

Analysis of Single and Contra-Rotating Propellers

Gregory G. Zilliac
Ames Research Center, Moffett Field, CA

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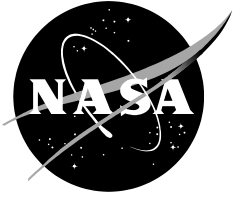
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Gregory G. Zilliac
Ames Research Center, Moffett Field, CA

National Aeronautics and
Space Administration

Ames Research Center
Moffett Field, CA 94035-0000

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Nomenclature and Acronyms

a, a'	Axial and tangential induction factors, respectively
A	Rotor annulus area [m ²]
AOA	Angle of attack
b	Blade element length in chord-wise direction [m]
B	Number of blades
BEMT	Blade Element Momentum Theory
CFD	Computational Fluid Dynamics
C_l, C_d, C_m	Airfoil section lift drag and pitching moment coefficients
C_N	Normal force coefficient
C_p	Rotor power coefficient, $C_p = \frac{P}{\rho n^3 D^5} = 2\pi C_Q$ or pressure coefficient
C_T	Rotor thrust coefficient, $C_T = T / \rho n^2 D^4$ or tangential force coef.
C_Q	Rotor torque coefficient, $C_Q = Q / \rho n^2 D^5$
C_x, C_y	Force coefficients in thrust and torque direction, respectively
CR	Contra-rotating variable pitch open rotor
CFM	CFM International is a General Electric and Safran joint venture
CLEEN	Continuous Lower Energy, Emissions, and Noise
D	Rotor tip diameter (of forward rotor for CR), [m]
EPFD	NASA Electrified Powertrain Flight Demonstration
F	Prandtl tip loss factor
FAA	Federal Aviation Administration
g	Axial distance between rotor disk planes for CR rotor system, [m]
G	Material shear modulus, [Pa]
h	Blade cross-section maximum thickness, [m]
GASP	General Aviation Synthesis Program
I	Area moment of inertia, [m ⁴]
J	Rotor advance ratio, $J = V/nD$ or polar area moment of inertia [m ⁴]
M	Mach number
MIT	Massachusetts Institute of Technology
n	Rotor Revolutions per Second [rev/s]
P	Power, [W] or pressure [pa]
PQA	$PQA = P / \rho n^3 D^3 A = C_p D^2 / A$
PAX	Number of Passengers
PQA/J^3	Rotor power loading parameter, $PQA/J^3 = PQA / J^3$
q	Dynamic pressure, [pa]
Q	Torque, [Nm] (note: Q' is torque/unit length, [N])
R	Rotor tip radius, [m]
r	Rotor radial position, [m]
Rn	Reynolds number
RANS	Reynolds-Averaged Navier Stokes
RISE	Revolutionary Innovation for Sustainable Engines
rpm	Revolutions Per Minute, [rev/min]
SHP	Shaft Horsepower, [hp]
SR	Single variable pitch open rotor
T	Thrust, [N] (note: T' is thrust/unit length, [N/m]) or temperature
TQA	$TQA = T / \rho n^2 D^2 A = C_T D^2 / A$
V	Velocity, [m/s]
v	Velocity component, [m/s]

VP	Variable Pitch
W	Resultant velocity, [m/s]
x	$x = r/R$ or axial distance from rotor or airfoil section coordinate
y	Airfoil section coordinate normal to chord
$y_{c,max}$	Airfoil section maximum camber
α	Angle of attack, [rad]
β	Blade pitch angle (subscript f: forward, b: aft), [rad]
β_{75}	Blade pitch angle at 75% radius, [rad]
γ	Specific heat ratio
δ	Dimensionless wake width
η	Rotor efficiency, $\eta = TQA / PQA = C_T J / C_P$
θ	Element torsional rotation angle, [rad]
κ	Interference coefficient
κ_A	Korn Factor
Λ	Blade leading edge sweep angle, [rad]
ρ	Air density, [kg/m ³]
σ	Rotor solidity
ϕ	Blade inflow angle, [rad]
Ω	Angular velocity, [rad/s]

Subscripts or Superscripts

b	Back or rear rotor
c	Camber
$comp$	Compressible
$crit$	Critical
d	Drag
dd	Drag divergence
eff	Effective
f	Front or forward rotor
i	Induced
inc	Incompressible
l	Local or lift
ld	Lift divergence
m	Pitching moment
p	Pressure
ref	Reference
sup	Supersonic
t	Thickness
$trans$	Transonic
0	Freestream
∞	Free stream value
x	Axial (directed along rotor rotational axis)
θ	Tangential (i.e., rotation direction)
$'$	Per unit length or local property
75	Characteristic blade element (typically at $x = 0.70$ to 0.80)

Summary

The high efficiency of open rotor propulsion is attractive to designers of sustainable transport aircraft configurations of the future. Modern open rotors for transonic aircraft are geometrically complex, can involve multiple stages (e.g., downstream stator or contra-rotating second rotor) and typically operate under conditions where the flow over a portion of the blade is transonic. The overall net efficiency of modern open rotors can approach 86% at a cruise Mach number of 0.8 leading to significant fuel savings in comparison to turbofan engines. Modern open rotors are considerably quieter than the propfans of the 1980s.

Modeling of open rotor configurations to predict the thrust produced and power required is believed by many to be outside the capabilities of traditional blade element analysis but this may be because of blade geometrical and aerodynamic modeling deficiencies not an inherent deficiency in the blade element approach. Presented in this memorandum is a description of the modeling approach behind a newly developed Blade Element Momentum Theory analysis code called PyRotor that is written in the Python programming language. The underlying theory and functionality of the code are described. Several rotors, ranging in complexity from a simple two-blade propeller up to a modern CR rotor system, are analyzed and compared with previously published measured data.

Introduction

The performance of modern aircraft rotor systems is challenging to predict. The most advanced rotor systems have up to a dozen highly loaded twisted, swept, raked, skewed and variable-pitch blades and in some cases are followed by an array of variable-pitch stators (e.g., CFM RISE currently under development and recently studied in [1]). Coaxial propellers in contra-rotation (CR) also known as counter-rotating propellers, have been implemented on a few aircraft such as the Tu-95. CR propellers have efficiency advantages over single propellers but suffer from excessive noise and mechanical complexity. Recent research summarized in [2] shows that CR noise can be significantly reduced through rotor system geometrical changes like making the aft rotor smaller in diameter and increasing the rotor disk separation. The focus of the current study is to develop a propeller performance predictive methodology that is accurate enough to be useful in the design and modeling of complex aircraft rotor systems.

The design of modern rotor systems requires the capability to predict the thrust produced, power required, propeller efficiency and noise over a range of operating conditions (motor rpm, flight Mach number and altitude). The main predictive capabilities in common use are broadly categorized as CFD, potential flow, or blade element momentum theory (BEMT) based. None of these approaches is fully satisfactory when designing a propeller or predicting aircraft performance. The number of predictions required to cover the design space is too costly and time consuming for CFD. Potential flow methods do not directly model compressible and viscous effects (typically, these effects are added in during the solution process). BEMT uses simplified models of the propeller wake rotational effects that have a first-order impact on the blade thrust. For the purposes of the current study, the BEMT approach is preferred because of the relative ease that geometrical changes can be made and the rapid case turnaround.

The BEMT approach to propeller analysis has been around since the dawn of aviation with many of the greats including Glauert, Prandtl, Goldstein, and von Karman, contributing. As described in [3-5], BEMT combines blade element theory (BET, aka strip theory) with the momentum theory of propellers (MT) that predicts the induced velocity increase in the wake of a

propeller. The blade is broken down into a number of two-dimensional elements each experiencing a local velocity that is the vector combination of the axial freestream velocity and radially varying rotational velocity. Momentum theory is used to predict the change in the local velocity induced primarily by the lift on the element. The difficulty with momentum theory is that the theory is ideal and therefore the induced velocity predicted is too high and leads to excessive thrust and torque. A so-called “tip loss” model is often incorporated to account for the fact that the induced velocity varies along the blade and must be zero at the blade tip and hub and additional modeling is required to model the energy losses in the wake. The most common tip loss model, and the one used in the current paper, is a model originally developed by Glauert [6]. In the BEMT analysis, lift and drag on each blade element are integrated over the length of the blade and thrust and power required are determined. The equation set is implicit in the induced velocity components and therefore an iterative solution process is required.

Many researchers have developed BEMT codes over the years and because of the rise of wind energy, there has been a resurgence in their development and use. Bergmann et al. [7] present a comparison of a few of the more well known BEMT codes with RANS CFD and measured results on two- and three-bladed propellers. The codes investigated include JavaProp [8], XROTOR [9], JBlade [10], RAALF [11] and PropCODE [7]. Worrisome is the 10% or higher spread in the power and thrust versus advance ratio comparison between the various codes and RANS predicted and measured results. The most accurate code appears to be XROTOR, developed by Mark Drela at MIT, that has the most sophisticated capabilities of the codes compared. The greatest discrepancies between the codes and the RANS results occurred at low advance ratio where the impact of the rotor wake on the blade is the greatest. For rotor predictions to be useful in propeller design, optimization and aircraft performance analyses, high accuracy is desirable particularly under cruise conditions where the propeller operates near peak efficiency and during climb when the propeller is highly loaded (i.e., not under low advance ratio conditions where BEMT is less accurate).

The motivation for this study was the realization that very few propeller maps (i.e., tables with propeller thrust coefficient as a function of advance ratio and power coefficient at different Mach numbers) are available for use in aircraft design studies. Aircraft analysis tools such as GASP [13] and FLOPS [14] require comprehensive propeller maps and often expediency leads the analyst to use what is available instead of what is desired. (For example, a propfan rotor map from the 80s was used in reference [15] as a substitute for a modern CR rotor because the BEMT tools necessary to generate a CR propeller map were not available.) There are additional challenges when attempting to model the performance of existing propellers. Often, the airfoil sections used and the blade geometry are proprietary and measured thrust and power data is limited, if available at all. Predicting noise is as important as performance but is not within the scope of the present effort.

In the current study, an effort is made to address several of the error sources found in other BEMT implementations. Specifically, blade geometrical fidelity including the effects of blade torsion are improved. Aerodynamic force modeling on the blade elements is also improved through implementation of a neural network based force model called NeuralFoil [12] that predicts incompressible airfoil lift, drag and pitching moment for any airfoil shape.

Presented in the following sections is a description of a newly developed BEMT analysis code called PyRotor that is written in the Python programming language. The main difference between PyRotor and other BEMT codes is the implementation of NeuralFoil to improve the modeling of the blade aerodynamics. The underlining theory and functionality of the code are

described. Several rotors, ranging in complexity from a simple two-blade propeller up to a modern CR rotor system, are analyzed and compared with previously published measured data.

Rotor Theory and Modeling

Aircraft propellers produce thrust by rotating a twisted wing to increase the momentum of air passing through the propeller disk plane. The Momentum Theory of Propellers says that the velocity through the propeller disk is the average of the axial velocity far upstream and downstream of the propeller disk. Simple momentum theory predicts the velocity for an ideal propeller that is uniformly loaded without losses. To rotate a propeller, a torque is applied and the propeller rotation imparts angular momentum to the flow field in addition to the axial velocity predicted by momentum theory. The cross-section of a propeller blade is airfoil shaped and therefore lift is produced (largely in the thrust direction). The vorticity produced by the lift is shed along the blade into the wake. The vorticity rolls up into a vortex that emanates from near the propeller tip and forms a helical-like structure trailing behind the blade. Therefore, in addition to the freestream axial velocity component V_0 and the tangential velocity component Ωr in the direction of blade rotation, induced velocity components are present. These induced velocity components originate from the variation of lift along the blade and wake structure that have a first order impact on the thrust produced and torque required.

Rotor Geometry

Modern rotors are geometrically complex! Blades can be twisted, swept, raked, leaned, cambered, skewed, comprised of multiple airfoil sections that vary in chord length, blended together and often held by a variable-pitch mechanism. Modern rotors often operate at transonic speeds and are highly loaded which can lead to deflections that further complicate the geometry. In the current study, blade rake, blade lean, blade camber (not to be confused with airfoil section camber that is modeled) and blade skew are not modeled. The torsional effect on blade pitch angle is modeled.

The unloaded blade geometry is modeled either by input of a regular grid of blade surface points or by polynomials that define the blade surface geometry. The polynomials (up to sixth-order) are functions of blade radial position r and include one polynomial each for blade chord-to-diameter ratio b/D , thickness-to-chord ratio h/b , blade pitch angle β , blade leading-edge sweep angle Λ and airfoil section design C_l . For each airfoil section, the cross-section geometry is modeled by two polynomials in x (the airfoil coordinate) that define the airfoil section thickness-to-chord ratio y_t/b , and camber distribution $y_c/b/C_{l_design}$. The airfoil section design $C_l(r)$ polynomial multiplies the $y_c/b(x)/C_{l_design}$ polynomial to enable blade radial camber variations for airfoils with analytically derived camber distributions (like the NACA series 16 airfoils where $y_c = -0.079577C_{l_design}[x \ln(x) + (1-x) \ln(1-x)]$ for unity chord) that are used in many propeller designs. The polynomial-based approach to blade geometry permits blade definition like that shown in Figure 1 for the two-blade propeller studied in [16]. Multiple blade segments made up of different airfoil sections can be defined each with their own set of polynomials and the segments are blended together over a specified transition length along the blade. Alternately, the blade geometry can be read directly from an input file that defines the non-dimensional airfoil coordinates at the blade element stations along with the blade element chord, twist and sweep. An auxiliary pre-processing application called Viewer is used to create

the PyRotor input files for blade geometrical data derived from a CFD grid file or point-cloud measurements of an existing blade.

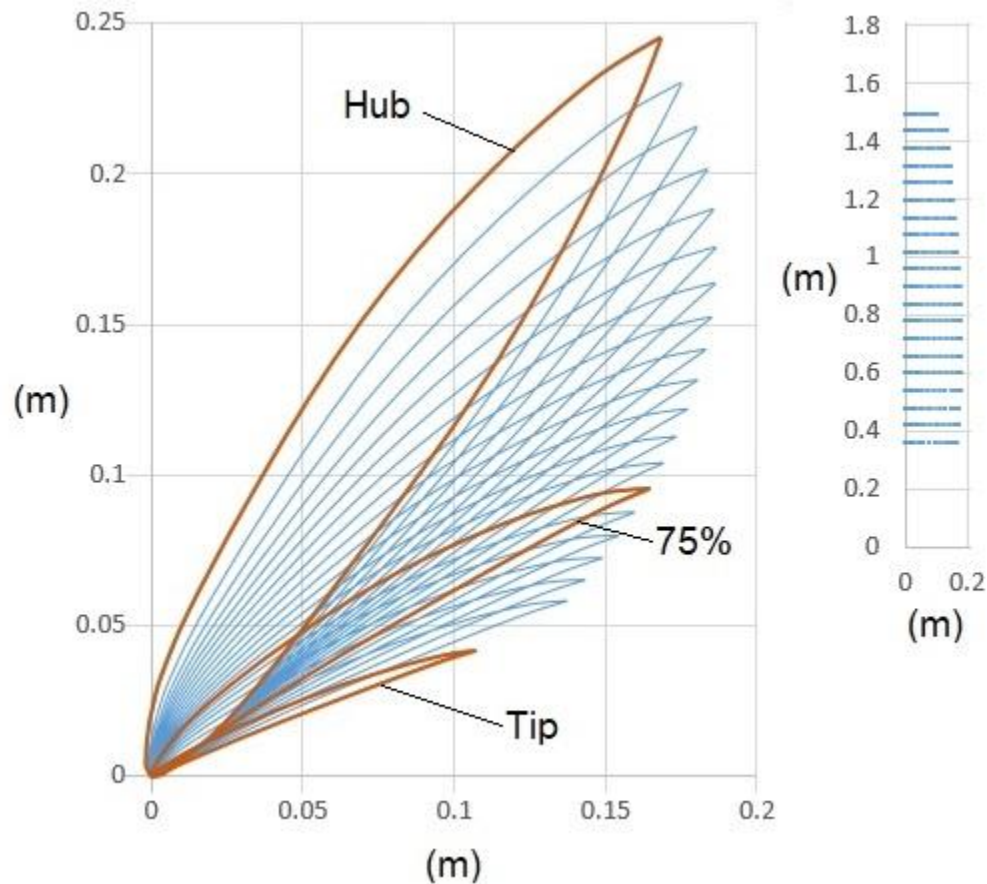


Figure 1. End and plan views of the modeled elements of a two-blade propeller of NACA TR-1309 [16].

Blade geometry defined by polynomials requires some up front fitting work but produce high fidelity geometry from which any number of the blade elements are easily defined. We have also used PyRotor to analyze propeller geometry that originated from point-cloud optical scans that required a similar level of up-front work. In any event, a greater degree of automation is desirable.

