Solving the Radiation Transport Equation for Fibrous Insulation Using FE-DOM

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AFOSR / NASA Collaboration IA1-42919

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Outline

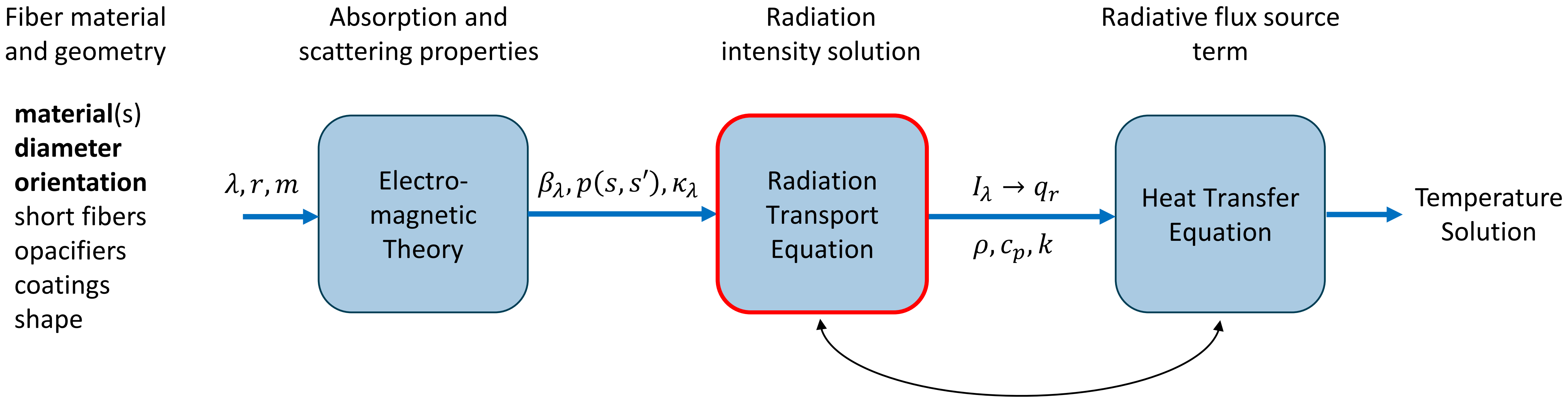
- Overview of research
- Radiation transport equation (RTE)
- Phase function expansion
- Solve the RTE using the Finite Element method and discrete ordinates method (FE-DOM)
- Verification with analytical solution
    - Isotropic and Rayleigh scattering
- Validation with measured reflectance
    - LI-900, Q-Fiber Felt
- Next Steps

Overview

Objective

Develop a predictive capability for the radiation heat transfer performance of high temperature fibrous insulation.

- Capability supports optimal design, increased reliability, and efficient manufacturing for entry thermal protection.
- Requires solving the radiation transport equation (RTE) in addition to the heat transfer equation.



Coupled Equations

- Fiber emission is a function of temperature
- Radiative flux is a function of intensity

RTE

Radiation Transport Equation

- The RTE defines the change in intensity as radiation propagates through an absorbing, emitting, and scattering medium.
- The intensity solution is a function of spatial (y or τ) and angular coordinates (ξ and φ).

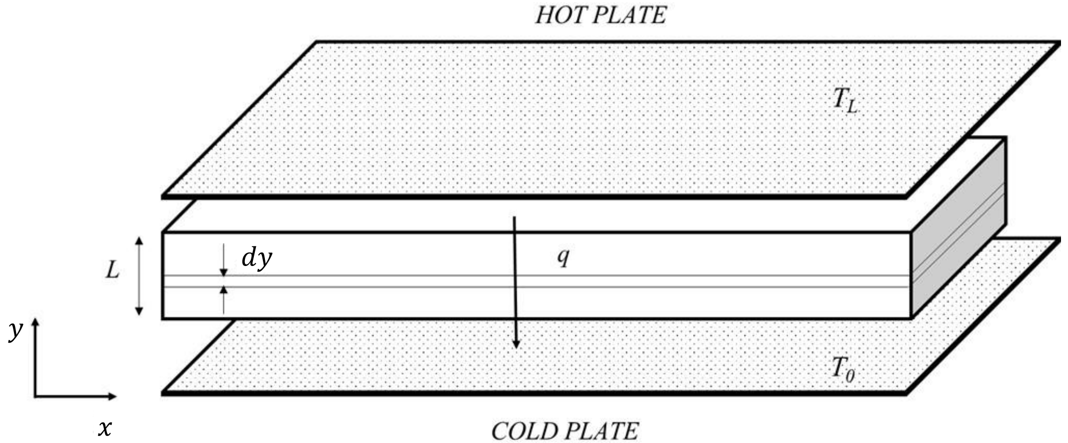
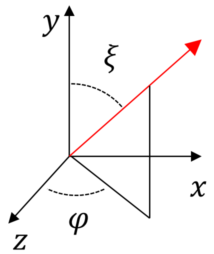


Image from Kamran Daryabeigi



Fibrous Medium Properties

κ_λ : absorption coefficient
$\sigma_{s,\lambda}$: scattering coefficient
$\beta_\lambda = \kappa_\lambda + \sigma_{s,\lambda}$: extinction coefficient
$p(\eta)$: scattering phase function

$\tau = \int_0^y \beta\, dy$: optical depth
$\mu = \cos\xi$: direction cosine
I_λ : intensity for wavelength λ
$I_{\lambda B}$: Planck black-body intensity
Ω : solid angle

Plane-parallel scattering medium

$$\boxed{\mu\frac{\partial I_\lambda(y,\mu,\varphi)}{\partial y} = -\beta_\lambda I_\lambda(y,\mu,\varphi) + \frac{\sigma_{s,\lambda}}{4\pi}\int_0^{4\pi} I_\lambda(y,\mu',\varphi')p(\mu,\varphi;\mu',\varphi')d\Omega' + \kappa_\lambda I_{\lambda B}(y) + \frac{\sigma_{s,\lambda}}{4\pi}Fe^{-\int_0^y \beta\,dy/\mu_0}p(\mu,\varphi;\mu_0,\varphi_0)}$$

Change in intensity over dx | absorption and scattering into medium (extinction) | scattering from medium | fiber emission (source) ⟷ OR ⟷ collimated irradiation (source) Magnitude: F Direction: (μ_0, φ_0)

Phase Function Expansion

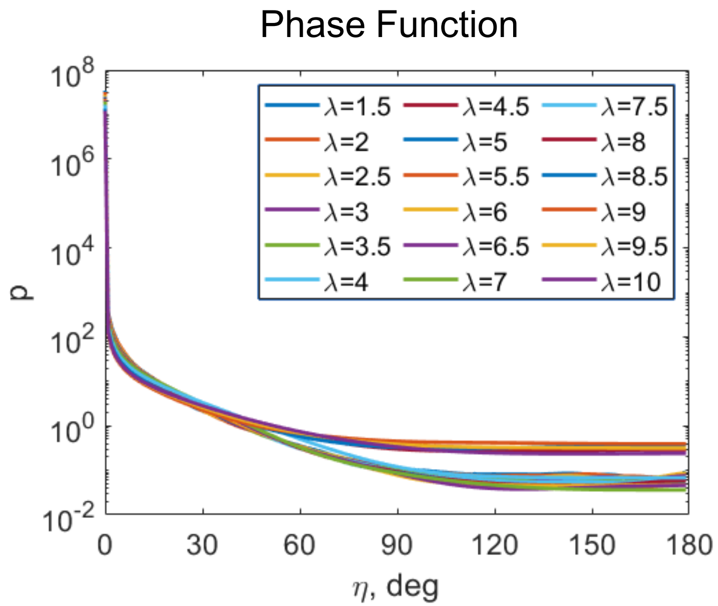
- Transform $p(\eta)$ $\quad \cos\eta = \mu_i\mu_s + \sqrt{(1-\mu_i^2)(1-\mu_s^2)}\cos(\varphi_s - \varphi_i)$

- Legendre polynomial expansion for phase function [Chandrasekhar 1960]

$$p(\cos\eta) = \sum_{l=0}^{N} w_l P_l(\cos\eta) \qquad w_l = \frac{2l+1}{2}\int_0^{\pi} p(\eta)P_l(\cos\eta)\sin(\eta)\,d\eta$$

- Addition theorem of spherical harmonics [Arfken 1985]

$$P_l(\cos\eta) = P_l(\mu_i)P_l(\mu_s) + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\mu_i)P_l^m(\mu_s)\cos m(\varphi_s - \varphi_i)$$



- Transformed phase function expansion

$$p(\eta) \to \sum_{m=0}^{M} p^{(m)}(\mu_i, \mu_s)\cos m(\varphi_s - \varphi_i)$$

$$p^{(m)}(\mu_i, \mu_s) = (2 - \delta_{0,m})\left\{\sum_{l=m}^{N} w_l \frac{(l-m)!}{(l+m)!}P_l^m(\mu_i)P_l^m(\mu_s)\right\}$$

- Similarly, expand intensity solution in the RTE

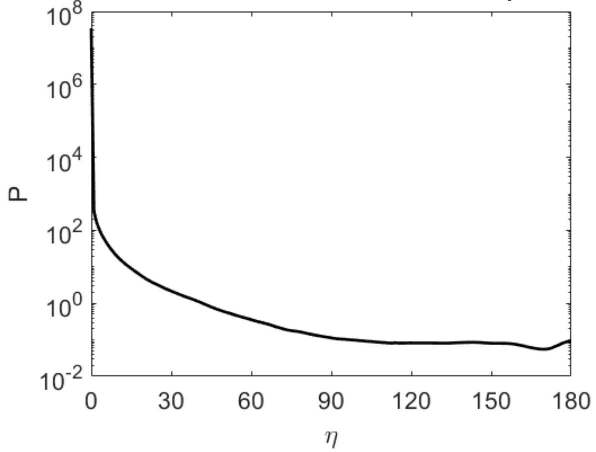
$$I(\tau, \mu, \varphi) = \sum_{m=0}^{M} I^{(m)}(\tau, \mu)\cos m(\varphi_0 - \varphi)$$

