

Modeling and Simulation of Parachute Apparent Mass Aerodynamics for the Dragonfly Mission

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Analytical expressions to model the apparent mass aerodynamic forces and moments for the Dragonfly drogue and main parachutes are presented. Model parameters are based on a reconstruction of subscale parachute dynamic motions assuming steady-state coning motion in a wind tunnel. The apparent mass model formulation and model parameters are implemented in a multibody simulation. The simulation is validated against the dynamic motions measured during subscale drop tests of the Dragonfly drogue and main parachute configurations. The apparent mass aerodynamics model is applied to the end-to-end entry, descent, and landing simulation of the Dragonfly mission. The aeroshell/lander dynamics (attitude and rotation rates) simulated using the apparent mass aerodynamics (tensor model) are more conservative than the dynamics simulated using heritage apparent mass (point mass) model.

I. Nomenclature

a_{ii}	= non-dimensional apparent mass coefficients
A_{ii}	= dimensional apparent mass parameters
C_T	= tangential force coefficient
C_N	= normal force coefficient
$C_{m_{TBCP}}$	= pitching moment coefficient at the Triple Bridle Confluence Point
$C_{m_x,static}, C_{m_y,static}, C_{m_z,static}$	= static aerodynamic moment coefficients about the (x, y, z) axes, respectively
D_0	= parachute nominal diameter
D_A	= aeroshell diameter
D_P	= parachute projected diameter
F_x, F_y, F_z	= force components about the (x, y, z) axes, respectively
I_{xx}, I_{yy}, I_{zz}	= parachute mass moments of inertia about the (x, y, z) axes, respectively
K_{APP}	= non-dimensional apparent mass multiplier
L_{CoV}	= parachute length
L_R	= riser length
L_S	= suspension lines length
M_x, M_y, M_z	= moment components about the (x, y, z) axes, respectively
p, q, r	= components of the angular velocity about the (x, y, z) axes, respectively
S_0	= parachute nominal area
S_P	= parachute projected area
t	= time
u, v, w	= components of the linear velocity about the (x, y, z) axes, respectively

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$\mathbf{v}_b$	= linear velocity vector
V_w	= wind tunnel speed
V_∞	= airspeed
X_{EV}, Y_{EV}, Z_{EV}	= entry vehicle coordinate system axes
$X_{Drogue_R}, Y_{Drogue_R}, Z_{Drogue_R}$	= drogue parachute coordinate system axes with origin at the triple bridle confluence point
y_k	= filtered rate at time step k
α	= smoothing coefficient
α_T	= total angle of attack
γ_k	= unfiltered rate at time step k
ρ	= freestream density
Θ	= elevation angle
τ	= filter time constant
$\dot{\phi}$	= coning rate
ω_b	= angular velocity vector

Acronyms

APL	= Johns Hopkins University Applied Physics Laboratory
ASNA	= Airborne Systems North America
CoM	= Center of Mass
CoV	= Center of Volume
DGB	= Disk-Gap-Band (parachute type)
DFLY	= Dragonfly
DoF	= Degrees-of-Freedom
EDL	= Entry, Descent, and Landing
EV	= Entry Vehicle
LaRC	= Langley Research Center
LMS	= Lockheed Martin Space
MSL	= Mars Science Laboratory
NC	= Non-Circulatory
ODE	= Ordinary Differential Equation
PDS	= Parachute Decelerator Subsystem
POST2	= Program to Optimize Simulated Trajectories II
TBCP	= Triple Bridle Confluence Point

II. Introduction

THE Dragonfly mission was selected in 2019 as the fourth mission of the NASA New Frontiers program [1]. The project was confirmed in 2024 with a target launch in 2028 and arrival at Titan (Saturn’s largest moon) by 2034. Dragonfly is a rotorcraft/lander designed to investigate the prebiotic chemistry, habitability and potential chemical biosignatures of Titan [1]. The Dragonfly mission is led by the Johns Hopkins University Applied Physics Laboratory (APL). NASA is leading the entry, descent and landing system development. Lockheed Martin Space (LMS) is developing the entry capsule and Airborne Systems North America (ASNA) is providing the Parachute Decelerator Subsystem (PDS).

An end-to-end simulation of the Dragonfly entry, descent, and landing (EDL) sequence was created at the NASA Langley Research Center using the Program to Optimize Simulated Trajectories II (POST2) [2]. The simulation tool is used to evaluate the PDS performance, design the subsystem, develop requirements, and verify and validate flight mechanics requirements. Among numerous models in the simulation, the parachute models are particularly important to simulate the dynamic performance of the system during atmospheric descent. The primary purpose of the parachute models are to support the POST2 simulation for obtaining sufficiently accurate values of the following responses of interest:

- System dynamics: attitude (e.g. Euler Angles), rotation rates

- Parachute loads: PDS and EDL Assembly (EA) design
- Trajectory analyses: Position and altitude versus time

Parachute models are kept as simple as possible to support the needs of the POST2 simulation. Various parachute models used for Dragonfly have heritage from Mars missions [3]; however, they are tuned to be specific to the Dragonfly PDS, Titan descent environments, and available experimental data. This paper describes the formulation and evaluation of the parachute apparent mass model used to evaluate the system dynamics during Titan descent.

III. Parachute Decelerator Subsystem

A. System Description

1. Drogue Parachute

The drogue parachute is a mortar-deployed Disk-Gap-Band (DGB) parachute of 8.25 m nominal diameter, scaled to maintain similar geometric proportions to the parachutes used by the Viking missions to Mars [4]. The parachute trailing distance, defined as the distance from the location of the maximum aeroshell diameter to the parachute skirt, is 10 aeroshell diameters (45 m) to minimize potential interactions between the wake of the aeroshell and the parachute. The drogue parachute diameter is sized to stabilize and decelerate the entry capsule through the low-supersonic and transonic regimes while sustaining acceptable peak inflation loads. The drogue parachute descent occurs over 108 minutes during which the parachute will traverse cryogenic environments. The drogue parachute includes a swivel designed to uncouple rotations of the parachute and the payload about its axis of symmetry.

2. Main Parachute

The main parachute is deployed by the drogue parachute at a dynamic pressure of 78 Pa. The main parachute canopy is a 20 degree conical ringslot with 16.7 m nominal diameter and 20 % geometric porosity. The suspension lines length is roughly equivalent to the parachute nominal diameter and the trailing distance is 8 aeroshell diameters (36 m). The main parachute also includes a swivel with the same purpose as that on the drogue. The main parachute is designed to be robust to opening in low-subsonic conditions, provide sufficient ballistic coefficient differential to release the heatshield, and achieve the required terminal velocity while maintaining acceptable dynamic motions for mid-air lander release. Prior to release, the lander will be attached to the backshell and main parachute assembly for roughly 20 minutes. During descent, the main parachute and backshell/lander will be flying in a high-density environment and can experience various modes of motion including glide and pendulum/coning oscillations.

B. Concept of Operation

Upon launch the spacecraft will go through a six year interplanetary cruise followed by the atmospheric entry, descent and landing phase. The EDL phase begins at cruise stage separation of the 2600 kg (Maximum Possible Value) entry vehicle consisting of the Dragonfly lander, the backshell, and the 4.5-m diameter heatshield. Following a ballistic entry and hypersonic/supersonic deceleration phase, the mortar will deploy the DGB drogue parachute. After 108 minutes of atmospheric descent, the drogue will act as a pilot to deploy the ringslot Main parachute at an altitude of approximately 4 km above the surface of Titan. The heatshield release occurs minutes after main parachute deployment which exposes the (stowed) lander to the Titan atmosphere. Afterwards the lander is lowered (posed) to prepare the lander rotors to initiate flight. The lander pose maneuver initiates the Preparation for Powered Flight (PPF) phase. During PPF, the lander rotors spin at an idle rate followed by a yaw despin maneuver. Lander LiDAR is expected to achieve ground lock during PPF and soon after begin the Transition to Powered Flight (TPF) phase in which the lander separates from the backshell/parachute at an altitude near 1 km and a terminal descent velocity of 2.9 m/s. The EDL concept of operations is shown in Figure 1.

IV. Modeling

The POST2 simulation of the Dragonfly EDL sequence includes numerous models such as atmosphere, gravity, aeroshell aerodynamics, etc. The scope of this paper is limited to parachute models relevant to the evaluation of system dynamics (attitude and rotation rates) during parachute descent. This section contains a background of parachute modeling assumptions and submodels relevant to system dynamics. The parachute apparent mass model formulation

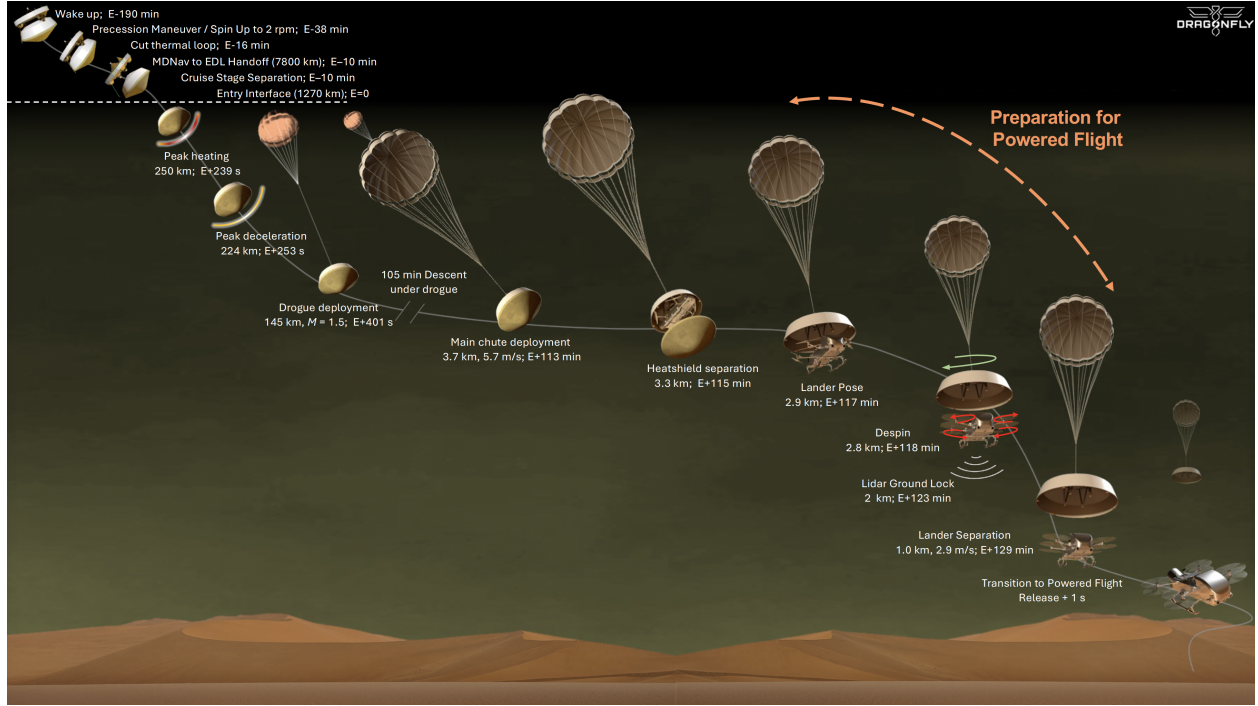


Fig. 1 EDL Concept of Operations

and model evaluation will be discussed in the following section.