Aerodynamic Model

The chief difficulty with BEMT lies in the aerodynamic modeling of the blade elements. Along a propeller blade virtually all aerodynamic parameters vary including the cross-section geometry, local Mach and Reynolds numbers and blade twist angle. Flow separation may be present on some blade elements and elements near the blade tip are often transonic. No single practical modeling approach has been identified to predict the aerodynamic forces on the blade elements across the Mach number range and so the aerodynamic modelling has been split into three regimes (compressible, transonic and supersonic). Coupled with a stall model, the methodology described below should result in reasonable lift, drag and pitching moment predictions for a two-dimensional cambered airfoil section at any angle of attack and Reynolds number over a Mach range from zero to approaching 1.4. At Mach numbers less than the drag

divergence Mach, airfoil geometry with attached flow is modeled with an accuracy equivalent to a compressibility-corrected XFOIL [17] prediction. In the Mach range between drag divergence and Mach one, lift is predicted by a surrogate model and a quadratic variation of drag is assumed. Above Mach one, thin airfoil similarity methods are applied. Post-stall force and moment predictions are modeled empirically.

The underpinning of the aerodynamic model in PyRotor is NeuralFoil [12], a neural network trained on 10 million XFOIL (a panel code coupled with an integral boundary layer method) runs that can predict the incompressible lift, drag, pitching moment and pressure distribution up to stall on arbitrary airfoils. NeuralFoil requires knowledge of the airfoil section x, y coordinate pairs, Reynolds number, Re_l and angle of attack, α_l . Note that the subscript l implies a “local” quantity pertaining to the blade element (i.e., includes the axial and rotational velocity components).

The Karman Tsien compressibility correction is applied to the incompressible $C_{p_{inc}}$ values, returned by NeuralFoil at each airfoil coordinate x, y location to predict the airfoil compressible $C_{p_{comp}}$ distribution:

$$C_{p_{comp}} = \frac{C_{p_{inc}}}{\beta + [M_\infty^2 / (1 + \beta)] C_{p_{inc}} / 2} \quad \text{where } \beta = \sqrt{1 - M_\infty^2}$$

To avoid the singularity of the Karman Tsien relation when $C_{p_{inc}}$ is strongly negative, the denominator is limited to a minimum of 0.25β . This affects only highly loaded elements near M_{dd} .

The lift and pitching moment coefficients $C_{l_{comp}}$ and $C_{m_{comp}}$ for M up to the lift divergence Mach number M_{ld} are:

$$C_{l_{comp}} = \int_0^1 (C_{p_{comp,lower}} - C_{p_{comp,upper}}) d\left(\frac{x}{c}\right)$$

$$C_{m_{comp,c/4}} = \int_0^1 - (C_{p_{comp,lower}} - C_{p_{comp,upper}}) \left(\frac{x}{c} - .25\right) d\left(\frac{x}{c}\right)$$

The drag coefficient $C_{d_{comp}}$ for M up to the drag divergence Mach number M_{dd} is:

$$C_{d_{comp}} = C_{d_{inc}} K \quad \text{where: } K = \frac{C_{p_{comp}}}{C_{p_{inc,min}}} = \frac{1}{\beta + [M_\infty^2 / (1 + \beta)] C_{p_{inc,min}} / 2}$$

The critical Mach number $M_{crit_{\Lambda=0}}$ for unswept elements is where sonic flow is first observed on the element and has typical values of 0.8 to 1.0. It is computed from a surrogate model [12] based on the minimum $C_{p_{inc}}$ generated by the unswept airfoil (i.e., returned by NeuralFoil).

$$M_{crit_{\Lambda=0}} = \left\{ 1.0083619 - \min(C_{p_{inc}}) + \left[-0.51058894 \min(C_{p_{inc}}) \right]^{0.6553655} \right\}^{-0.5536965}$$

As pointed out in reference [12], use of the M_{crit} surrogate model avoids the necessity of iterative solution that may not converge. The surrogate model has a negligible impact on M_{crit} accuracy.

$M_{crit_{\Lambda}}$ is the critical Mach number for a swept element and is the freestream Mach number at which a sonic velocity component first appears in the direction perpendicular to the wing's leading edge.

$$M_{crit_{\Lambda}} = \frac{M_{crit_{\Lambda=0}}}{\cos \Lambda}$$

Lift and drag divergence Mach numbers (M_{ld} , M_{dd} , respectively) depend on many factors including angle of attack, Reynolds number and section camber and thickness. The drag divergence Mach number (M_{dd} always $> M_{crit}$) is the freestream Mach number at which the drag begins to rapidly increase with increasing Mach. M_{dd} is determined from the Korn equation [18] that incorporates the effect of wing sweep.

$$M_{dd} = \frac{\kappa_A}{\cos \Lambda} - \frac{h/b}{\cos^2 \Lambda} - \frac{C_{l_{inc}}}{10 \cos^3 \Lambda} \quad \text{where } \kappa_A = 0.87$$

Lift increases with increasing Mach number up to the lift divergence Mach number (point where a shock appears on the airfoil upper surface) and lift falls off with increasing Mach number beyond M_{ld} as the shocks move towards the leading edge. The lift divergence Mach number M_{ld} is modeled here simply as:

$$M_{ld} = \min(M_{dd} + 0.025, 0.925) \quad \text{with } M_{dd} \text{ limited to } 0.90$$

The compressible lift and pitching moment coefficients ($C_{l_{comp}}$, $C_{m_{comp}}$) at the local Mach M_l and Reynolds numbers Rn_l are computed by integrating the Karman Tsien corrected NeuralFoil predicted pressure coefficients. For drag $M_l > M_{dd}$ or lift and pitching moment at $M_l > M_{ld}$ (i.e., outside the validity range of the Karman Tsien correction), additional modeling is required.

The transonic Mach range is challenging to model and is often overly approximate in BEMT codes. Lift and pitching moment coefficients decrease over the range $M_{ld} < M_l < 1.0$ caused by the formation and movement of shocks on the airfoil surfaces and shock induced separation. Lift is modeled using a surrogate model that was derived from measured lift on eight airfoils (NACA 4-006, 23015, 4415, 2315, 2312, 2309, 2306, 66-210 as reported in references [19, 20]). Presented in Figure 2 is the measured lift $C_l - C_{l_{ld}}$ versus $M - M_{ld}$ overlaid by the surrogate model prediction of the same data (note α is in radians in the surrogate equation). $C_{l_{ld}}$ is the Karman Tsien corrected lift coefficient predicted at M_{ld} .

$$\begin{aligned} C_l - C_{l_{ld}} = & [-702.6(M - M_{ld})^5 + 681.1(M - M_{ld})^4 - 230(M - M_{ld})^3 + 29.17(M - M_{ld})^2 \\ & + 0.523(M - M_{ld}) - 0.0039] (4.262 - 2.282\alpha) \left(\frac{h}{b}\right)^{0.597} \left[-0.551 \right. \\ & \left. - 20.535 \left(\frac{y_{c,max}}{b}\right) \right] \end{aligned}$$

The surrogate model should perform reasonably well for airfoils in the range $M_{ld} < M_l < 1.0$, $-6^\circ < \alpha < 10^\circ$, $0.06 < \frac{h}{b} < 0.15$, and $0.0 < \frac{y_{c,max}}{b} < 0.04$. If the range of the surrogate model is exceeded, a simplified model of $C_l - C_{l_{ld}} = -1.634 (M - M_{ld})$ is used in the post-lift divergence transonic regime.

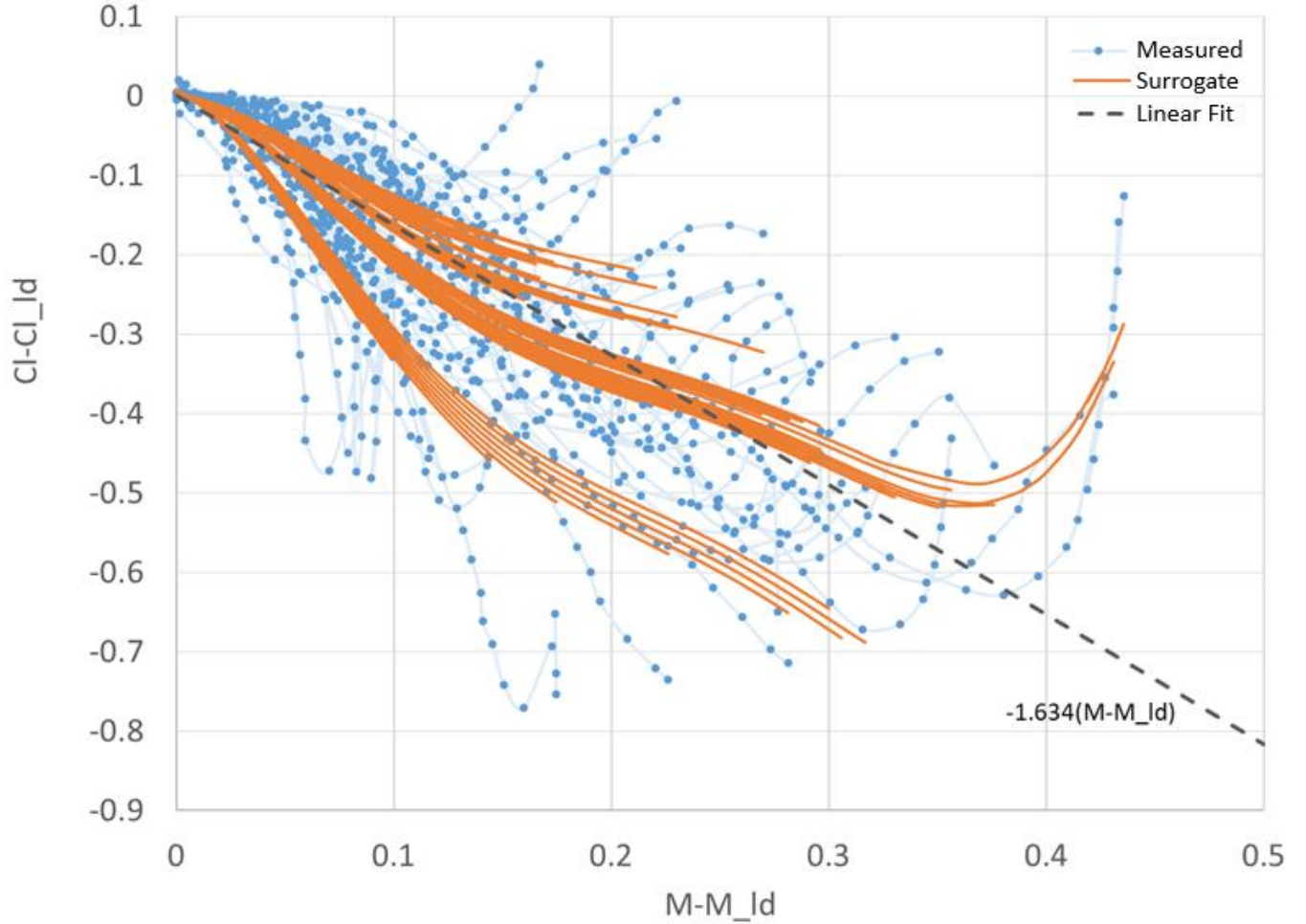


Figure 2. Measured and modeled post-lift-divergence lift coefficient versus Mach number compared with a surrogate model.

In the transonic range $M_{ld} < M_l < 1.0$ lift (i.e., C_l) is calculated using the predicted lift $C_{l_{ld}}$ at M_{ld} and the surrogate model prediction of $C_l - C_{l_{ld}}$ at $M_l - M_{ld}$. Pitching moment is approximated as a linear variation between $C_{m_{ld}}$ at M_{ld} and the thin airfoil theory result $C_{m_{sup1}}$ at $M_l = 1$ (discussed below).

$$C_{l_{trans}} = C_{l_{ld}} + (C_l - C_{l_{ld}})_{surrogate}, \quad C_{m_{trans}} = C_{m_{ld}} + \left(\frac{C_{m_{sup1}} - C_{m_{ld}}}{1 - M_{ld}} \right) (M_l - M_{ld})$$

For $M_{dd} < M_l < 1.0$, drag is calculated [21] (note v is capped at $\alpha = 15^\circ$) as:

$$C_{d_{trans}} = C_{d_{inc}} + v \left(\frac{M_l - M_{dd}}{1 - M_{dd}} \right)^2 \quad \text{where } v = 4.8\alpha^2 + 0.15\alpha + 0.5 * \frac{h}{b} \quad (\alpha \text{ in rad})$$

In the above expression $C_{d_{inc}}$ is the incompressible drag coefficient predicted by NeuralFoil at α and Rn_l . The v polynomial is an approximation of the pressure drag coefficient at Mach 1 obtained by fitting the measured pressure drag coefficient of a 4, 6 and 9% thick, slightly-cambered NACA 65A series airfoils at $M_\infty = 1.0$ and $Rn = 2.7$ million over a range of angle of attack [22].

For most propellers, the blade elements experiencing transonic flow are near the tip and are relatively thin airfoils. A similarity law by Liu [23] for supersonic thin airfoils is used to model the element forces for $M_l > 1.0$. Liu's similarity law relates the thin airfoil results predicted by Ackeret [24] at

$M_\infty = \sqrt{2}$ to those at lower Mach numbers ($1 < M_\infty < 1.4$) where Ackeret's theory is inaccurate in comparison to measured results.

$$C_{l_{sup}} = C_{l_{\sqrt{2}}} \exp \left[\frac{2}{\gamma+1} (\sqrt{2} - M_l) \right] \quad \text{where: } C_{l_{\sqrt{2}}} = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} = 4\alpha \text{ at } M_\infty = \sqrt{2}$$

$$C_{m_{le, sup}} = C_{m_{\sqrt{2}}} \exp \left[\frac{2}{\gamma+1} (\sqrt{2} - M_l) \right] \quad \text{where: } C_{m_{\sqrt{2}}} = \frac{-2\alpha}{\sqrt{M_\infty^2 - 1}} = -2\alpha \text{ at } M_\infty = \sqrt{2}$$

$$C_{d_{sup}} = C_{d_{inc}} + (C_{d_{\sqrt{2}}} - \delta_l) \exp \left[\frac{2}{\gamma+1} (\sqrt{2} - M_l) \right] + \delta_l$$

where: $C_{d_{\sqrt{2}}} = 4 \left[\alpha^2 + \left(\frac{h}{b} \right)^2 \right] / \sqrt{M_\infty^2 - 1} = 4 \left[\alpha^2 + \left(\frac{h}{b} \right)^2 \right]$ at $M_\infty = \sqrt{2}$ and $\delta_l = 0.74/Rn_l^{1/5}$, α in rad.

At incidence angles above the thin airfoil validity limit of 5° , regressions of CFD generated biconvex airfoil results [25] are blended with the lower incidence angle model.

$$C_{l_{sup}} = (13.676 - 4.590M)\alpha \exp \left(\frac{-\alpha}{-0.377 + 0.720M} \right) \quad \alpha > 5^\circ$$

$$C_{d_{sup}} = 0.0503 + 2.8428 \alpha^{1.584} M^{-0.801}$$

The drag $C_{d_{sup}}$ is the sum of the wave drag (largest component of supersonic drag) and $C_{d_{inc}}$ which approximates the drag other than wave. The value δ above accounts for the wake width that is modeled as twice the non-dimensional turbulent boundary layer thickness at the trailing edge of an equivalent flat plate ($\delta_l = 2 \times 0.37/Rn_l^{1/5}$ for $Rn_l < 10^7$). In the code, the constraint that $C_{d_{sup}}$ must be greater than $C_{d_{inc}}$ is enforced to prevent degenerate cases. The pitching moment coefficient about the quarter chord, needed to calculate torsion effects, is simply $C_{m_{sup}} = C_{m_{le, sup}} + 0.25C_{l_{sup}}$. Note that the pitching moment coefficient similarity described above is assumed to be valid by extension but has not been thoroughly tested. The thin airfoil

assumption is generally assumed to be reasonable for airfoils with $\frac{h}{b} < 0.06$ and at low angle of attack.

In summary, the aerodynamic model regimes for a two-dimensional blade element in free air (i.e., prior to sweep, solidity and tip losses and stall considerations) are:

$$\begin{array}{lll}
 \text{if } M_l < M_{ld} & C_l = C_{l_{comp}}, & C_m = C_{m_{comp}} \\
 \text{if } M_l < M_{dd} & C_d = C_{d_{comp}} & \\
 \text{if } M_{ld} < M_l < 1.0 & C_l = C_{l_{trans}}, & C_m = C_{m_{trans}} \\
 \text{if } M_{dd} < M_l < 1.0 & C_d = C_{d_{trans}} & \\
 \text{if } M_l > 1.0 & C_l = C_{l_{sup}}, & C_d = C_{d_{sup}}, \quad C_m = C_{m_{sup}}
 \end{array}$$

Blending is implemented between the different speed regimes and with the stall model ensuring that the composite lift and drag curves remain continuous across the full transition from low speed through supersonic flow.

