

National Aeronautics and Space Administration

NASA

A Systems Approach to Modeling Uncertain ODE Systems

Benjamin W. L. Margolis, Kenneth R. Lyons, Carlos A. Natividad
NASA Ames Research Center

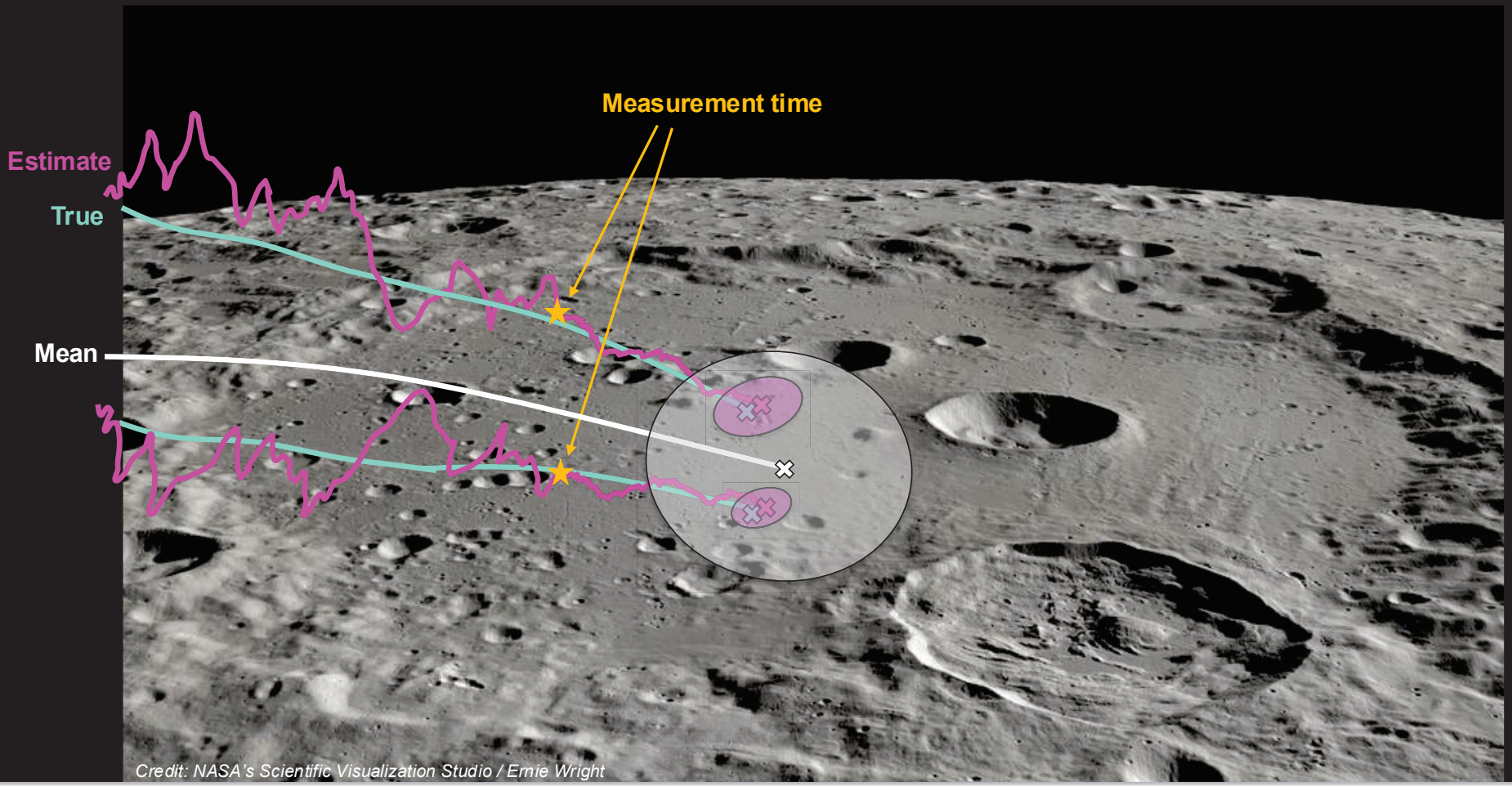
Collin E. York, David C. Woffinden
NASA Johnson Space Center