- Solve for $I_\lambda^{(m)}(\tau, \mu)$ for M terms

$$\mu\beta\frac{\partial I^{(m)}(\tau,\mu)}{\partial\tau} + \beta I^{(m)}(\tau,\mu) = \frac{1}{2}\int_{-1}^{1}\sigma_s p^{(m)}(\mu,\mu')I^{(m)}(\tau,\mu')d\mu' + \frac{1}{4\pi}\sigma_s p^{(m)}(\mu,\mu_0)e^{-\tau/\mu_0}$$

Example: $p^{(0)}(\mu_i, \mu_s)$ for LI-900, λ=1.5 μm

Phase function for λ=1.5 μm



- Compute the zeroth order phase function for LI-900 (silica fibers, type B2.3), λ=1.5 μm [Lee 1996]

- Transform $p(\eta) \rightarrow p^{(0)}(\mu_i, \mu_s)$
- Three approaches:

1) Azimuthally averaged [Lee 1998]

$$p^{(0)}(\mu_i, \mu_s) = \frac{1}{2\pi}\int_0^{2\pi} p(\eta)\, d\varphi_s$$
$$= \frac{1}{\pi}\int_{\eta_1}^{\eta_2} \frac{\sigma p(\xi_i, \omega_i, \eta)\sin(\eta)}{\sqrt{(1-\mu_i^2)(1-\mu_s^2)-(\cos(\eta)-\mu_i\mu_s)^2}}\, d\eta$$

$$\cos(\eta_1) = \mu_i\mu_s + \sqrt{(1-\mu_i^2)(1-\mu_s^2)}$$
$$\cos(\eta_2) = \mu_i\mu_s - \sqrt{(1-\mu_i^2)(1-\mu_s^2)}$$

2) Eddington (m=0, N=1 legendre expansion) [Joseph 1976]

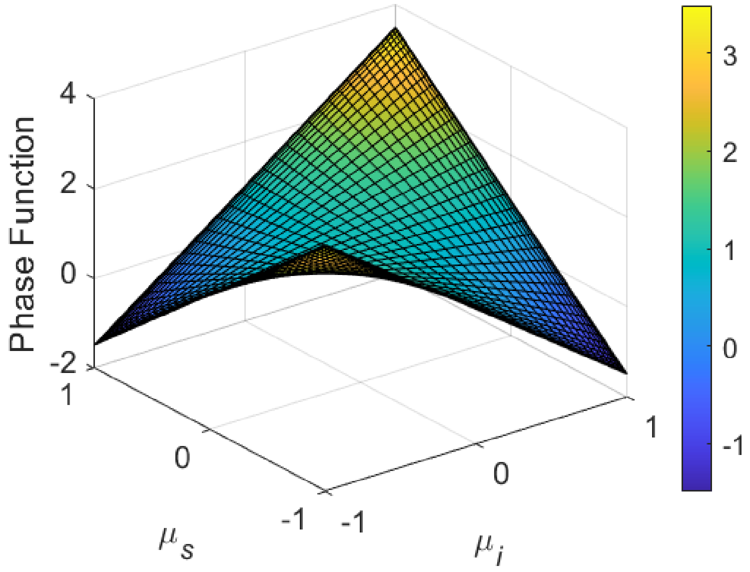
$$p^{(0)}(\mu_i, \mu_s) = 1 + 3g\mu_i\mu_s$$

3) Legendre polynomial expansion

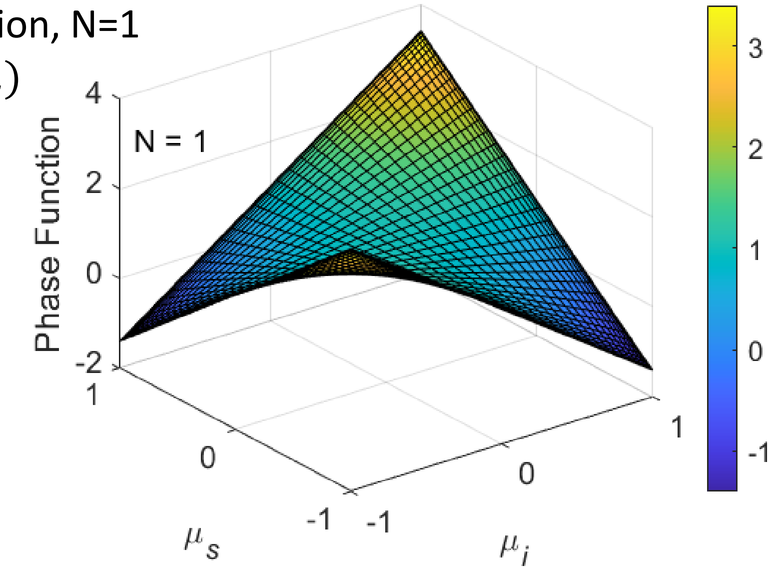
$$p^{(0)}(\mu_i, \mu_s) = \sum_{l=0}^{N} w_l P_l(\mu_i) P_l(\mu_s) \qquad w_l = \frac{2l+1}{2}\int_0^{\pi} p(\eta) P_l(\cos\eta)\sin(\eta)\, d\eta$$

Example: $p^{(0)}(\mu_i, \mu_s)$ for LI-900, λ=1.5 μm

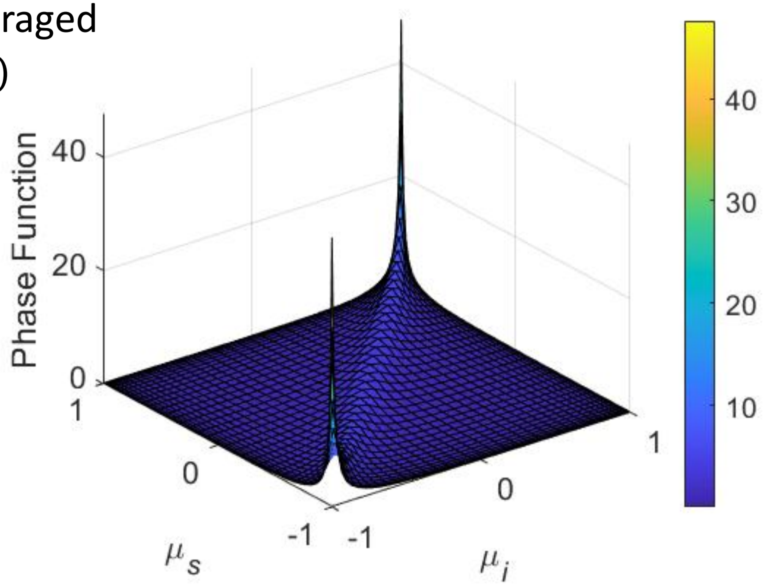
Eddington
$p^{(0)}(\mu_i, \mu_s)$



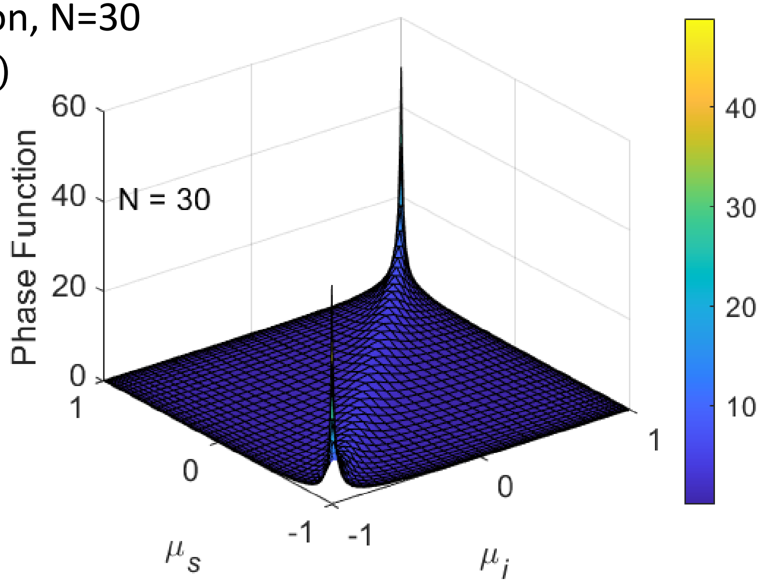
Legendre Expansion, N=1
$p^{(0)}(\mu_i, \mu_s)$