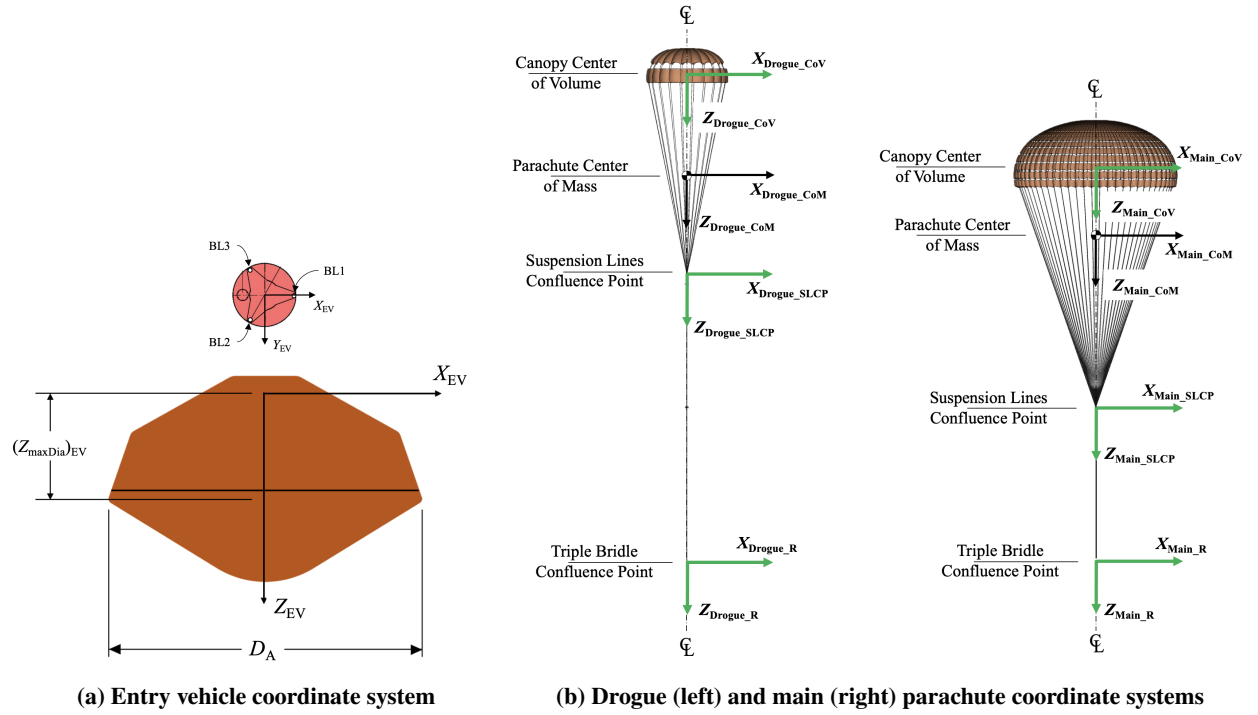


Fig. 2 Coordinate system definitions

A. Basic Modeling Assumptions

The POST2 simulation models the parachute and aeroshell as two rigid bodies connected by three flexible bridles. The canopy, suspension lines, and the riser are treated as a single rigid body with 6 degrees-of-freedom (6DOF). The triple bridle legs are modeled as massless, tension-only, linear spring dampers. The parachute wind angles (i.e., angle of attack and angle of sideslip) and dynamic pressure are calculated at the canopy center of volume (CoV) (shown in Figure 2b). The mass of the drogue and main parachutes are 8.1 kg and 13.3 kg, respectively.

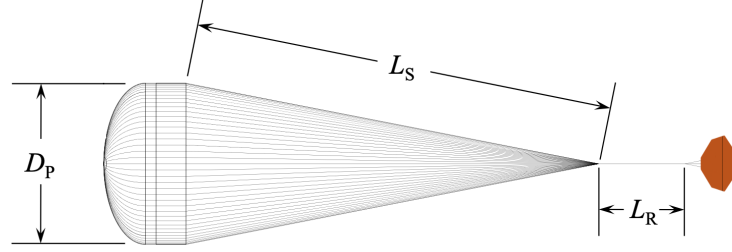


Fig. 3 Parachute geometry definitions

B. Geometric Parameters

The key geometric parameters for the drogue and main parachutes are shown in Tables 1-2, and Figure 3. Most geometric parameters are defined in terms of the nominal parachute diameter (D_0) and the ratio of projected to nominal parachute diameter (D_P/D_0). Additional parameters such as the distance from the bridle confluence point to the center of mass (CoM) and to the CoV are defined assuming that the CoM and CoV are aligned with the parachute axis of symmetry (rigid parachute assumption).

Table 1 Geometric Parameters: Drogue Parachute

Item	Symbol	Value
Nominal parachute diameter	D_0	8.25 m
Nominal parachute area	S_0	$S_0 = \pi D_0^2/4$
Ratio of projected to nominal parachute diameter	D_P/D_0	0.735
Projected parachute area	S_P	$S_P = \pi D_P^2/4$
Center of mass from bridle confluence point	$(Z_{CoM})_{Drogue_R}$	-29.1 m
Center of volume from bridle confluence point	$(Z_{CoV})_{Drogue_R}$	-41.4 m

Table 2 Geometric Parameters: Main Parachute

Item	Symbol	Value
Nominal parachute diameter	D_0	16.66 m
Nominal parachute area	S_0	$S_0 = \pi D_0^2/4$
Ratio of projected to nominal parachute diameter	D_P/D_0	0.704
Projected parachute area	S_P	$S_P = \pi D_P^2/4$
Center of mass from bridle confluence point	$(Z_{CoM})_{Main_R}$	-32.9 m
Center of volume from bridle confluence point	$(Z_{CoV})_{Main_R}$	-35.1 m

C. Aerodynamic Coefficients

The aerodynamic coefficients are modeled using the tangential (C_T), normal (C_N), and pitching moment coefficients ($C_{m_{TBCP}}$) which are defined as functions of the total angle of attack, α_T (calculated at the CoV, see Figure 4). The three

coefficients can be used to describe the static aerodynamics of an axisymmetric parachute. The static aerodynamic coefficients are dispersed within predefined bounds for the Monte Carlo analysis. The dispersed nonlinear function is constant during the entire simulation and uses α_T as the independent variable for interpolation. This static aerodynamic model formulation has heritage from models used for Mars missions [3]. Focus is given to the pitching moment coefficient defined at the suspension lines confluence point (Figures 5a and 5b). The pitching moment coefficient can be used to determine α_T at trim (determined from the point at which the pitching moment is zero and the slope is negative). Based on the α_T at trim extracted from the nominal pitching moment coefficient it can be inferred that the drogue parachute is more statically stable (restoring aerodynamic moment is greater in magnitude) than the Ringsail main parachute. The nominal α_T at trim of the Ringsail parachute is 10.7° , while the nominal α_T at trim of the DGB parachute is 3.8° .

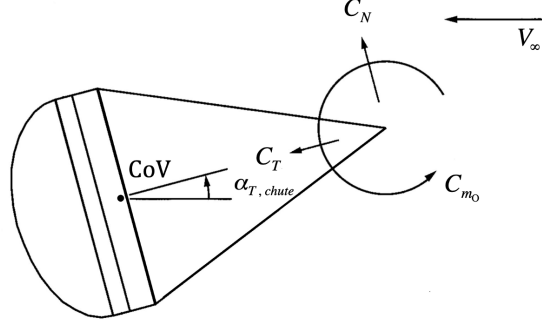


Fig. 4 Parachute static aerodynamic coefficients and wind angle definition

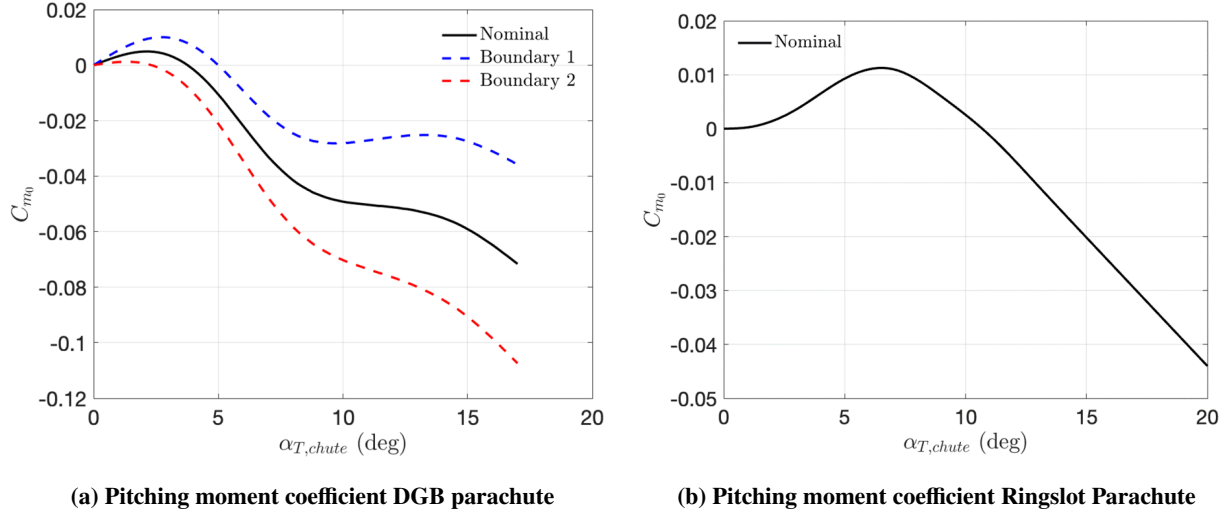


Fig. 5 Pitching moment coefficients for the Dragonfly Drogue and Main parachutes

V. Apparent Mass Aerodynamics

A. Background

The apparent mass aerodynamics refers to the fluid-generated forces and moments on a body due to the non-circulatory component of the flow. Apparent mass terms are particularly important to either lightweight bodies (e.g. parachutes) in motion through an atmosphere or bodies moving through very high density environments (e.g. water). Apparent mass models used for past Mars missions assume that the air surrounding the parachute adds inertia to the parachute, added as

a point mass, but do not account for the shape of the body or its translational/rotational motion [3]. While the simplified point mass apparent mass models were deemed acceptable for low-density atmospheric environments, their validity has been questioned for Titan atmospheric descent given its high atmospheric density. A closed form apparent mass formulation that accounts for the parachute geometry and its translational and rotational motion is presented below.