Michaela L. Tarpley
Amentum Space Exploration Division

Sagar Bhatt, Stephen R. Steffes
The Charles Stark Draper Laboratory

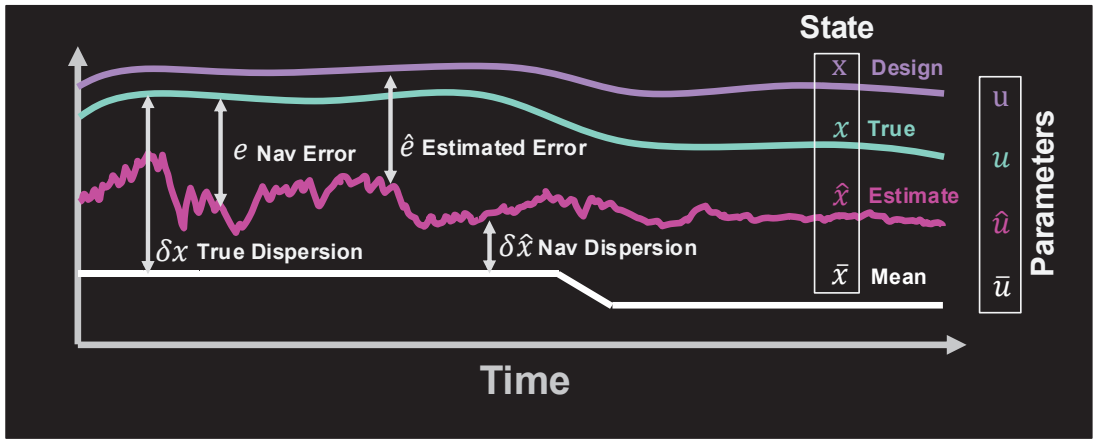
NDA-10/GNC-39
Uncertainty Quantification in Multi-Disciplinary Design
January 15, 2026, SciTech Forum

Motivation


Measurement time
Estimate
True
Mean
Credit: NASA's Scientific Visualization Studio / Ernie Wright

Uncertainty Metrics Quantifying GN&C Performance

- **Uncertainty metrics used to quantify the closed-loop GN&C performance, include:**
  - –Navigation Errors
  - –Trajectory Dispersions
- **Generally, it is the 3-standard deviation of the true navigation covariance (P) and the true dispersion covariance (D) that are displayed and used to quantify and serve as performance metrics.**



The general lunar GN&C performance metrics include:

- **True dispersion covariance ($\mathbf{D}$):** measures the variation of the true state from the mean physical state
- **Navigation dispersion covariance ($\hat{\mathbf{D}}$):** measures the variation of the estimated state from the mean physical state
- **True navigation covariance ($\mathbf{P}$):** represents the actual navigation performance
- **Onboard navigation covariance ($\hat{\mathbf{P}}$):** characterizes the flight computer's predicted navigation accuracy

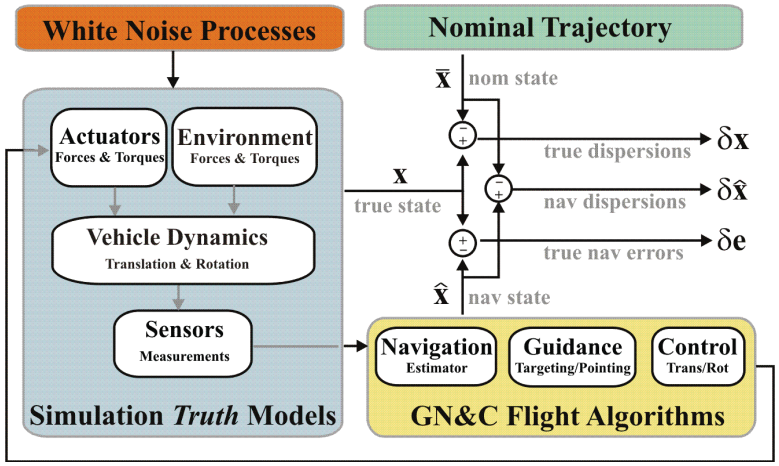
$$\delta\mathbf{x} = \mathbf{x} - \bar{\mathbf{x}}$$
$$\delta\hat{\mathbf{x}} = \hat{\mathbf{x}} - \bar{\mathbf{x}}$$
$$\delta\boldsymbol{e} = \mathbf{x} - \hat{\mathbf{x}}$$
$$\delta\hat{\boldsymbol{e}} = \mathbf{x} - \hat{\mathbf{x}}$$