Azimuthally Averaged
$p^{(0)}(\mu_i, \mu_s)$



Legendre Expansion, N=30
$p^{(0)}(\mu_i, \mu_s)$

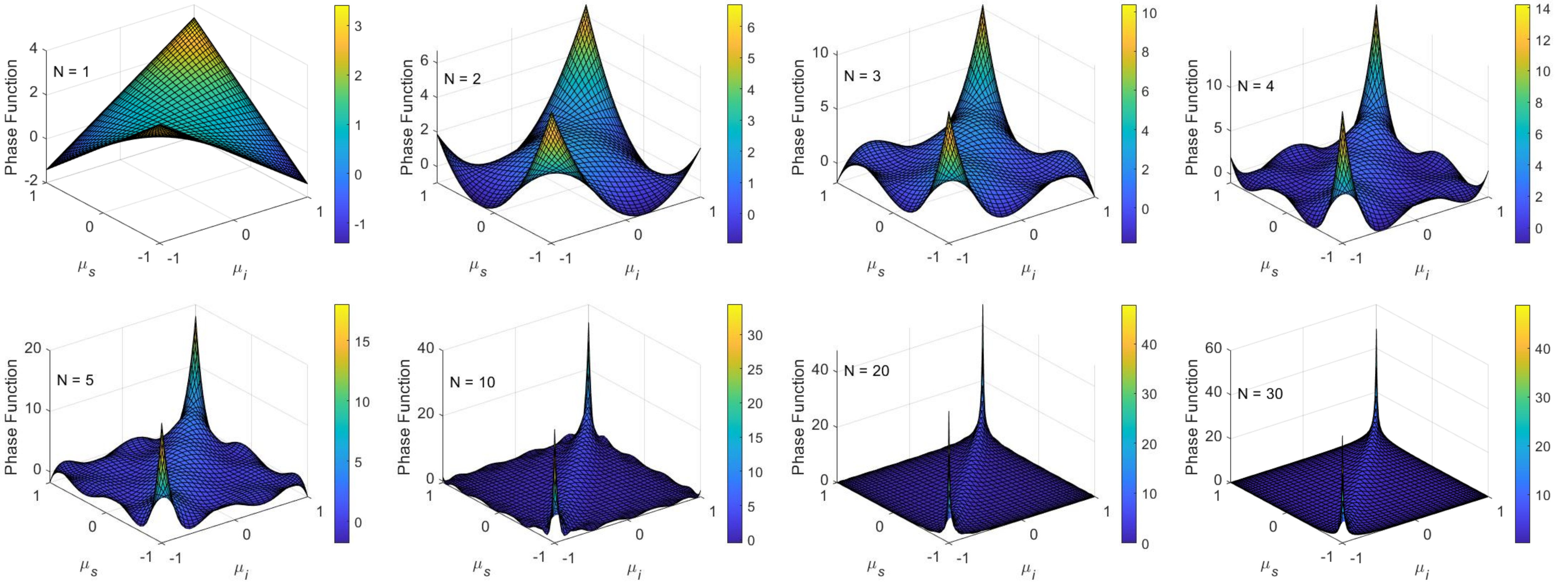


Example: $p^{(0)}(\mu_i, \mu_s)$ for LI-900, λ=1.5 μm

- Zeroth term Legendre expansion (m=0) for $N \geq 1$ and $\lambda = 1.5\ \mu m$

$$p^{(0)}(\mu_i, \mu_s) = \sum_{l=0}^{N} w_l P_l(\mu_i) P_l(\mu_s)$$

Equivalent to Eddington



FE-DOM

$$\boxed{\mu\beta\frac{\partial I(\tau,\mu)}{\partial\tau}+\beta I(\tau,\mu)=\frac{1}{2}\int_{-1}^{1}\sigma_s p(\mu;\mu')I(\tau,\mu')d\mu'+\frac{\sigma_s}{4\pi}Fp\,(\mu;\mu_0)e^{-\tau/\mu_0}}$$

Ω: Angular Domain, $\mu \in [-1,1]$
$\mathcal{D}$: Spatial Domain, $\tau \in [0,\tau_1]$

- The intensity solution, $I(\tau,\mu)$, is a function of $\tau \in \mathcal{D}$ and $\mu \in \Omega$
- Discretize the angular domain, Ω, by the Discrete Ordinates Method (DOM)
  - Replace the integration by Gauss-Legendre quadrature of *n* discrete directions

Direction cosine
$\mu = \cos(\xi)$

$$\int_{-1}^{1} p(\mu;\mu')I(\tau,\mu')d\mu' = \sum_{j=1}^{n} w_j p(\mu,\mu_j)I(\tau,\mu_j)$$

  - Leads to a set of *n* partial differential equations

$$R_d\left(\{I_j\}_{j=1}^{n}\right)=\mu_d\beta\frac{\partial I_d}{\partial\tau}+\beta I_d-\frac{\sigma_s}{2}\sum_{j=1}^{n}w_j p(\mu_d,\mu_j)I_j(\tau)-\frac{\sigma_s}{4\pi}Fp\,(\mu_d;\mu_0)e^{-\tau/\mu_0}=0 \qquad \text{for } d=1,2,\ldots,n$$

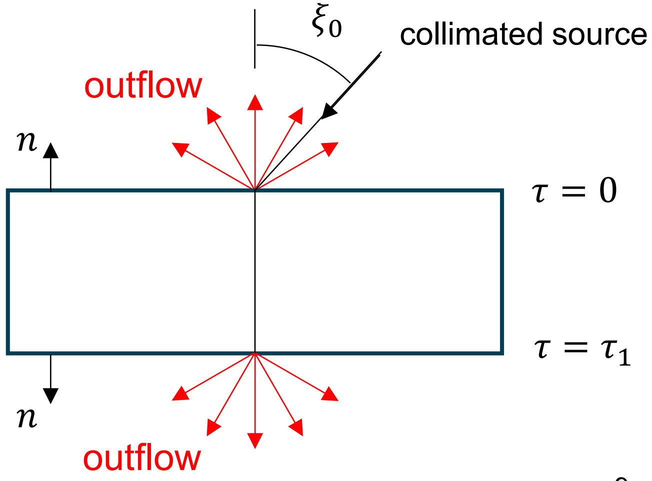
- Define inflow Γ^- and outflow Γ^+ boundaries with outward normal n

  Inflow Γ^-: $\partial\mathcal{D}_d^- = \{\tau \in \partial\mathcal{D} |\ \mu_d \cdot n < 0\}$

  Outflow Γ^+: $\partial\mathcal{D}_d^+ = \{\tau \in \partial\mathcal{D} |\ \mu_d \cdot n > 0\}$

- Dirichlet boundary conditions

$$I = I^- \quad \forall(\tau,\mu) \in \Gamma^-$$



FE-DOM

- Finite element method (FE)
  - Define spatial mesh $\mathcal{D}^h$ which approximates $\mathcal{D}$ and $\mathcal{V}_d^h = \{v \in L^2(\mathcal{D}^h) \text{ and } \mu_d \nabla v \in L^2(\mathcal{D}^h)\}$
  - Linear basis functions

- Streamline Upwind Petrov-Galerkin (SUPG) weak formulation:

$$\forall d = 1,2,\dots,n \qquad \int_{\mathcal{D}^h} R_d\left(\{I_j^h\}_{j=1}^n\right) \times \left(w^h + \gamma \mu_d \nabla w^h\right) d\tau = 0$$

Test function $w^h \in \mathcal{V}_d^h$
Trial function $I_d^h \in \mathcal{V}_d^h$
Streamline diffusion parameter γ

- Integration by parts on the first term introduces the boundary conditions

$$-\int_{\mathcal{D}^h} \mu_d I_d^h \nabla w^h \, d\tau + \underbrace{\boxed{\int_{\partial\mathcal{D}_d^+} \mu_d \cdot n I_d^h w^h \, d\tau}}_{\text{Outflow Boundary}} + \underbrace{\boxed{\int_{\mathcal{D}^h} \mu_d \nabla I_d^h \gamma \mu_d \nabla w^h \, d\tau}}_{\text{Artificial diffusion}} + \int_{\mathcal{D}^h} I_d^h\left(w^h + \gamma \mu_d \nabla w^h\right) d\tau$$

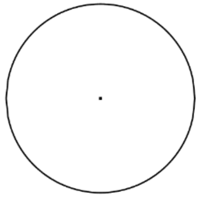
$$-\frac{\sigma_s}{2} \int_{\mathcal{D}^h} \sum_{j=1}^{n} w_j I_j(\tau) \left(w^h + \gamma \mu_d \nabla w^h\right) d\tau = \underbrace{\boxed{-\int_{\partial\mathcal{D}_d^-} \mu_d \cdot n I^- w^h \, d\tau}}_{\text{Inflow Boundary}} + \frac{\sigma_s}{4\pi} \int_{\mathcal{D}^h} Fp\,(\mu_d; \mu_0) e^{\left(-\frac{\tau}{\mu_0}\right)} \left(w^h + \gamma \mu_d \nabla w^h\right) d\tau$$

- Scattering term is included in the bilinear function which leads to a linear system

Isotropic Scattering

- Semi-infinite plane parallel medium
- Isotropic scattering and a collimated source

$p(\cos\eta) = 1$



$$\mu\frac{\partial I}{\partial \tau} + I = \frac{1}{2}\frac{\sigma_s}{\beta}\int_{-1}^{1} I(\tau,\mu')\mathrm{d}\mu' + \frac{1}{4}\frac{\sigma_s}{\beta}Fe^{-\tau/\mu_0}$$

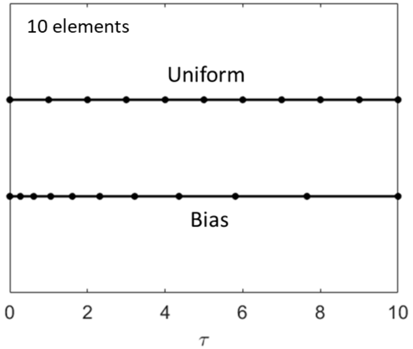
$I(0,\mu) = 0$ for $0 \le \mu < 1$
$S(\tau,\mu)e^{-\tau} \to 0$ as $\tau \to \infty$ boundedness

- An analytical solution exists and depends on n

- Use the following for discontinuity at $(\tau,\mu) = (0,0)$
  - Double Gauss

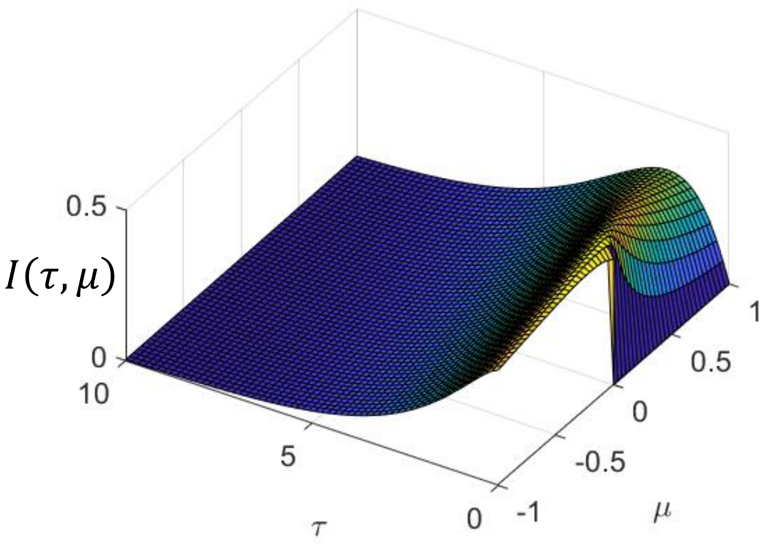
$$\int_{-1}^{1} p(\mu;\mu')I(\tau,\mu')d\mu' = \int_{-1}^{0} p(\mu;\mu')I(\tau,\mu')d\mu' + \int_{0}^{1} p(\mu;\mu')I(\tau,\mu')d\mu'$$