Sweep Model

Sweeping a propeller increases the critical Mach number thereby delaying the negative impacts of sonic flow on the aerodynamic force and moment coefficients to a higher freestream Mach. Blade element sweep is modeled using simple sweep theory. Typically, sweep is introduced as either a sheared (i.e., pushed back) or swept (i.e., rotated) airfoil cross-section depending on how the propeller is designed and simple sweep theory distinguishes between the two. The sweep of a blade element is defined by the leading edge sweep angle Λ .

For swept configurations:

$$C_{l_\Lambda} = C_{l_{\Lambda=0}} \cos^2 \Lambda \quad C_{d_\Lambda} = C_{d_{\Lambda=0}} \cos^3 \Lambda \quad C_{m_\Lambda} = C_{m_{\Lambda=0}} \cos^2 \Lambda$$

For sheared configurations:

$$C_{l_\Lambda} = C_{l_{\Lambda=0}} \cos \Lambda \quad C_{d_\Lambda} = C_{d_{\Lambda=0}} \cos^2 \Lambda \quad C_{m_\Lambda} = C_{m_{\Lambda=0}} \cos \Lambda$$

In the BEMT modeling, aerodynamic coefficient values are computed as if the blade is unswept followed by application of the simple sweep theory rules.

Post Stall Model

Blade element stall angle is modeled based on the stall-angle Reynolds number dependency of a generic airfoil. The change in stall angle as a function of Reynolds number (i.e., $\partial \alpha_{stall} / \partial Rn$) is assumed to be the same for all airfoils. It is further assumed that the airfoil at 75% of the blade radius is representative of the airfoils at other blade radii. The measured stall angle α_{s75} at Rn_{s75} for the representative airfoil is used in the regression shown below and in Figure 3 to determine α_{stall} . For best results Rn_{s75} should be within an order of magnitude of Rn_l

$$\alpha_{stall,base} = [0.00000273(Rn_l - Rn_{s75}) + \alpha_{s75}] / \cos \Lambda \quad (\alpha \text{ in deg.})$$

At low supersonic Mach numbers, the same angle of attack produces a progressively weaker pressure disturbance as Mach increases (per linearized theory's $1/\sqrt{M^2 - 1}$ scaling), delaying stall to higher angles of attack. For $M_l \geq 1.0$ the baseline stall angle is further multiplied by a Mach-dependent delay factor that was determined from computed RANS solutions [25] of a biconvex circular arc airfoil at low supersonic speeds:

$$f_d = 1 + k_d \sqrt{M_l^2 - 1} \quad \text{where } k_d = 0.3837$$

The complete stall angle used is:

$$\begin{aligned} \alpha_{stall} &= \alpha_{stall,base} & M_l < 1.0 \\ \alpha_{stall} &= \alpha_{stall,base} \cdot f_d & M_l \geq 1.0 \end{aligned}$$

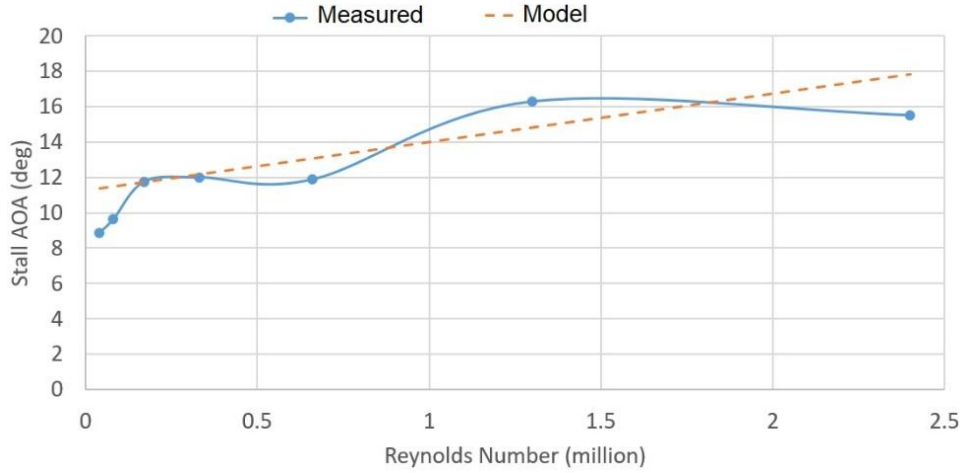


Figure 3. Airfoil baseline stall angle model Reynolds number dependency.

Above the stall angle, the aerodynamic coefficient behavior is dictated by a method developed by Truong [26]. The model is based on key features observed in the topology of post-stall lift and drag curves. The validity of the post-stall model is demonstrated on a NACA 0012 airfoil in Figure 4 (measured data from [27]) and this model is assumed to apply to the post-stall characteristics of all BEMT airfoil elements. Normal and tangential coefficients are defined as:

$$C_N = (2.08 + 0.0836 \cos \alpha + 0.406 \cos^2 \alpha) \sin \alpha$$

$$C_T = (0.09 - 0.178 \cos \alpha - 0.298 \cos^3 \alpha) \sin^2 \alpha$$

The lift and drag are related to the normal and tangential coefficients by the coordinate transformation relations $C_l = C_N \cos \alpha + C_T \sin \alpha$ and $C_d = C_N \sin \alpha - C_T \cos \alpha$.

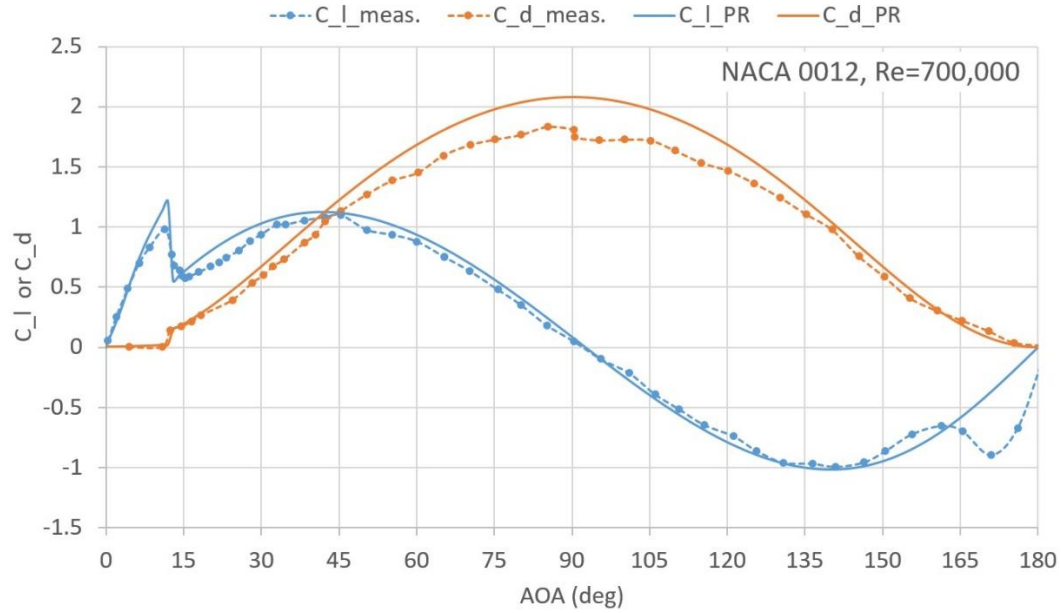


Figure 4. Comparison of aerodynamic model lift and drag coefficient predictions with measured data on a NACA 0012 airfoil (Measurements from [27]).

Aerodynamic Model Validation

The NACA series 16 airfoil has long been used in propeller design because its shape is derived from an analytical function containing the section design lift coefficient which makes it easy to design a propeller blade with a specified lift distribution. Shown in Figure 5 is the NACA 16-1009 that has a design lift coefficient of 1.0 and is 9% thick. Wind tunnel measurements of series 16 airfoils are limited and the primary source is [28]. Presented in Figures 6 and 7 are comparisons of the measured and predicted lift and drag coefficients over a range of Mach numbers and angle of attack. The measurements were performed on an airfoil with a 5 inch chord resulting in a Reynolds number range of 0.85 million to 2 million over the test Mach range of 0.3 to 0.8 (conditions are the same for the predictions).

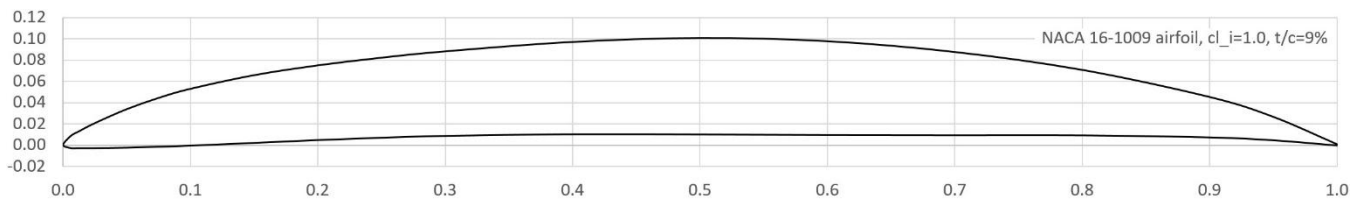


Figure 5. NACA 16-1009 airfoil with a design $C_l = 1.0$ and a thickness of 9%.

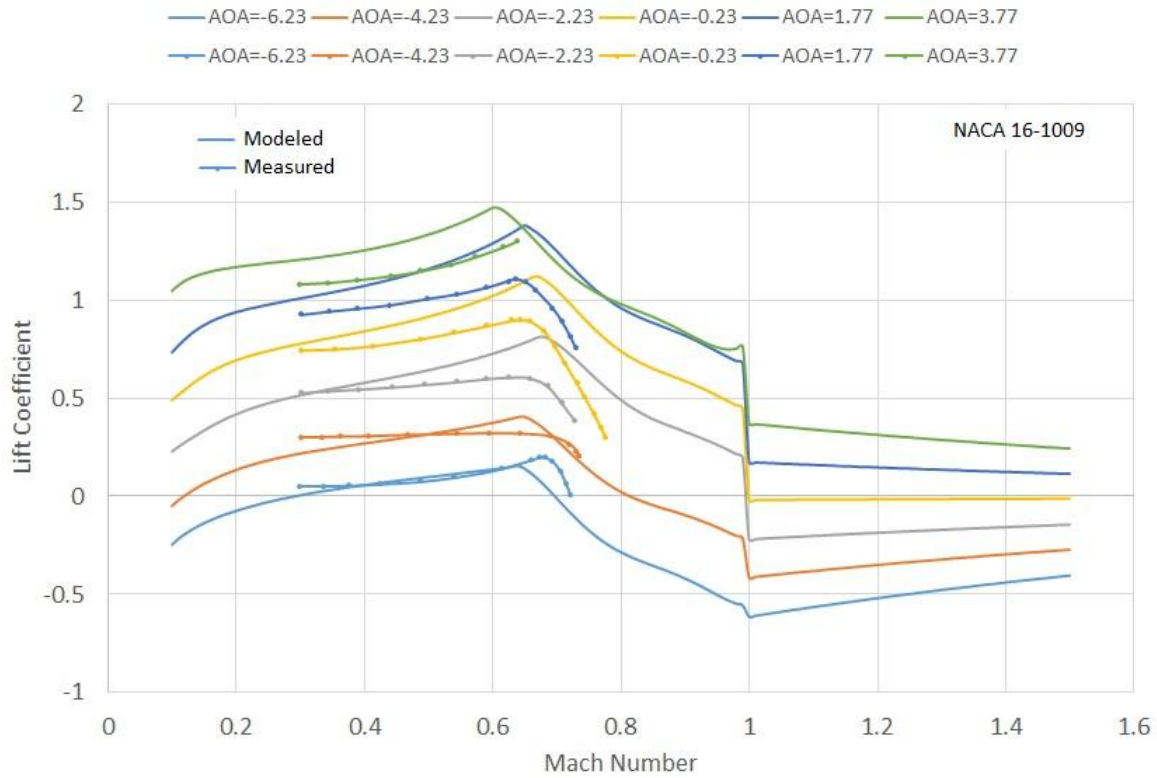


Figure 6. Comparison of aerodynamic model lift coefficient predictions with measured data on a NACA 16-1009 airfoil (Measurements from NACA TN 1546).

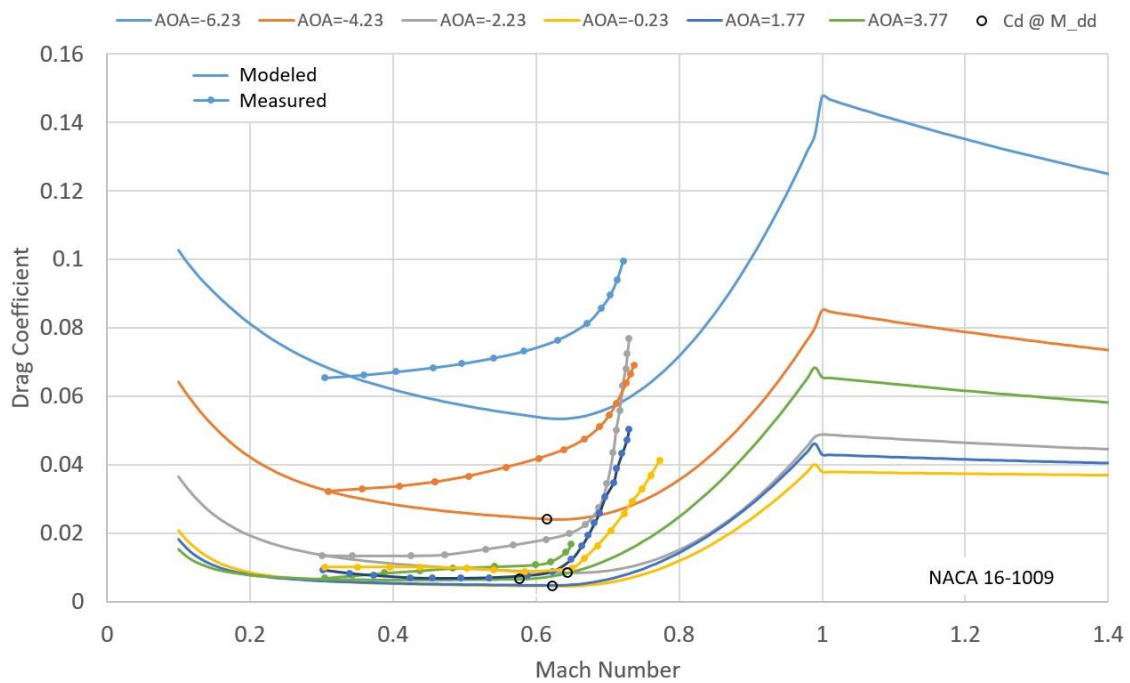


Figure 7. Comparison of aerodynamic model drag predictions with measured data on a NACA 16-1009 airfoil (Measurements from NACA TN 1546).

As seen in Figure 6 the two-dimensional element lift coefficient predicted by the aerodynamic model is reasonably close to the measured results from wind tunnel tests [28] up to the lift divergence Mach number. At greater Mach numbers, no data is available to compare with. On the other hand, the drag coefficient does not compare as well. The drag divergence of the measurements begins at around 0.6 and the measured drag rise is steeper than the modeled rise on this particular airfoil.

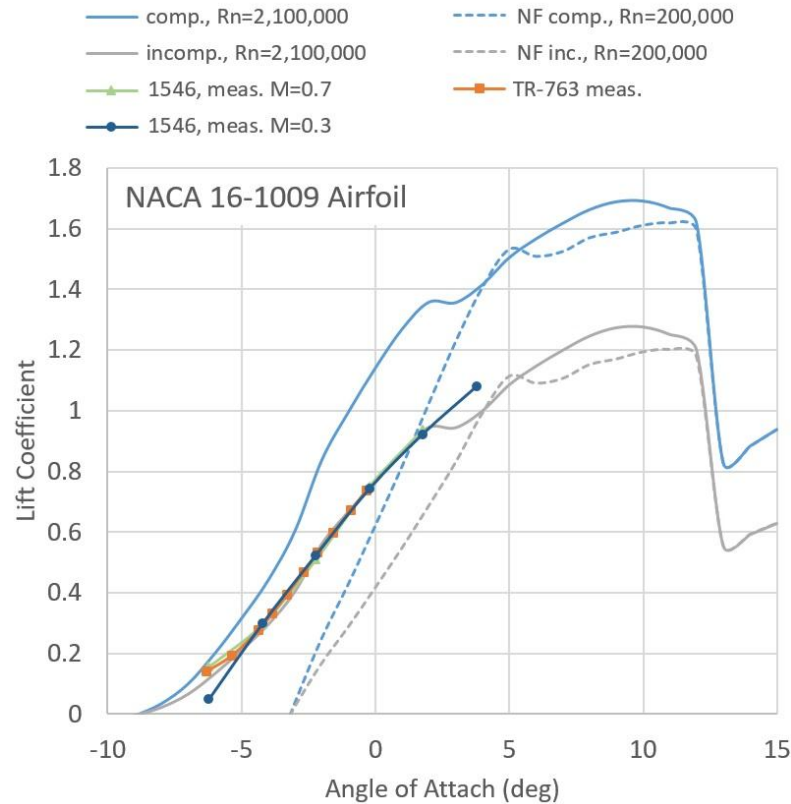


Figure 8. Variation of NACA 16-1009 airfoil predicted lift coefficient over a range of angle of attack, Reynolds and Mach numbers compared with measured data (from NACA TN 1546 and TR 763).