$$\mathbf{D} = E\left[\delta\mathbf{x}\,\delta\mathbf{x}^{\mathrm{T}}\right]$$
$$\hat{\mathbf{D}} = E\left[\delta\hat{\mathbf{x}}\,\delta\hat{\mathbf{x}}^{\mathrm{T}}\right]$$
$$\mathbf{P} = E\left[\delta\boldsymbol{e}\,\delta\boldsymbol{e}^{\mathrm{T}}\right]$$
$$\hat{\mathbf{P}} = E\left[\delta\hat{\boldsymbol{e}}\,\delta\hat{\boldsymbol{e}}^{\mathrm{T}}\right]$$

Generating GN&C Performance Metrics

- **These performance metrics capturing the system uncertainty are typically obtained using:**
  - –Monte Carlo Analysis (MCA)
  - –Linear Covariance Analysis (LCA)
- **Both MCA and LCA methods provide similar statistical information using different but complementary approaches.**
- **Due to its speed and accuracy, LCA has shown it can reliably be incorporated into a variety of design and analysis tools**

Monte Carlo



$$\mathbf{D} \approx \frac{1}{N-1}\sum_{i=1}^{N} \delta\mathbf{x}\,\delta\mathbf{x}^{\mathrm{T}}$$

$$\hat{\mathbf{D}} \approx \frac{1}{N-1}\sum_{i=1}^{N} \delta\hat{\mathbf{x}}\,\delta\hat{\mathbf{x}}^{\mathrm{T}}$$

$$\mathbf{P} \approx \frac{1}{N-1}\sum_{i=1}^{N} \delta\boldsymbol{e}\,\delta\boldsymbol{e}^{\mathrm{T}}$$

$$\hat{\mathbf{P}} = \mathbf{P}_{\text{filter}}$$

LinCov

The performance metrics are generated by directly propagating, updating, and correcting an onboard and augmented state covariance matrix:

$$\mathbf{C} = E\left[\delta\mathbf{X}\delta\mathbf{X}^{\mathrm{T}}\right]$$

The augmented state includes the true dispersion and navigation dispersion state:

$$\delta\mathbf{X}^{\mathrm{T}} = \left[\delta\mathbf{x}^{\mathrm{T}}\ \delta\hat{\mathbf{x}}^{\mathrm{T}}\right]$$

$$\mathbf{D} = [\mathbf{I}, 0]\,\mathbf{C}\,[\mathbf{I}, 0]^{\mathrm{T}}$$

$$\hat{\mathbf{D}} = [0, \mathbf{I}]\,\mathbf{C}\,[0, \mathbf{I}]^{\mathrm{T}}$$

$$\mathbf{P} = [\mathbf{I}, -\mathbf{I}]\,\mathbf{C}\,[\mathbf{I}, -\mathbf{I}]^{\mathrm{T}}$$

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}$$

Historical modeling



Physical System | GNC System

Continuous Propagation

$$\dot{x} = f(x, \hat{u}, t) + w \qquad \dot{\hat{x}} = \hat{f}(\hat{x}, \hat{u}, \tilde{y}, t)$$
$$\tilde{y} = c(x, t) + \eta \qquad \hat{u} = \hat{g}(\hat{x}, t)$$
$$\dot{\hat{P}} = \left[\hat{F}_{\hat{x}} + \hat{F}_{\tilde{y}}\hat{C}_{\hat{x}}\right]\hat{P} + \hat{P}\left[\hat{F}_{\hat{x}} + \hat{F}_{\tilde{y}}\hat{C}_{\hat{x}}\right] + \hat{F}_{\tilde{y}}\hat{S}_{\eta}\hat{F}_{\tilde{y}}^{T} + \hat{S}_{w}$$