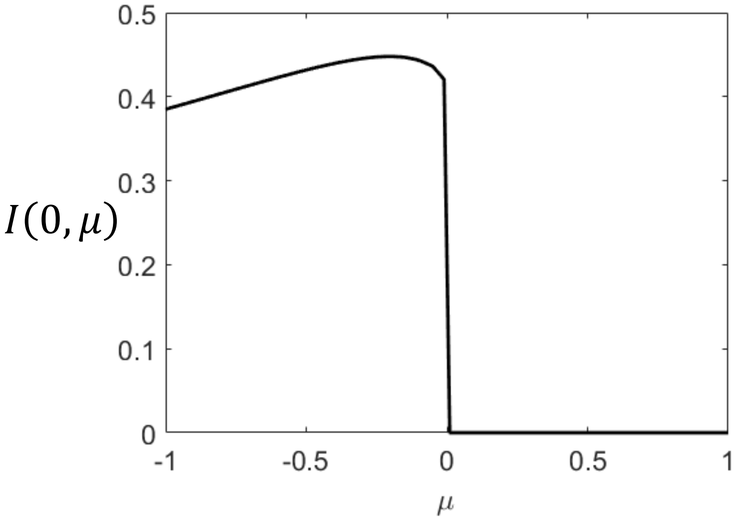
  - Biased node spacing (log)



Analytical Solution

$\frac{\sigma_s}{\beta} = 0.9$, $\mu_0 = 1$, $F = 1$, $\tau_1 = 10$, n=10





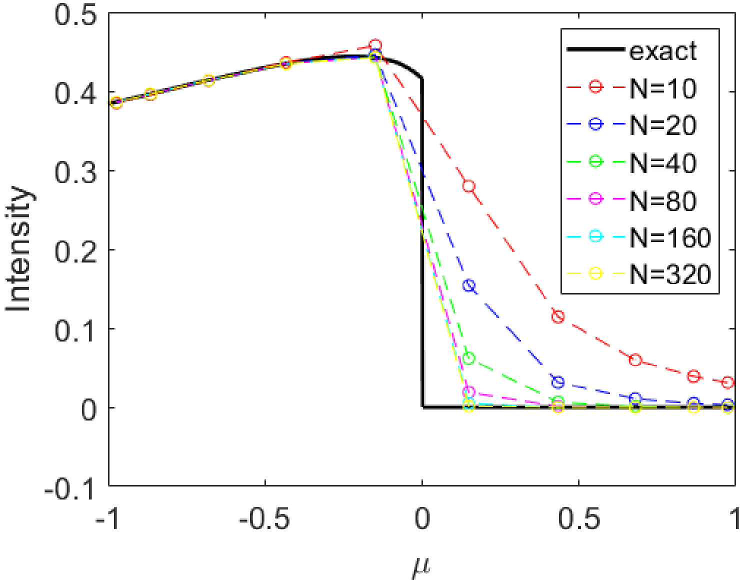
Isotropic Scattering

Semi-infinite plane parallel medium and isotropic scattering

- $\frac{\sigma_s}{\beta} = 0.9, \mu_0 = 1, F = 1, \tau_1 = 10, \gamma = 0$
- n=10 discrete ordinates, N: number of elements
- Uniform node spacing, $h = \tau_1 / N$
- Bias node spacing, $h = 10^{i \cdot log(\tau_1 + 1)/N} - 10^{(i-1) \cdot log(\tau_1 + 1)/N}$
- FE-DOM approximation converges to analytical solution, L2 error convergence is $O(h^2)$
- Bias node spacing and Double Gauss increase accuracy for the same n and N

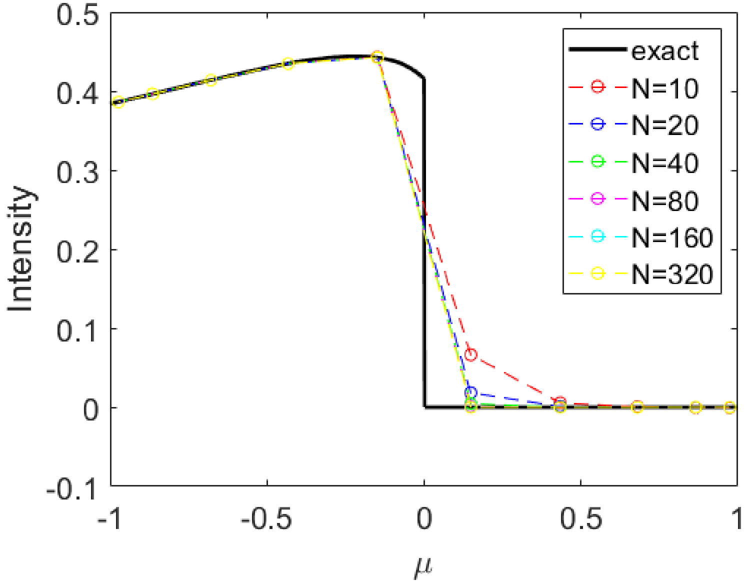
Uniform Node Spacing

$I(0, \mu)$



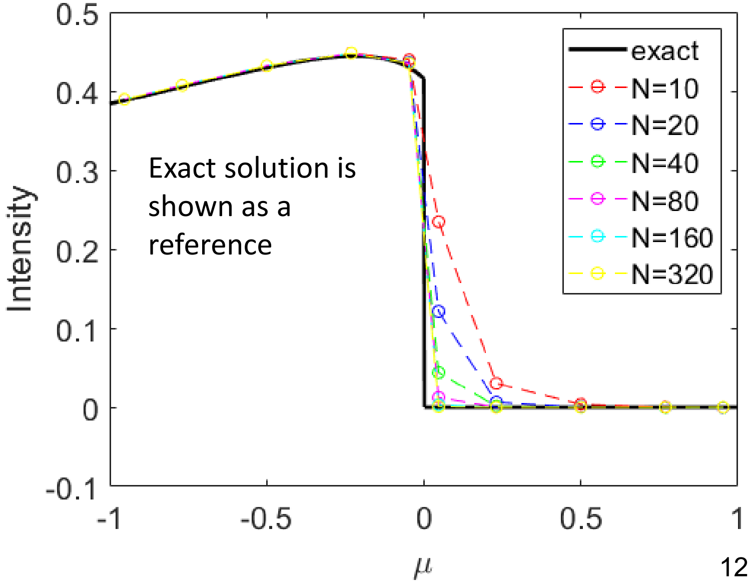
Bias Node Spacing

$I(0, \mu)$



Bias Node Spacing, Double Gauss

$I(0, \mu)$



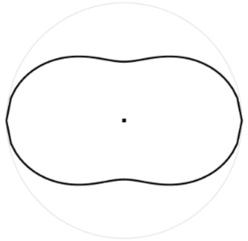
Rayleigh Scattering

- Semi-infinite plane parallel medium
- Rayleigh scattering and a collimated source

$$\mu\frac{\partial I^{(m)}}{\partial\tau}+I^{(m)}=\frac{1}{2}\frac{\sigma_s}{\beta}\int_{-1}^{1}I^{(m)}(\tau,\mu')p^{(m)}(\mu,\mu')\mathrm{d}\mu'+\frac{1}{4}\frac{\sigma_s}{\beta}Fp^{(m)}(\mu,\mu_0)e^{-\tau/\mu_0}$$

- $\frac{\sigma_s}{\beta}=1,\ \mu_0=0.8,\ F=1,\ \tau_1=200,\ \gamma=0$
- n=10 discrete ordinates, N: number of elements
- FE-DOM approximation converges to analytical solution with expected convergence rate

$$p(\cos\eta)=\frac{3}{4}(1+\cos^2\eta)$$



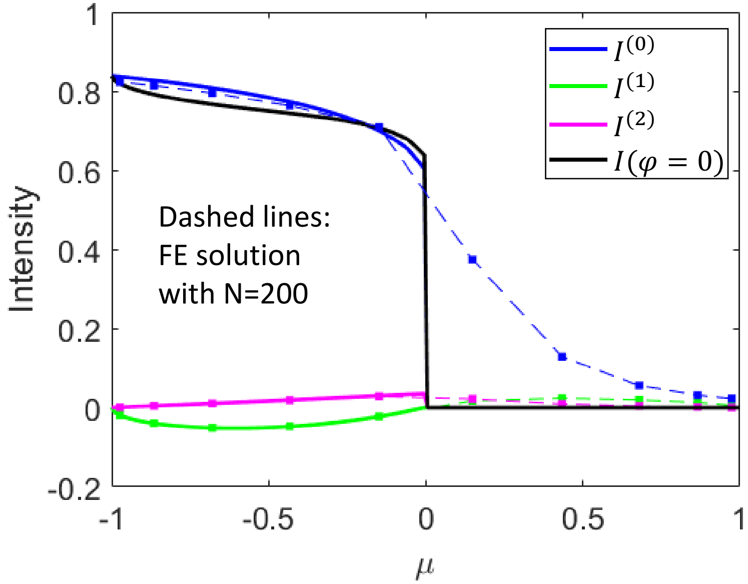
$$p^{(0)}(\mu,\mu')=\frac{3}{8}[3-\mu^2-\mu'^2+3\mu^2\mu'^2]$$

