (\text{D3b})$$

$$\begin{aligned} x = & x_o + \left[v_{x_o} - v_e \sin(\phi_i) \sin(\theta_i) \right] \cdot \Delta t \\ & + v_e \sin(\phi_i) \sin(\theta_i) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] , \end{aligned} \quad (\text{D3c})$$

$$\begin{aligned} y = & y_o + \left[v_{y_o} - v_e \cos(\phi_i) \sin(\theta_i) \right] \cdot \Delta t \\ & + v_e \cos(\phi_i) \sin(\theta_i) \cdot \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] , \end{aligned} \quad (\text{D3d})$$

with variances calculated using

$$\begin{aligned} \sigma_{v_x}^2 = \sigma_{v_{x_o}}^2 + & \left\{ \sigma_{v_e}^2 \cdot \sin^2(\phi_i) \sin^2(\theta_i) + v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \sin^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \\ & + \frac{v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \Delta t^2}{[m_o + \dot{m} \cdot \Delta t]^2} \cdot \left[\sigma_{\dot{m}}^2 + \frac{\dot{m}^2}{m_o^2} \cdot \sigma_{m_o}^2 \right], \end{aligned} \quad (D4a)$$

$$\begin{aligned} \sigma_{v_y}^2 = \sigma_{v_{y_o}}^2 + & \left\{ \sigma_{v_e}^2 \cdot \cos^2(\phi_i) \sin^2(\theta_i) + v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \cos^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \cdot \ln^2 \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \\ & + \frac{v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \Delta t^2}{[m_o + \dot{m} \cdot \Delta t]^2} \cdot \left[\sigma_{\dot{m}}^2 + \frac{\dot{m}^2}{m_o^2} \cdot \sigma_{m_o}^2 \right], \end{aligned} \quad (D4b)$$

$$\begin{aligned} \sigma_x^2 = \sigma_{x_o}^2 + \sigma_{v_{x_o}}^2 \cdot \Delta t^2 + & \left\{ \sigma_{v_e}^2 \cdot \sin^2(\phi_i) \sin^2(\theta_i) + v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \sin^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \\ & \times \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 \\ & + v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \left[\frac{\sigma_{\dot{m}}^2}{\dot{m}^2} + \frac{\sigma_{m_o}^2}{m_o^2} \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2, \end{aligned} \quad (D4c)$$

$$\begin{aligned} \sigma_y^2 = \sigma_{y_o}^2 + \sigma_{v_{y_o}}^2 \cdot \Delta t^2 + & \left\{ \sigma_{v_e}^2 \cdot \cos^2(\phi_i) \sin^2(\theta_i) + v_e^2 \sin^2(\phi_i) \sin^2(\theta_i) \cdot \left[\sigma_{\phi_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\phi}^2 \cdot \Delta t^2 \right] \right. \\ & \left. + v_e^2 \cos^2(\phi_i) \cos^2(\theta_i) \cdot \left[\sigma_{\theta_o}^2 + \frac{1}{4} \cdot \sigma_{\omega_\theta}^2 \cdot \Delta t^2 \right] \right\} \\ & \times \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2 \\ & + v_e^2 \cos^2(\phi_i) \sin^2(\theta_i) \cdot \left[\frac{\sigma_{\dot{m}}^2}{\dot{m}^2} + \frac{\sigma_{m_o}^2}{m_o^2} \right] \cdot \left\{ \Delta t - \frac{m_o}{\dot{m}} \cdot \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \cdot \ln \left[1 + \frac{\dot{m}}{m_o} \cdot \Delta t \right] \right\}^2. \end{aligned} \quad (D4d)$$