Impulsive State Maneuver

$$x_j^{+c} = x_j^{-c} + d\left(x_j^{-c}, \Delta\hat{u}_j, t_j\right) + \Delta w_j \qquad \hat{x}_j^{+c} = \hat{x}_j^{-c} + \hat{d}\left(\hat{x}_j^{-c}, \Delta\hat{u}_j, \Delta\tilde{y}_j, t_j\right)$$
$$\Delta\tilde{y}_j = \Delta c(x_j, t_j) + \Delta\eta_j \qquad \Delta\hat{u}_j = \Delta\hat{g}\left(\hat{x}_j^{-c}, t_j\right)$$
$$\hat{P}\left(t_j^{+c}\right) = \left[I + \hat{D}_{\hat{x}}(t_j) + \hat{D}_{\Delta\tilde{y}}(t_j)\Delta\hat{C}_{\hat{x}}(t_j)\right]\hat{P}\left(t_j^{-c}\right)\left[I + \hat{D}_{\hat{x}}(t_j) + \hat{D}_{\Delta\tilde{y}}(t_j)\Delta\hat{C}_{\hat{x}}(t_j)\right]^{T}$$
$$+ \hat{D}_{\Delta\tilde{y}}(t_j)\hat{S}_{\Delta\eta}\hat{D}_{\Delta\tilde{y}}^{T}(t_j) + \hat{S}_{\Delta w}(t_j)$$

Discrete Measurement

$$\tilde{z}_k = h(x_k, t_k) + \nu_k \qquad \hat{x}_k^{+} = \hat{x}_k^{-} + \hat{K}(t_k)\left[\tilde{z}_k - \hat{h}(\hat{x}_k, t_k)\right]$$

$$\hat{P}\left(t_k^{+}\right) = \left[I - \hat{K}(t_k)\hat{H}_{\hat{x}}(t_k)\right]\hat{P}\left(t_k^{-}\right)\left[I - \hat{K}(t_k)\hat{H}_{\hat{x}}(t_k)\right]^{T} + \hat{K}(t_k)\hat{R}_{\nu}\hat{K}^{T}(t_k)$$
$$\hat{K}(t_k) = \hat{P}(t_k)\hat{H}_{\hat{x}}^{T}(t_k)\left[\hat{H}_{\hat{x}}(t_k)\hat{P}(t_k)\hat{H}_{\hat{x}}^{T}(t_k) + \hat{R}_{\nu}(t_k)\right]^{-1}$$

Historical modeling

Augmented State Dispersion Covariance Matrix

Continuous Propagation

$$\dot{C} = \mathcal{F}C + C\mathcal{F}^T + \mathcal{G}S_\eta\mathcal{G}^T + \mathcal{W}S_w\mathcal{W}^T$$

$$\mathcal{F} = \begin{bmatrix} F_x & F_{\hat{u}}\hat{G}_{\hat{x}} \\ \hat{F}_{\tilde{y}}C_x & \hat{F}_{\hat{x}} + \hat{F}_{\hat{u}}\hat{G}_{\hat{x}} \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 0 \\ \hat{F}_{\tilde{y}} \end{bmatrix}, \text{ and } \quad \mathcal{W} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Augmented state $X = \left[x^T(t), \hat{x}^T(t)\right]^T$

Mean augmented state $\overline{X} = \left[\overline{x}^T(t), m^T(\overline{x})\right]^T$

Augmented state dispersions $\delta X \triangleq X - \overline{X}$

Augmented state dispersion covariance matrix $C = E\left[\delta X\, \delta X^T\right]$

Impulsive State Maneuver

$$C\left(t_j^{+c}\right) = \mathcal{D}_j C\left(t_j^{-c}\right)\mathcal{D}_j^T + \mathcal{M}_j S_{\Delta\eta}(t_j)\mathcal{M}_j^T + \mathcal{N}_j S_{\Delta w}(t_j)\mathcal{N}_j^T$$

$$\mathcal{D}_j = \begin{bmatrix} I + D_x(t_j) & D_{\Delta\hat{u}}(t_j)\Delta\hat{G}_{\hat{x}}(t_j) \\ \hat{D}_{\Delta\tilde{y}}(t_j)\Delta C_x(t_j) & I + \hat{D}_{\hat{x}}(t_j) + \hat{D}_{\Delta\hat{u}}(t_j)\Delta\hat{G}_{\hat{x}}(t_j) \end{bmatrix}, \quad \mathcal{M}_j = \begin{bmatrix} 0 \\ \hat{D}_{\Delta\tilde{y}}(t_j) \end{bmatrix}, \text{ and } \quad \mathcal{N}_j = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Discrete Measurement

