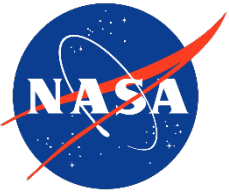


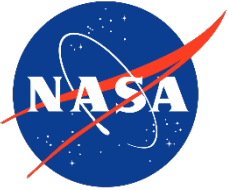
1pPA7: Solver for complex-valued scattering wavenumber for sonic booms in turbulence

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May 11, 2026

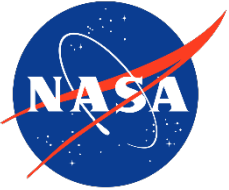


Multiple scattering theory (MST) has recently been applied to sonic boom propagation through atmospheric turbulence to predict the ensemble-averaged waveform [J. Acoust. Soc. Am. 154, 3078–3088 (2023)]. In that work, the mean acoustic wavefield was expressed in terms of the acousto-turbulent self-energy, whose contribution modifies the effective wavenumber, with the dispersion relation defined by the zeros of the mean-field denominator. The present work focuses on obtaining these scattering wavenumbers numerically in the presence of both velocity and sound-speed fluctuations, a combination that cannot be treated analytically within the original formulation. A Newton–Raphson iteration scheme is used to solve the complex dispersion relation for exponential turbulence, yielding the turbulence-modified wavenumber predicted by MST. This work also extends the formulation to von Kármán turbulence, which was not treated in the original study, enabling evaluation of scattering behavior for an additional turbulence spectrum. The resulting scattering wavenumbers provide a basis for comparing theoretical predictions with previous N-wave sonic boom waveforms obtained in a flight test, and for quantifying how turbulence influences the mean field and the associated uncertainty in sonic boom loudness.

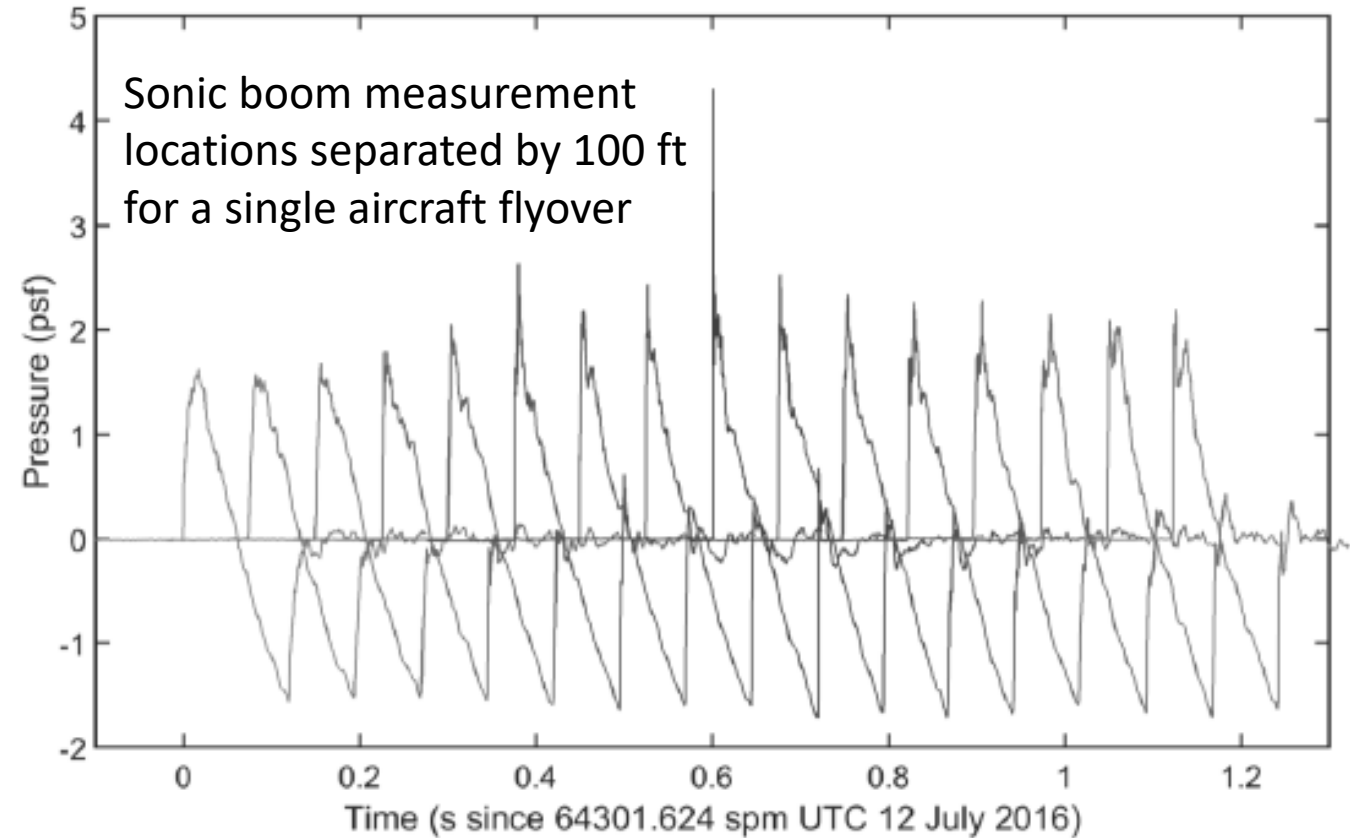
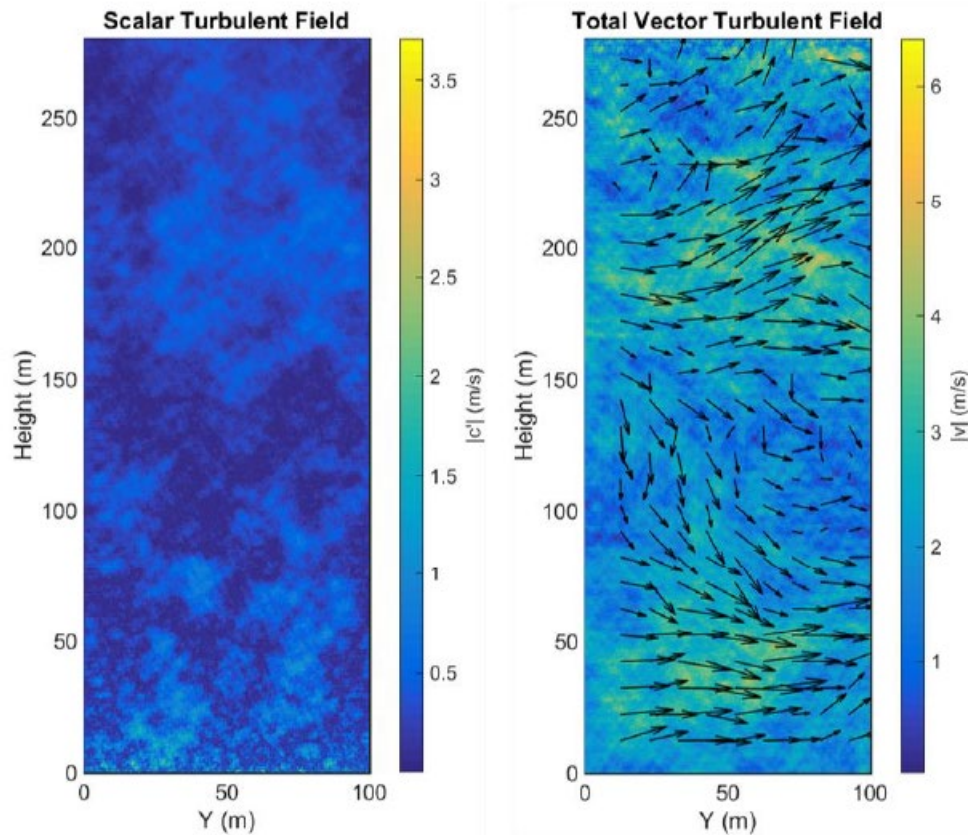