To get the best results from a BEMT method, the blade geometry should be well represented and the blade element aerodynamic model as accurate as possible over a wide range of conditions. Shown in Figure 8 is a comparison of the NACA 16-1009 airfoil lift coefficient over a range of conditions that are typically encountered by a propeller. As can be seen, the predicted element lift can vary by over 50% at a given angle of attack depending on the Mach and Reynolds Numbers for this cambered airfoil commonly used in propeller designs.

The Karman Tsien compressibility correction used for $M_l < M_{dd}$ and also Liu's similarity law applied for $M_l > 1$ assume a thin airfoil ($< 6\%$ thick) at an angle of attack within the lift curve linear range. When applied to thicker airfoils, these approximations typically result in an under prediction of lift and drag.

Blade Element Momentum Theory

The Blade Element Momentum Theory combines the Momentum Theory of Propellers (MT) with Blade Element Theory (BET) to predict the performance of real propellers (i.e., those with non-uniform flow, induced velocity and losses). In BEMT, the propeller is broken down into a number of independent elements. The local resultant velocity (including the induced velocity from MT) and incidence angle that each element experiences are used with the airfoil section lift and drag coefficients and “tip loss models” that model wake effects to determine the forces on the individual blade elements. The force components, resolved in the axial and tangential directions, are integrated along the blade to determine the thrust and torque. The flow fields created by CR rotors (contra rotating or a rotor followed by a stator) mutually interfere with each other leading to consideration of additional induced velocity components.

Following the BEMT derivation of Ning [29], we start with the Momentum Theory (MT) of propellers for a single propeller disk. Figure 9 shows a propeller within a stream tube.

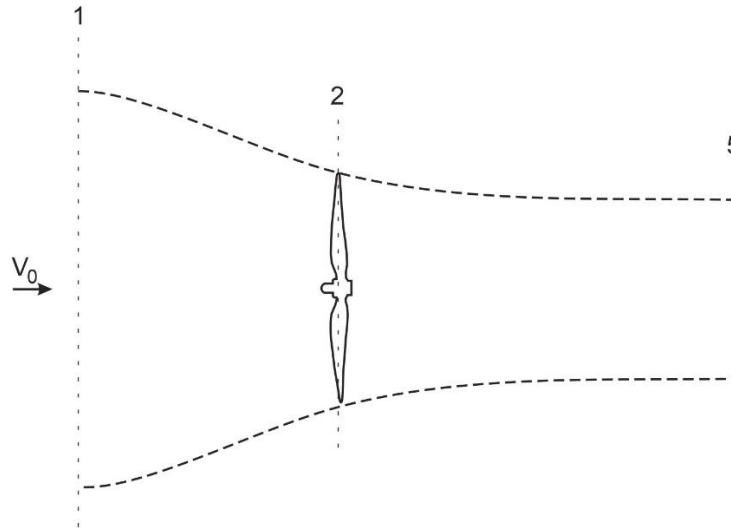


Figure 9. SR rotor system axial stations.

For a velocity field $\vec{V} = v_x \vec{e}_x + v_\theta \vec{e}_\theta$ generated by a single rotor (SR), conservation of linear momentum and mass combined with the Bernoulli equation can be used to determine the thrust/unit blade length T' on a propeller disk annulus (or B blade elements within the annulus). Similarly, torque/unit blade length Q' can be determined by applying conservation of angular momentum:

From linear momentum balance:
$$T' = 4\pi\rho \left(\frac{v_{ix2}}{V_0} \right) \left(1 + \frac{v_{ix2}}{V_0} \right) V_0^2 r F$$

From angular momentum balance:
$$Q' = 4\pi\rho \left(\frac{v_{i\theta 2}}{\Omega r} \right) \left(1 + \frac{v_{ix2}}{V_0} \right) V_0 \Omega r^3 F$$

In the above expressions, it is assumed that v_{x2} is constant across the blade element, $P_5 = P_1$ and v_{ix} and $v_{i\theta}$ are the axial and tangential induced velocity components. These expressions also include a tip-loss-function $F(r)$ multiplier that accounts for the influence of tip and hub losses from flow three-dimensionality as discussed in the next section.

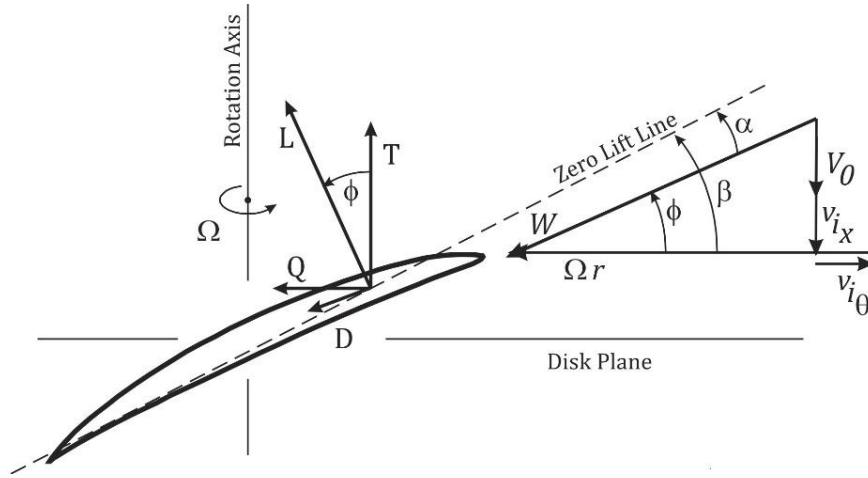


Figure 10. SR rotor velocity triangle and forces on an element at radial location r .

In BET, a single rotor blade is subdivided into elements that are independent of each other. Shown in Figure 10 is the cross-section of a blade element at r from the rotation axis. The thrust and torque on B blade elements each of area ($b \, dr$) is:

$$T' = Bq_l [C_l \cos \phi - C_d \sin \phi] b$$

$$Q' = Bq_l [C_l \sin \phi + C_d \cos \phi] b r$$

Where q_l is the local dynamic pressure based on the element resultant velocity W (as seen in Figure 10) and C_l and C_d are the local airfoil lift and drag coefficients determined at $\alpha = (\beta - \phi)$, local M_l and Re_l .

$$W = \sqrt{(V_0 + v_{ix2})^2 + (\Omega r - v_{i\theta2})^2}$$

The chief difficulty of relating the induced velocity to a specific propeller geometry is overcome by equating the MT and BET thrust and torque equations resulting in the BEMT equations.

$$\text{Thrust eq.} \quad 4\pi\rho \left(\frac{v_{ix2}}{V_0} \right) \left(1 + \frac{v_{ix2}}{V_0} \right) V_0^2 r F = Bq_l [C_l \cos \phi - C_d \sin \phi] b$$

$$\text{Torque eq.} \quad 4\pi\rho \left(\frac{v_{i\theta2}}{\Omega r} \right) \left(1 + \frac{v_{ix2}}{V_0} \right) V_0 \Omega r^3 F = Bq_l [C_l \sin \phi + C_d \cos \phi] b r$$

It is common to express the BEMT equations in terms of induction factors $a = v_{ix2}/V_0$, and $a' = v_{i\theta2}/\Omega r$ and a local solidity defined as $\sigma' = Bb/2\pi r$. From the velocity triangle shown in Figure 10, $W \sin \phi = V_0(1 + a)$ and $W \cos \phi = \Omega r(1 - a')$, therefore:

$$\text{Thrust eq.} \quad a = \frac{1}{\frac{4F \sin^2 \phi}{\sigma' [C_l \cos \phi - C_d \sin \phi]} - 1}$$

$$\text{Torque eq.} \quad a' = \frac{1}{\frac{4F \sin \phi \cos \phi}{\sigma' [C_l \sin \phi + C_d \cos \phi]} + 1}$$

The above two SR equations with unknown induction factors a and a' are implicit equations because everything on the right hand side except σ' depends on the induction factors and therefore an iterative solution approach is necessary. Rather than iterating both equations one can establish a single residual equation $R(\phi) = 0$ and use root finding methods (e.g., Newton's method, a bracket method like Brent's method or a simple fixed-point iteration like that implemented in PyRotor) to determine the value of ϕ that drives the residual function to zero. Ning [29] uses the following residual function that comes from the velocity triangle of Figure 10 (i.e., $\tan \phi = V_0(1 + a)/\Omega r(1 - a')$):

$$R(\phi) = \frac{\sin \phi}{1 + a} - \left(\frac{V_0}{\Omega r} \right) \frac{\cos \phi}{1 - a'} = 0$$

Like many root finding solution procedures, the solution process may diverge because of poor initial ϕ guess. Cases can also be defined for which no roots exist (e.g., specifying operating conditions outside the range of the propellers capability).

The thrust and torque of B blades in coefficient form are:

$$C_T = \frac{T}{\rho n^2 D^4} = \frac{B}{\rho n^2 D^4} \int_{r_{hub}}^{r_{tip}} q_l [C_l \cos \phi - C_d \sin \phi] b \, dr$$

$$C_Q = \frac{Q}{\rho n^2 D^5} = \frac{B}{\rho n^2 D^5} \int_{r_{hub}}^{r_{tip}} q_l [C_l \sin \phi + C_d \cos \phi] b \, r \, dr$$

The power coefficient, efficiency and advance ratio are:

$$C_p = \frac{P}{\rho n^3 D^5} = 2\pi C_Q \quad \eta = \frac{C_T J}{C_p} \quad J = \frac{V_0}{nD}$$

Blade Interaction

On propellers with three or fewer blades, typically blade-to-blade aerodynamic interference (not modeled in most BEMT implementations) is small and often ignored. For increased thrust, power and reduced noise, modern rotors often have many blades at the cost of reduced efficiency. As the number of blades increase, so does the blade-to-blade aerodynamic interference. A reasonable approach to modeling blade aerodynamic interaction is to consider an annulus of a propeller circumferentially unrolled and modeled as an infinite linear cascade of blades with blade trailing edge separation of $2\pi r/B$ as shown in Figure 11.

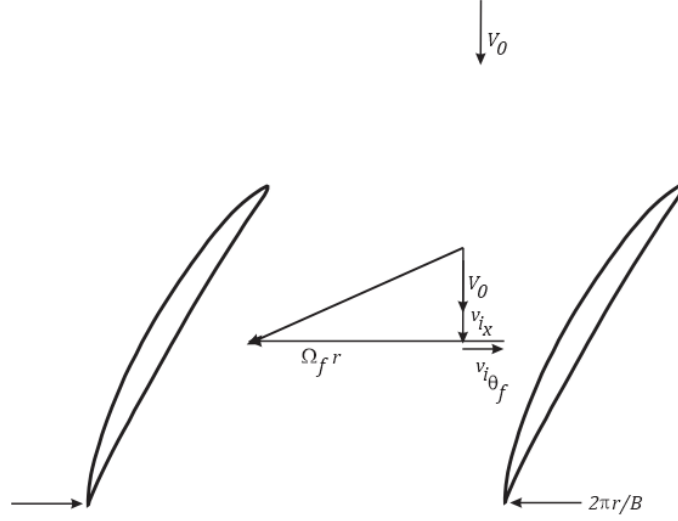


Figure 11. Velocity in a cascade representation of a propeller.

The rotor solidity, which is the ratio of total blade area to rotor disk area, is defined here as $\sigma = B(b/D)_{75}/\pi x_{75}$. The blade-solidity-dependent empirical function f_σ is used to adjust the elemental lift and drag through f_M described below.

$$f_\sigma = \frac{1}{1+(\sigma/\sigma_0)^p} \quad \sigma_0 = 0.618, \quad p = 20$$

Values of f_σ range from 1.0 for a typical two-blade aircraft propeller down to 0.04 for a 12-blade propfan rotor.

For modern high-solidity prop-fan rotors operating at transonic Mach numbers, as the local Mach number exceeds the drag divergence Mach number, shock-induced wave drag and shock/boundary-layer interactions couple with blade-to-blade interference resulting in a highly nonlinear three-dimensional flow. The solidity and local Mach number dependent f_M modification shown below is a simple model of these effects calibrated against the three available validation cases discussed in the results section of this memorandum.

$$C_l = C_{l0}/f_M, \quad C_d = C_{d0} \cdot f_M \quad \text{where:} \quad \begin{array}{ll} \text{if } M_l < 0.3 & f_M = 1.0 \\ \text{if } 0.3 \leq M_l < 1.0 & f_M = 1 + 1.5 (M_l - 0.3) f_\sigma \\ \text{if } M_l \geq 1.0 & f_M = 1 + 1.05 f_\sigma \end{array}$$

Inclusion of this solidity-related correction improved the comparison of predicted and measured thrust and power over a range of conditions for all of the cases studied. For $M_l > 0.3$, the f_M factor reduces lift and increases drag. It should be noted that f_M has no effect on lift or drag for $M_l < 0.3$ and the effect gradually increases with increasing M_l and increasing blade solidity σ . Hopefully, the utility of this model will lead to more rigorous examination and improved blade interference modeling in the future.

Loss Modeling

Since a propeller blade is of finite length, the load on a propeller blade vanishes at the tip. Pressure equalization between the suction and pressure side of the blade at the tip causes the vorticity shed along the propeller blade to roll up into a vortex that follows a helical path in the blade wake (caused by the blade rotation). These effects are commonly modeled as “tip losses” (see [30]) and the particular model used here for F_{tip} is Glauert’s model [6] from 1935 which is a simplification of Prandtl’s tip loss factor. Defining $x = 2r/D$, the tip loss factor is:

$$F_{tip} = \left(\frac{2}{\pi}\right) \cos^{-1} \left(\exp \left(-\frac{B}{2} \left(\frac{1-x}{x} \right) / \sin \phi \right) \right)$$

The thrust generated by the blade near the root decreases substantially in comparison to the outer portion of the blade and so the impact of hub losses is minimal. Typically, hub losses are modeled as:

$$F_{hub} = \left(\frac{2}{\pi}\right) \cos^{-1} \left(\exp \left(-\frac{B}{2} \left(\frac{x-x_h}{x} \right) / \sin \phi \right) \right)$$

The tip and hub factors are multiplied together to determine the aggregate impact on the blade loading distribution:

$$F_l = F_{tip} F_{hub}$$

A basic shortcoming of most tip loss models, including Glauert’s, is that the wake is often modeled as a rigid helical screw surface of constant pitch and diameter rather than a contracting wake like that predicted by momentum theory. Additionally, radial velocity components that take energy from the wake are neglected completely in the classic BEMT formulation (and in the tip loss function). The tip loss function was derived under the limiting assumption of light to moderate loading (i.e., $a < 0.4$ or $C_T < 0.4$). Finally, unaccounted for blade interference effects increase with increasing blade count. These effects (among others) directly impact the blade induced velocity and therefore thrust and torque.

Blade Deformation

Blade torsion can have a small effect on the local blade angle and therefore impact the propeller thrust and power. A simple model of the torsional deformation is to consider the blade as an elastic beam loaded by the aerodynamic pitching moment C_m and the central twisting moment M_{ct} (caused by centrifugal forces to predict the torsional rotation of each blade element about its elastic axis). The total pitch angle change at a given element location is the sum of the elemental rotation from the blade root to the element of interest.

The section elastic axis and center of gravity are assumed to be at 35% of the airfoil chord and therefore $c_{m_ea} = c_{m_1/4} + (0.35 - 0.25)c_l$ and the aerodynamic pitching moment applied about the element elastic axis is $M_{aero} = c_{m_ea} q b^2 \Delta r$ for an element of span width Δr . The central twisting moment is $M_{ct} = -(\Omega^2/2)(\sin 2\beta) r \rho_{prop} \Delta r \int x dA$. The first moment of area integral $\int x dA \cong 0.05 \left(\frac{h}{b}\right) b^3$ for an airfoil profile approximated by two triangles (each with a base of $(h/b)b$ and height of $0.35b$ and $0.65b$ straddling the elastic axis). The net twisting torque acting on the blade element is $Q = M_{aero} + M_{ct}$.

$$Q = [c_{m_{1/4}} + (0.35 - 0.25)c_l]qb^2\Delta r - 0.05 (\Omega^2/2)(\sin 2\beta) r \rho_{prop} \left(\frac{h}{b}\right) b^3 \Delta r$$

The contribution to change in the pitch angle of a blade element can be determined using the solid mechanics angle of twist equation $\Delta\theta = Q \Delta r/GJ$. The approximate polar moment of area J of a solid blade cross section about an axis through its centroid is:

$$J = I_{xx} + I_{yy} \cong 0.036b^4 \left(\frac{h}{b}\right) \left[\left(\frac{h}{b}\right)^2 + \left(\frac{y_{c_max}}{b}\right)^2\right] + 0.043b^4 \left(\frac{h}{b}\right)$$

The reduction in blade pitch angle β is the summation of the $\Delta\theta$ from the blade root to the blade-wise element location of interest. If non-solid or composite blades are used, the torsional rigidity GJ should be adjusted accordingly.