B. General Assumptions

A closed form expression for the non-circulatory forces and moments can be derived based on the following assumptions about the fluid, flowfield, and body (vehicle).

- 1) The fluid is inviscid and incompressible (ideal fluid)
- 2) Signals propagate through the fluid instantaneously
- 3) The flow is attached (no flow separation)
- 4) The body motion may be steady or accelerated
- 5) The body moving through the fluid is rigid

The general expressions (forces and moments) for the apparent mass aerodynamic terms were introduced by Imlay [5]. The general form was re-casted using notation relevant to parachute literature (see Appendix).

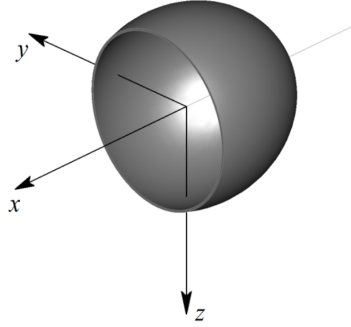


Fig. 6 Coordinate system definition for rigid hemispherical shell

C. Apparent Mass Expression for a Parachute

It stands to reason that apparent mass aerodynamics is very relevant to the flight mechanics of parachutes due to their comparable low mass with respect to the fluid through which they traverse. The general form of the apparent mass force and moment equations by Imlay can be simplified to express the fluid flow around a rigid hemispherical shell and assuming that the coordinate system is at its center of volume and with a coordinate system orientation as shown in Figure 6.

$$F_{x,NC} = -A_{11}\dot{u} - A_{22}(wq - vr) \quad (1)$$

$$F_{y,NC} = -A_{22}(\dot{v} - wp) - A_{11}ur \quad (2)$$

$$F_{z,NC} = -A_{22}(\dot{w} + vp) + A_{11}uq \quad (3)$$

$$M_{x,NC} = 0 \quad (4)$$

$$M_{y,NC} = -A_{55}(\dot{q} - pr) - (A_{11} - A_{22})uw = -A_{55}(\dot{q} - pr) \quad (5)$$

$$M_{z,NC} = -A_{55}(\dot{r} + pq) + (A_{11} - A_{22})uv = -A_{55}(\dot{r} + pq) \quad (6)$$

The expressions above assume that static aerodynamics terms are accounted for separately, therefore nonlinear terms that depend solely on linear velocity can be dropped (that is, $uw = uv = 0$). The resulting expression for apparent mass

can be used to calculate the non circulatory forces and moments on the parachute. The terms A_{11} and A_{22} have units of physical mass (e.g. kg), while the term A_{55} has units of mass moment of inertia (e.g. kg·m²). Due to the various inertial terms involved (mass, moments of inertia, etc.) and the matrix symmetry, the term tensor apparent mass model is used to refer to the proposed apparent mass formulation. For a parachute, the apparent mass terms (A_{ii}) can be expressed as a function of the parachute geometry, fluid density, and non-dimensional coefficients:

$$A_{11} = a_{11}\rho D_p^3 \quad (7)$$

$$A_{22} = a_{22}\rho D_p^3 \quad (8)$$

$$A_{55} = a_{55}\rho D_p^5 \quad (9)$$

The coefficients a_{11} , a_{22} , and a_{55} are independent model parameters that can be estimated from experimental data.

VI. Estimating Model Parameters

While the theoretical foundation for apparent mass is well established, determining the apparent mass model coefficients from experimental data requires specialized facilities and test equipment [6]. As a consequence, the amount of data available to estimate the apparent mass parameters of parachutes is very limited. Of the three parachute apparent mass coefficients; a_{11} and a_{22} are the most important parameters for the apparent mass model. For this analysis, the a_{55} term is assumed to have a negligible contribution ($a_{55} = 0$). This section outlines a parameter estimation technique to estimate a_{11} and a_{22} from the parachute coning rate, airspeed and density. The parachute dynamic motions test data used for the evaluation correspond to the subscale ringsail parachute in the NASA Langley Transonic Dynamics Tunnel [7].

The parameter estimation technique is based on the assumptions below (illustrated in Figure 7):

- 1) The parachute is rigid and is attached by a spherical joint at the triple bridle confluence point (TBCP).
- 2) The wind tunnel operates with a constant windspeed V_w .
- 3) The parachute is in steady-state coning motion with a constant coning rate, $\dot{\phi}$, and elevation angle, Θ .
- 4) The length from the attachment point to the center of volume is L_{CoV} .
- 5) The effects of gravity are assumed to be negligible.

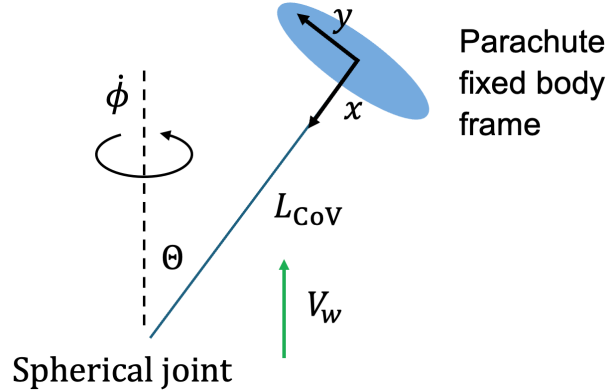


Fig. 7 Rigid parachute in coning motion under constant wind

The parachute wind relative velocity and parachute body rate vectors in the body coordinate system can be defined as a function of the coning rate ($\dot{\phi}$), the elevation angle (Θ), and the wind tunnel velocity (V_w).

$$\mathbf{v}_b = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} V_w \cos \Theta \\ -V_w \sin \Theta \\ \dot{\phi} L_{CoV} \sin \Theta \end{bmatrix} \quad (10)$$

$$\omega_b = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\dot{\phi} \cos \Theta \\ \dot{\phi} \sin \Theta \\ 0 \end{bmatrix} \quad (11)$$

The static aerodynamic moments are defined by assuming that the airflow produces a quasi-static aerodynamic moment in the parachute fixed frame (excludin apparent mass effects):

$$\begin{bmatrix} M_{x,\text{static}} \\ M_{y,\text{static}} \\ M_{z,\text{Static}} \end{bmatrix} = \begin{bmatrix} C_{m_x,\text{static}} \\ C_{m_y,\text{static}} \\ C_{m_z,\text{static}} \end{bmatrix} \frac{1}{2} \rho V_\infty^2 D_0 S_0 \quad (12)$$

Where D_0 is the parachute reference diameter, S_0 is the parachute reference area, ρ is the density during the wind tunnel test, and V_b is the airspeed at the parachute center of volume:

$$V_\infty^2 = |\mathbf{v}_b|^2 = L_{\text{CoV}}^2 \dot{\phi}^2 \sin^2 \Theta + V_w^2 \quad (13)$$

The axisymmetric pitching moment coefficient at the TBCP, $C_{m\text{TBCP}}$, can be decomposed into the x, y, and z axes using the following relationships:

$$C_{m_x,\text{static}} = 0 \quad (14)$$

$$C_{m_y,\text{static}} = C_{m\text{TBCP}} \cos \left[\arctan \left(\frac{v}{w} \right) \right] \quad (15)$$

$$C_{m_z,\text{static}} = -C_{m\text{TBCP}} \sin \left[\arctan \left(\frac{v}{w} \right) \right] \quad (16)$$

The static aerodynamic moments can be redefined by replacing Equation 10 into Eqs. 15-16, back-substituting into Equation 12, and rearranging terms:

$$\begin{bmatrix} M_{x,\text{static}} \\ M_{y,\text{static}} \\ M_{z,\text{Static}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{L_{\text{CoV}} \dot{\phi}}{\sqrt{L_{\text{CoV}}^2 \dot{\phi}^2 + V_w^2}} \\ \frac{V_\infty}{\sqrt{L_{\text{CoV}}^2 \dot{\phi}^2 + V_w^2}} \end{bmatrix} \frac{\pi}{8} \rho V_\infty^2 D_0^3 C_{m\text{TBCP}} \quad (17)$$

Assuming that the parachute is coning at a steady-state rate, the kinematic and kinetic equations for rotational motion about the parachute TBCP can be defined in terms of the static moment coefficients due to the airflow, the moment induced by the non-circulatory (apparent mass) forces at the TBCP, the parachute body rates, and the parachute moments of inertia (MoI) at the TBCP:

$$\begin{bmatrix} M_{x,\text{static}} \\ M_{y,\text{static}} \\ M_{z,\text{Static}} \end{bmatrix} + \begin{bmatrix} -L_{\text{CoV}} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} F_{x,\text{NC}} \\ F_{y,\text{NC}} \\ F_{z,\text{NC}} \end{bmatrix} = \omega_b \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{yy} \end{bmatrix} \omega_b \quad (18)$$

In Equation 18 it is assumed that the non-circulatory moments are negligible, which is equivalent to $a_{55} = 0$. Furthermore, the MoI are defined such that I_{xx} corresponds to the MoI about the parachute axis of symmetry (x-axis). $I_{yy} = I_{zz}$ are the components of MoI perpendicular to the parachute axis of symmetry. The expression above can be further expanded using Equations 1, 2, 3, 10, 11, and 17. Additional simplifications can be made by neglecting the linear accelerations (steady-state assumption: $\dot{u} = \dot{v} = \dot{w} = 0$) and assuming that $I_{yy} \gg I_{xx}$ (valid for MoI measured at TBCP). After various algebraic manipulations, the expressions for the y and z axes can be rearranged into Equations 19 and 20, respectively:

$$(a_{11} - a_{22}) V_\infty \sin(2\Theta) = -C_{m\text{TBCP}} \left(\frac{D_0}{D_P} \right)^3 \frac{\pi}{4} \left(\frac{L_{\text{CoV}}^2 \dot{\phi}^2 \sin^2 \Theta + V_w^2}{\sqrt{L_{\text{CoV}}^2 \dot{\phi}^2 + V_w^2}} \right) \quad (19)$$

$$\left(a_{22} + \frac{I_{yy}}{\rho D_P^3 L_{CoV}^2}\right) L_{CoV} \dot{\phi}^2 \sin(2\Theta) = -C_{mTBCP} \left(\frac{D_0}{D_P}\right)^3 \frac{\pi}{4} \left(\frac{L_{CoV}^2 \dot{\phi}^2 \sin^2 \Theta + V_w^2}{\sqrt{L_{CoV}^2 \dot{\phi}^2 + V_w^2}}\right) \frac{V_w}{L_{CoV}} \quad (20)$$