$$C\left(t_k^+\right) = \mathcal{A}_k C\left(t_k^-\right)\mathcal{A}_k^T + \mathcal{B}_k R_\nu(t_k)\mathcal{B}_k^T$$

$$\mathcal{A}_k = \begin{bmatrix} I & 0 \\ \hat{K}(t_k)H_x(t_k) & I - \hat{K}(t_k)\hat{H}_{\hat{x}}(t_k) \end{bmatrix} \quad \text{and} \quad \mathcal{B}_k = \begin{bmatrix} 0 \\ \hat{K}(t_k) \end{bmatrix}$$

Initial Condition

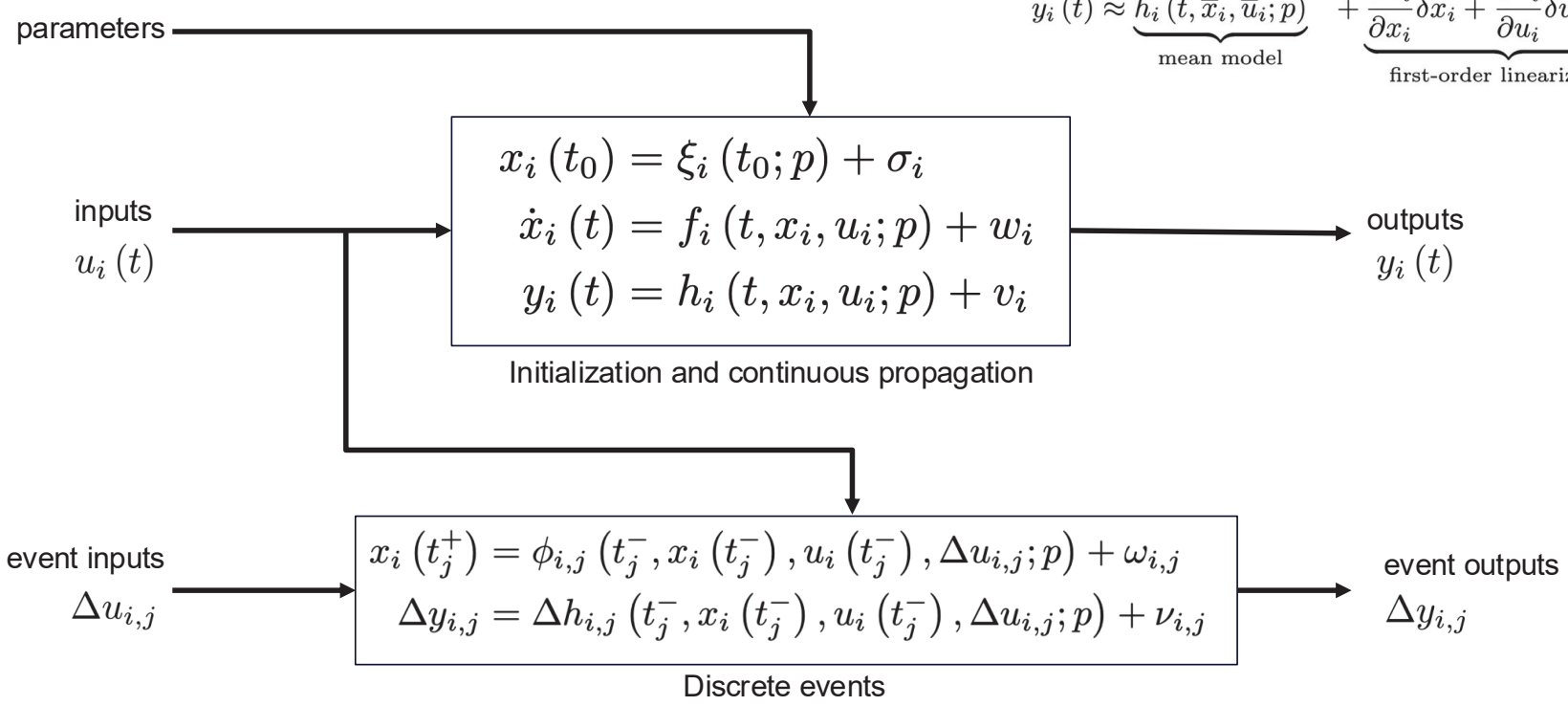
$$C(t_0) = \begin{bmatrix} D_{\text{true}} & D_{\text{true}}M_x^T \\ M_x D_{\text{true}} & M_x D_{\text{true}} M_x^T + P_{\text{true}} \end{bmatrix}$$