- **Sonic boom pressure waveforms are affected by turbulence near earth's surface**
- **Current modeling approaches ignore backscattering of acoustic energy from turbulence**
- **Multiple scattering theory (MST) is used to obtain the mean waveform in turbulence**
 - Accounts for backscattering
- **This paper describes a solver for the MST interaction energy between turbulent and acoustic fields**
 - Exponential turbulence spectrum
- **Discussion of von Kármán turbulence spectrum solution**
- **Backscattering attenuation can be incorporated into the PCBoom Burgers solver**

Turbulence effects on sonic boom

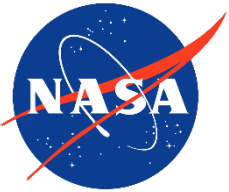


➤ Random focusing and defocusing of waveforms



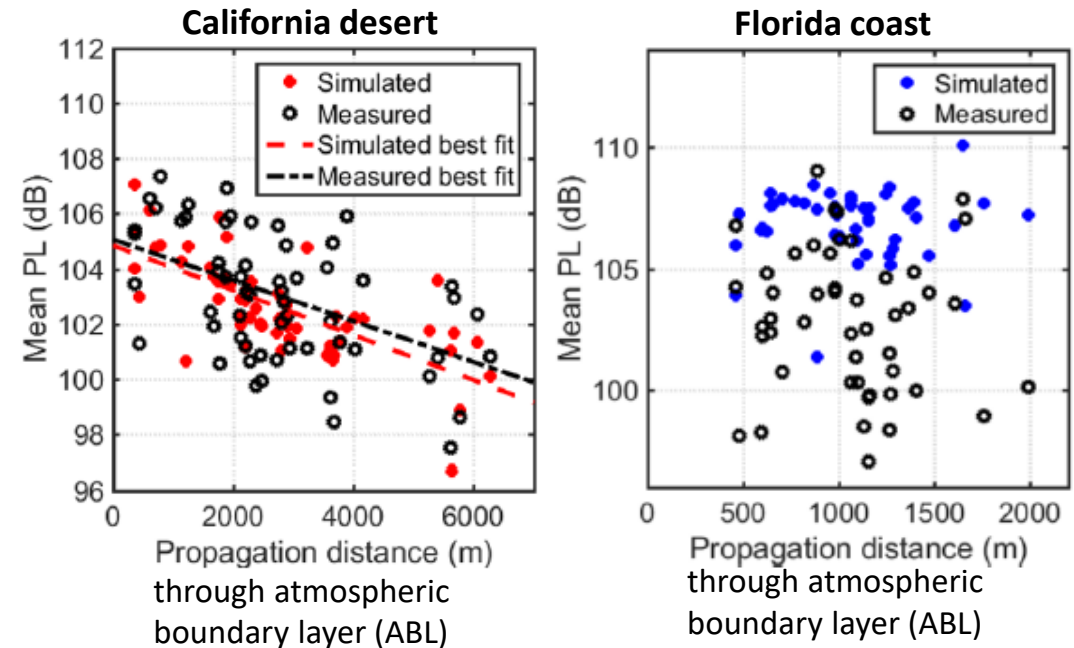
Credit: NASA/TM-2020000002482

Current turbulent propagation model limitations



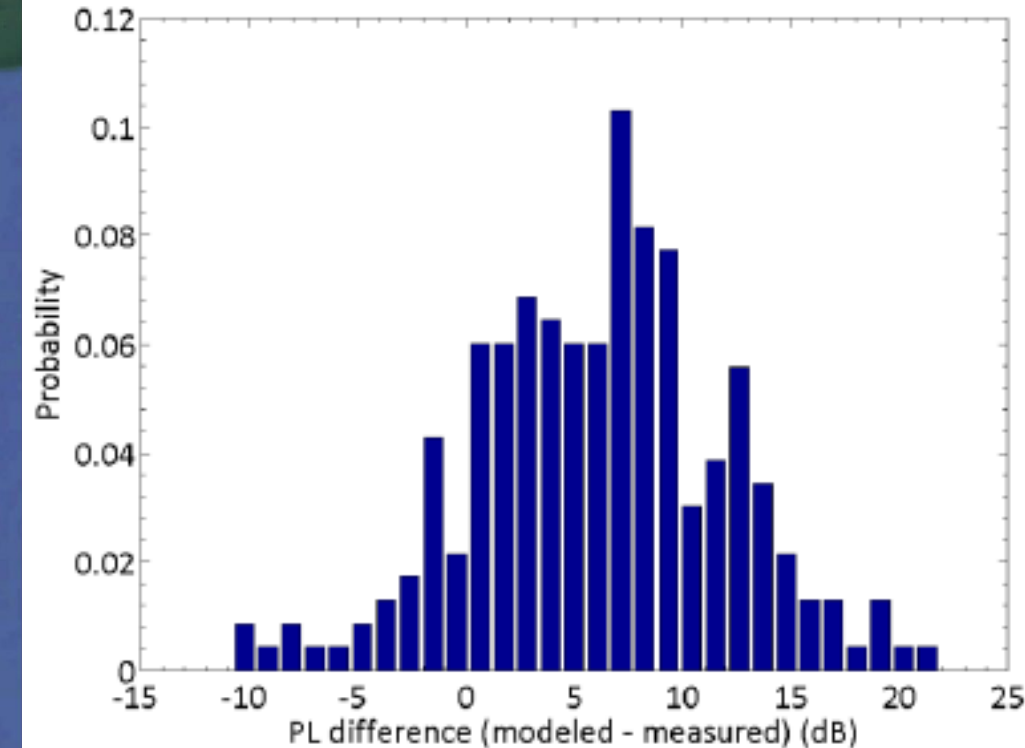
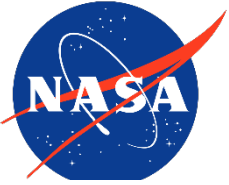
- **KZK and nonlinear FLHOWARD solvers are “one-way” equations**
 - Account for absorption, but not backscattering
- **How significant is backscattering?**
 - Additional loss mechanism
 - Enables SODAR, for example

SonicBAT Flight Test



Credit: NASA/CR-2020-220509

Example from previous flight test in Galveston, TX 2018



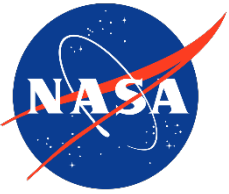
- Shallow arrival angles → long propagation path through turbulent boundary layer → rounded N-waves