$$p^{(1)}(\mu,\mu')=\frac{3}{8}[4\mu\mu'\sqrt{(1-\mu)^2(1-\mu')^2}]$$

$$p^{(2)}(\mu,\mu')=\frac{3}{8}[(1-\mu^2)(1-\mu'^2)]$$

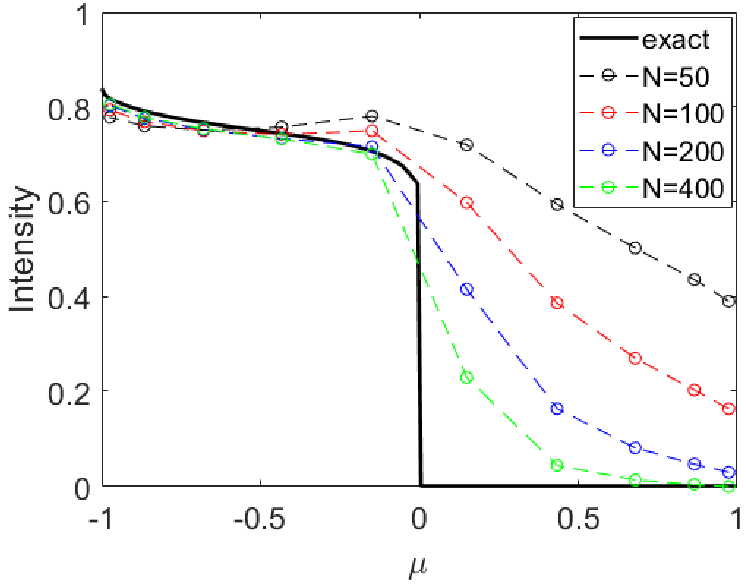
Exact Solution

$I(0,\mu)$



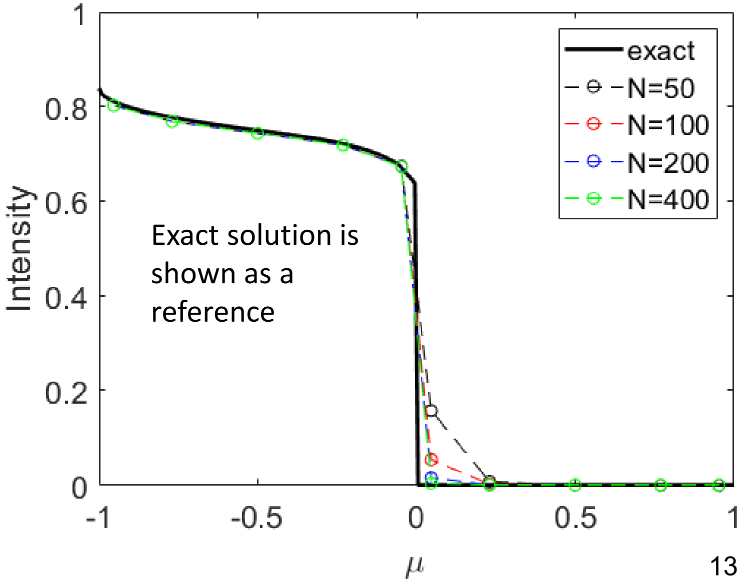
Uniform Node Spacing

$I(0,\mu)$



Bias Node Spacing, Double Gauss

$I(0,\mu)$



Case Study: LI-900

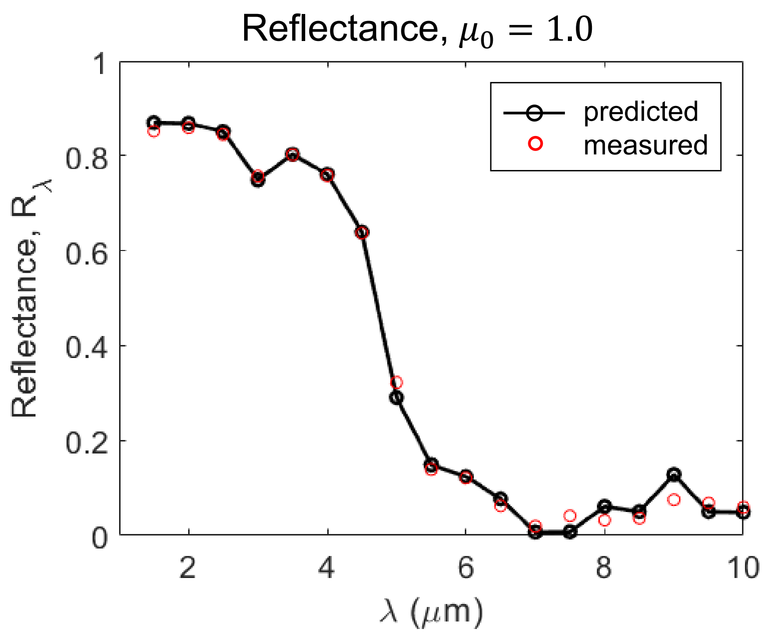
LI-900: bonded silica fibers

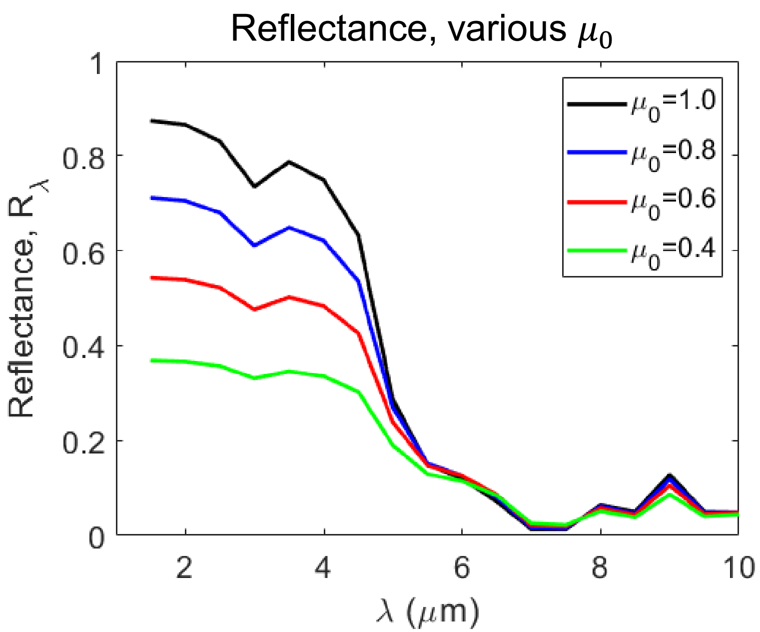
- Bulk density 145 kg/m^3, fiber volume fraction 0.0677
- Fibers randomly oriented in space, mean fiber diameter $4.0\ \mu m$
- Collimated source and non-reflecting boundaries
- Solve the RTE and compute hemispherical reflectance (R_λ) for each wavelength

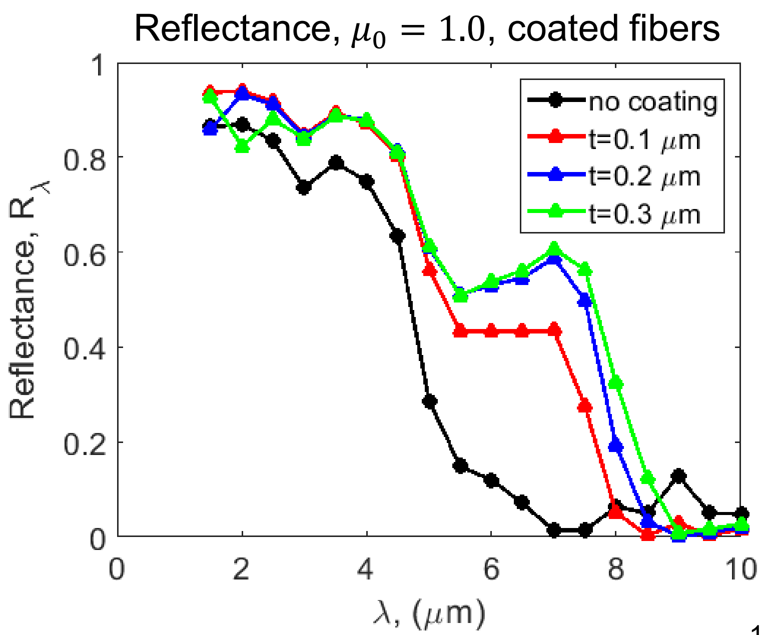
$$R_\lambda = \int_0^{2\pi}\int_{-1}^{0} I_\lambda(0,\mu,\varphi)\mu\, d\mu d\varphi = 2\pi \int_{-1}^{0} I_\lambda^{(0)}(0,\mu)\mu\, d\mu$$

- Solution agrees with measured results for $\mu_0 = 1.0$ [Lee 1996]
- Add silicon coating to fibers
    - Absorption and scattering data provided by AFOSR partner

SEM image [Lee 2000]