REFERENCES

- ¹A. Miele, *The calculus of variations in applied aerodynamics and flight mechanics*, In: Optimization Techniques with Applications to Aerospace Systems,” Academic Press, pp. 99-170 (1962).
[https://doi.org/10.1016/S0076-5392\(08\)62092-5](https://doi.org/10.1016/S0076-5392(08)62092-5)
- ²J. Meditch, *On the Problem of Optimal Thrust Programming for a Lunar Soft Landing*, IEEE Trans. Automat. Contr., Vol. **9**, No. 4, pp. 477-484 (1964).
<https://doi.org/10.1109/TAC.1964.1105758>
- ³A. W. Wilhite, J. Wagner, R. Tolson, and M. M. Moen, *Lunar Module Descent Mission Design*, AIAA/AAS Astrodynamics Specialist Conference and Exhibit (2012).
<https://doi.org/10.2514/6.2008-6939>
- ⁴F. Gazzola and E. M. Marchini, *The Moon Lander Optimal Control Problem Revisited*, Eng. Math., Vol. **3**, Iss. 5, pp. 1-14 (2021).
<https://doi.org/10.3934/mine.2021040>
- ⁵P. Lu, *Augmented Apollo Powered Descent Guidance*, J. Guid. Control Dyn., Vol. **42**, No. 3, pp. 447-457 (2019).
<https://doi.org/10.2514/1.G004048>
- ⁶P. Lu, *Theory of Fractional-Polynomial Powered Descent Guidance*, J. Guid. Control Dyn., Vol. **43**, No. 3, pp. 398-409 (2020).
<https://doi.org/10.2514/1.G004556>
- ⁷S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge UK (2004).
<https://doi.org/10.1017/CB09780511804441>
- ⁸D. Malyuta, T. P. Reynolds, M. Szmuk, B. Açıkmeşe, and M. Mesbahi, *Fast Trajectory Optimization via Successive Convexification for Spacecraft Rendezvous with Integer Constraints*, AIAA Scitech 2020 Forum (2020).
<https://doi.org/10.2514/6.2020-0616>
- ⁹M. Szmuk and B. Açıkmeşe, *Successive Convexification for 6-DoF Mars Rocket Powered Landing with Free-Final-Time*, AIAA Scitech Forum, Session: Entry, Descent and Landing GN&C Technology I (2018).
<https://doi.org/10.2514/6.2018-0617>
- ¹⁰T. Reynolds, D. Malyuta, M. Mesbahi, B. Açıkmeşe, and J. M. Carson, *A Real-Time Algorithm for Non-Convex Powered Descent Guidance*, AIAA Scitech Forum, Session: Entry, Descent and Landing GN&C Technology III (2020).
<https://doi.org/10.2514/6.2020-0844>
- ¹¹R. Mu, Y. Deng, and P. Wu, *Adaptive Convex Optimization Guidance for Lunar Landing*, Aerospace, **10**(7), 634 (2023).
<https://doi.org/10.3390/aerospace10070634>
- ¹²A. L. Garcia, *Numerical Methods for Physics*, Prentice Hall, Englewood Cliffs NJ (2000).
<https://archive.org/details/numericalmethods0000garc>
- ¹³D.-L. Yang, B. Xu, and L. Zhang, *Optimal Low-Thrust Spiral Trajectories Using Lyapunov-Based Guidance*, Acta Astronaut., **126**, pp. 175-185 (2016).
<http://dx.doi.org/10.1016/j.actaastro.2016.04.028>
- ¹⁴F. G. R. Lemoine, D. E. Smith, M. T. Zuber, G. A. Neumann, and D. D. Rowlands, *A 70th Degree Lunar Gravity Model (GLGM-2) from Clementine and Other Tracking Data*, J. Geophys. Res., Vol. **102**, No. E7, pp. 16339-16359 (1997).
<https://ntrs.nasa.gov/citations/19980018465>
- ¹⁵A. S. Konopliv, R. S. Park, D.-N. Yuan, S. W. Asmar, M. M. Watkins, J. G. Williams, E. Fahnestock, G. Kruizinga, M. Paik, D. Strelakov, N. Harvey, D. E. Smith, and M. T. Zuber, *The JPL Lunar Gravity Field to Spherical Harmonic Degree 660 from the GRAIL Primary Mission*, J. Geophys. Res. Planets, Vol. **118**, pp. 1415-1434 (2013).
<https://doi.org/10.1002/jgre.20097>
- ¹⁶К. Е. Тсиолковский, *Исследование Мировых Пространств Реактивными Приборами, Переиздание Работ 1903 и 1911 Г. С Некоторыми Изменениями и Дополнениями*, Калуга (1926).
<https://epizodsspace.airbase.ru/bibl/tsiolkovskiy/issledovanie/tsiolkov-issled-1926.pdf>
- ¹⁷J. Orphee, N. Olson, K. Clements, M. Hawkins, A. Summers, W. Wu, M. Fritzinger, and K. Miller, *Evaluation of Precision Landing Performance Using a Generalized Aerospace Simulation in SIMULINK Framework*, AAS 23-094 (2023).
<https://ntrs.nasa.gov/citations/20230000011>
- ¹⁸J. Pei, W. J. Elke III, A. D. Hayes, C. Proctor, J. M. Munoz-Burgos, V. Cabezas-Tapia, M. Hawkins, H. W. Bae, and S. Buckner, *NESC Trajectory Optimization Workshop - EDL Challenge Problem*, NESC Trajectory Optimization Workshop, NASA Langley Research Center (2024).
<https://ntrs.nasa.gov/citations/20250005784>