Near Field Wake Model

In a CR rotor system, the back rotor is situated in the contracting wake of the front rotor with typical rotor separation distances (g/R) between 0.2 and 0.8. As shown in [31] and in Figure 12, which depicts the normalized wake radius r versus distance downstream of an isolated propeller disk, the degree of wake contraction can be quite significant in the near field particularly at low advance ratio.

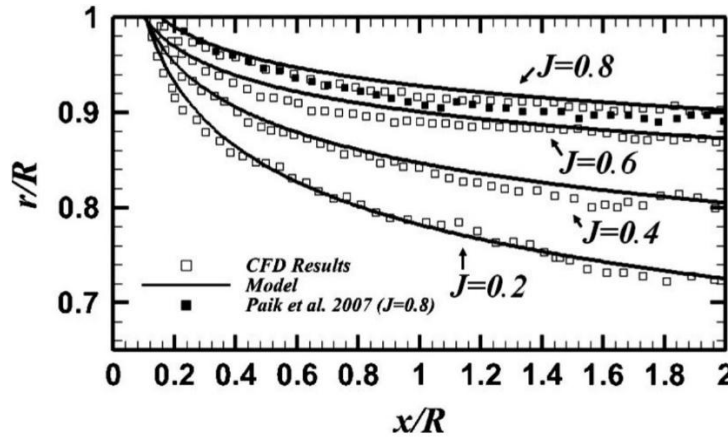


Figure 12. Wake contraction downstream of a ship propeller (from [31], used with permission).

A propeller wake model, originally derived by Theodorsen and discussed by Veldhuis [32], that predicts the wake contraction r_x/r of an isolated actuator disk at a location x downstream based on the axial induction factor is given by the following equation:

$$\frac{r_x}{r} = \sqrt{\frac{1 + a_f}{1 + a_f \left(1 + \frac{x}{\sqrt{r^2 + x^2}}\right)}} \quad \text{for } \frac{x}{R} > 0.11$$

From continuity, the velocity increase in an annulus of the wake is $v_x/v_{x_{annulus}} = (r/r_x)^2$. Upstream of an actuator disk, the velocity is predicted by Bastankhah et al. [33] to be:

$$\frac{v_x}{v_{x_{annulus}}} = \sqrt{a_b(a_b + 2) \left(1 + \frac{x}{\sqrt{r^2 + x^2}}\right) + 1} \quad \text{for } \frac{x}{R} < 0$$

Note that in the above equation, the sign on a_b has been flipped because it was originally derived for a turbine not a rotor.

In the current CR application the $v_x/v_{x_{annulus}}$ relations of interest are the downstream velocity induced by the forward rotor and the upstream velocity induced by the aft rotor (hence a_f and a_b induction factors differ in the above equations). The $v_x/v_{x_{annulus}}$ relations are used in interference factor expressions to model the impact of wake contraction on the CR propeller induced velocity as discussed in the next section.

Contra-Rotating Rotors BEMT

The approach to BEMT for contra-rotating rotors is similar to that of SR but with additional induced velocity components. The axial (x) and tangential (θ) velocity components for a CR system that comprise the resultant velocity (e.g., W_f) are shown in Figure 13. In this figure the subscript i stands for induced, “ f ” for front and “ b ” for the back rotor (or stator). The subscript combination “ fb ” means the induced interference of the front rotor on the back rotor, for example. It is assumed that tangential velocity effects do not propagate upstream and so $v_{i\theta_{bf}} = 0$. Note that the model of a single rotor (Figure 10), is equivalent to that depicted in Figure 13 with the axially induced velocity interference of the nonexistent back blade set to zero (i.e., $v_{ix_{bf}} = 0$).

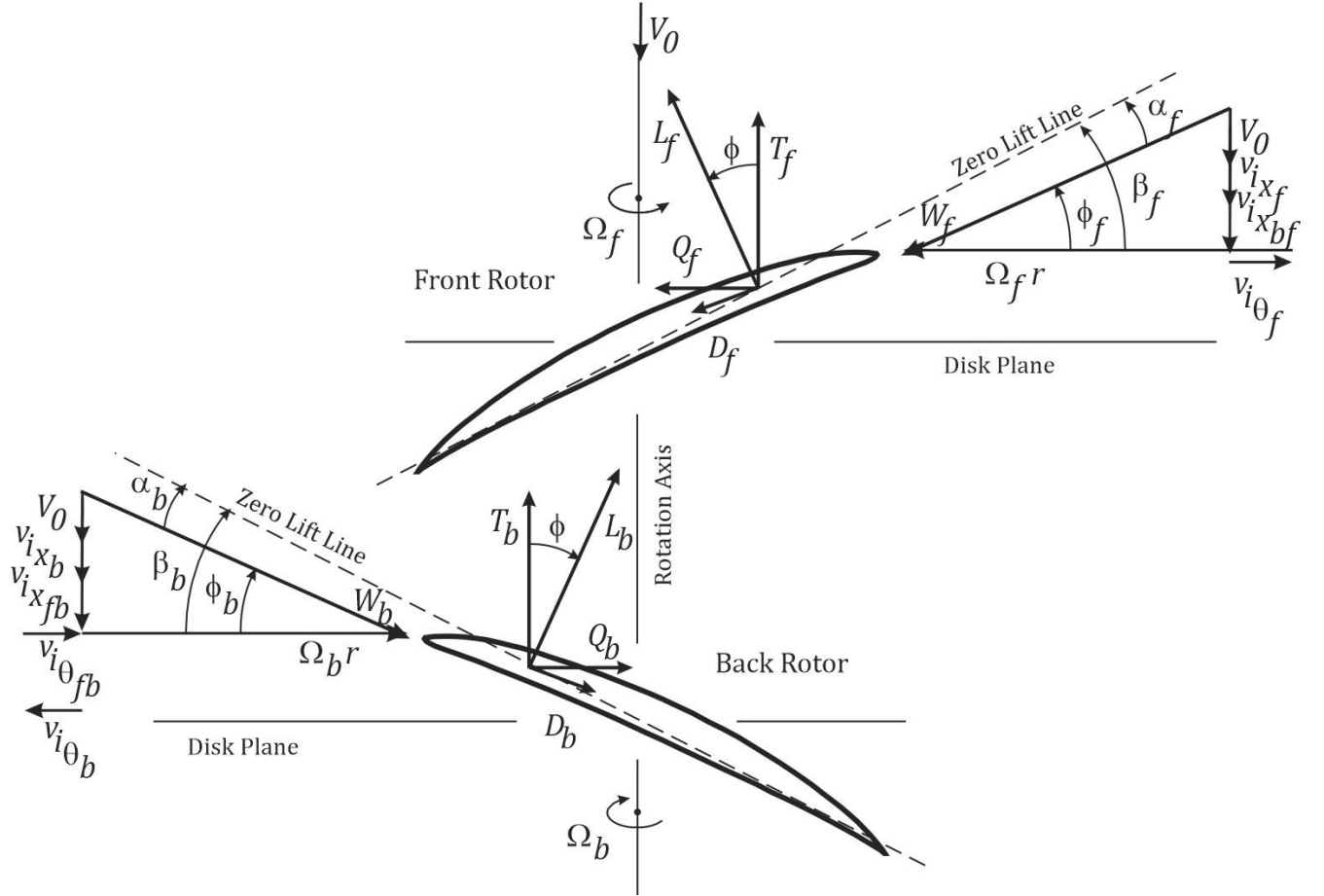


Figure 13. CR rotor system velocity triangles.

The self-induced velocity components v_{ix_f} , $v_{i\theta_f}$, v_{ix_b} , $v_{i\theta_b}$ are determined as part of the BEMT solution process like they are for the SR case. The mutually-induced velocity components $v_{ix_{bf}}$, $v_{ix_{fb}}$, $v_{i\theta_{fb}}$ are incorporated as an interference coefficient (e.g., $\kappa_{x_{fb}}$) acting on the self-induced velocity components similar to the approach taken by Smith [34]. However, it is assumed that the induced velocity component v_{ix} in the near field at g/R behind the rotor scale with v_x/v_{x_disk} (determined from the $v_x/v_{x_annulus}$ relations discussed in the previous section) rather than the fixed-geometry scaling implemented by Smith et al.

$$v_{ix_{bf}} = \kappa_{x_{bf}} v_{ix_b} \quad v_{ix_{fb}} = \kappa_{x_{fb}} v_{ix_f} \quad v_{i\theta_{bf}} = \kappa_{\theta_{bf}} v_{i\theta_b} \quad v_{i\theta_{fb}} = \kappa_{\theta_{fb}} v_{i\theta_f}$$

$$\kappa_{x_{fb}} = 1 + \frac{a_f}{1 + a_f} \cdot \frac{g}{\sqrt{r_f^2 + g^2}}, \quad \kappa_{x_{bf}} = \sqrt{a_b(a_b + 2) \left(1 - \frac{g}{\sqrt{r^2 + g^2}}\right) + 1}, \quad \kappa_{\theta_{fb}} = \kappa_{x_{fb}}, \quad \kappa_{\theta_{bf}} = 0,$$

$$\kappa_{D_b} = 2 \quad \text{all for } \left|\frac{g}{R}\right| > 0.11$$

The interference coefficient for the induced swirl influence of the front rotor on the back ($\kappa_{\theta_{fb}}$) is modeled equivalent to the axial velocity model and the swirl influence of the back rotor

on the front rotor is assumed to be zero. It should be noted that the mutually induced interference expressions above represent an attempt to model a complex interaction and may be over simplified for high fidelity simulations.

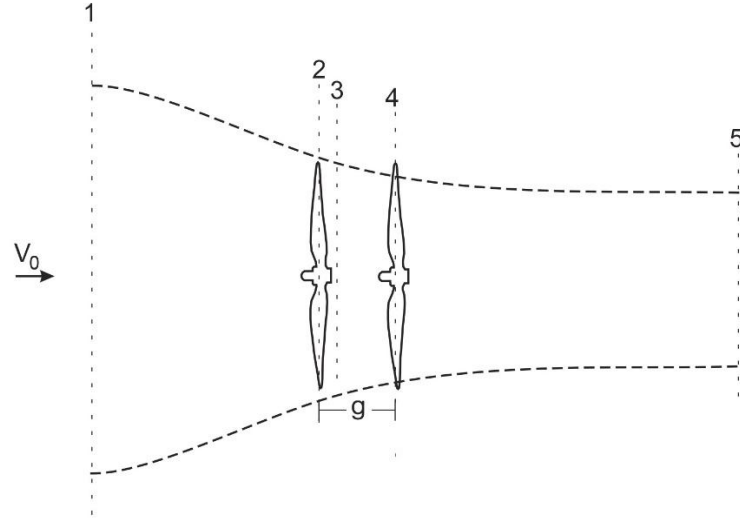


Figure 14. CR rotor system axial stations.

When analyzing a CR system, the velocity at several stream wise locations (see Figure 14) is required. The equations used to compute the velocity components are listed in table 1.

Table 1. CR rotor system axial and tangential velocity components.

Location	Axial Velocity	Tangential Velocity
1	$v_{x_1} = V_0$	$v_{\theta_1} = 0$
2	$v_{x_2} = V_0 + (v_{ix_f} + v_{ix_{bf}})$	$v_{\theta_2} = \Omega_f r - (v_{i\theta_f} - v_{i\theta_{bf}})$
3	$v_{x_3} = V_0 + (v_{ix_f} + v_{ix_{bf}})$	$v_{\theta_3} = \Omega_f r - (v_{i\theta_f} - v_{i\theta_{bf}})$
4	$v_{x_4} = V_0 + (v_{ix_b} + v_{ix_{fb}})$	$v_{\theta_4} = \Omega_b r - (v_{i\theta_b} - v_{i\theta_{fb}})$
5	$v_{x_5} = V_0 + \kappa_{D_b} (v_{ix_{fb}} + v_{ix_{bf}})$	$v_{\theta_5} = \kappa_{D_b} (v_{i\theta_{fb}} - v_{i\theta_{bf}})$

Note that the velocity components in Table 1 are all positive and the signs preceding the components reflect the opposing rotation directions of the front and back rotors (as shown in Figure 13).

The BEMT equations for CR propellers are derived similarly to the process for SR propellers. The MT equations for thrust and torque on an annulus are:

From linear momentum balance:
$$T' = 4\pi\rho \left(\frac{v_{ix2}}{V_0} \right) \left(1 + \frac{v_{ix2}}{V_0} \right) V_0^2 r F$$

From angular momentum balance:
$$Q' = 4\pi\rho \left(\frac{v_{i\theta2}}{\Omega r} \right) \left(1 + \frac{v_{ix2}}{V_0} \right) V_0 \Omega r^3 F$$

For the CR system, the BEMT equations are similar to those for a SR except with different induction factors and element resultant velocity for both the front and back rotors:

$$\begin{aligned} a_f &= (v_{ix_f} + v_{ix_{bf}})/V_0 & a'_f &= (v_{i\theta_f} - v_{i\theta_{bf}})/|\Omega_f|r \\ a_b &= (v_{ix_b} + v_{ix_{fb}})/V_0 & a'_b &= (v_{i\theta_b} - v_{i\theta_{fb}})/|\Omega_b|r \end{aligned}$$

Similar to the approach for a single rotor, the CR system of equations is solved iteratively, but because the induction factors a_f , a'_f , a_b , a'_b of the front and rear rotor are mutually dependent on each other, the front and rear rotor equations are solved simultaneously (i.e., within a single iteration loop).

Modeling Deficiencies

Although the propeller modeling described in the previous sections would appear to be comprehensive, it is lacking in several notable ways.

- Nacelle aerodynamic effects (i.e., non-uniform inflow along the blade leading to $\eta \approx 0.05$ differences) are not included.
- Rotor geometrical features such as blade rake, lean and camber are not modeled.
- Impacts of blade span-wise flow are not modeled.
- Blade deformations, other than simple torsion effects, are not included.
- Non-helical blade wakes are not modeled.
- Flow choking caused by shocks between blades is not rigorously modeled.
- Solidity model is very basic.

Aside from blade deformation, the deficiencies listed above can be addressed using CFD but at substantial cost for the number of cases required by comprehensive simulation studies.

PyRotor Solution Procedure

The BEMT solution procedure implemented in PyRotor is as follows:

- 1) Define forward rotor airfoil and blade geometry (i.e., read input file)
- 2) At a specified RPM, altitude, velocity and blade pitch angle β_{75}
 - a. Divide the blade into a number of blade elements
- 3) Specify initial guess for blade element inflow angle ϕ and iterate
 - a. Determine the blade element local velocity components, M_l , and Rn_l
 - b. Obtain airfoil data $C_l(x)$, $C_d(x)$, $C_m(x)$ from NeuralFoil at each blade section location
 - c. Correct unseparated NeuralFoil C_l , C_d for local Mach (Karman Tsien correction)
 - d. Check for supersonic flow and stall and apply any additional corrections
 - e. Solidity correction
 - f. Calculate blade torsion
 - g. Compute induction factors
 - h. Compute new inflow angle and repeat a. through h. until ϕ is converged
- 4) Calculate propeller C_p , C_T , C_Q , η

Under some conditions, the fixed-point iteration is susceptible to bouncing between two values of ϕ for which the residual is outside the convergence criteria of $\Delta\phi < 0.0000001$ rad. When this occurs, often an acceptable root is found midway between the two oscillating values of ϕ . The midpoint ϕ value is tested and accepted if $\Delta\phi < 0.02$ rad. In the CR case, both rotors are iterated simultaneously (i.e., alternately in a single loop) until the coupled solution converges for both rotors.

Results

Two-Blade Propeller

The first test case studied is a two-bladed propeller designated as NACA 10-(10)(08)-03 and tested in the Langley 16-ft High Speed Tunnel in the 1950s [16]. The propeller blade uses NACA series 16 airfoils (see [16]) and the blade geometry is shown in Figure 15. During the test, the advance ratio was varied by changing the tunnel speed from 27 to 224 m/s while maintaining the propeller rpm at 1140 for blade pitch angles ranging between 20 and 55 degrees (pitch angle specified at 75% of the blade radius).