The final expression relating the apparent mass terms, wind tunnel test conditions and the parachute dynamic motions, can be derived using algebraic elimination:

$$\left(\frac{a_{11} - a_{22}}{a_{22} + \frac{I_{yy}}{\rho D_P^3 L_{CoV}^2}}\right) V_w \sin(2\Theta) = \left(\frac{L_{CoV} \dot{\phi}}{V_w}\right)^2 \quad (21)$$

Equation 21 can be solved by least-squares. The required inputs are the parachute geometry, mass properties, the coning rate, freestream density, and wind speed. Given various test conditions, the least-squares solution results in the best linear-fit to the apparent mass parameters (a_{11} and a_{22}).

$$\begin{bmatrix} 1 & -\left(1 + \left(\frac{L_{CoV} \dot{\phi}_1}{V_{w1}}\right)^2\right) \\ 1 & -\left(1 + \left(\frac{L_{CoV} \dot{\phi}_2}{V_{w2}}\right)^2\right) \\ \vdots & \vdots \\ 1 & -\left(1 + \left(\frac{L_{CoV} \dot{\phi}_N}{V_{wN}}\right)^2\right) \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{22} \end{bmatrix} = \frac{I_{yy}}{D_P^3 L_{CoV}^2} \begin{bmatrix} \frac{1}{\rho_1} \left(\frac{\dot{\phi}_1}{V_{w1}}\right)^2 \\ \frac{1}{\rho_2} \left(\frac{\dot{\phi}_1}{V_{w1}}\right)^2 \\ \vdots \\ \frac{1}{\rho_N} \left(\frac{\dot{\phi}_N}{V_{wN}}\right)^2 \end{bmatrix} \quad (22)$$

Equation 22 was used to solve for values of a_{11} and a_{22} using four independent test points ($N = 4$) with varying wind tunnel airspeed and density. The parachute coning rate, $\dot{\phi}$, was extracted from the test data in which both the elevation Θ and $\dot{\phi}$ were nearly constant [7]. The least-squares solution to the four data points resulted in $a_{11} = 0.094$ and $a_{22} = 0.033$, which results in a ratio of $a_{11}/a_{22} = 2.85$. The four data points and the best fit curve are shown in Fig 8. The apparent mass parameter ratio from the reconstruction (a_{11}/a_{22}) is higher than what has been previously reported in the literature [8, 9]. However, it is expected that the apparent mass parameters (and by extension their ratio) can vary with the parachute canopy design.

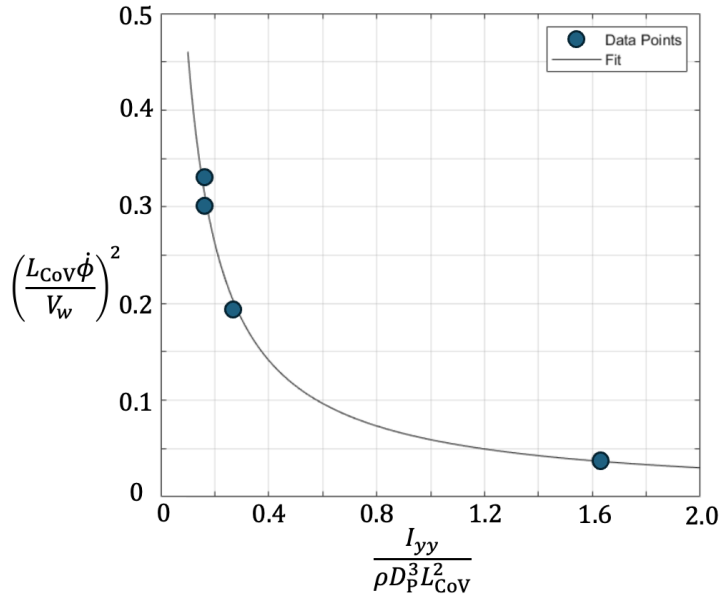


Fig. 8 Test data and model parameter fit

The approach presented can also be extended to estimate a_{55} , although the contribution of this term for the parachutes being evaluated is small and is thus neglected. Using the apparent mass terms derived from this study in a simulation of a rigid parachute in a wind tunnel results in a clean coning motion. An extended evaluation of the dynamic motions of the subscale ringsail parachute during the wind tunnel test and its simulation are outside the scope of this document.

VII. Model Implementation

A. Coordinate System Reconciliation

The coordinate system definition of the Dragonfly PDS assumes that z is the axis of symmetry (2b). On the other hand, the coordinate system used to define the apparent mass aerodynamics assumes that x is the axis of symmetry. Consequently, the Equations used to define the apparent mass aerodynamics, 1, 2, 3, 4, 5, 6 are re-defined using the PDS coordinate system:

$$F_{x,NC_DFLY} = -A_{11}(\dot{u} - vr) - A_{33}wq \quad (23)$$

$$F_{y,NC_DFLY} = -A_{11}(\dot{v} + ur) + A_{33}wp \quad (24)$$

$$F_{z,NC_DFLY} = -A_{33}\dot{w} - A_{11}(vp - uq) \quad (25)$$

$$M_{x,NC_DFLY} = -A_{44}(\dot{p} - rq) \quad (26)$$

$$M_{y,NC_DFLY} = -A_{44}(\dot{q} + rp) \quad (27)$$

$$M_{z,NC_DFLY} = 0 \quad (28)$$

Where the apparent mass parameters are given by $A_{33} = a_{33}\rho D_p^3$, $A_{11} = a_{11}\rho D_p^3$ and $A_{44} = a_{44}\rho D_p^5$. As stated earlier, the non-circulatory moments are neglected for the Dragonfly parachutes ($a_{44} = 0$). It can be observed that the non-circulatory forces and moments are defined as a function of the time rate of change of translational and rotational velocity states ($\dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r}$), which implies that the apparent mass model is defined implicitly. The implicit parachute apparent mass model is denoted as the tensor model.

B. Apparent Mass Model Parameters

1. Heritage Model

The heritage model, used for Mars Science Laboratory (MSL), applies a point mass (m_{App}) to the parachute center of mass while simultaneously adding buoyancy to counteract the force of gravity. Thus the point mass model only accounts for additional inertial forces [3].

$$m_{App} = K_{App}\pi\rho D_0^3 \quad (29)$$

The term K_{App} is defined with a nominal value of 0.035, a minimum value of zero and a maximum value of 0.07. A connection between the mass term in the heritage model and the mass term in the tensor model can be made by grouping relevant terms as follows:

$$a_{33} = K_{App}\pi\left(\frac{D_p}{D_0}\right)^{-3} \quad (30)$$

The range of a_{33} for the heritage model is on the order of zero at the minimum, 0.28 at the mean and a maximum of 0.56.

2. Tensor Model

The parameters used to inform the apparent mass model are based on the parameter reconstruction presented in the previous section. While the parachute canopy used for the wind tunnel tests (ringsail) was not a direct match to the Dragonfly PDS (DGB and ringslot), the magnitude (~ 0.094) and ratios ($a_{11} = \frac{1}{3}a_{33}$) of the apparent mass terms derived from the ringsail canopy were deemed acceptable to conduct model verification and sensitivity analyses for the Dragonfly simulation.

C. Addressing Numerical Instabilities

The model is implemented into POST2 as defined in Equations 23 through 28. Numerical instabilities were introduced from implementing an implicit model using an explicit ODE Solver (Runge-Kutta 4th order). A sensitivity showed that numerical instability was driven by the linear and angular acceleration terms ($\dot{u}$, $\dot{v}$, $\dot{w}$, $\dot{p}$, $\dot{q}$, $\dot{r}$). The POST2 analysis is done with a integration rate of 200 - 1000 Hz depending on the state of the simulation, for example, if the entry vehicle is under free fall or under parachute. Increasing the integration rate did not resolve the numerical instability. It was determined that numerical filtering was required to address this instability using a first order filter shown in Equation 31 - 32.

$$y_k = (1 - \alpha) \cdot y_{k-1} + \alpha \cdot \gamma_k \quad (31)$$

$$\alpha = \frac{\Delta t}{\Delta t + \tau} \quad (32)$$

The filtered rate at the current time step, y_k , is a function of the filtered rate at the previous time step, y_{k-1} , the current unfiltered rate, γ_k , and the smoothing coefficient, α , (set between 0 and 1). A higher α adds more weight to the non-filtered inputs and inversely, a lower α adds more weight to the filtered input. The smoothing coefficient is a function of the time step and the time constant, τ . A time constant of 0.01 had numerical issues, whereas 0.02 and 0.03 resolved the numerical issues and produced similar solutions. A time constant of 0.03 was selected for this analysis. The implemented solution includes two sequential first-order filters to filter the translational and rotational rate terms as shown in Figure 9. The accelerations are then determined using a forward differencing method. The filtered rates and accelerations were implemented as described in Equations 23 through 28.

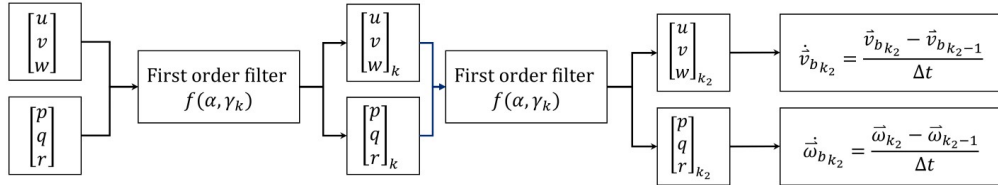


Fig. 9 Overview of the sequential first-order filter implementation of the translational and rotational rates

VIII. Results

A. Model Validation - Subscale Drop Test Simulation

The tensor apparent mass model and model parameters are validated by evaluating the model performance against heritage apparent mass modeling techniques in predicting (and/or bounding) the system dynamics of subscale drop tests of Dragonfly during parachute descent.

A series of subscale drop tests were performed to investigate the system dynamics (attitude and rotation rates) of the entry vehicle in four different configurations tested in three phases. Phase I consisted of a closed aeroshell under the drogue parachute; Phase II of a closed aeroshell under the main parachute [10]; and Phase III of the open backshell in two distinct subphases, with the lander in either its stowed or posed configuration [11].