➤ PCBoom 6 was consistently overpredicting PL

- Many modeling improvements made since 2018, but this is what sparked the backscattering hypothesis

Credit: NASA/TM-2020000003224

Description of Multiple Scattering Theory



- **Goal: obtain scattering attenuation for the Burgers equation in PCBoom**
- **Lonzaga derived the mean pressure waveform of a plane wave including scattering effects:**

- $\langle p(\mathbf{r}, t) \rangle = \int \frac{d\omega dk_1}{(2\pi)^2} \frac{S(\omega) e^{-i(\omega t - k_1 x)}}{k_1^2 - \kappa_o^2 - \tilde{\Sigma}(k_1)},$

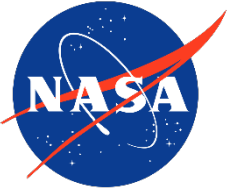
where S is the source amplitude, k_1 is the wavenumber in the propagation direction, $\kappa_o = \omega/c$, and

- $\tilde{\Sigma}$ is the interaction energy (“self-energy”) between acoustic wavefield and turbulence field
 - $\tilde{\Sigma}(k) = 4\kappa_o \int \frac{d^3 k'}{(2\pi)^3} \frac{\Phi_e(k-k')}{k^2 - \kappa_o^2}$, where $\Phi_e(k)$ is the 3D Fourier transform of the turbulence field’s correlation function
 - $\Phi(k) = \frac{a\sigma^2 \ell^3}{(k^2 \ell^2 + 1)^m}$, where σ^2 is the nondimensionalized variance and ℓ is the correlation length of turbulence. a and m are dimensionless variables that describe different turbulence spectra, e.g., for exponential turbulence $a = 8\pi$ and $m = 2$.