The BEMT and measured results compare reasonably well as shown in Figures 16-18. The C_T , C_P and η results follow the right trends and the peak efficiency of the measured results is about 90% which is a reasonable value for a two-blade propeller. The peak efficiency of the modeled results is a percent or so lower than the measured results even though the C_T and C_P results look pretty good especially at low pitch angles. Since $\eta = C_T J / C_P$, small differences in these quantities can lead to meaningful differences in efficiency, which ideally we'd like to predict within a percent. The 10 ft diameter blade tested in the 16 ft diameter wind tunnel may have been influenced by the tunnel walls (under the right conditions, ducted fans can have higher efficiency than open rotors) although the reported wind tunnel data was corrected for wall interference effects and forces on the spinner. The maximum helical-tip Mach number of the test series is 0.85 at $\beta = 55^\circ$ and so it is possible that error in the transonic drag model of the outer blade elements (comprised of NACA series 16 airfoils, see Figure 7) contributed to the inaccuracy.

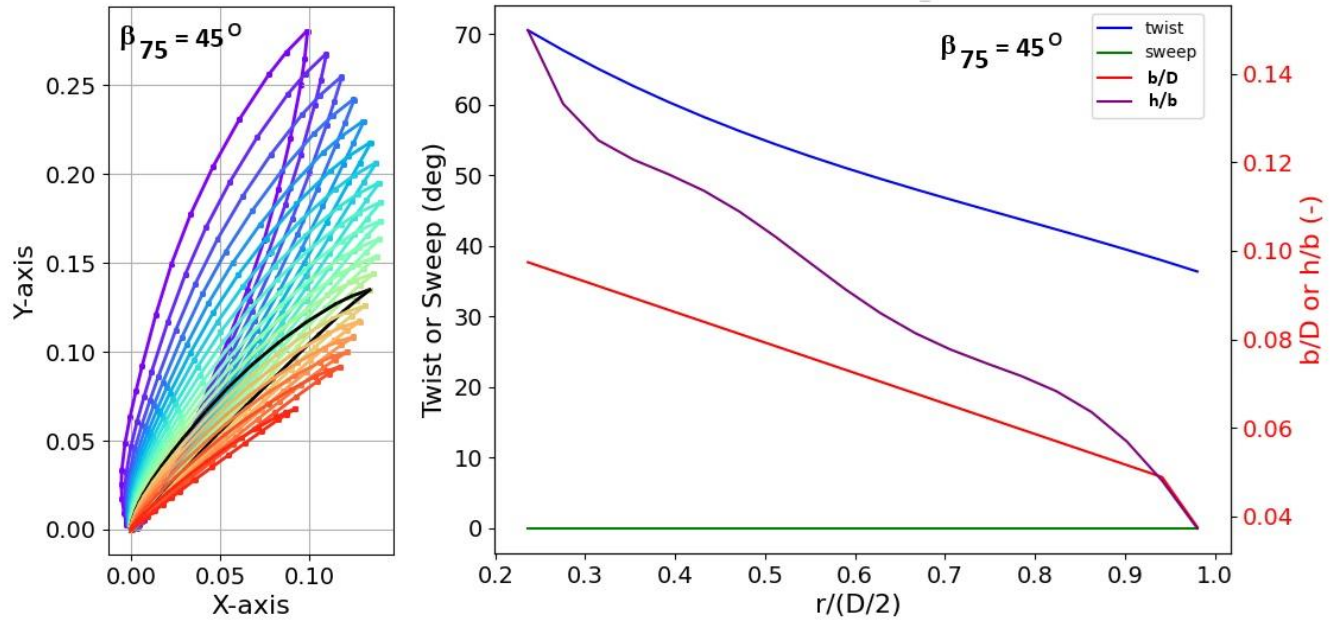


Figure 15. NACA 10-(10)(08)-03 two-blade propeller of NACA TR-1309 [16] shown at a pitch angle of 45° (airfoil at 75% span shown in black).

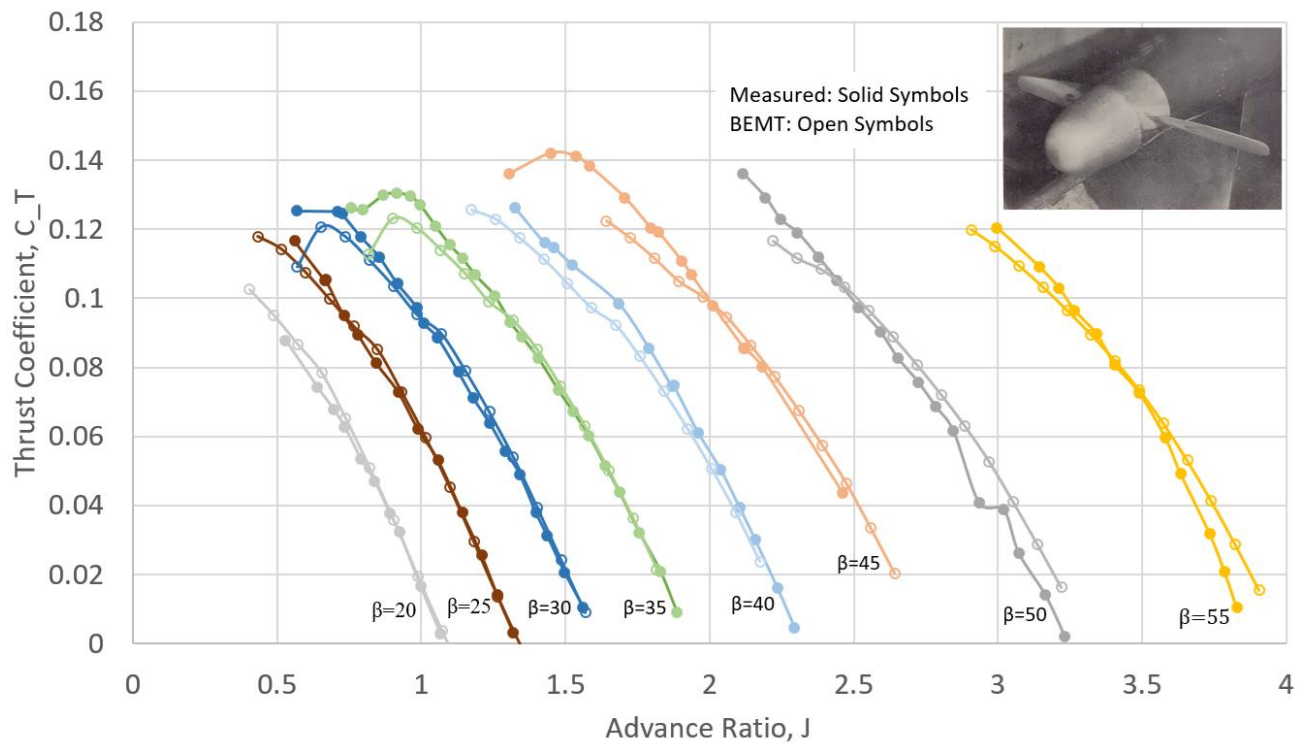


Figure 16. Thrust coefficient comparison of the modeled two-blade propeller of NACA TR-1309 [16] at RPM=1140 over a range of advance ratios and blade pitch angles. Closed symbols are measured, open symbols are BEMT predictions.

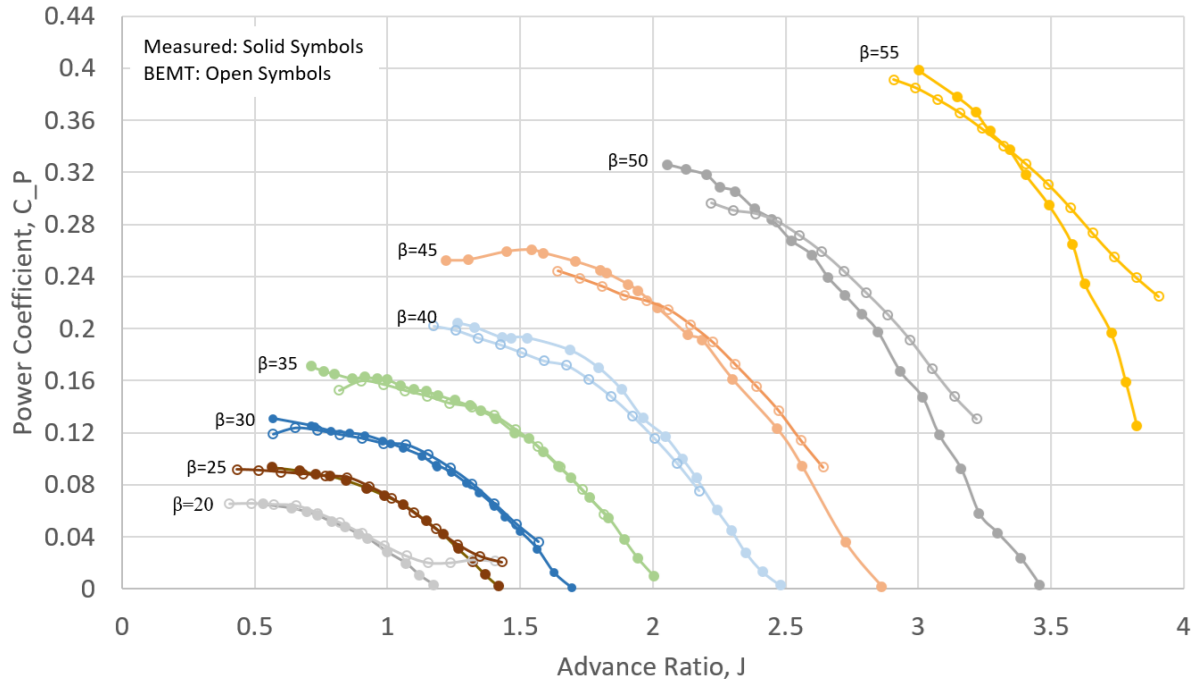


Figure 17. Power coefficient comparison of the modeled two-blade propeller of NACA TR-1309 [16] at RPM=1140 over a range of advance ratios and blade pitch angles. Closed symbols are measured, open symbols are BEMT predictions.

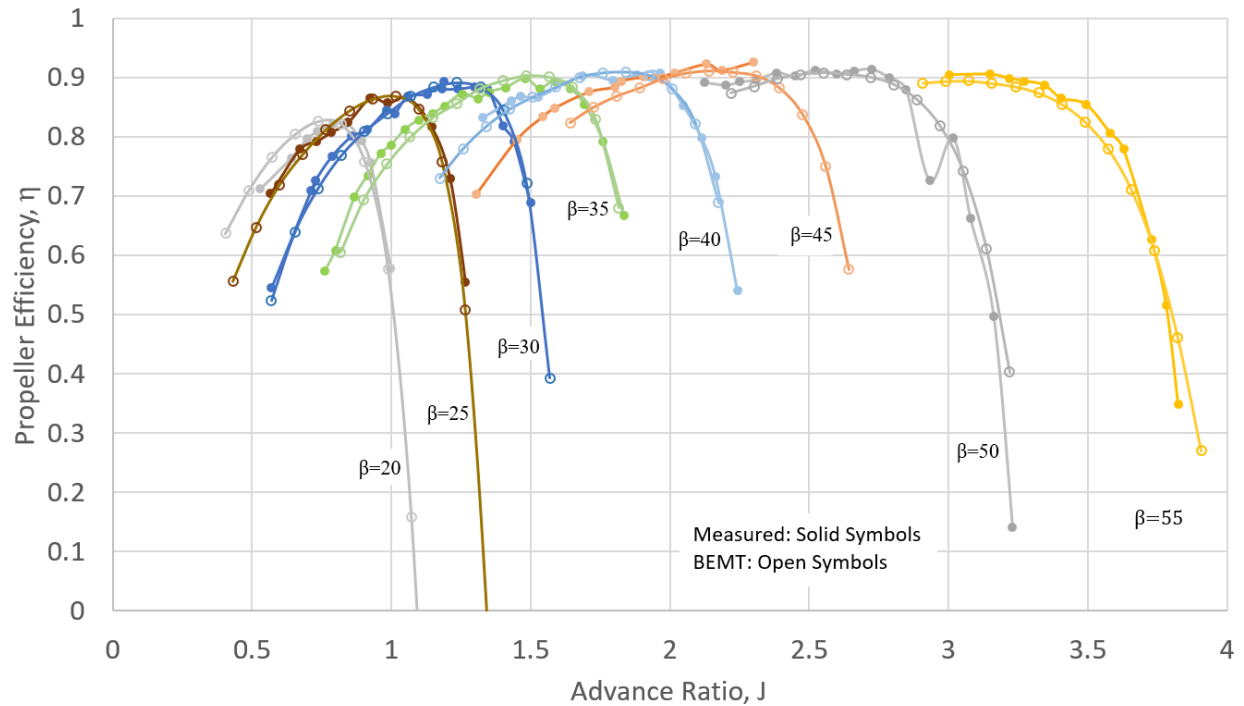


Figure 18. Propeller efficiency comparison of the modeled two-blade propeller NACA TR-1309 [16] at RPM=1140 over a range of advance ratios and blade pitch angles. Closed symbols are measured, open symbols are BEMT predictions.

Eight-Blade Rotor

Wind tunnel tests were conducted on the SR-3 eight-blade rotor in the NASA Glenn 8x6 wind tunnel run in open circuit mode. The swept blade is comprised of series 65 with circular arc-camber-line airfoils near the blade root and NACA series 16 airfoils outboard (see [35]). The blade geometry is shown in Figure 19. During the test, the advance ratio was varied by changing the blade rpm at a fixed Mach number and blade pitch setting. Data for Mach number ranging between 0.45 and 0.85 were reported in reference [35] but only the Mach 0.45 data is compared to BEMT in the current effort. Blade pitch angles ranged between 45.5 and 57.0 degrees (pitch angle specified at 75% of the blade radius). The rotor power and efficiency data reported in [35] are “net” values that represent the efficiency of the propeller operating in the velocity field of a nacelle.

Presented in Figures 20, 21 and 22 is a comparison between the BEMT predicted and measured thrust and power coefficients and efficiency at $M_\infty=0.45$. The measured thrust coefficient C_T shown in Figure 20 was calculated from the power and efficiency numbers using $C_T = \eta C_P / J$ because the measured C_T results were not published.

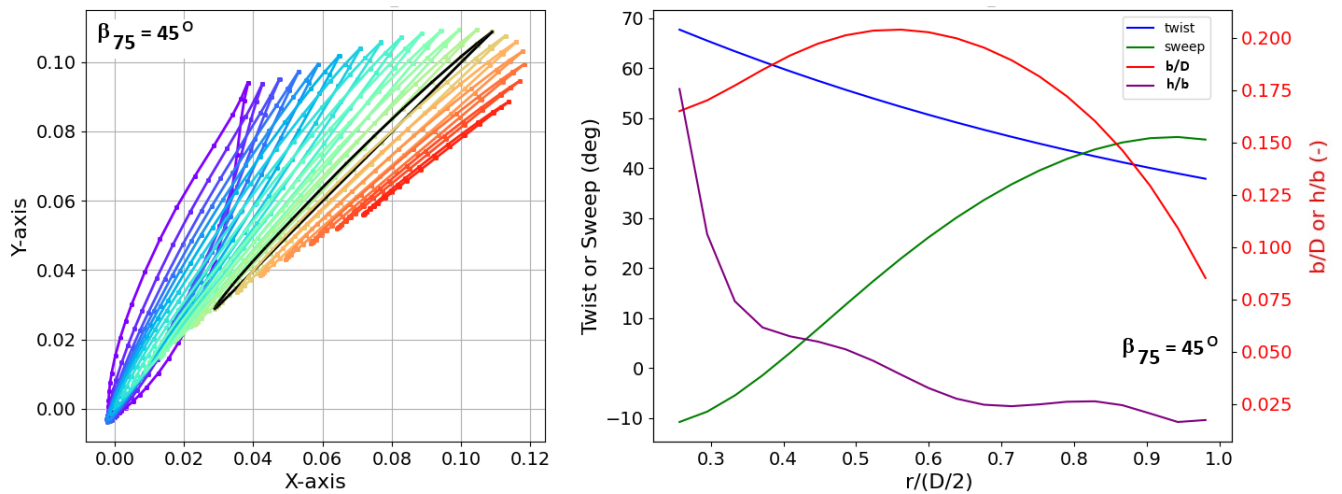


Figure 19. SR-3 eight-blade rotor of NASA CR-3505 [35] shown at a pitch angle of 45° (airfoil at 75% span shown in black).