A Systems Approach

- **To facilitate flexible modeling, a systems approach was developed so physical and filter prediction models could be built-up modularly**
- **Uncertain ODE System characterized by**

$$\dot{x}_i(t) \approx f_i(t, \overline{x}_i, \overline{u}_i; p) + \frac{\partial f_i}{\partial x_i}\delta x_i + \frac{\partial f_i}{\partial u_i}\delta u_i + w_i$$

$$y_i(t) \approx \underbrace{h_i(t, \overline{x}_i, \overline{u}_i; p)}_{\text{mean model}} + \underbrace{\frac{\partial h_i}{\partial x_i}\delta x_i + \frac{\partial h_i}{\partial u_i}\delta u_i + w_i}_{\text{first-order linearization}}$$



Uncertain Block Diagram Algebra

When systems (or events) are connected, upstream output noise propagates to downstream process (or update) noise

$$\begin{aligned} x_i(t_0) &= \xi_i(t_0;p) + \sigma_i \\ \dot{x}_i(t) &= f_i(t, x_i, u_i; p) + w_i \\ y_i(t) &= h_i(t, x_i, u_i; p) + v_i \end{aligned} \quad \xrightarrow{u_j = g_j(y_i)} \quad \begin{aligned} x_j(t_0) &= \xi_j(t_0;p) + \sigma_j \\ \dot{x}_j(t) &= f_j(t, x_j, u_j; p) + w_j \\ y_j(t) &= h_j(t, x_j, u_j; p) + v_j \end{aligned}$$

$$x(t) = \begin{bmatrix} x_i(t) \\ x_j(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} f_i(t, x_i, u_i; p) + w_i \\ f_j(t, x_j, g_j(y_i); p) + w_j \end{bmatrix}$$

$$\dot{x}(t) \approx \begin{bmatrix} f_i(t, \overline{x}_i, \overline{u}_i; p) \\ f_j(t, \overline{x}_j, g_j(\overline{y}_i); p) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_i}{\partial x_i} & 0 \\ \frac{\partial f_j}{\partial u_j}\frac{\partial g_j}{\partial y_i}\left(I - \frac{\partial h_j}{\partial u_j}\frac{\partial g_j}{\partial y_i}\right)^{-1}\frac{\partial h_i}{\partial x_i} & \frac{\partial f_j}{\partial x_j} \end{bmatrix} \delta x + \begin{bmatrix} w_i \\ w_j + \underbrace{\frac{\partial f_j}{\partial u_j}\frac{\partial g_j}{\partial y_i}\left(I - \frac{\partial h_j}{\partial u_j}\frac{\partial g_j}{\partial y_i}\right)^{-1} v_i}_{\text{propagated noise}} \end{bmatrix}$$

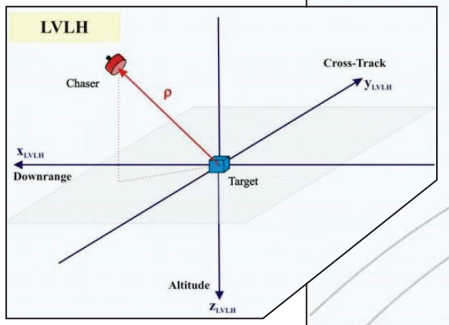
Example Scenario: Low Earth Orbit Rendezvous