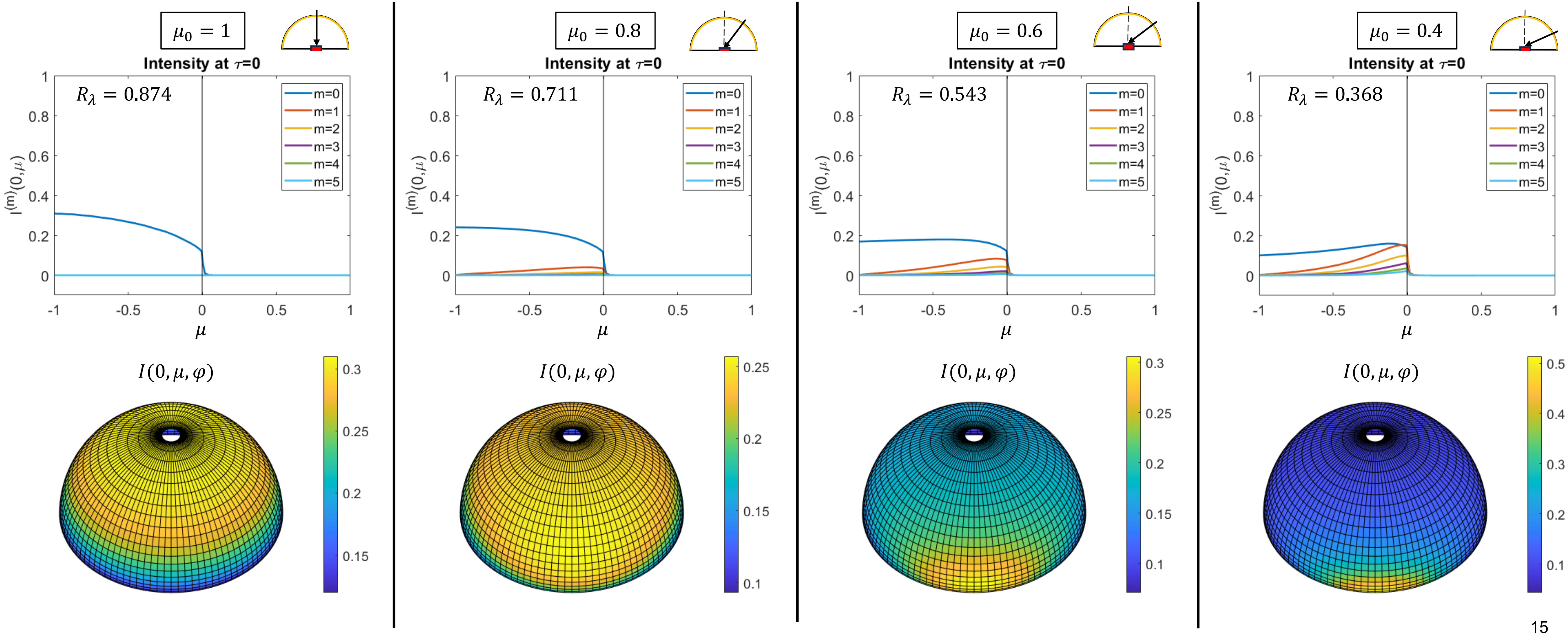


Case Study: LI-900

Phase function expansion

- Intensity solution at $\tau = 0$, up to m=5 higher order terms
- Axially symmetric for normal incidence
- Only the zeroth order term contributes to the hemispherical reflectance

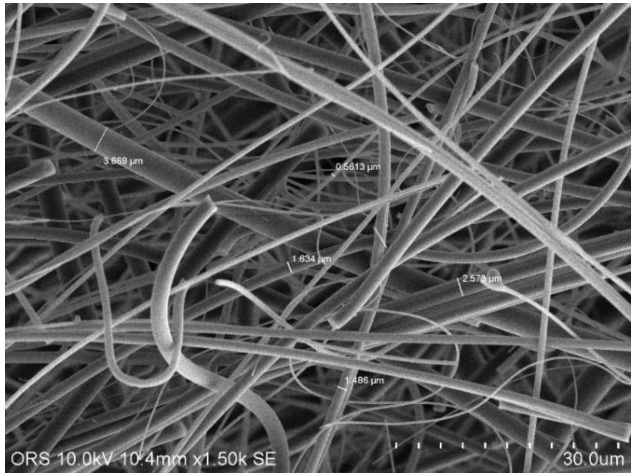
$$I(\tau,\mu,\varphi) = \sum_{m=0}^{N} I^{(m)}(\tau,\mu)\cos m(\varphi_0 - \varphi)$$



Q-Fiber Felt

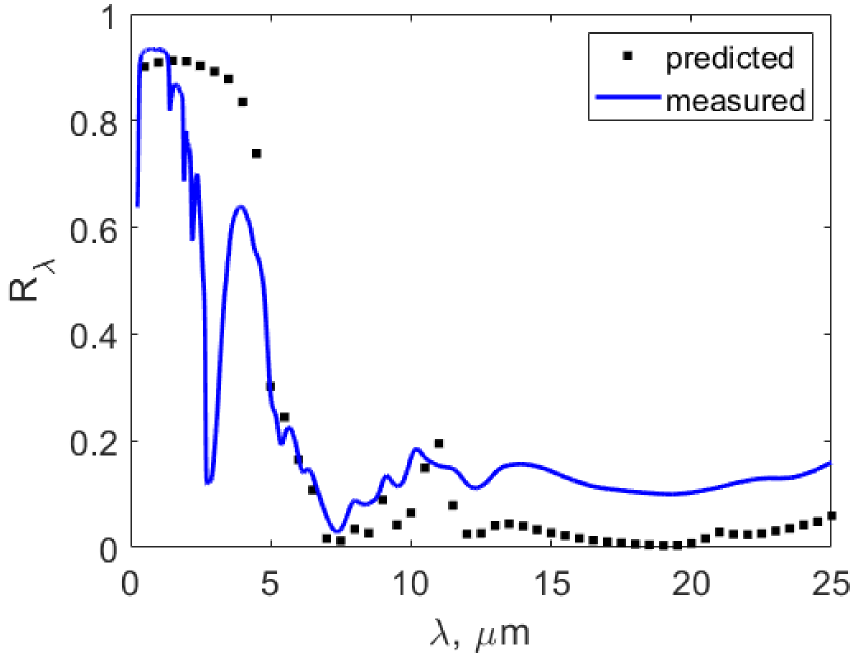
Q-Fiber Felt: pure silica fiber sheets manufactured by Johns Manville

- Bulk density 48 kg/m^3, fiber volume fraction 0.022
- Fibers randomly oriented in space, mean fiber diameter 1.8 μm
- Collimated source ($\mu_0 = 1.0$) and non-reflecting boundaries
- Solve the RTE and compute hemispherical reflectance for each wavelength
- Add moisture absorption, modeled as fiber coating at 0.8 wt % [Carjaval 2026]

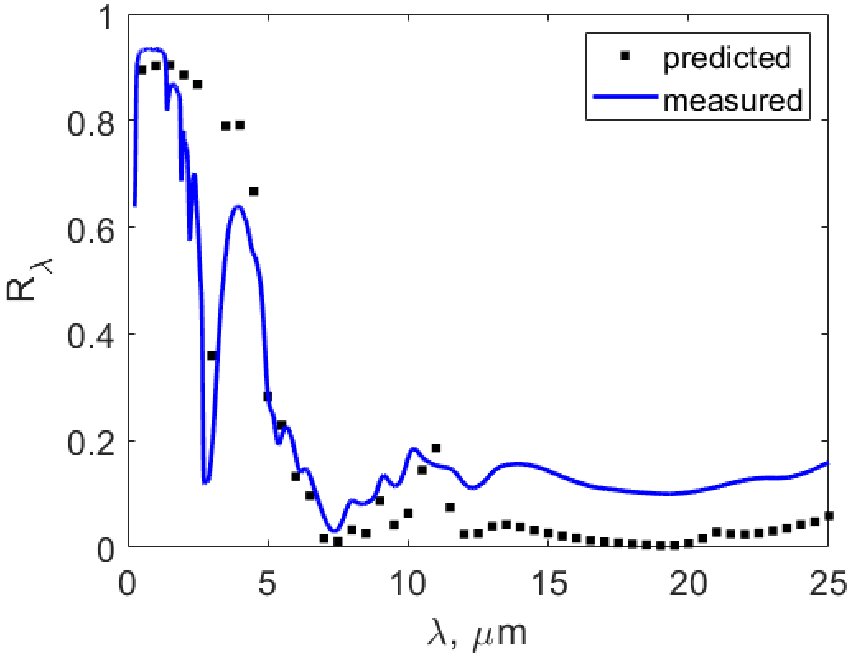


SEM image from Oneida Research Services

No Moisture



Moisture Content 0.8 wt %



Next Steps

- Investigate difference in measured and predicted hemispherical reflectance at higher wavelengths

- Validate RTE solution for additional measurements of reflectance and transmittance for Q-Fiber Felt
  - Measurements for three densities of Q-Fiber Felt
  - Reflectance measurements at multiple incident angles
  - Measured bidirectional reflectance distribution function (BRDF)

- Couple the RTE and heat transfer equations for temperature solution

- Internal research funded
  - Extend RTE solver to 2-D in space
  - Add predictive capability for short fibers and opacifiers
  - Study Opacified Fibrous Insulation (OFI), manufactured by Heetshield, Inc.
  - Reflectance and transmittance measurements at Surface Optics
  - Thermal vacuum testing at NASA Langley of Q-Fiber Felt and OFI

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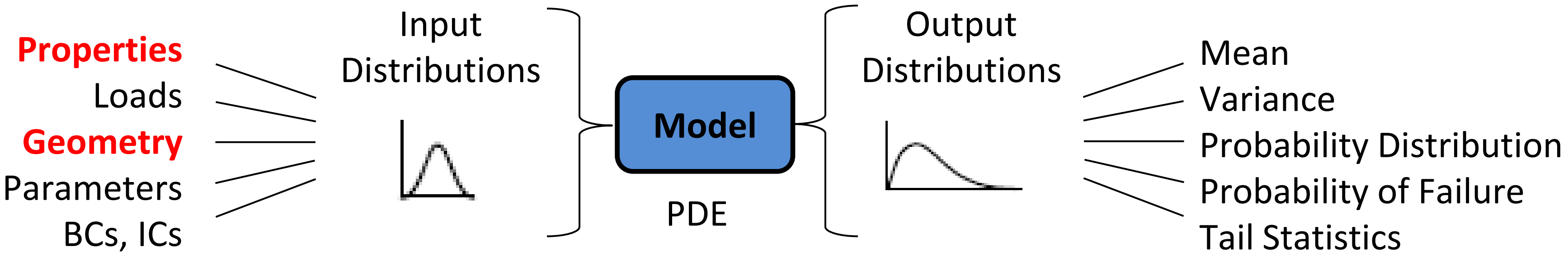
Thank you for your attention