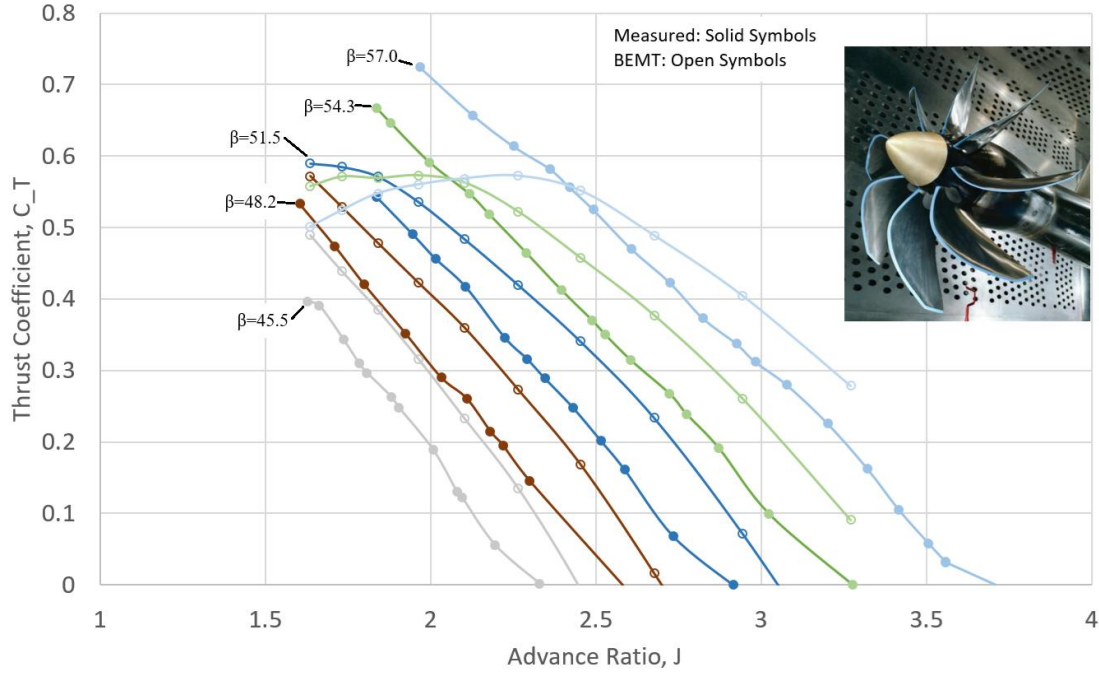


Figure 20. Thrust coefficient comparison of the modeled eight-blade propfan of NASA CR-3505 [35] over a range of advance ratios and blade pitch angles at $M_\infty=0.45$. Closed symbols are measured, open symbols are BEMT predictions.

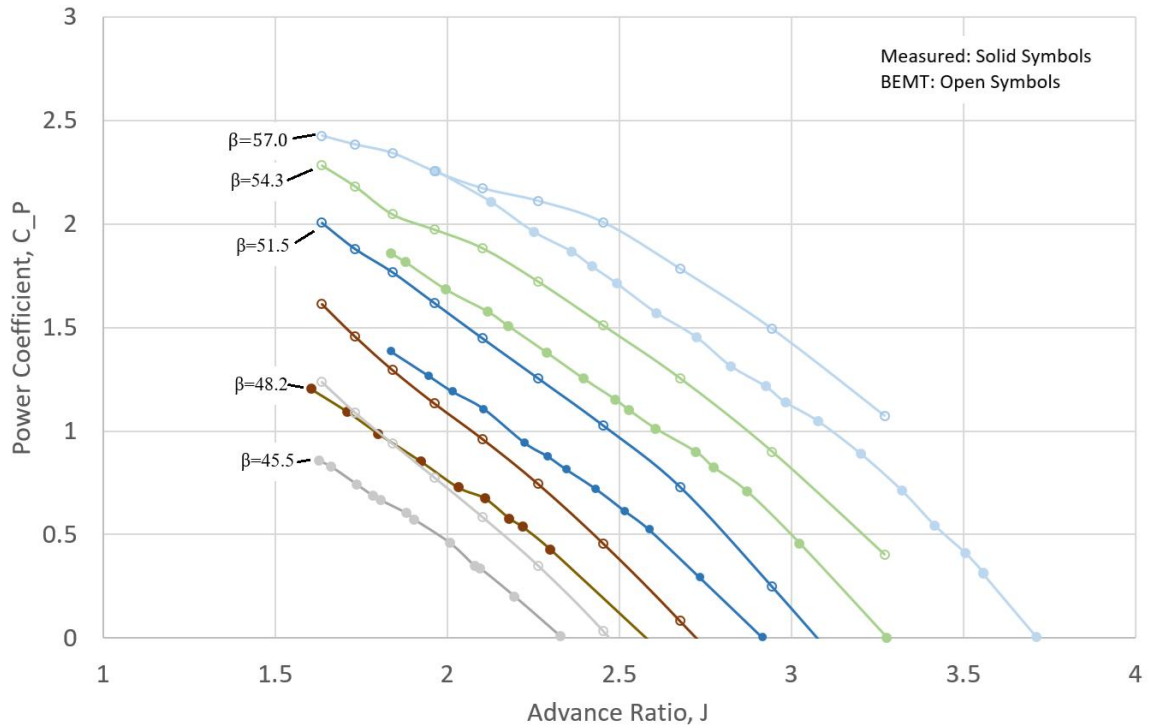


Figure 21. Power coefficient comparison of the modeled eight-blade propfan of NASA CR-3505 [35] over a range of advance ratios and blade pitch angles at $M_\infty=0.45$. Closed symbols are measured, open symbols are BEMT predictions.

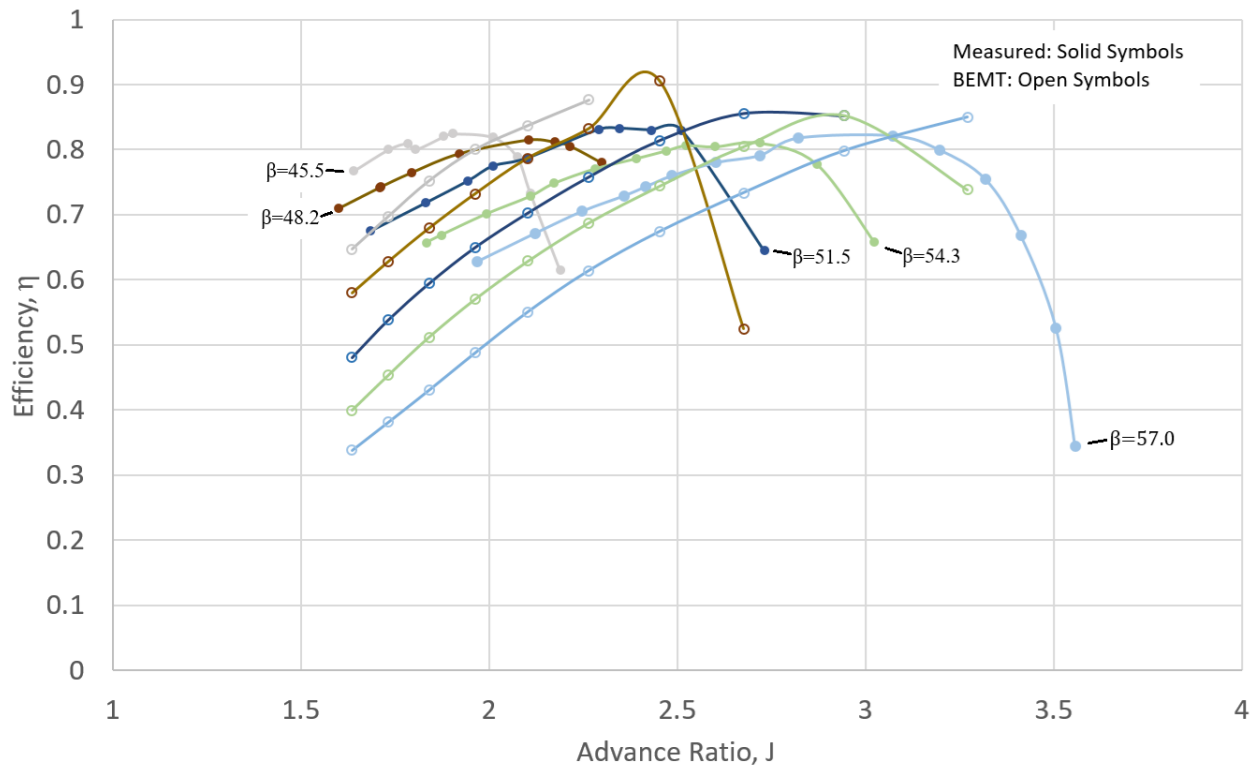


Figure 22. Propeller efficiency comparison of the modeled eight-blade propfan of NASA CR-3505 [35] at $M_\infty=0.45$. Closed symbols are measured, open symbols are BEMT predictions.

As can be seen in Figure 20, the predicted thrust coefficient falls well below the measured values at thrust coefficient values greater than about 0.5. One might believe that this non-agreement is caused by violation of the lightly-loaded assumption of Glauert's tip-loss model, which might contribute but doesn't explain the magnitude of the difference. The dominant effect is that at these highly loaded conditions, the outboard elements operate at $\alpha \approx 9 - 12^\circ$, where partially separated flow, amplified by the compressibility correction, produces a steep drag rise that limits the thrust. No elements stall, and the helical tip Mach number remains below 1. For this case the predicted zero-thrust advance ratio J is about 0.1–0.15 higher than measured at every β , which is only about a 1° effective pitch difference.

Contra-Rotating Rotors

The F31/A31 is a CR rotor featuring a 12-blade front rotor and a 10-blade rear rotor with a design tip speed of 750 ft/s (230 m/s). The rotors were designed in the late 1980s and tested in the NASA Glenn 9×15 and 8×6 wind tunnels in 2009–2011. The measured data set has been used as a benchmark for validating new CROR designs and noise prediction models since.

The blade geometry was extracted from a CFD grid used in the RANS performance predictions of [37] shown in Figure 23 with surface grids shown in Figure 24. The BEMT blade stations in Figures 25 and 26 are oriented parallel to the rotation axis. The tip station runs off the swept blade tip and so it is slightly truncated in the chord-wise direction. The pitch angle of the front blade is specified at 75% of the blade radius and the rear blade at 77.4% of the blade radius.

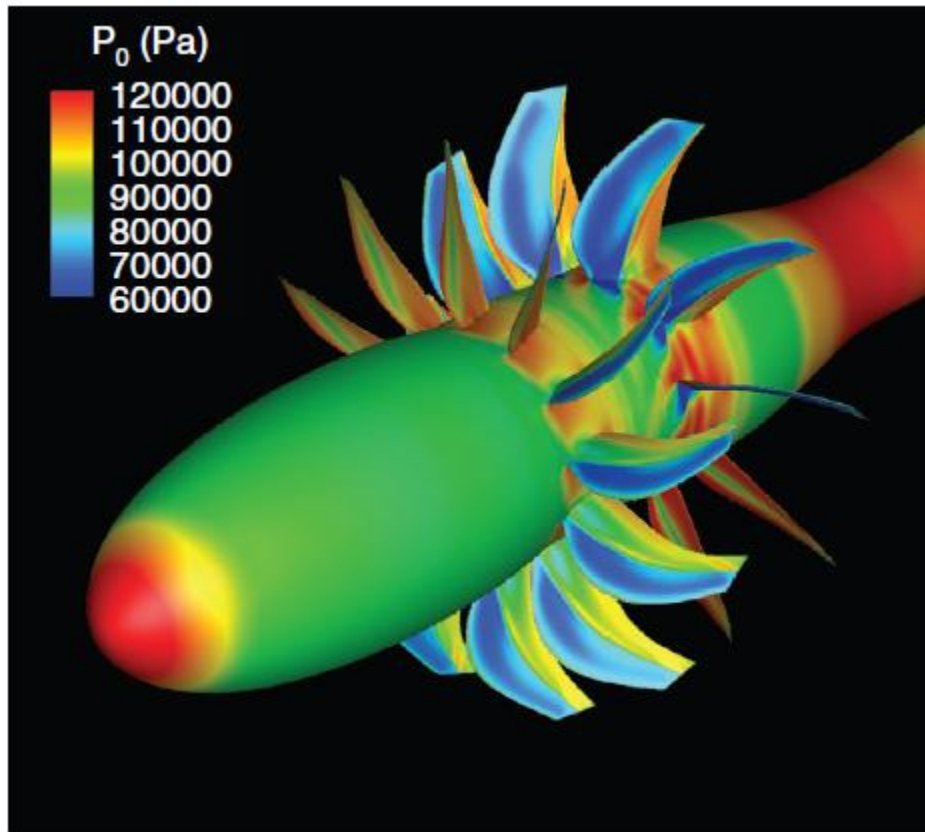


Figure 23. Mean pressure distribution on F31/A31 rotor at $M_\infty = 0.78$, rpm = 6848 and $\beta_f/\beta_b = 64.4^\circ/61.8^\circ$. Front rotor rotates clockwise (looking downstream) and the aft rotor rotates counter-clockwise (from ref. [37], used with permission).

A computer program called Viewer was written to take in either structured or unstructured surface grid files like those shown in Figure 24 and produce the blade element geometrical data and airfoil section normalized coordinates points as shown in Figure 25 and 26 for the F31 and A31 rotors.

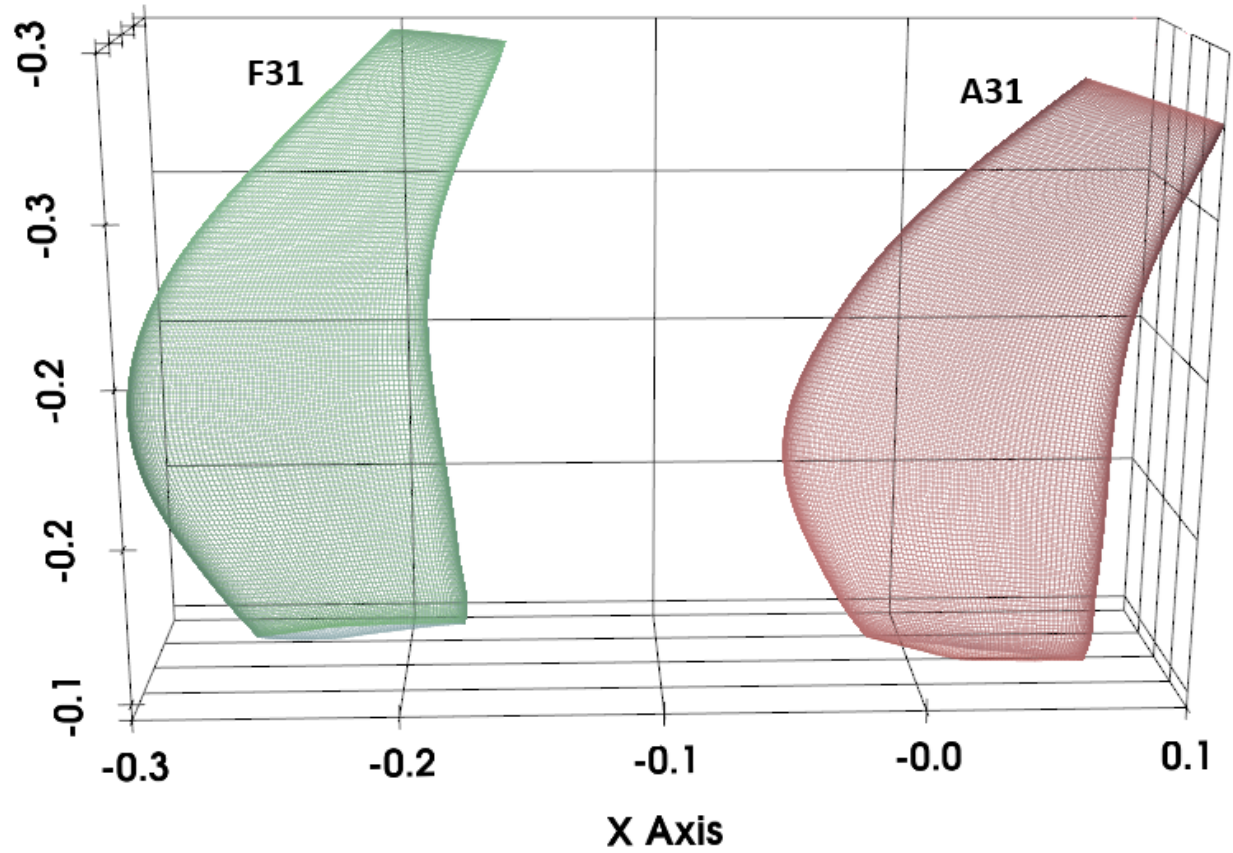


Figure 24. F31/A31 CFD surface grids.