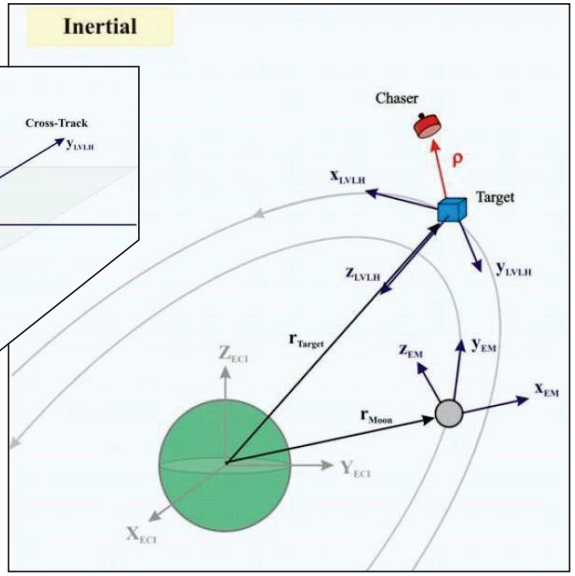


target

$$u_{\text{grav}} = [p_{\text{grav}}]$$
$$y_{\text{grav}} = [g(p_{\text{grav}})] = \left[-\mu_{\text{Earth}} \frac{p_{\text{grav}}}{|p_{\text{grav}}|^3}\right]$$

$$x_{\text{vehicle}} = \begin{bmatrix} p_{\text{vehicle}} \\ v_{\text{vehicle}} \end{bmatrix}$$
$$u_{\text{vehicle}} = \begin{bmatrix} g_{\text{vehicle}} \\ a_{\text{vehicle}} \end{bmatrix}$$
$$\dot{x}_{\text{vehicle}} = \begin{bmatrix} v_{\text{vehicle}} \\ g_{\text{vehicle}} + a_{\text{vehicle}} + w_{\text{vehicle}} \end{bmatrix}$$





chaser

$$u_{\text{Sband}} = \left[p_{\text{target}}^T, v_{\text{target}}^T, p_{\text{chaser}}^T, v_{\text{chaser}}^T\right]^T$$
$$\Delta y_{\text{Sband}} = \begin{bmatrix} |p_{\text{target}} - p_{\text{chaser}}| + \nu_{\text{range}} \\ \frac{(p_{\text{target}} - p_{\text{chaser}})^T (v_{\text{target}} - v_{\text{chaser}})}{|p_{\text{target}} - p_{\text{chaser}}|} + \nu_{\text{range-rate}} \end{bmatrix}$$

$$u_{\text{grav}} = [p_{\text{grav}}]$$
$$y_{\text{grav}} = [g(p_{\text{grav}})] = \left[-\mu_{\text{Earth}} \frac{p_{\text{grav}}}{|p_{\text{grav}}|^3}\right]$$

$$x_{\text{vehicle}} = \begin{bmatrix} p_{\text{vehicle}} \\ v_{\text{vehicle}} \end{bmatrix}$$
$$u_{\text{vehicle}} = \begin{bmatrix} g_{\text{vehicle}} \\ a_{\text{vehicle}} \end{bmatrix}$$
$$\dot{x}_{\text{vehicle}} = \begin{bmatrix} v_{\text{vehicle}} \\ g_{\text{vehicle}} + a_{\text{vehicle}} + w_{\text{vehicle}} \end{bmatrix}$$