Backup Charts

Research Plans

Uncertainty Quantification

Collaboration with Kursat Sendur at Sabanci University



- Characterization of input uncertainty
  - Choose a statistical model based on observations, judgement, experience
  - Develop statistical model based on available data

- Propagation of input uncertainty
  - Monte Carlo – simple and robust, requires a large number of samples
  - Reduced Order Model (ROM) – minimize the number of deterministic solutions required
  - Stochastic Methods – intrusive and non-intrusive generalized Polynomial Chaos

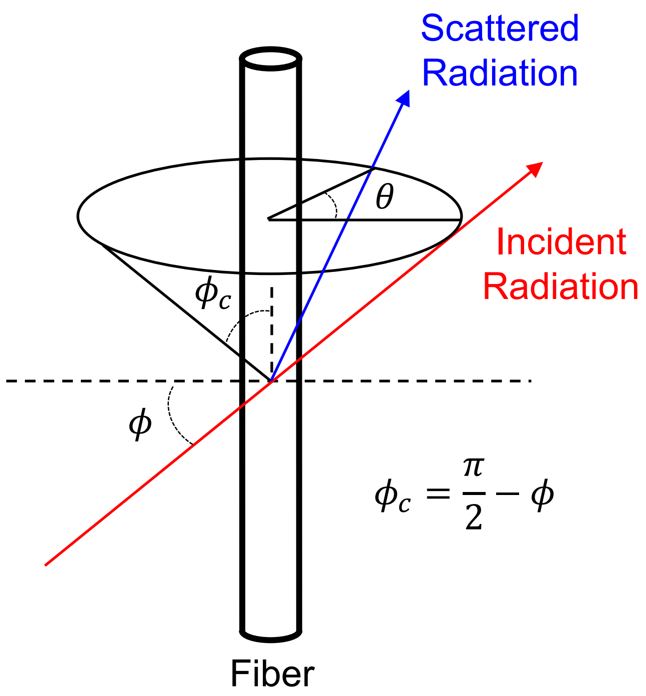
- Compute statistics of output distributions

Fiber Distribution Properties

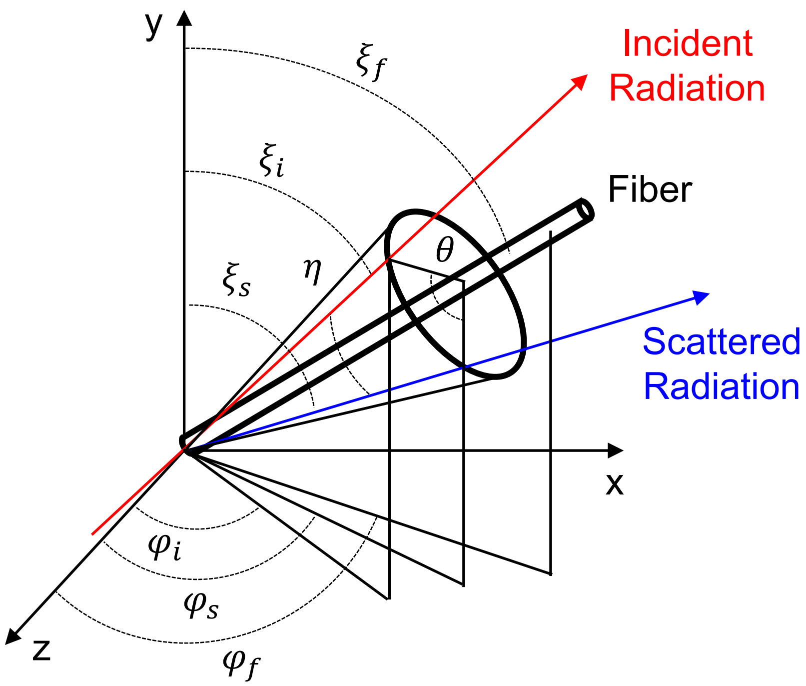
Absorption and scattering

- Assume infinitely long cylinders and independent scattering
- Utilize MatScat for single fiber properties [Schafer 2016]
- Integrate over fiber radius and orientation for fiber distribution [Lee 1998]
- Properties depend on wavelength, λ

<u>Fiber Coordinate System</u>



<u>Global Coordinate System</u>



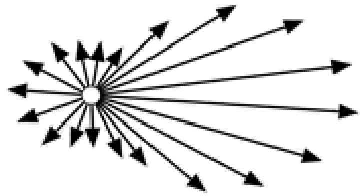
<u>Fiber Distribution Properties</u>

κ_λ : absorption coefficient

$\sigma_{s,\lambda}$: scattering coefficient

$\beta_\lambda = \kappa_\lambda + \sigma_{s,\lambda}$: extinction coefficient

$p(\eta)$: scattering phase function



<u>Orientation of fibers</u>

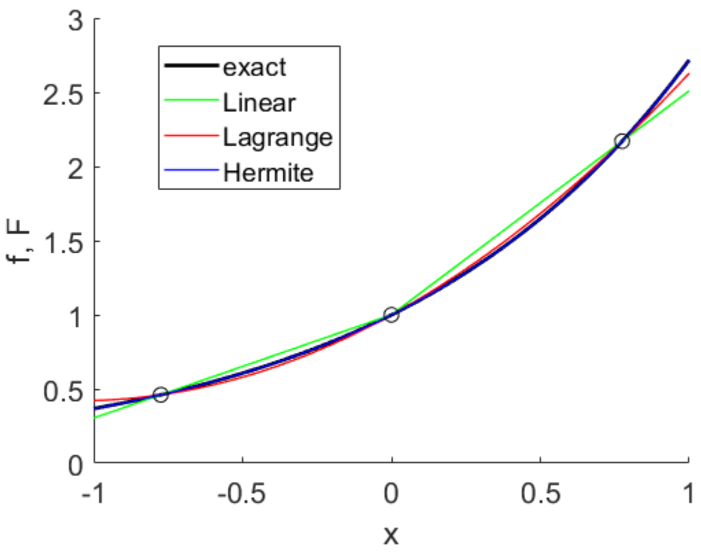
- Random in space
- Random in a plane inclined at ξ_{fo}
- Preferentially oriented at (ξ_{fo}, φ_{fo})

Solution Interpolation for Angular Discretization

- Interpolation schemes for $F(x)$ using Gauss-Legendre integration points
  - Linear – piecewise polynomial of degree 1 which interpolates $f(x)$
  - Lagrange – unique polynomial of degree n-1 which interpolates $f(x)$
  - Hermite – unique polynomial of degree 2n-1 which interpolates $f(x)$ and $f'(x)$
- Example, choose $f(x) = e^x$

<u>Interpolation Function</u>

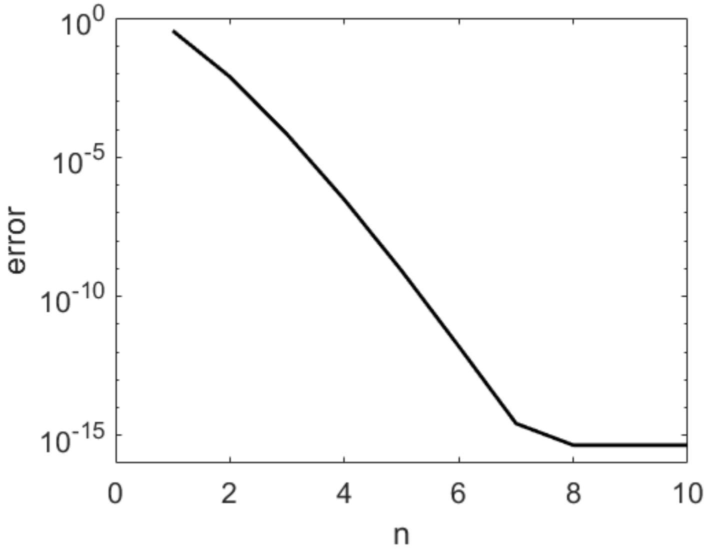
n=3 Gauss-Legendre points



<u>Integration Error</u>

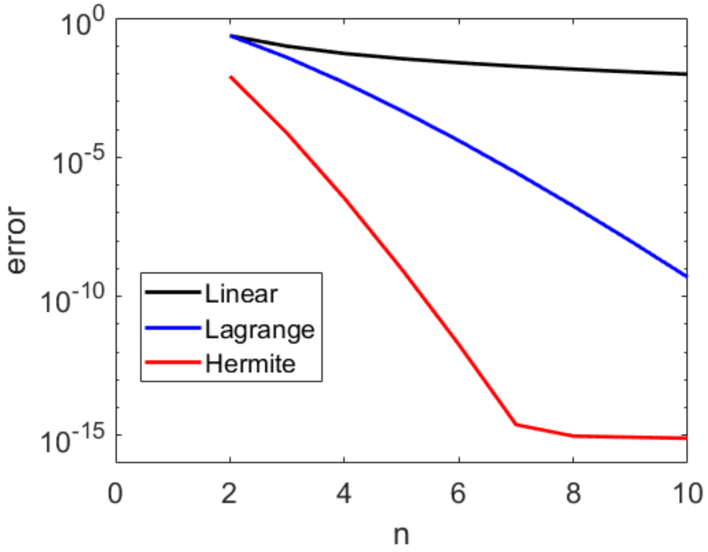
$$I = \int_{-1}^{1} e^x \, dx = e - \frac{1}{e} = 2.3504$$

$$\text{error} = \left| I - \sum_{i=1}^{n} f(x_i) w_i \right|$$



<u>Interpolation Error</u>

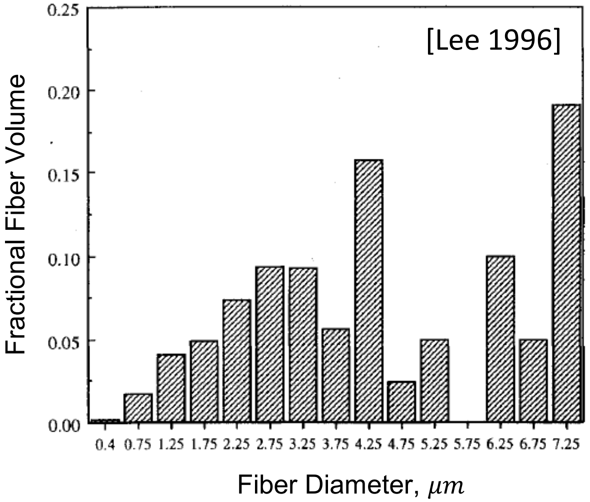
$$\text{error} = \|f(x) - F(x)\|_2 = \left(\int_{-1}^{1} |f(x) - F(x)|^2 dx \right)^{1/2}$$