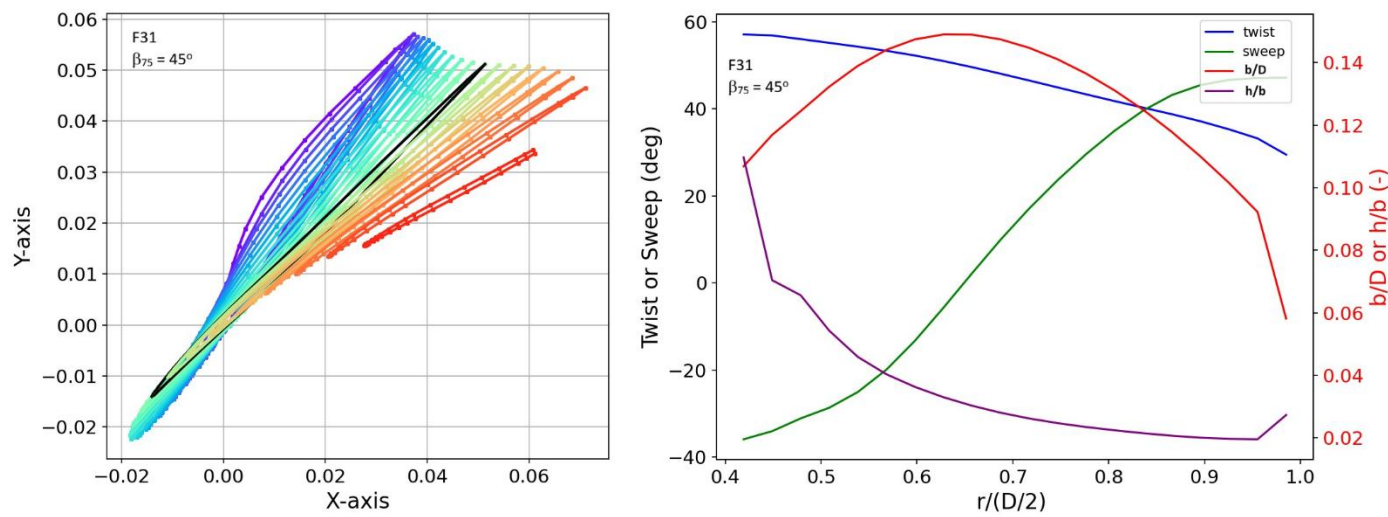


Figure 25. End view of front F31 rotor at $\beta_{75} = 45^\circ$ extracted geometry at 20 blade-wise stations (airfoil at 75% radius shown in black) and associated element twist, sweep, normalized chord length, b/D and profile maximum thickness, h/b .

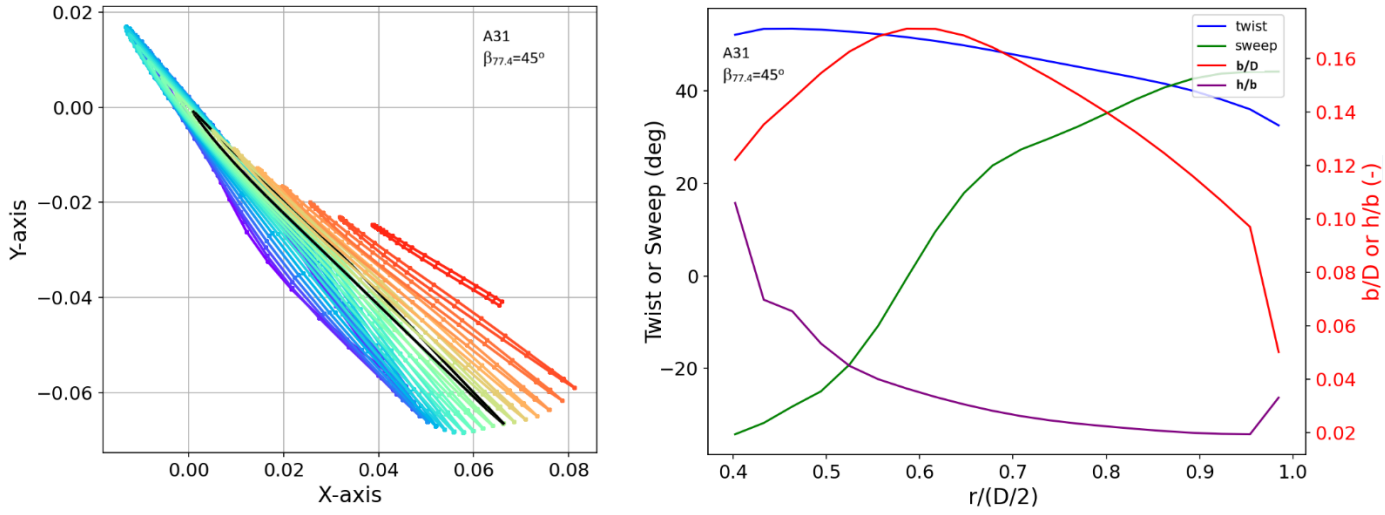


Figure 26. End view of aft A31 rotor at $\beta_{77.4} = 45^\circ$ extracted geometry at 20 blade-wise stations (airfoil at 77.4% radius shown in black) and associated element twist, sweep, normalized chord length, b/D and profile maximum thickness, h/b .

Rotor data from a NASA Glenn 8-ft x 6-ft wind tunnel test at Mach 0.73 ($V_0 = 834$ ft/s) and pitch angles $\beta_{f_{75}}/\beta_{b_{77.4}} = 60.5/59$ degrees over blade corrected rpm ranging between about 6130–7450 were compared to BEMT predictions in Figures 27 through 29. The wind tunnel test report (reference [36]), presents rotor thrust and torque in engineering units rather than coefficients and so we have followed suit. The measured thrust and torque values are the “apparent” values reported in the companion wind tunnel data spreadsheet released with [36]. Apparent thrust includes the thrust increase from the nacelle interaction, which BEMT doesn't model. Therefore, the apparent thrust is expected to exceed what an isolated-rotor code predicts. The nacelle interaction raises thrust, but the size of the increase isn't known here. Corrected rpm accounts for wind tunnel freestream static temperature variations (corrected $rpm = rpm/\sqrt{T_s/T_{ref}}$). In [36], measured power data is not presented for the individual rotors (power could be computed from torque, if desired). In figure 29, the apparent thrust and the actual rpm are used to calculate the propeller efficiency as $\eta = 60TV_0/2\pi rpm Q$. So that the efficiency can be compared with other references, the data is plotted against corrected rpm.

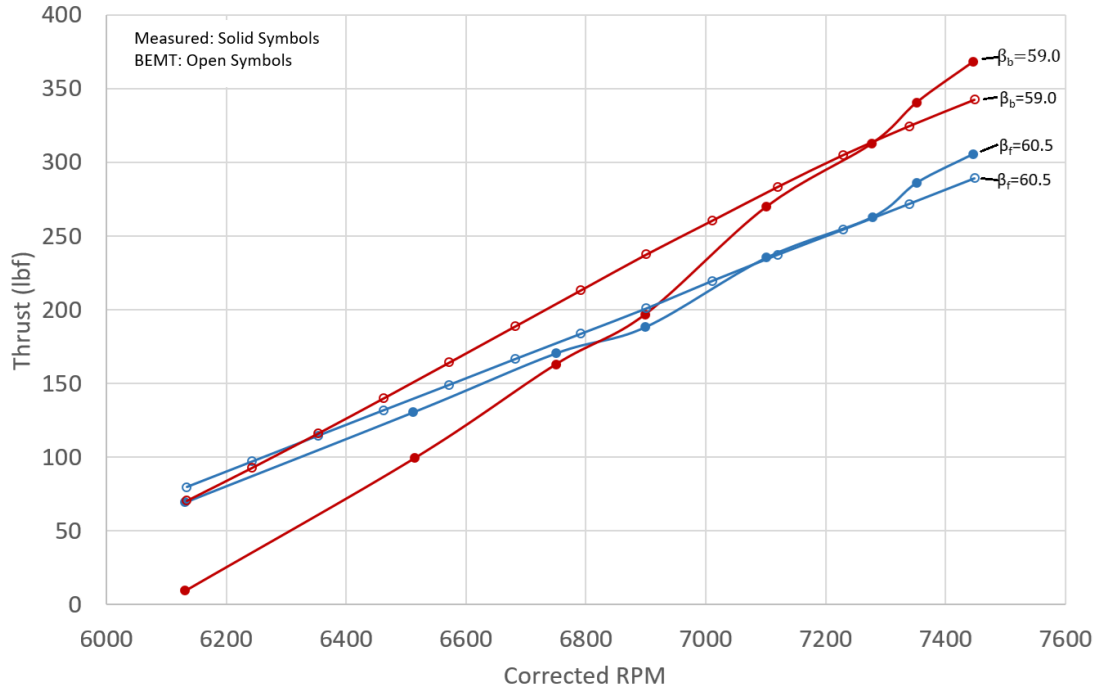


Figure 27. Thrust comparison of the modeled F31/A31 CROR of NASA TM 2014-216676 [36] over a range of corrected rpm at pitch angles $\beta_{f_{75}}/\beta_{b_{77.4}} = 60.5/59$ degrees and $M_{\infty}=0.73$. Closed symbols are measured, open symbols are BEMT predictions.

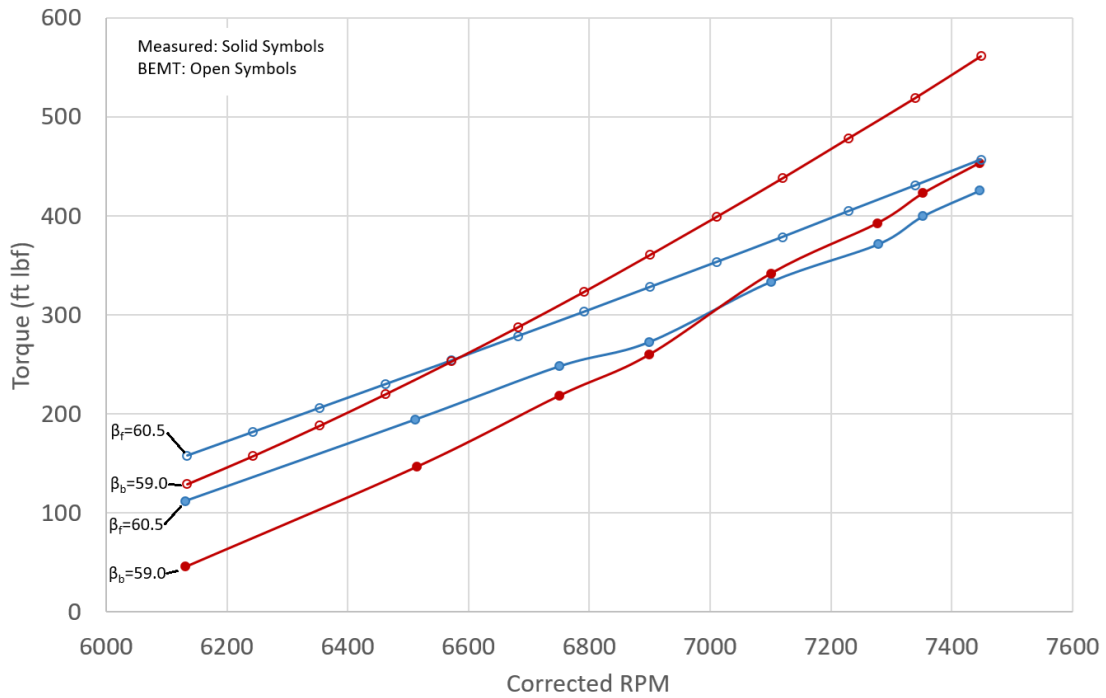


Figure 28. Torque comparison of the modeled F31/A31 CROR of NASA TM 2014-216676 [36] over a range of corrected rpm at pitch angles $\beta_{f_{75}}/\beta_{b_{77.4}} = 60.5/59$ degrees and $M_{\infty}=0.73$. Closed symbols are measured, open symbols are BEMT predictions.

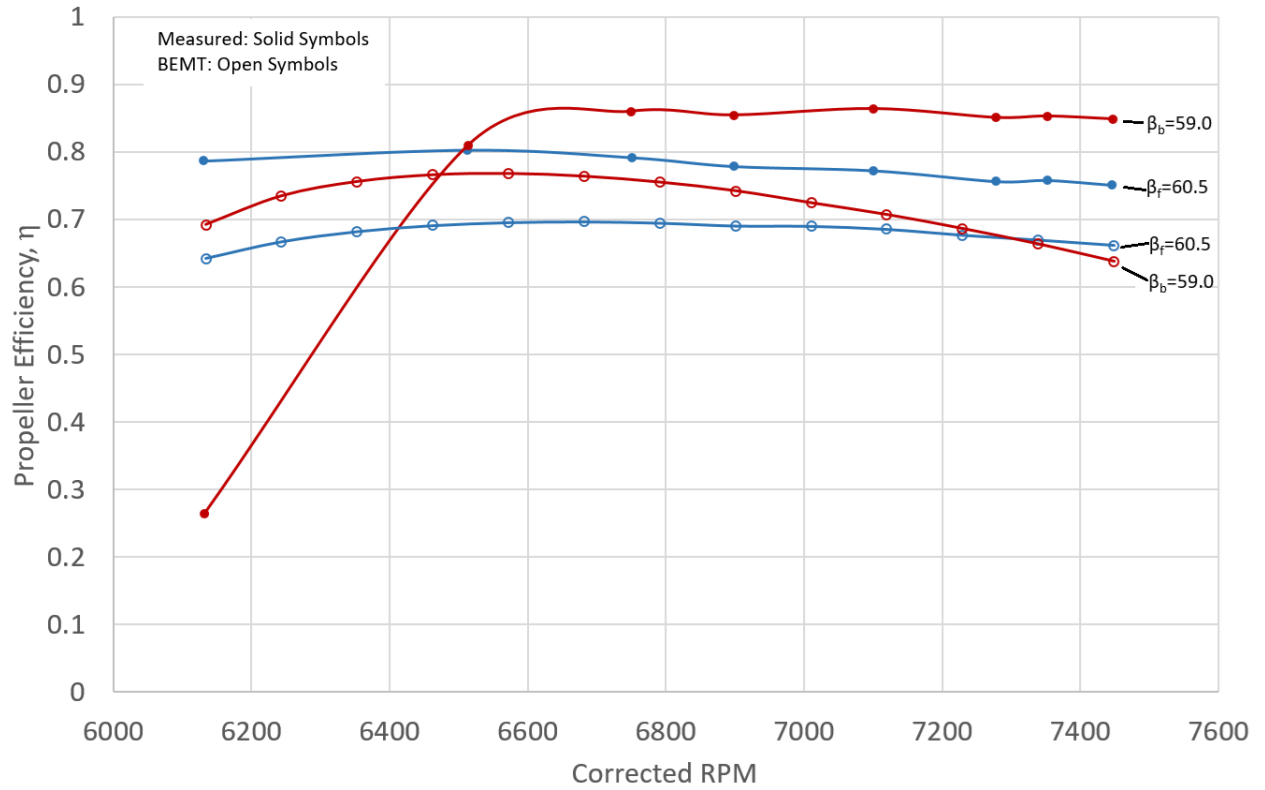


Figure 29. Propeller efficiency comparison of the modeled F31/A31 CROR of NASA TM 2014-216676 [36] over a range of corrected rpm at pitch angles $\beta_{f_{75}}/\beta_{b_{77.4}} = 60.5/59$ degrees and $M_{\infty}=0.73$. Closed symbols are measured, open symbols are BEMT prediction.

As can be seen in the comparisons, the thrust is over-predicted at the lower rpm settings, and becomes under-predicted as rpm increases. The torque is over-predicted at all rpm settings. The predicted thrust and torque discrepancies lead to gross under-prediction of the propeller efficiency as shown in Figure 29. The causes of the poor agreement between measurement and prediction are many. The thrust, torque and efficiency discrepancies most likely stem from deficient inflow and blade-interference modeling at high solidity and from oversimplified modeling of transonic airfoil section data. Nevertheless, the BEMT modeling of a CR rotor is close enough to warrant further development. Possibly, the model could be grounded at a few judiciously chosen measured base points to enable prediction of a complete CR performance map.

Conclusions

In this memorandum, the Blade Element Momentum Theory is explored for rotor systems of varying complexity. The premise of this effort was that improved blade geometrical fidelity and blade element aerodynamic modeling would lead to improved rotor thrust and power predictions. The principal conclusions are:

- 1.) Excellent agreement between measured and BEMT thrust, power and efficiency predictions was obtained on a two-blade propeller operating at freestream Mach numbers up to 0.66
- 2.) The agreement on the SR-3 eight-blade swept rotor was less than stellar but the trends between the measured and predicted results matched well. The difference between measured and modeled peak efficiency was roughly 3% but the peaks were at different advance ratios and therefore the modeling needs additional improvement (e.g., inclusion of nacelle effects).
- 3.) A framework for BEMT modeling of contra-rotating rotors has been established. Attempts at modeling the F31/A31 rotors using BEMT were reasonably successful but fell short of the desired accuracy goal because of the challenges of modeling rotor interference effects and transonic and low supersonic flow on the outboard portions of the rotor and other blade geometrical differences not modeled.
- 4.) At low freestream Mach numbers, the modeling fidelity was reasonably good. At Mach numbers approaching 0.8, the overall approach to aerodynamic modeling (i.e., NeuralFoil-based aerodynamic model corrected for compressibility coupled with the transonic lift-divergence surrogate model and similarity methods at supersonic Mach numbers) was not sufficient to model the changes in blade aerodynamic behavior caused by compressibility. A neural network tool similar to NeuralFoil but valid up to Mach 1.3, possibly built on results using Multi-element Streamline Euler Solver (e.g., MSES by Mark Drela of MIT), is desirable.

Acknowledgement

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