In each phase of the comparison, the statistics of two data products were compared between the test and simulation: nadir angle and lateral attitude (lat. att.) rates. These are defined in Figure 10. For both the test and simulation, data are

collected only from the quasi-steady-state period of the drop, after the initial release transients have died off and before the vehicle impacts the ground. The data from each drop within a phase is then stitched together and percentiles are generated from this aggregate data. For the tests, 12 separate drops are stitched together [10], while the simulation makes use of 100 drops to generate statistics.

The simulation for each drop test configuration made use of both a point mass and tensor parachute model, in order to determine the sensitivity of the system to the apparent mass model. In addition, the phase I and II simulations utilize the Earth Global Reference Atmosphere Model (GRAM) along with two variations of the winds model: Baseline and patchy. More detail regarding the simulation setup, model parameters, etc. can be found in Ref [12].

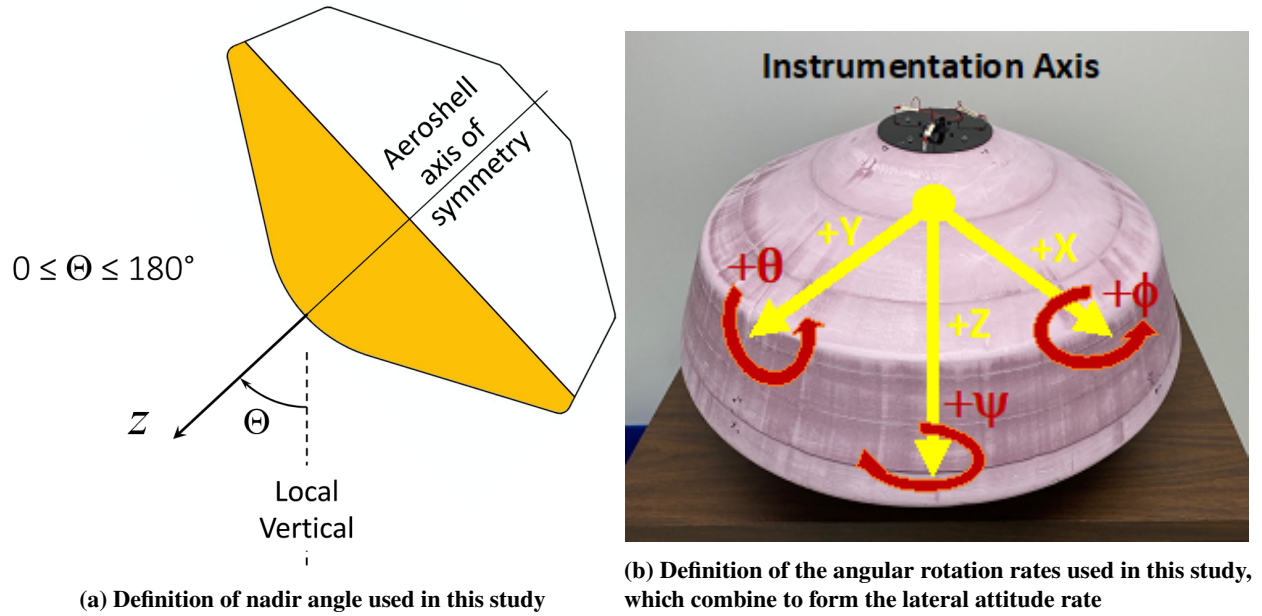


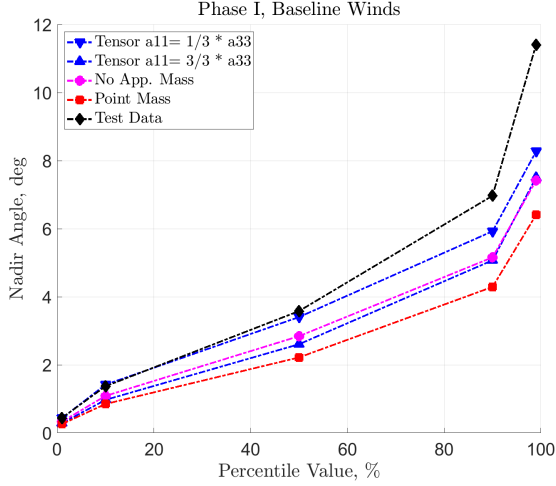
Fig. 10 Definitions of angles and angular rates used in this study

1. Phase I

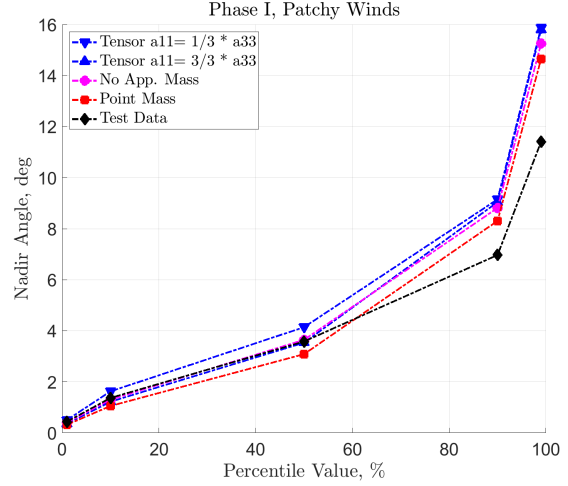
Results for the drogue parachute drop test case (Phase I) nadir statistical results are summarized in Figure 11. Figure 11a shows the results for baseline winds, while Figure 11b shows patchy winds. Results for the lateral rates are shown in Figure 12, with similar subfigure breakdown. Results show that the nadir angles and lateral rates have a secondary sensitivity to the apparent mass model, while the wind model has a first-order effect. This is partly due to the diameter of the drogue parachute being about half that of the main parachute, since the forces due to apparent mass are expected to be less than 10 % of those on the main parachute (assuming equal density). No combination of wind or apparent mass model was able to bound the lateral attitude rates from test, however the patchy atmospheric model was able to approach the test data more closely, indicating that unmodeled wind-induced dynamics could be a driving factor.

2. Phase II

Results for the main parachute closed-aeroshell drop test case (Phase II) nadir statistical results are summarized in Figure 13, while results for the lateral attitude rates are shown in Figure 14. Results here indicate a much greater sensitivity to the apparent mass model than in the Phase I study. The point mass results under-predict the test data everywhere except the 99-th percentile in the patchy results, while the no apparent mass and tensor apparent mass results bound the test data nearly everywhere. The tensor results indicate a sensitivity to the ratio of A_{11}/A_{33} , with a lower ratio driving higher results in both the nadir and lateral attitude rates across the percentile range. The $A_{11}/A_{33} = 1$ results for the lateral attitude rate in the baseline wind configuration show the best agreement to test in the baseline wind case of all of the Phase II results.

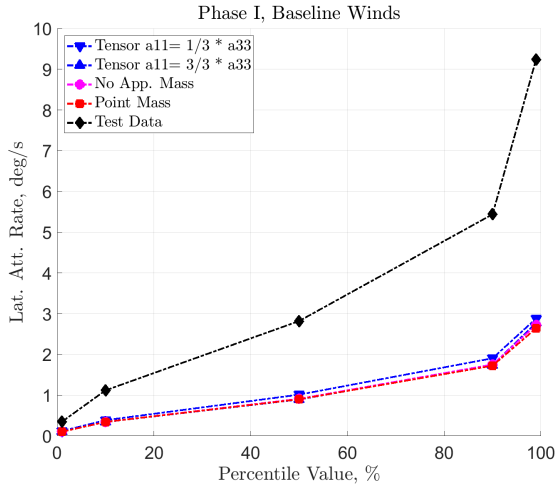


(a) Nadir Baseline Sensitivities to Apparent Mass

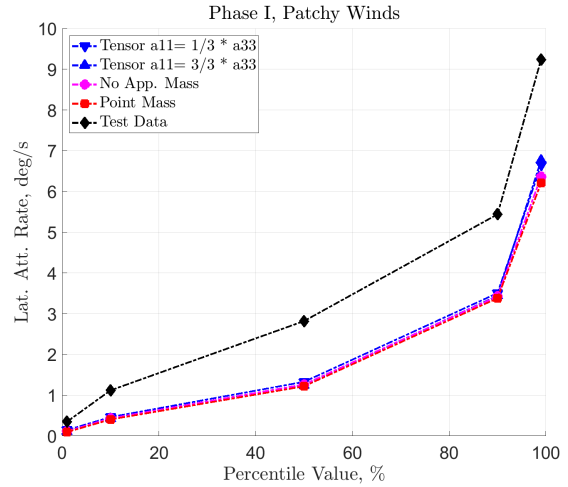


(b) Nadir Patchy Sensitivities to Apparent Mass

Fig. 11 Phase I Nadir Sensitivities to Apparent Mass



(a) Lat. Att. Rate Baseline Sensitivities to Apparent Mass



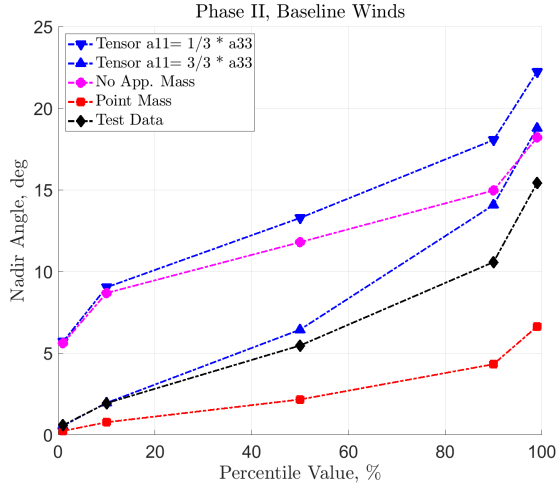
(b) Lat. Att. Rate Patchy Sensitivities to Apparent Mass

Fig. 12 Phase I Lat. Att. Rate Sensitivities to Apparent Mass

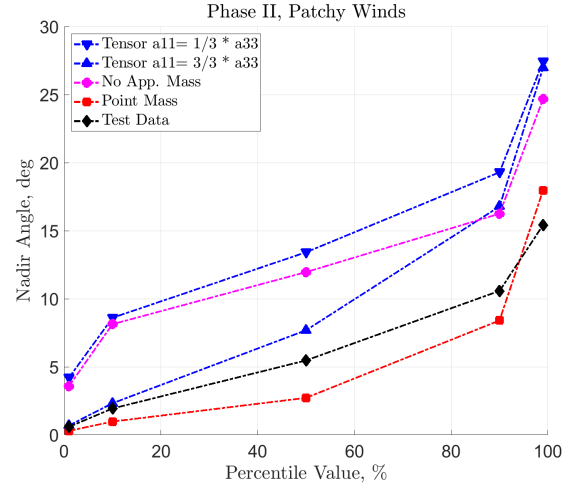
3. Phase III

Stowed Results for the open backshell stowed (Phase III, Stowed) drop test case are shown in Figure 15. Results indicate the tensor model creates the closest agreement to test in nadir angle above roughly the 50-th percentile, which is the area of more concern. The lateral attitude rate indicates a good agreement between the $A_{11}/A_{33} = 1$ configuration and test data across the entire percentile range, while the point mass model configuration overpredicts the test data by a large margin. The $A_{11}/A_{33} = 1/3$ tensor configuration also overpredicts the test data.