- **Dispersion relation: $k_e^2 - \kappa_o^2 - \tilde{\Sigma}(k_e) = 0$**

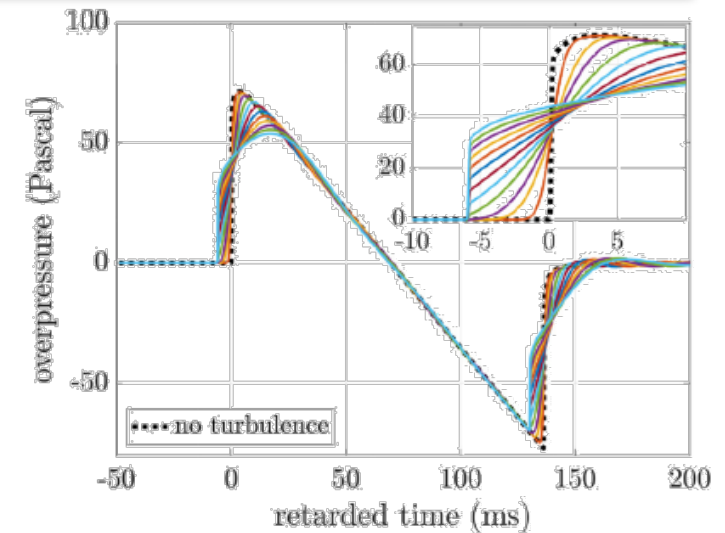
- Effective wavenumber k_e is solved numerically, with the numerical solution described later
- The scattering wavenumber is given by $k_s = k_e - \kappa_o = \alpha_r + i\alpha_i$
 - dispersion α_r (real part) and attenuation α_i (imaginary part) due to backscattering
 - attenuation → loudness reduction

Multiple Scattering Theory Simulation Example



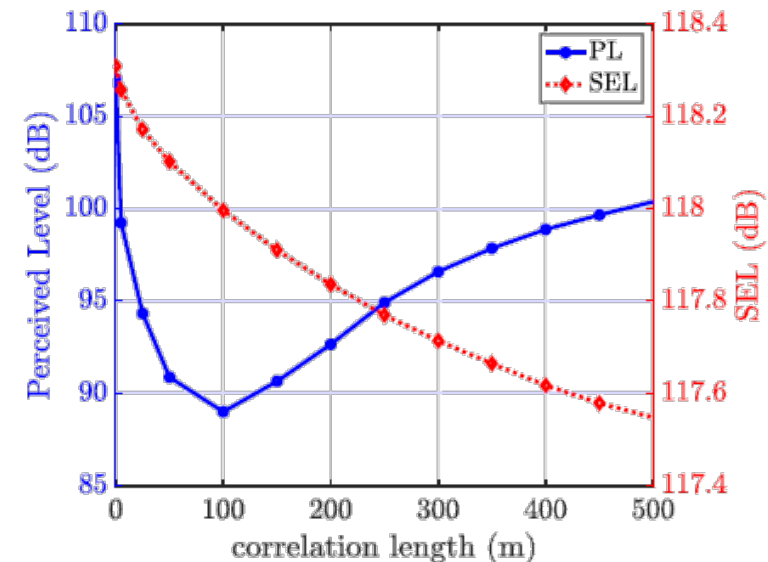
➤ For an N-wave moving through turbulence

- Exponential correlation function
- 700 m propagation distance
- σ of 1/340
 - Nondimensional strength of turbulence
- Varied correlation length from 0 to 500 m



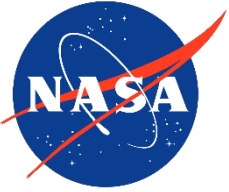
(a)

➤ Lonzaga found that significant reduction in loudness is possible due to backscattering



(b)

Importance to NASA Quesst mission



➤ Phase 1: Planning, X-59 aircraft built & safe to fly

- First flight: Oct. 28, 2025

➤ Phase 2: Acoustic Validation

- X-59 designed to produce a sonic “thump” with Perceived Level of 75 dB
- Near- and Far-field propagation tools validated
 - If predicted noise levels are biased high compared to measurements, models may need to include additional physical effects such as backscattering



Credit: NASA

➤ Phase 3: Community noise survey flight tests with X-59

Transition to the numerical solver

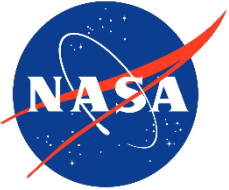
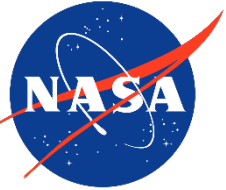


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Approach to numerical solution for scattering wavenumber k_s



- Goal: solve for roots of dispersion relation $k_e^2 - \kappa_o^2 - \tilde{\Sigma}(k_e) = 0$ to obtain scattering wavenumber $k_s = k_e - \kappa_o = \alpha_r + i\alpha_i$
- Solved numerically for each frequency using the Newton-Raphson iteration method:

- $0 = f(x) = f(x_0 + h) \approx f(x_0) + h f'(x_0)$

- Rearranging for h yields $h \approx -\frac{f(x_0)}{f'(x_0)}$, and

- then $x = x_0 + h \approx x_0 - \frac{f(x_0)}{f'(x_0)}$

- Repeated iterations move $f(x_{n+1})$ closer to 0:

- $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \dots$

- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

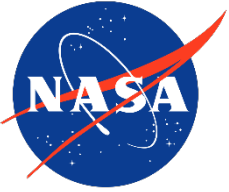
- Algorithm terminates

- when $f(x_{n+1})$ is within a user-selectable tolerance near zero: $|f(x)| \approx |f(x_{n+1})| \leq d_{tolerance}$
 - after a user-selectable maximum number of iterations

- PCBoom is a fortran program that NASA develops and maintains, so root-finder code written in-house

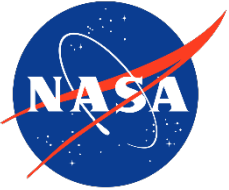


Solution for exponential turbulence spectrum

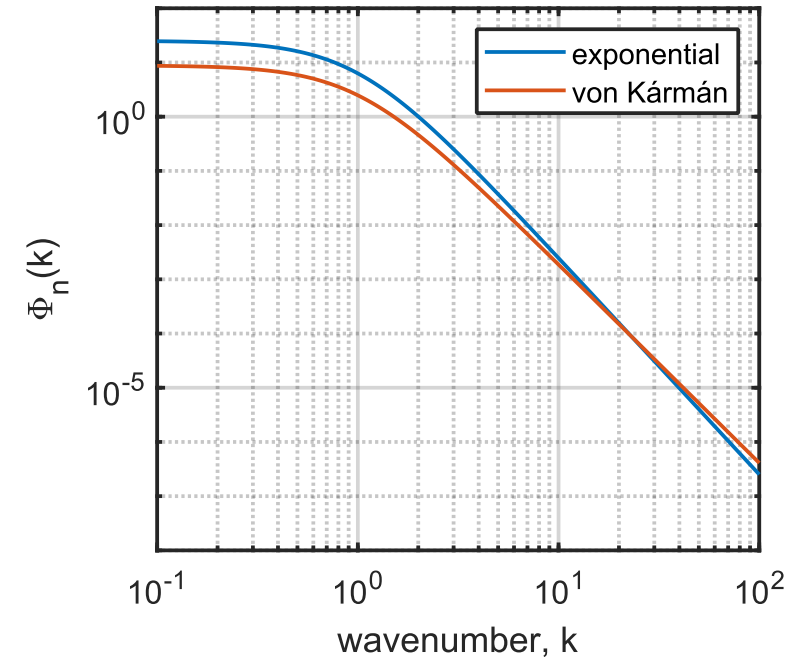


- **Newton-Raphson method:** $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- **Insert our function:** $f(x) \rightarrow D(k_e) = k_e^2 - \kappa_o^2 - \tilde{\Sigma}(k_e)$
 - Need the derivative: $f'(x) \rightarrow D'(k_e)$
 - The derivative is $D'(k_e) = 2k_e - \tilde{\Sigma}'(k_e)$ because $\kappa_o = \frac{\omega}{c}$
 - Need the derivative $\tilde{\Sigma}'$
 - Lonzaga derived $\tilde{\Sigma}(k) = \sum_v \frac{4 \kappa_o^4 \ell_v^2 \sigma_v^2}{(k^2 - \kappa_o^2) \ell_v^2 - 2i \kappa_o \ell_v + 1}$,
with a 3D turbulence field where index v is 1, 2, or 3 corresponding to scalar, shear, and buoyant turbulence.
 - The derivative is $\tilde{\Sigma}'(k_e) = \frac{8\omega^4}{c^4} k_e \sum_v \left(\frac{-\ell_v^4 \sigma_v^2}{\left(k_e \ell_v^2 + \frac{(c - i\omega \ell_v)^2}{c^2} \right)^2} \right)$
- **Our implementation solves an array of m user-supplied frequencies, ω_m**
 - A user-specified initial guess of the complex solution is used for the first frequency
 - e.g., a naive first guess could be $x_0(\omega_0) = 1 + 1i$
 - For subsequent frequencies, the previous frequency's solution is used as the initial guess
 - adjacent frequencies are typically close in value, i.e., $x_0(\omega_{m+1}) = x_n(\omega_m)$
 - Solution runtime scales linearly with m

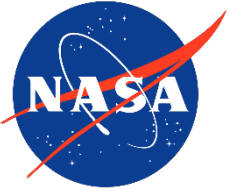
Obtaining an analytic solution for von Kármán turbulence spectrum



- Exponential spectrum was implemented first due to its analytic solution
- von Kármán spectrum of turbulence is more realistic than exponential
- Analytic solution of the von Kármán interaction energy is needed for rapidly solving for the dispersion relation roots



Expanding to von Kármán turbulence spectrum



General form: $\tilde{\Sigma}(k) = 4\kappa_o \int \frac{d^3 k'}{(2\pi)^3} \frac{\Phi_e(k-k')}{k^2 - \kappa_o^2}$, where $\Phi(k) = \frac{a\sigma^2 \ell^3}{(k^2 \ell^2 + 1)^m}$