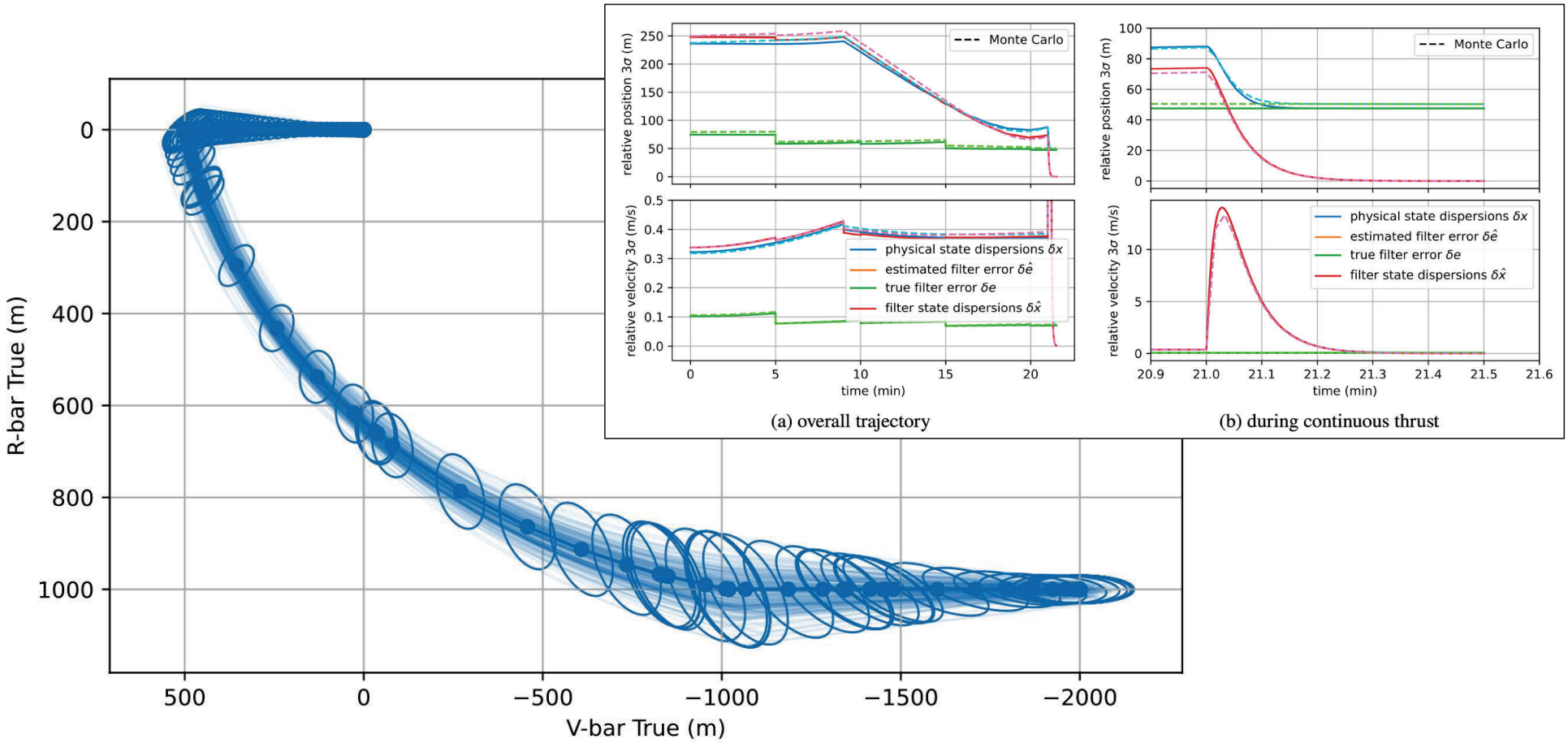
$$x_{\text{thruster}} = \begin{bmatrix} b \\ s \\ \psi \end{bmatrix}$$
$$u_{\text{thruster}} = [\hat{a}]$$
$$y_{\text{thruster}} = [(I + [s\backslash])(I - [\psi\times])\hat{a} + b + \eta]$$

measurement

GNC system

$$x_{\text{GNC}} = \left[\hat{p}^T \ \hat{v}^T\right]^T = [p_x \ p_y \ p_z \ v_x \ v_y \ v_z]^T$$
$$u_{\text{GNC}} = [T_x \ T_y \ T_z]^T$$
$$\dot{x}_{\text{GNC}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 2\omega v_z + T_x \\ -\omega^2 p_y + T_y \\ 3\omega^2 p_z - 2\omega v_x + T_z \end{bmatrix}$$
$$y_{\text{GNC}} = [-K_p(\hat{p}(t) - p_r(t)) - K_v(\hat{v}(t) - v_r(t))]$$

Scenario Results



Conclusion



Developed uncertain block diagram algebra for Swift Integrated GNC Mission Analysis

Implemented in code using Condor, a mathematical modeling framework for engineers with deadlines

(NASA 2025 Software of the Year)

Demonstrated a scenario showing…

- Dual inertial physical models
- Relative motion GNC system model with nonlinear mapping from physical state
- Discrete measurements with navigation updates
- Time delays between targeting and maneuver execution
- Impulsive maneuver execution
- Continuous thrust governed by GNC system control law

Evaluated using both Linear Covariance Analysis and Monte Carlo Analysis