Posed Results for the open backshell posed (Phase III, Posed) drop test case are shown in Figure 16. Nadir results indicate the tensor model creates a closer match to test than the point mass model at higher percentiles. The lateral attitude rate results indicate that the $A_{11}/A_{33} = 1$ tensor configuration creates the best comparison, while the $A_{11}/A_{33} = 1/3$ tensor configuration and the point mass model result in an over-prediction of test data by roughly the same margin.

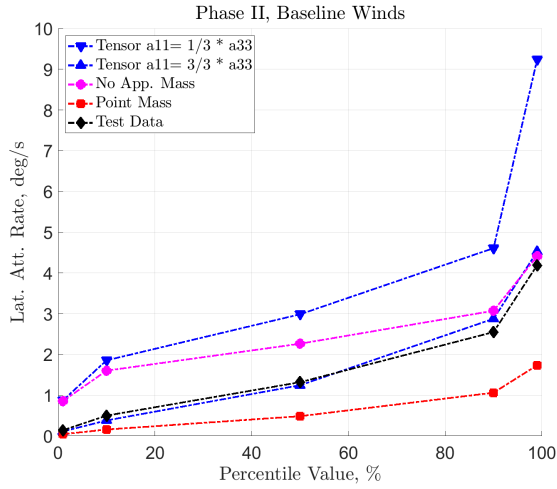


(a) Nadir Baseline Sensitivities to Apparent Mass

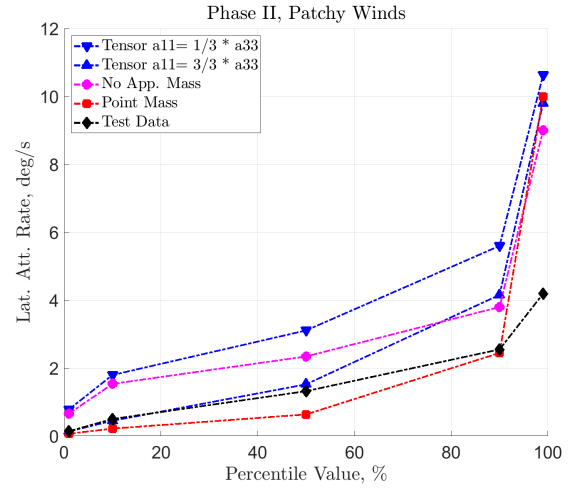


(b) Nadir Patchy Sensitivities to Apparent Mass

Fig. 13 Phase II Nadir Sensitivities to Apparent Mass



(a) Lat. Att. Rate Baseline Sensitivities to Apparent Mass

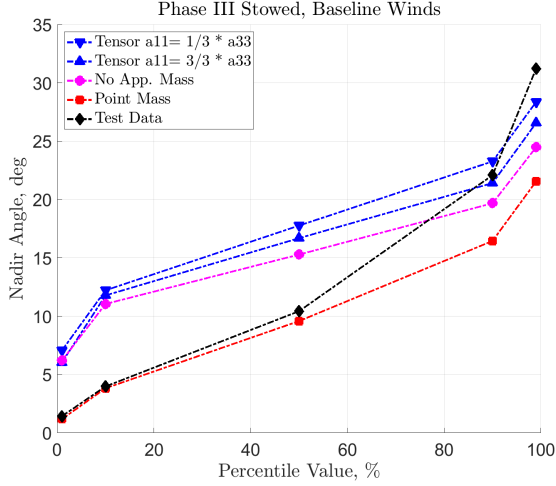


(b) Lat. Att. Rate Patchy Sensitivities to Apparent Mass

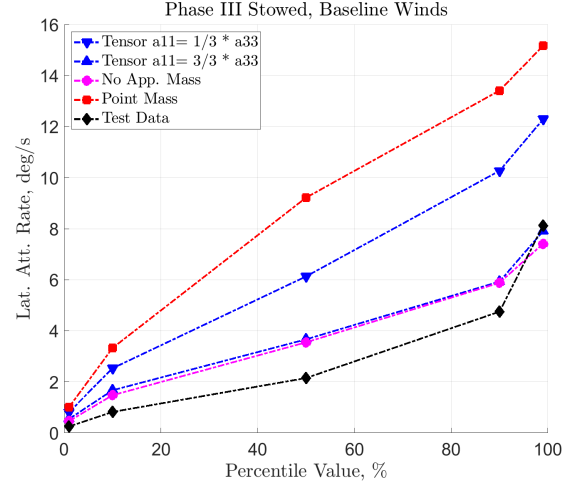
Fig. 14 Phase II Lat. Att. Rate Sensitivities to Apparent Mass

4. Summary of Parachute Apparent Mass Model Validation

Overall, the apparent mass model validation indicates that the simulated system dynamics for the drogue parachute with the entry vehicle (Phase I) are relatively insensitive to the apparent mass model. Variations in the system dynamics for drogue parachute descent are driven by the wind model. On the other side, for the main parachute descent (Phases II and III), the system dynamics are influenced by the apparent mass model and vary considerably between the heritage (point mass) model and the two variations of the tensor model. While none of the apparent mass models can reproduce test data for all parachute/payload configurations evaluated, the tensor model was able to bound the system dynamics from Phase II (main parachute and entry vehicle with heatshield) and, compared to the point mass model, improved the system dynamics predictions for Phase III. Based on these observations, it was decided to proceed with implementing the tensor apparent mass model in the Dragonfly EDL simulation.

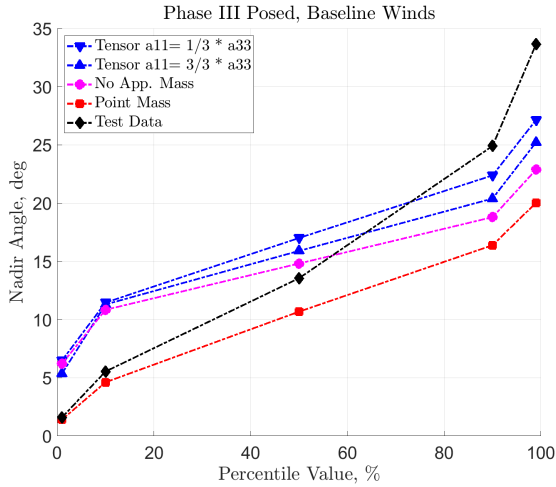


(a) Nadir Sensitivities to Apparent Mass

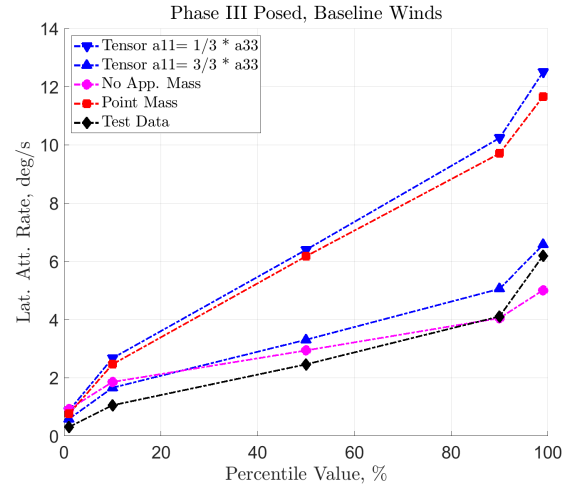


(b) Lat. Att. Rate Sensitivities to Apparent Mass

Fig. 15 Phase III Stowed Sensitivities to Apparent Mass



(a) Posed Nadir Sensitivities to Apparent Mass



(b) Posed Lat. Att. Rate Sensitivities to Apparent Mass

Fig. 16 Phase III Posed Sensitivities to Apparent Mass

B. Titan Simulation

The impact of the tensor apparent mass model was evaluated in the POST2 simulation of the Dragonfly mission using the Monte Carlo setup described in Ref [2]. The apparent mass model parameters for the drogue and main parachutes were chosen based on findings from the model validation. The model parameters are summarized as follows:

- 1) Model parameters for the drogue parachute are constant such that $a_{11} = a_{33} = 0.094$.
- 2) For the main parachute a_{33} is constant with a value of 0.094. a_{11} is dispersed and is correlated to a_{33} .
- 3) a_{11} is dispersed uniformly between $a_{11} = a_{33} \times \left[\frac{1}{3}, 1\right]$.

The results in this section are focused on the metrics evaluated in assessing the tensor apparent mass with respect to the heritage (point mass) model, in particular, the impact on the entry vehicle system dynamics.

1. Impact of Drogue Apparent Mass Model

Figure 17 - 19 compare the main deployment conditions with respect to the nadir angle, lateral rates ($\dot{q}$, $\dot{r}$ components), and the atmosphere relative velocity of the vehicle. The tensor apparent mass model changes parachute

mode from mostly pendulum to a coning mode which leads to a higher nadir angle at main deploy (Fig 17). The total drag force the parachute imparts is also higher with the tensor model which produces higher deceleration and leads to a slower descent rate at main deploy (Fig 19). However, the lateral rates of the entry vehicle are relatively unchanged.