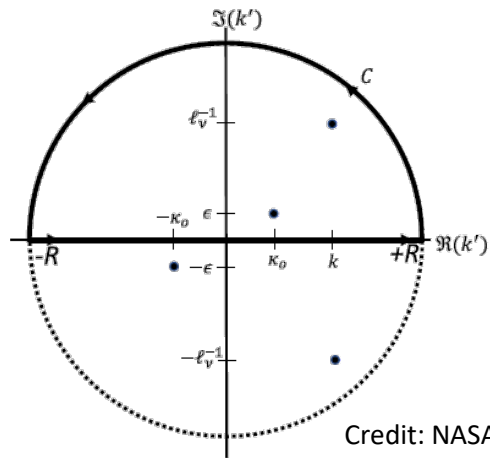
➤ Exponential turbulence self-energy

- $a = 8\pi, m = 2$

➤ Rewritten as

$$\tilde{\Sigma}(k) = \frac{8\kappa_o^4}{k} \sum_v \frac{\sigma_v^2}{\ell_v} \int \frac{dk'}{2\pi} \frac{k'}{k'^2 - \kappa_o^2} \frac{1}{((k' - k)^2 + \ell_v^{-2})}$$

➤ Simple poles, solved with contour integration and residue theorem



Credit: NASA/Lonzaga

➤ Solution:

$$\tilde{\Sigma}(k) = \sum_v \frac{4 \kappa_o^4 \ell_v^2 \sigma_v^2}{(k^2 - \kappa_o^2) \ell_v^2 - 2i\kappa_o \ell_v + 1}$$

➤ von Kármán turbulence self-energy

- $a = \frac{8\pi^{3/2}\Gamma(11/6)}{\Gamma(1/3)}, m = 11/6$

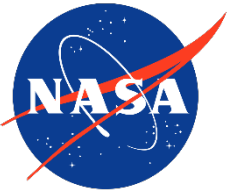
➤ Rewritten as

$$\tilde{\Sigma}(k) = \frac{6a\kappa_o^4}{5\pi k} \sum_v \frac{\sigma_v^2}{\ell_v^{2/3}} \int \frac{dk'}{2\pi} \frac{k'}{k'^2 - \kappa_o^2} \frac{1}{((k' - k)^2 + \ell_v^{-2})^{5/6}}$$

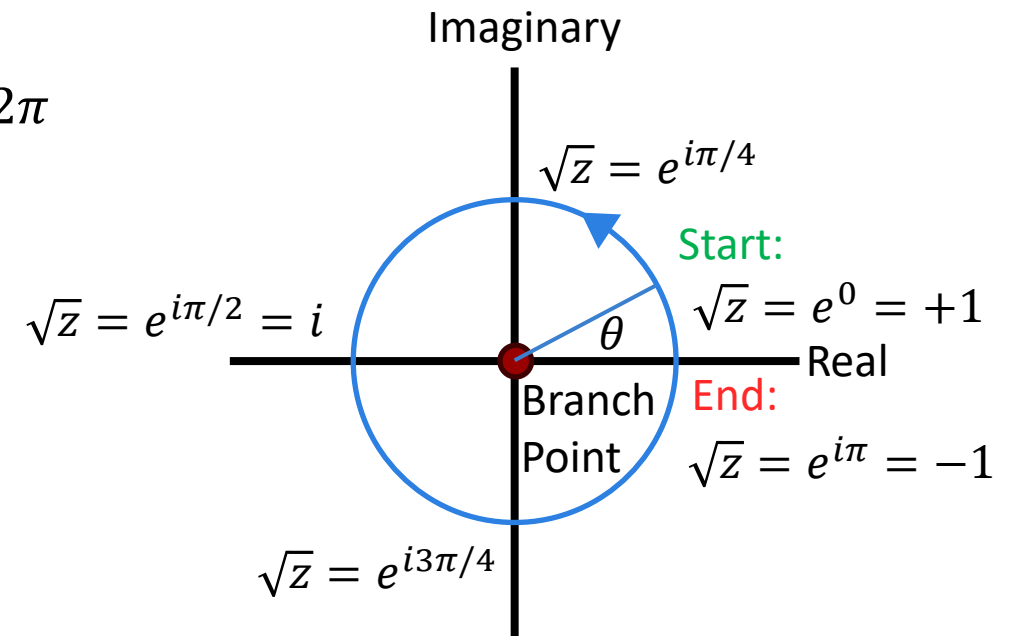
➤ Simple poles

➤ 5/6 power results in branch points and branch cuts, complicating the solution

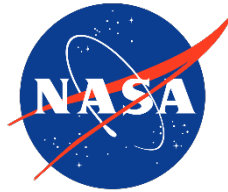
Branch point and branch cuts



- To see if a point is a branch point of a function, compute the function “walking” in a circle around it. Branch points have different values at the start and end of the circle.
- **Example:** $z = 0$ is a branch point for $f(z) = \sqrt{z}$
 - Complex numbers can be represented by $re^{i\theta} = r(\cos \theta + i \sin \theta)$
 - $f(z) = \sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r} e^{i\theta/2}$
 - Start at $r = 1$ and encircle $z = 0$, moving θ from 0 to 2π
- **Branch cuts connect branch points**
 - Needed to maintain single-valuedness
 - Integration contour avoids branch cuts