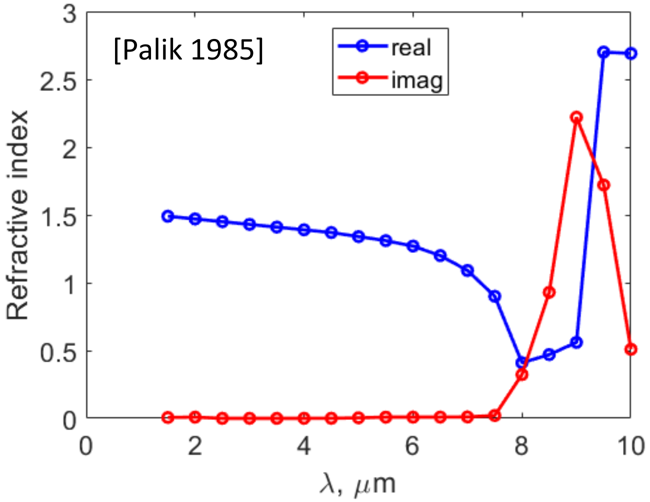


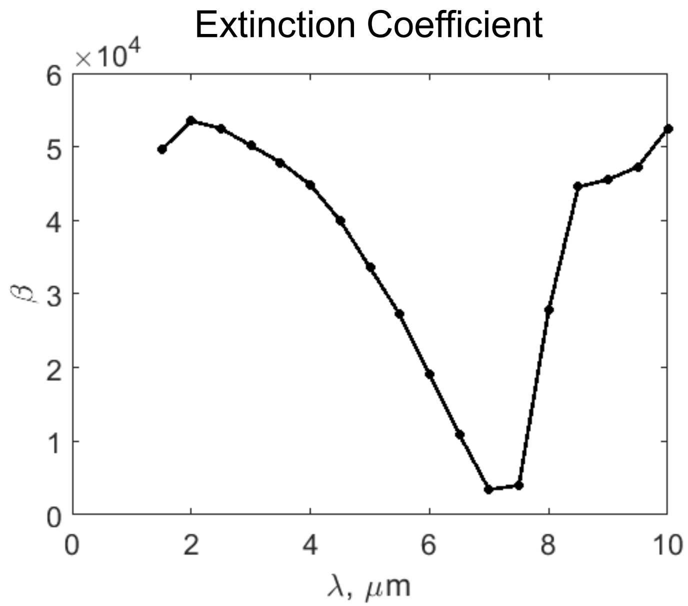
Case Study: LI-900

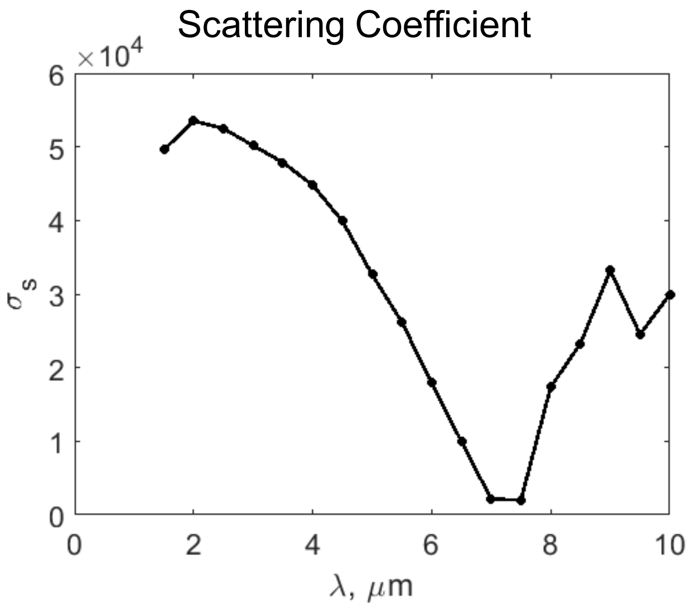
Validate RTE solution

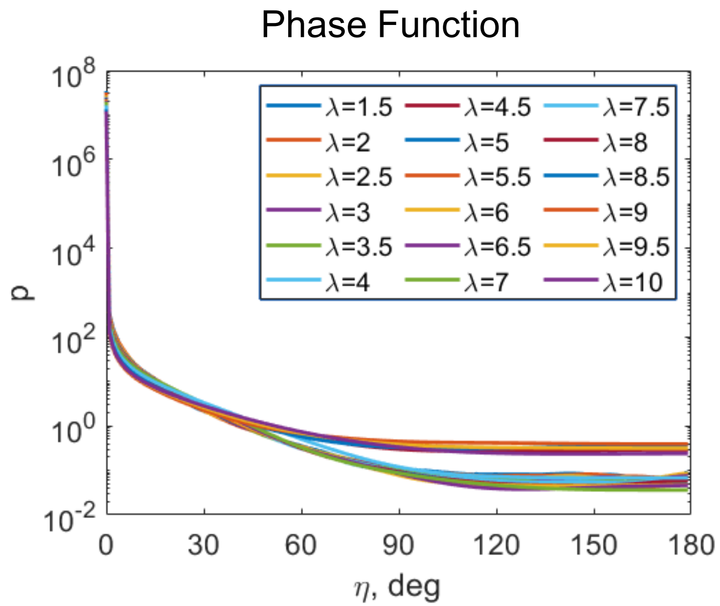
- LI-900 – bonded silica fibers
- Bulk density 145 kg/m^3, specimen thickness 0.15 cm
- Fiber volume fraction 0.0677, fibers randomly oriented in space
- Compute extinction and scattering properties for $\lambda \in [1.5,10]\ \mu m$











Coupled Solution

Case study [Ozisik 1972]

- Coupled RTE and heat transfer equation
- Isotropic scattering
- Solution agrees with published results
  - Non-dimensionalized solution for $N = 0.1, \frac{\sigma_s}{\beta} = 0.5$ shown
- Equations are strongly coupled and nonlinear
- Solve by Newton-Raphson method

- Radiation transport, emissive BCs

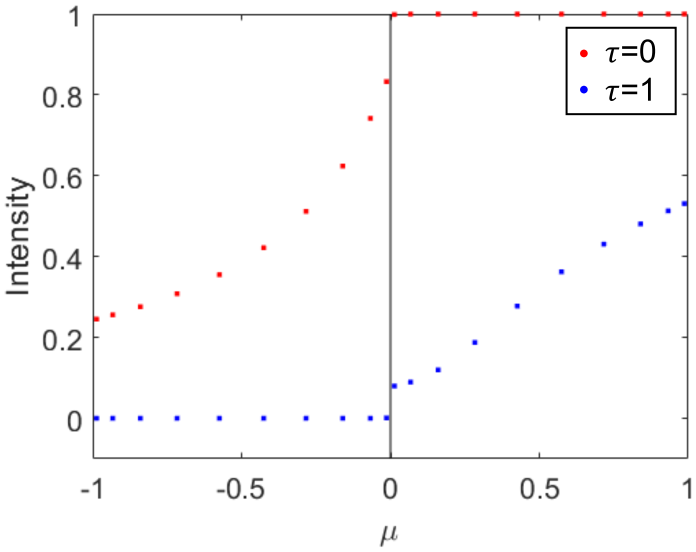
$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{\sigma_s}{2\beta} \int_{-1}^{1} I(\tau,\mu') d\mu' + \kappa T^4(\tau)$$

- Heat transfer, fixed temperature BCs

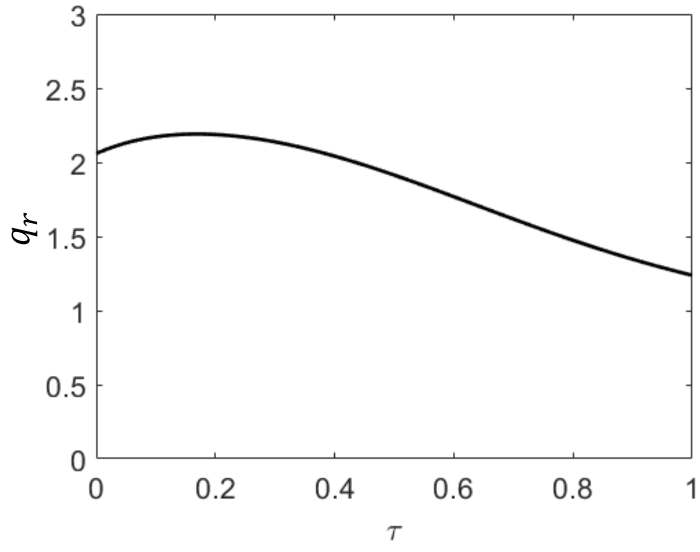
$$k\beta \frac{\partial^2 T(\tau)}{\partial \tau^2} - \frac{\partial q_r(\tau)}{\partial \tau} = 0 \qquad q_r(\tau) = 2\pi \int_{-1}^{1} I(\tau,\mu')\mu' d\mu'$$

$$N = \frac{k\beta}{4\sigma n^2 T_{ref}^3}$$

Intensity solution



Radiative heat flux



Temperature solution