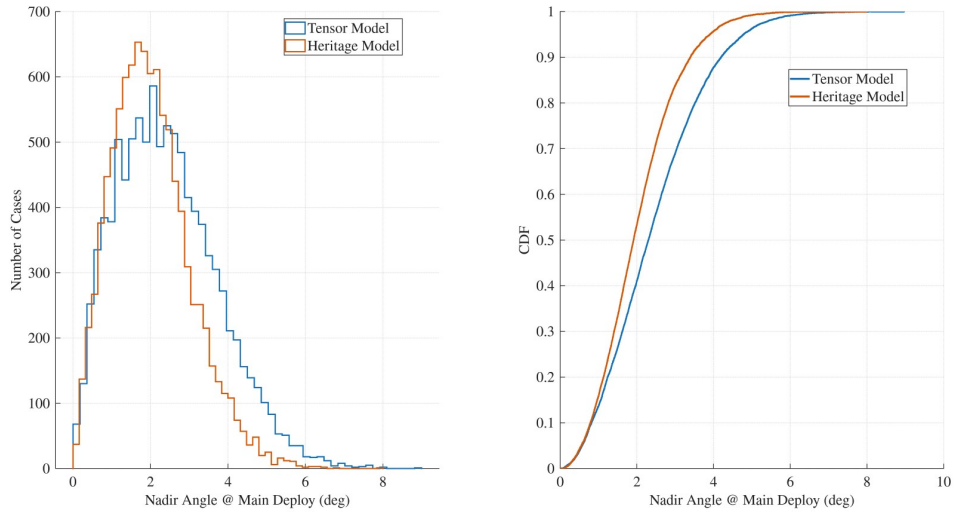


Fig. 17 Entry vehicle nadir angle at main deploy

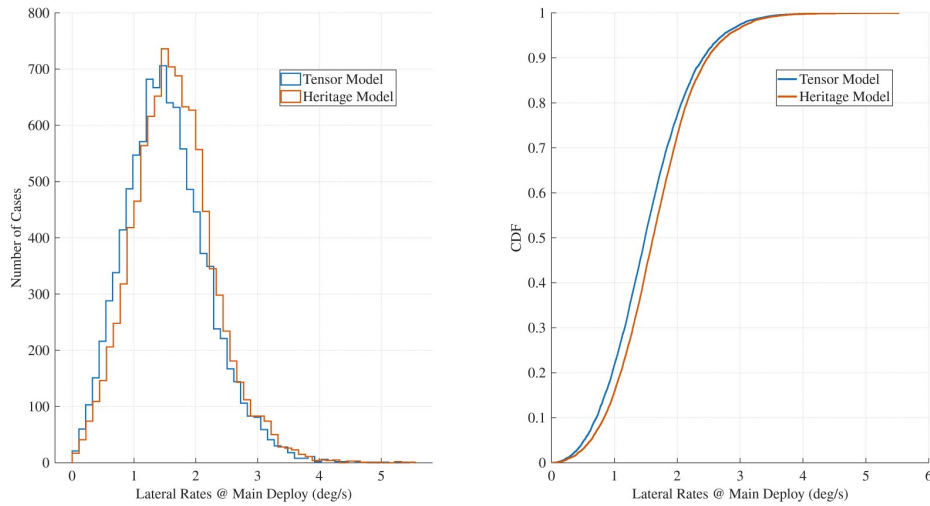


Fig. 18 Entry vehicle lateral rate at main deploy

2. Impact of Main Apparent Mass Model

The critical event in the Dragonfly EDL Phase is the lander deploy event, in which the lander will separate from the entry vehicle and begin its first flight. To ensure safe separation, understanding the dynamics of the entry vehicle is paramount. Figure 20 - 22 highlight some of the key elements. At this point of the descent, the entry vehicle's descent rate is approximately the same as the horizontal winds which leads to high α_T . This metric becomes less insightful to characterize how dynamic the vehicle is at lander separation. To this end, the vehicle is evaluated using the nadir definition (Figure 10a) rather than α_T . Figure 20 - 21 shows a much stronger mode change in which the vehicle is coning significantly more with the tensor model. This is expected due to the larger canopy creating larger apparent mass effects. These conditions are provided to the lander team to design and evaluate the performance of the vehicle. Finally, the atmosphere descent rate of the vehicle at lander release (critical for safe initiation of powered flight) is required to be

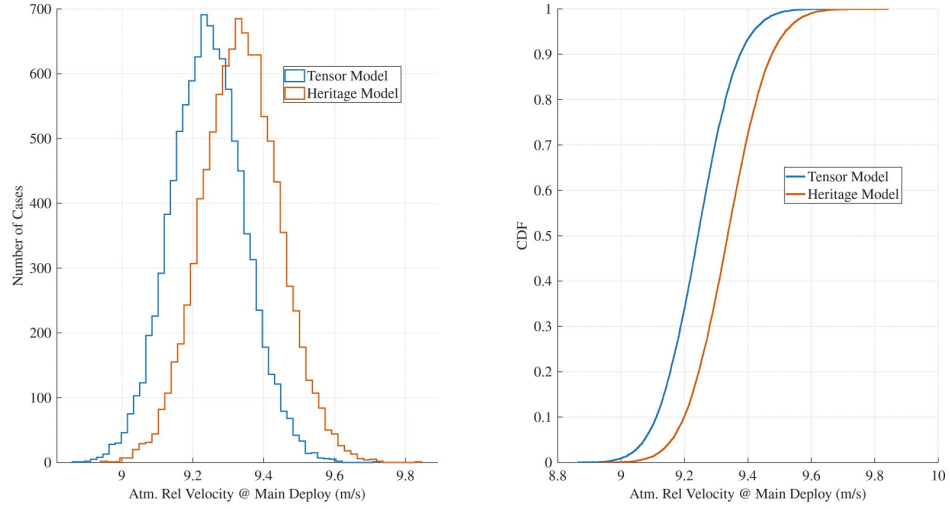


Fig. 19 Entry vehicle atmosphere relative velocity at main deploy

below 3.1 m/s at the 99 %-tile. Figure 22 shows a slightly wider distribution, but still within the requirement.

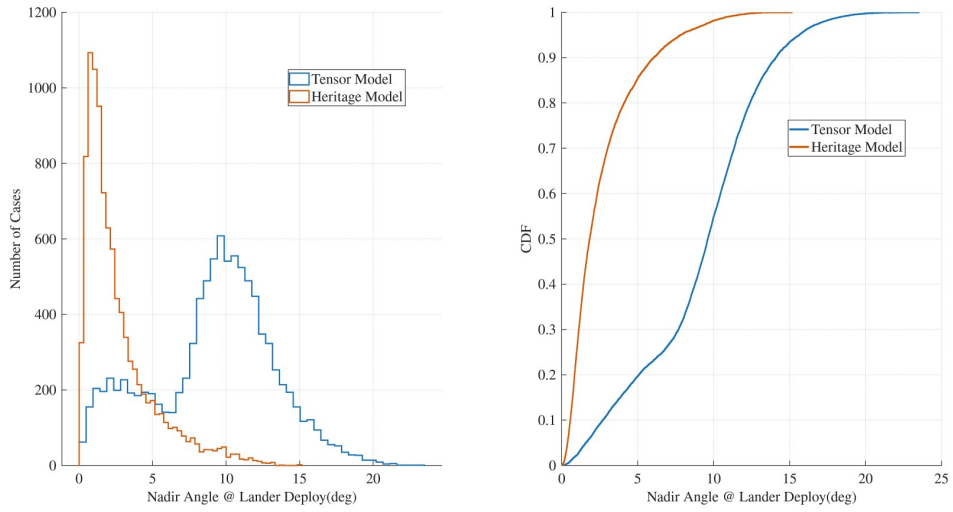


Fig. 20 Entry vehicle nadir angle at lander deploy

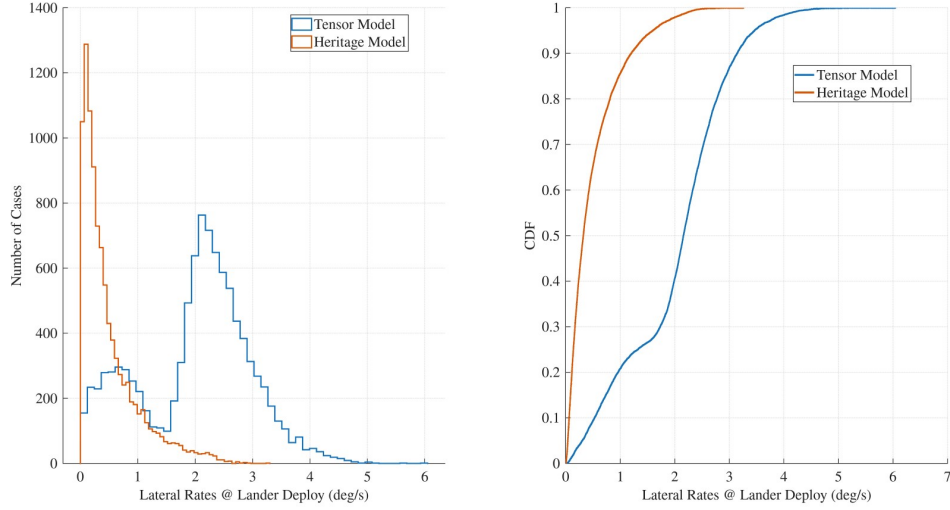


Fig. 21 Entry vehicle lateral rate at lander deploy

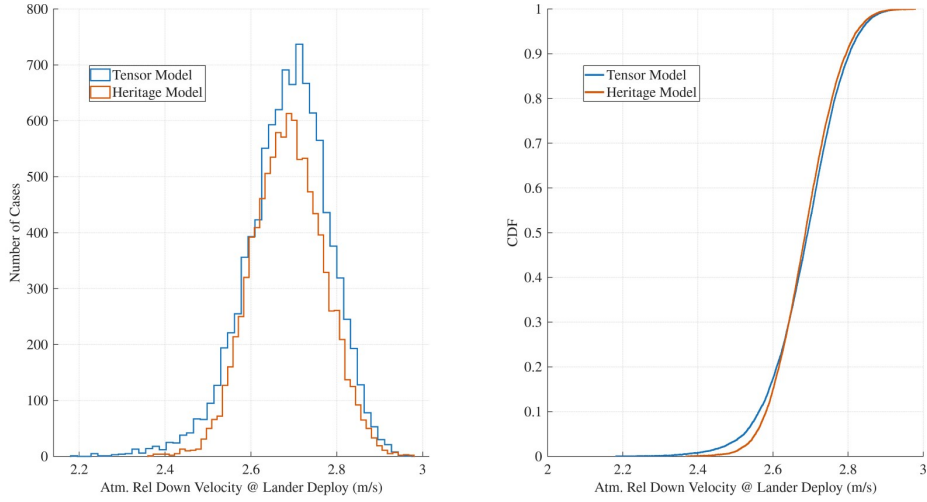


Fig. 22 Entry vehicle atmosphere relative down velocity at lander deploy

IX. Conclusion

Quantifying the dynamic motions of the Dragonfly vehicle during key entry, descent, and landing events (main parachute deployment, heatshield separation, and lander release) is critical to develop a EDL system capable of delivering the lander safely to the surface of Titan. While parachute models from past Mars missions have served as a foundation to build models to evaluate the Dragonfly system dynamics, the validity of the heritage point apparent mass model for parachute descent has been questioned due to the increased sensitivity to high-atmospheric density compared to the low-density Martian atmospheric environment. The tensor apparent mass model is a physics derived formulation that defines the non-circulatory forces and moments acting on a parachute as a function of three model parameters. The apparent mass model parameters are estimated from test data of a subscale parachute in a wind tunnel. The parameter estimation technique was developed from the assumption that the parachute oscillates in steady-state coning motion at a constant coning rate and elevation angle given a particular windtunnel air density and windspeed. Before implementing the model on the Dragonfly simulation, the tensor apparent mass formulation was re-defined in terms of the coordinate system used by the Dragonfly mission. Since the tensor model is defined implicitly (translational and rotational accelerations are model variables) the linear and angular accelerations are determined from filtered linear and angular velocity components to ensure that the explicit ODE integration method (Runge-Kutta) is numerically