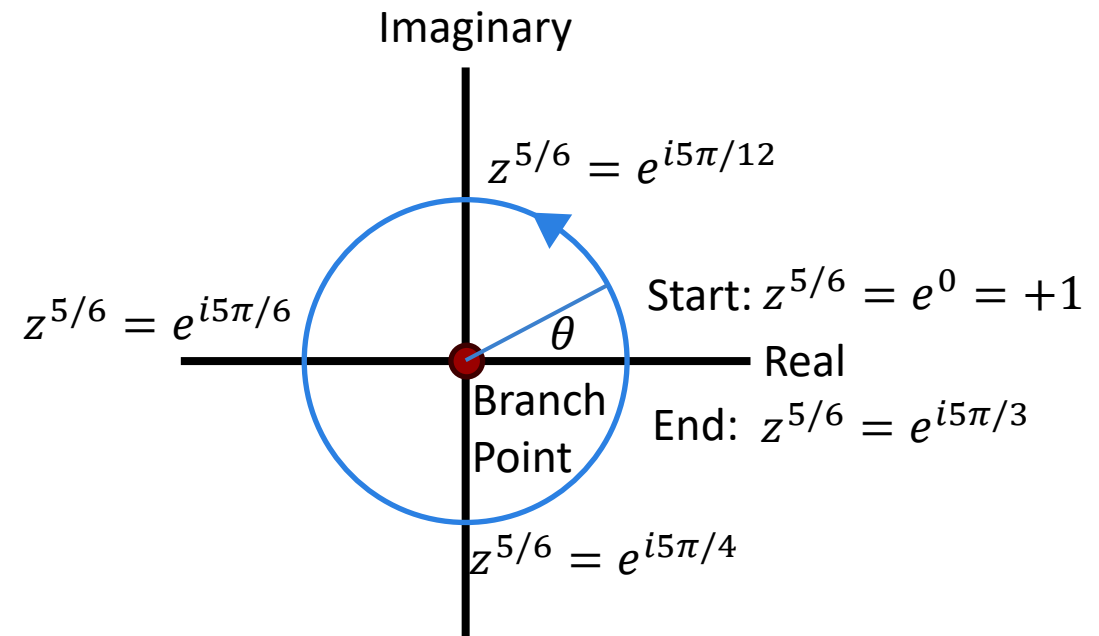


Branch point: our case has $((k' - k)^2 + \ell_v^{-2})^{5/6}$

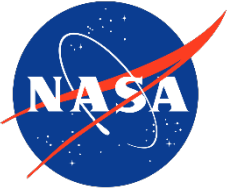


➤ **Example:** $z = 0$ is a branch point for $f(z) = z^{5/6}$

- Complex numbers can be represented by $re^{i\theta} = r(\cos \theta + i \sin \theta)$
- $f(z) = z^{5/6} = r^{5/6} e^{i5\theta/6}$
- Start at $r = 1$ and encircle $z = 0$, moving θ from 0 to 2π



Identifying poles and branch points in the integral



$$\tilde{\Sigma}(k) \sim I = \int \frac{dk'}{2\pi} \frac{k'}{k'^2 - \kappa_o^2} \frac{1}{((k' - k)^2 + \ell^{-2})^{5/6}}$$

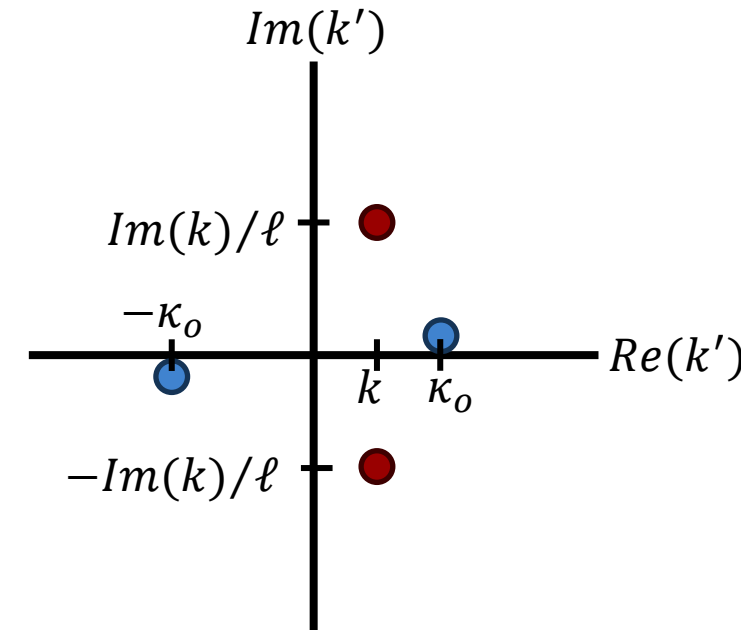
➤ **Solve with contour integration in the upper complex plane**

➤ $\frac{1}{k'^2 - \kappa_o^2}$ **term gives simple poles at $k'^2 - \kappa_o^2 = 0$**

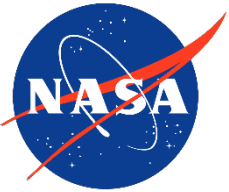
- Solving for k' gives two poles: $k_{pole\pm} = \pm\kappa_o \pm i\epsilon$ (limit as $\epsilon \rightarrow 0$)

➤ $\frac{1}{((k' - k)^2 + \ell_v^{-2})^{5/6}}$ **term gives branch points at $(k' - k)^2 + \ell_v^{-2} = 0$**

- Solving for k' gives two branch points: $k_{branch\pm} = k \pm i/\ell$
 - Further complicated because k can be complex
 - $k_{branch} = Re(k) \pm i Im(k)/\ell$
 - Branch points can intersect the poles

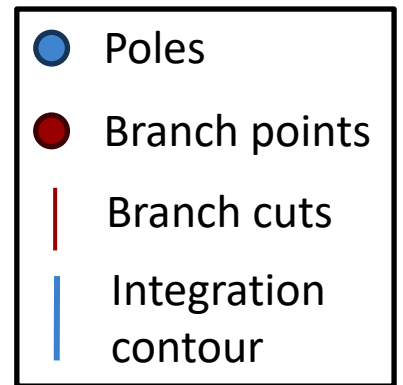
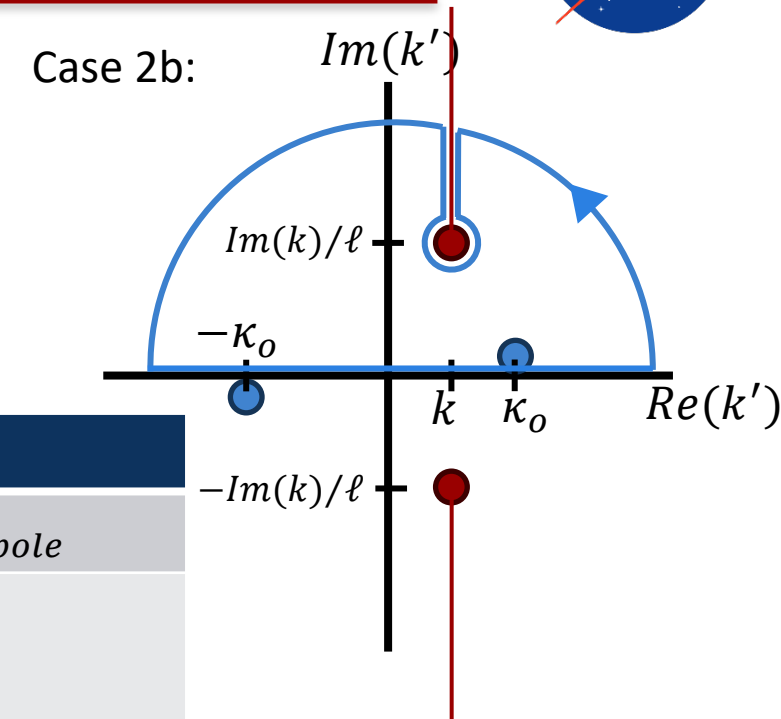


Branch cuts require the solution to be broken into several cases



$$\tilde{\Sigma}(k) \sim I = \int \frac{dk'}{2\pi} \frac{k'}{k'^2 - \kappa_o^2} \frac{1}{((k' - k)^2 + \ell^{-2})^{5/6}}$$

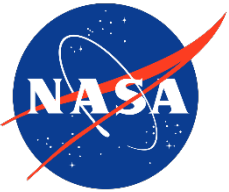
- Solve with contour integration in the upper complex plane
- Eight pole/branch point cases identified, up to 5 case solutions



Case	Condition	Branch cut locations	Solution
1	$Im(k) < -1/\ell$	Both lower half-plane	$I = I_{pole}$
2a	$Im(k) < -1/\ell$, branch point and pole are non-coincident (BPPNC)	Real axis and lower half-plane	$I = I_{pole} + I_{branch}$
2b	$-1/\ell < Im(k) < 1/\ell$, BPPNC	Lower and upper half- plane	
2c	$Im(k) = 1/\ell$, BPPNC	Real axis and upper half-plane	
2d	$Im(k) > 1/\ell$, BPPNC	Both upper half-plane	
3	$k = \kappa_o - i/\ell$	$k_{branch+}$ coincident with κ_o	In progress
4	$k = -\kappa_o - i/\ell$	$k_{branch+}$ coincident with $-\kappa_o$	
5	$k = \kappa_o + i/\ell$	$k_{branch-}$ coincident with κ_o	

$$k_{branch\pm} = k \pm i/\ell$$

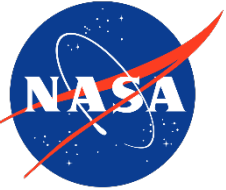
Emergence from the weeds



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- Analytical solution of von Kármán interaction energy enables rapid computation of the attenuation due to backscattering
- Backscattering attenuation can then be incorporated into the PCBoom Burgers solver

Summary and next steps



- **Backscattering may be important for modeling long-range propagation of sonic booms through turbulence**
- **Multiple Scattering Theory accounts for backscattering, which is generally not the case in one-way models**
- **Implemented a solver for scattering wavenumber using interaction energy between the acoustic and turbulence fields with exponential turbulence**
- **Working on von Kármán turbulence spectrum solution and solver**
 - Finish self-energy integration solution
 - Implement Newton-Raphson solver for von Kármán scattering wavenumber
- **Backscattering effects can then be incorporated in the PCBoom Burgers solver**