stable. The tensor apparent mass model is validated by assessing the simulated system dynamics (nadir angle and lateral rotation rates) against those measured during subscale drop tests of the Dragonfly drogue and main parachute descent configurations. The model validation showed that the sensitivity of the system dynamics to the apparent mass model vary significantly between the drogue parachute descent (Phase I) and the main parachute descent (Phases II and III). The simulated entry vehicle system dynamics during drogue parachute descent had a small sensitivity to the apparent mass model. In contrast, the wind model had a first-order effect on the simulated entry vehicle dynamics under drogue parachute. This is partly due to the diameter of the drogue parachute being about half of the main parachute, since the forces due to apparent mass are expected to be less than 10 % of those on the main parachute (assuming equal density). Neither the wind model nor the apparent mass model were able to bound the lateral attitude rates from test, however the patchy atmospheric model was able to approach the test data more closely, indicating that unmodeled dynamics could be due to unmodeled winds. The main parachute drop test case (Phases II and III) shows a more noticeable influence from the apparent mass model on the simulated system dynamics. However, while the Phase II test was bounded by the tensor model using both wind models, the 99 %-tile percentile nadir angle from the Phase III tests (stowed and posed) were not bounded by any of the apparent mass models. Overall, the model validation indicates that, compared to the point apparent mass model, the tensor apparent mass model results in an improvement in the predicted entry vehicle system dynamics for the main parachute. However, compared to the test data, the nadir angles are under-predicted, which indicates that the tensor apparent mass model does not account for all dynamics observed during drop tests. The tensor model was applied to the Dragonfly EDL simulation to perform a side-by-side comparison with the heritage (point mass) model. Overall, the entry vehicle dynamics (nadir angle and lateral angular rates) increase using the tensor model (particularly for the main parachute) which is consistent with results of the simulated drop test (model validation). Tensor apparent mass model is shown to be more conservative than the heritage point mass model and is therefore more likely to bound the system dynamics that the entry vehicle/lander could experience at Titan. While the tensor apparent mass model improves the system dynamics predictions over the heritage point mass model, it does not account for all the dynamics of the parachute/payload system. Furthermore, certain assumptions made to derive the non-circulatory forces and moments are not valid for parachutes. Additional evaluations should be conducted to investigate additional mechanisms (intrinsic to the parachute) that can excite the parachute/payload system dynamics.

Acknowledgments

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References

- [1] Barnes, J. W., Turtle, E. P., Trainer, M. G., Lorenz, R. D., MacKenzie, S. M., Brinckerhoff, W. B., Cable, M. L., Ernst, C. M., Freissinet, C., Hand, K. P., et al., "Science goals and objectives for the dragonfly titan rotorcraft relocatable lander," *The Planetary Science Journal*, Vol. 2, No. 4, 2021, p. 130.
- [2] Pensado, A. R., Winski, R. G., Williams, J. W., and Robb, C., "Dragonfly Mission Entry and Descent Modeling and Simulation Overview," AIAA 2025-2237, *AIAA SciTech 2025*, Orlando, FL, 2025. <https://doi.org/10.2514/6.2025-2237>.
- [3] Cruz, J. R., Way, D., Shidner, J., Davis, J. L., Powell, R. W., Kipp, D., Adams, D. S., Sengupta, A., Witkowski, A., and Kandis, M., "Parachute models used in the Mars Science Laboratory entry, descent, and landing simulation," *AIAA Aerodynamic Decelerator Systems (ADS) Conference*, 2013, p. 1276.
- [4] Marietta, M., "Viking Lander "As Built" Performance Capabilities, NAS1-9000," 1976.
- [5] Imlay, F. H., "The complete expressions for added mass of a rigid body moving in an ideal fluid," Tech. rep., 1961.
- [6] Ibrahim, S. K., "Experimental Determination of the Apparent Moment of Inertia of Parachutes," Tech. rep., 1965.
- [7] Zumwalt, C., Cruz, J., O'Farrell, C., and Keller, D., "Wind tunnel test of subscale ringsail and disk-gap-band parachutes," *34th AIAA Applied Aerodynamics Conference*, 2016, p. 3882.
- [8] Eaton, J. A., "Validation of a Computer Model of a Parachute," Ph.D. thesis, University of Leicester, 1982.
- [9] Ginn, J. M., Clark, I. G., and Braun, R. D., "Parachute dynamic stability and the effects of apparent inertia," *AIAA Atmospheric Flight Mechanics Conference*, 2014, p. 2390.

- [10] Cruz, J., Shafner, J., Duvall, B., Gray, M., Khasawneh, I., Meek, C., Miller, J., and Chamberlain, B., “An Investigation of the Dragonfly Mission Aeroshell/Parachute Dynamics Through Subscale Drop Tests,” *AIAA AVIATION FORUM AND ASCEND 2024*, 2024, p. 4329.
- [11] Shafner, J., Duvall, B., Khasawneh, I., and Meek, C., “An Investigation of the Dragonfly Mission Lander/Backshell/Parachute Dynamics through Subscale Drop Tests,” *AIAA Aerodynamic Decelerator Systems Conference and Seminar 2026*, 2026.
- [12] Williams, J., Pensado, A. R., Winski, R., Robb, C. S., Manwell, M., Guecha-Ahumada, N. G., and Shafner, J., “Simulation of the Dragonfly Subscale Drop Tests,” *AIAA SciTech 2026 Forum*, 2026. <https://doi.org/10.2514/6.2026-1957>.

Appendix

A. General Expression for Apparent Mass Aerodynamics

The general expressions (forces and moments) for the apparent mass aerodynamic terms are based on work by Imlay[5]. The expressions were recasted by Juan R. Cruz using parameters relevant to parachute literature:

$$\begin{aligned}
 \begin{Bmatrix} F_{x,NC} \\ F_{y,NC} \\ F_{z,NC} \end{Bmatrix} &= \begin{bmatrix} -A_{11} & -A_{12} & -A_{13} \\ -A_{12} & -A_{22} & -A_{23} \\ -A_{13} & -A_{23} & -A_{33} \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} + \begin{bmatrix} -A_{14} & -A_{15} & -A_{16} \\ -A_{24} & -A_{25} & -A_{26} \\ -A_{34} & -A_{35} & -A_{36} \end{bmatrix} \begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} \\
 &+ \begin{bmatrix} 0 & -A_{34} & A_{24} & -A_{35} & (A_{25} - A_{36}) & A_{26} \\ A_{34} & A_{35} & (A_{36} - A_{14}) & 0 & -A_{15} & -A_{16} \\ -A_{24} & (A_{14} - A_{25}) & -A_{26} & A_{15} & A_{16} & 0 \end{bmatrix} \begin{Bmatrix} p^2 \\ pq \\ pr \\ q^2 \\ qr \\ r^2 \end{Bmatrix} \\
 &+ \begin{bmatrix} 0 & -A_{13} & A_{12} & 0 & -A_{23} & A_{22} & 0 & -A_{33} & A_{23} \\ A_{13} & 0 & -A_{11} & A_{23} & 0 & -A_{12} & A_{33} & 0 & -A_{13} \\ -A_{12} & A_{11} & 0 & -A_{22} & A_{12} & 0 & -A_{23} & A_{13} & 0 \end{bmatrix} \begin{Bmatrix} up \\ uq \\ ur \\ vp \\ vq \\ vr \\ wp \\ wq \\ wr \end{Bmatrix}
 \end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} M_{x,\text{NC}} \\ M_{y,\text{NC}} \\ M_{z,\text{NC}} \end{pmatrix} &= \begin{bmatrix} -A_{14} & -A_{24} & -A_{34} \\ -A_{15} & -A_{25} & -A_{35} \\ -A_{16} & -A_{26} & -A_{36} \end{bmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} + \begin{bmatrix} -A_{44} & -A_{45} & -A_{46} \\ -A_{45} & -A_{55} & -A_{56} \\ -A_{46} & -A_{56} & -A_{66} \end{bmatrix} \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} + \begin{bmatrix} 0 & -A_{46} & A_{45} & -A_{56} & (A_{55} - A_{66}) & A_{56} \\ A_{46} & A_{56} & (A_{66} - A_{44}) & 0 & -A_{45} & -A_{46} \\ -A_{45} & (A_{44} - A_{55}) & -A_{56} & A_{45} & A_{46} & 0 \end{bmatrix} \begin{pmatrix} p^2 \\ pq \\ pr \\ q^2 \\ qr \\ r^2 \end{pmatrix} \\
&+ \begin{bmatrix} 0 & -A_{13} & A_{12} & -A_{23} & (A_{22} - A_{33}) & A_{23} \\ A_{13} & A_{23} & (A_{33} - A_{11}) & 0 & -A_{12} & -A_{13} \\ -A_{12} & (A_{11} - A_{22}) & -A_{23} & A_{12} & A_{13} & 0 \end{bmatrix} \begin{pmatrix} u^2 \\ uv \\ uw \\ v^2 \\ vw \\ w^2 \end{pmatrix} \\
&+ \begin{bmatrix} 0 & -A_{16} & A_{15} & -A_{34} & -(A_{26} + A_{35}) & (A_{25} - A_{36}) & A_{24} & (A_{25} - A_{36}) & (A_{26} + A_{35}) \\ (A_{16} + A_{34}) & A_{35} & (A_{36} - A_{14}) & A_{26} & 0 & -A_{24} & (A_{36} - A_{14}) & -A_{15} & -(A_{16} + A_{34}) \\ -(A_{15} + A_{24}) & (A_{14} - A_{25}) & -A_{26} & (A_{14} - A_{25}) & (A_{15} + A_{24}) & A_{16} & -A_{35} & A_{34} & 0 \end{bmatrix} \begin{pmatrix} up \\ uq \\ ur \\ vp \\ vq \\ vr \\ wp \\ wq \\ wr \end{pmatrix}
\end{aligned}$$